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On absolute moment-based upper bounds for L-moments

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ABSTRACT

A number of absolute moment-based upper bounds for Gini's mean difference are extended to general L-moments. Improvement of some bounds by alternative choice of centre for the absolute moments is explored. Different bounds are compared numerically. The distribution for which upper bounds for Gini's mean difference are attained is given. Extension is made to trimmed L-moments and hence to probability weighted moments.

1. Introduction

Let X follow a continuous distribution F on $\mathbb{R}$ with mean μ and, if finite, variance σ^2. Then, with due acknowledgement of the work of Sillitto (1969), Hosking (1990) defined the L-moments, λ_i, $i = 1, 2, \ldots$, of F, all of which exist provided μ does. The first of these is $\lambda_1 = \mu$. The second L-moment, a measure of spread that is one-half of Gini's mean difference $\mathcal{G}$, is defined by

$$\lambda_2 = \tfrac{1}{2}\mathbb{E}(|X_{2:2} - X_{1:2}|) = \tfrac{1}{2}\mathbb{E}(|X_1 - X_2|) = \tfrac{1}{2}\mathcal{G} = \int_0^1 F^{-1}(u)(2u-1)\,du. \tag{1}$$

For other L-moments, which are measures of skewness, "kurtosis", etc., see Section 2 below. The *Wikipedia* page "L-Moment" gives a useful overview of the L-moments and their uses.

Yin et al. (2023) provided absolute moment-based upper bounds for Gini's mean difference. Following Yin et al., since $\int_0^1 (2u-1) = 0$, we can write

$$\lambda_2 = \int_0^1 \{F^{-1}(u) - \mu\}(2u-1)\,du. \tag{2}$$

Then, take $p > 1$ and $q = p/(p-1)$ and apply Hölder's inequality to see that

$$\begin{aligned}\lambda_2 &\le \int_0^1 |F^{-1}(u) - \mu \,\|\, 2u-1|\,du \le \left\{\int_0^1 |F^{-1}(u) - \mu|^p\,du\right\}^{1/p} \left\{\int_0^1 |2u-1|^q\,du\right\}^{1/q} \\ &= \{\mathbb{E}(|X-\mu|^p)\}^{1/p}/(q+1)^{1/q} \equiv A_p^{1/p} P_q,\end{aligned} \tag{3}$$

say. This is Theorem 3.2 of Yin et al. (2023), although claimed there to pertain to $\mathcal{G}$ not λ_2; see also Cerone and Dragomir (2005). In particular, taking $p = q = 2$ and noting that $A_2 = \sigma^2$,

$$\lambda_2 \le \sigma/\sqrt{3}$$

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(Cerone and Dragomir, 2005; Eisenberg, 2005; Haye and Zizler, 2019; Yin et al., 2023).

In Section 2, we generalise (3) to other L-moments in (7). In Section 3, we generalise (3) and (7) to (10) and (11), respectively, by improving on the term $A_p^{1/p}$, and look at special cases thereof. A further idea of Yin et al. (2023) is employed and expanded upon in Section 4, resulting in the new upper bound (18). In Section 5, we look at the distributions which attain the upper bounds for λ_2. Extension of upper bounds to trimmed L-moments is made in Section 6, especially the general (21) and its special case (22); separate consideration of the first trimmed L-moment leads to its upper bound (23) and thence to a novel upper bound, (24), for probability weighted moments. Some brief concluding remarks constitute Section 7.

By way of further notation, write $\mu_i = \mathbb{E}(X^i)$, $i = 2, 3, \ldots$, and $A_r(c) = \mathbb{E}(|X - c|^r)$ so that $A_r \equiv A_r(\mu)$, $r > 0$. Moments are assumed to exist as and when necessary. Computations and figures in the article were produced with the assistance of *Maxima*.

2. Upper bounds for L-moments

Further L-moments (Hosking, 1990) are given by the skewness measure

$$\lambda_3 = \tfrac{1}{3}\mathbb{E}(X_{3:3} - 2X_{2:3} + X_{1:3}) = \int_0^1 F^{-1}(u)(6u^2 - 6u + 1)\,du, \tag{4}$$

the "kurtosis" measure

$$\lambda_4 = \tfrac{1}{4}\mathbb{E}(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}) = \int_0^1 F^{-1}(u)(20u^3 - 30u^2 + 12u - 1)\,du \tag{5}$$

and in general by

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k}\mathbb{E}(X_{r-k:r}) = \int_0^1 F^{-1}(u)P^*_{r-1}(u)\,du, \tag{6}$$

$r = 1, 2, \ldots$. Here, $P^*_{r-1}(u)$ is the $(r-1)^{\text{st}}$ shifted Legendre polynomial given by

$$P^*_{r-1}(u) = \sum_{k=0}^{r-1}(-1)^{r-1-k}\binom{r-1}{k}\binom{r-1+k}{k}u^k, \qquad 0 < u < 1$$

(*e.g.* Lima, 2022). Orthogonality of the shifted Legendre polynomials implies that, for $r = 2, 3, \ldots$, each of (4), (5) and (6) can, like (1), have a constant subtracted from $F^{-1}(u)$ without changing the value of λ_r. When that constant is μ, it follows that (3) can newly be generalised to

$$|\lambda_r| \le A_p^{1/p}\left\{\int_0^1 |P^*_{r-1}(u)|^q\,du\right\}^{1/q} \equiv A_p^{1/p} Q_{r,q}, \tag{7}$$

say.

Further explicit progress is available in the case $p = q = 2$ (when the Hölder inequality is the Cauchy–Schwarz inequality). The shifted Legendre polynomials are orthogonal but not orthonormal. In fact,

$$\int_0^1 |P^*_{r-1}(u)|^2\,du = \frac{1}{2r-1}$$

so that

$$|\lambda_r| \le \sigma/\sqrt{2r-1}. \tag{8}$$

Notice that these upper bounds decrease as r increases. This is in line with the fact that $|\lambda_r| \le \lambda_2$, $r = 3, 4, \ldots$, (Hosking, 1990) which yields the looser upper bounds $|\lambda_r| \le \sigma/\sqrt{3}$, $r = 3, 4, \ldots$.

The quantity $Q_{r,q}$ is not always explicitly available. Table 1 shows values for some choices of r and q, evaluated numerically in those cases where exact fractions are not given. The most important are probably those for integer p since the corresponding bounds involve integer absolute moments. Values are also given for integer q: these involve fractional absolute moments but have the advantage of requiring the existence of less high moments. Limiting cases are also given in the table. That for $p \to \infty$, $q \to 1$ seems less interesting – because $\lim_{p\to\infty} A_p^{1/p} = \sup(|X|)$ – than that for $p \to 1$, $q \to \infty$; the latter corresponds to

$$|\lambda_r| \le \mathbb{E}(|X - \mu|) \tag{9}$$

because $\lim_{q\to\infty} Q_{r,q} = 1$. Inequality (9) can be seen directly from (6) with μ subtracted because $0 < |P^*_{r-1}(u)| < 1$ for $0 < u < 1$. It is in Theorem 2.1 of Yin et al. (2023) for $r = 2$; see also Cerone and Dragomir (2005).

3. Improving on the term $A_p^{1/p}$

Notice newly that it is just as valid to replace μ in (2) and its higher order L-moment equivalents by any finite constant c, so that (3) and (7) become

$$\lambda_2 \le A_p(c)^{1/p}/(q+1)^{1/q} \tag{10}$$

Table 1
Selected values of $Q_{r,q}$ for $r = 2, 3, 4$.

p	q	$P_q = Q_{2,q}$	$Q_{3,q}$	$Q_{4,q}$
1	∞	1	1	1
4/3	4	$\left(\frac{1}{5}\right)^{1/4} \simeq 0.668740$	$\left(\frac{3}{35}\right)^{1/4} \simeq 0.541082$	$\left(\frac{241}{5005}\right)^{1/4} \simeq 0.468439$
3/2	3	$\left(\frac{1}{4}\right)^{1/3} \simeq 0.629961$	0.497488	0.424278
2	2	$\frac{1}{\sqrt{3}} \simeq 0.577350$	$\frac{1}{\sqrt{5}} \simeq 0.447214$	$\frac{1}{\sqrt{7}} \simeq 0.377964$
3	3/2	$\left(\frac{2}{5}\right)^{2/3} \simeq 0.542884$	0.418236	0.352942
4	4/3	$\left(\frac{3}{7}\right)^{3/4} \simeq 0.529685$	0.407716	0.344053
∞	1	$\frac{1}{2} = 0.5$	$\frac{2}{3\sqrt{3}} \simeq 0.384900$	$\frac{13}{40} = 0.345$

and

$$|\lambda_r| \le A_p(c)^{1/p}\, Q_{r,q}, \tag{11}$$

respectively.

When $p = 2$ (and $q = 2$), $A_2(c)$ is minimised by choosing $c = \mu$, so nothing is gained by this.

For $p \neq 2$, $A_p(c)$ is minimised by $c \neq \mu$, hence improving upon the upper bounds in (3) and (7). When $p = 1$ (and $q = \infty$), $A_1(c)$ is minimised by choosing $c = m$, where m is the median of F. So, bound (9) is improved to

$$|\lambda_r| \le \mathbb{E}(|X - m|)$$

(Yin et al., 2023, Theorem 2.2, when $r = 2$).

Improvement can also be made explicitly when $p = 4$. Then, in a neat exercise in algebraic manipulation performed by Blest (2003), the value c_4 that minimises $A_4(c)$ over c is given by what Blest calls the meson,

$$c_4 = \mu + 2\sigma \sinh\left\{\tfrac{1}{3}\sinh^{-1}(t)\right\} = \mu + \sigma\left\{\left(\sqrt{1+t^2}+t\right)^{1/3} - \left(\sqrt{1+t^2}-t\right)^{1/3}\right\} \equiv \mu + \sigma\mathcal{F}(t),$$

say. Here, $t = E(X-\mu)^3/(2\sigma^3)$ is one-half the usual coefficient of skewness. The corresponding minimised value of $A_4(c)$ is then given by

$$\mathcal{K}_4 \equiv \mathbb{E}\{(X - c_4)^4\} = A_4(c_4) = \sigma^4[\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}] \tag{12}$$

where κ is the classical kurtosis measure $\kappa = E(X-\mu)^4/\sigma^4$. Of course, $c_4 = \mu$ and $\mathcal{K}_4 = \sigma^4\kappa$ when the distribution of X has zero (classical) skewness, $s \equiv 2t$.

Something not noted by Blest (2003) is that the classical inequality between skewness and kurtosis, given in our notation by $\kappa \ge 1 + 4t^2$, implies that $\mathcal{K}_4 \ge \sigma^4[1 + 4t^2 - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}]$. Moreover, it is the case that $\mathcal{F}^3(t) = 2t - 3\mathcal{F}(t)$, so the inequality becomes

$$\mathcal{K}_4 \ge \sigma^4\{1 + \mathcal{F}^2(t)\}^3 \ge \sigma^4.$$

Blest (2003) proposed $\mathcal{K}_4$ as a "new measure of kurtosis adjusted for skewness" and the proposal has been pursued elsewhere since. Here, however, we use it just to decrease upper bounds. When $p = 4$ (and $q = 4/3$), we have

$$|\lambda_r| \le A_4(c_4)^{1/4}\, Q_{r,4/3} = \sigma\left[\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}\right]^{1/4} Q_{r,4/3} \equiv \sigma\, U_1(s,\kappa), \tag{13}$$

say. When $r = 2$, (13) reduces further to

$$\lambda_2 \le \left(\tfrac{3}{7}\right)^{3/4} \sigma\left[\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}\right]^{1/4}. \tag{14}$$

These bounds improve on (7) and (3), respectively, when $p = 4$ and skewness is present. Indeed, Fig. 1 shows the ratio of the upper bound in (14) to that in (3) as a function of $\kappa > 1$ when $p = 4$ and the skewness term $2t$ is at its greatest possible value, $\kappa - 1$, namely $\{1 + \mathcal{F}^2(\sqrt{\kappa - 1}/2)\}^{3/4}/\kappa^{1/4}$. The improvement is clear, reducing to a minimum value of about 0.825379 when $\kappa \simeq 11.1962$ and then increasing again, tending very slowly to 1 as $\kappa \to \infty$.

4. Improving bounds using the Cauchy-Schwarz inequality when $p = 3$

A further idea of Yin et al. (2023) is to apply the Cauchy–Schwarz inequality to A_3, namely $A_3^2 \le A_2 A_4 = \sigma^6\kappa$, to obtain a different upper bound for λ_2 when κ exists. For general $|\lambda_r|$, (7) with $p = 3$ gives rise to

$$|\lambda_r| \le \sigma\kappa^{1/6}\, Q_{r,3/2} \equiv \sigma U_2(\kappa), \tag{15}$$

say. In comparison with (7) with $p = 4$, (15) is the tighter upper bound for all $\kappa > (Q_{r,3/2}/Q_{r,4/3})^{12} \equiv R_r$, say. Yin et al. (2023) note that when $r = 2$ this is the case for κ greater than a little above 1.34, while the values given for $Q_{r,3/2}$ and $Q_{r,4/3}$ in Table 1

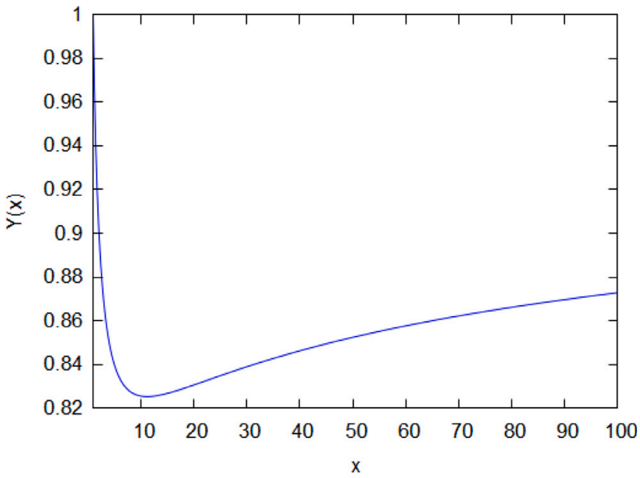

Fig. 1. The function $Y(x) \equiv \{1 + \mathcal{F}^2(\sqrt{x-1}/2)\}^{3/4}/x^{1/4}$ as a function of $x = \kappa$.

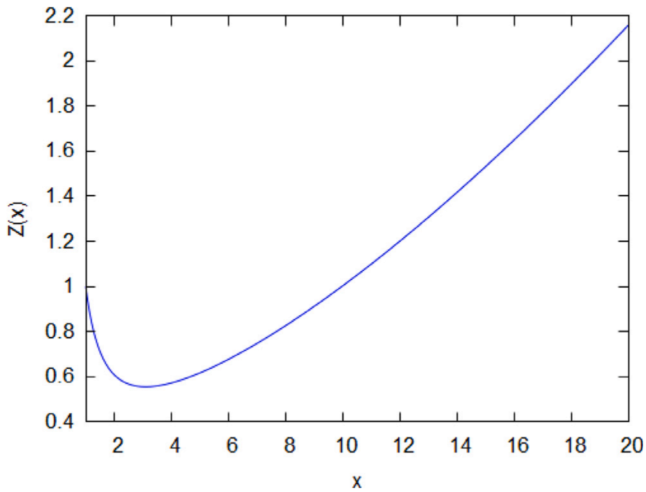

Fig. 2. The function $Z(x) \equiv \{1 + \mathcal{F}^2(\sqrt{x-1}/2)\}^9/x^2$ as a function of $x = \kappa$.

show that this value changes (increases) only in the second decimal place for $r = 3, 4$. In rough calculations to follow, we shall take $R_r \simeq 1.35$ to (approximately) cover all three cases. Note, in any case, that for almost all practical distributions, upper bound (15) is better than upper bound (7) with $p = 4$.

However, inequality (13) makes inroads into the superiority of inequality (15) when skewness is present. Then, (15) is the smaller whenever $\left[\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}\right]^3/\kappa^2 > R_r$. In the case of most extreme skewness, the left-hand side of this inequality amounts to $\{1 + \mathcal{F}^2(\sqrt{\kappa - 1}/2)\}^9/\kappa^2$, which is shown in Fig. 2. Then, (13) is the smaller – to a minimum of 0.555598 at $\kappa = 3.104136$ – until reaching $R_r \simeq 1.35$ at $\kappa \simeq 13.39$.

The Cauchy–Schwarz inequality continues to apply to $A_3(c)$, namely $A_3^2(c) \le A_2(c)A_4(c) \equiv P(c)$, say, for $c \ne \mu$ and, in particular, for $c = c_4$. It does not seem possible to explicitly minimise $P(c)$ over c. But, since $A_2(c_4) = A_2 + (c_4 - \mu)^2 = \sigma^2\{1 + \mathcal{F}^2(t)\}$ and with $A_4(c_4)$ given in (12),

$$P(c_4) = A_2(c_4)A_4(c_4) = \sigma^6 \left(\{1 + \mathcal{F}^2(t)\}\{\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}\}\right).$$

This is smaller than $P(\mu) = \sigma^6\kappa$ whenever

$$\kappa < 3\{1 + \mathcal{F}^2(t)\}\{2 + \mathcal{F}^2(t)\} \equiv u(s), \tag{16}$$

say, the right-hand side of which is at least 6. For κ satisfying (16), (15) can be replaced by

$$|\lambda_r| \le \sigma \left(\{1 + \mathcal{F}^2(t)\}[\kappa - 3\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\}]\right)^{1/6} Q_{r,3/2} \equiv \sigma U_3(s, \kappa), \tag{17}$$

say. Comparing this with (13), (17) is smaller whenever

$$\kappa > R_r + (R_r + 3)\mathcal{F}^2(t)\{2 + \mathcal{F}^2(t)\} \equiv \ell_r(s),$$

say. Although very much in favour of (17) when $s = 0$, increasing skewness yields an increasing range of superiority of (13) over (17).

In summary, when absolute moments up to the fourth exist,

$$|\lambda_r| \le M_r \equiv \sigma \min\{U_1(s,\kappa), U_2(\kappa), U_3(s,\kappa)\} = \sigma \begin{cases} U_1(s,\kappa), & 0 < \kappa \le \ell_r(s), \\ U_2(\kappa), & \ell_r(s) < \kappa \le u(s), \\ U_3(s,\kappa), & u(s) < \kappa. \end{cases} \tag{18}$$

Note that $\ell_r(s) \ge R_r \simeq 1.35$ and $u(s) \ge 6$, meaning that the components of M_r can be thought of as corresponding to low, moderate and high kurtosis, respectively.

5. Attaining bounds for λ_2

The distribution for which upper bounds are attained – and therefore the bounds are sharp – appears to be available only for λ_2 (or equivalently for Gini's mean difference).

Suppose, for convenience, that the median of F is zero. Then,

$$\lambda_2 = \int_0^1 F^{-1}(u)(2u-1)\,du = \int_0^1 |F^{-1}(u)|\,|2u-1|\,du \le A_p(0)^{1/p}/(q+1)^{1/q}. \tag{19}$$

Now, Hölder's inequality becomes equality when, in this case, $|F^{-1}(u)| \propto |2u-1|^{q-1} = |2u-1|^{1/(p-1)}$. This distribution is the reflected power-law distribution with parameter $p-1>0$ on $-1<x<1$ which has density

$$f(x) = \frac{1}{2(p-1)}\,|x|^{p-2},$$

distribution function

$$F(x) = \frac{1}{2}\left\{1 - \mathrm{sgn}(x)\,|x|^{p-1}\right\}$$

and quantile function

$$F^{-1}(u) = \mathrm{sgn}(2u-1)\,|2u-1|^{1/(p-1)}.$$

Also, for this distribution $A_p(0) = \mathbb{E}(|X|) = (p-1)/(2p-1)$ and the right-hand side of (19) is therefore $(p-1)/(2p-1)$ too.

Note that the above applies only to (10) with c equal to the centre of the distribution, that is, the median which is also the mean because of the symmetry of the reflected power-law distribution, the latter implying relevance also to (3).

Unfortunately, a similar analysis does not apply for $|\lambda_r|$, $r = 3, 4, \ldots$. This is because of the non-monotonicity of $P^*_{r-1}(u)$. It means both (i) that the second equals sign in (19) remains an inequality and (ii) that equality in Hölder's inequality would need $|F^{-1}(u)| \propto |P^*_{r-1}(u)|$, which does not give a valid quantile function.

6. Extension to trimmed L-moments

Extension to trimmed L-moments (Elamir and Seheult, 2003) is now considered. Trimmed L-moments are robustified versions of L-moments obtained by regarding a sample of size n as having been obtained by trimming the lowest s order statistics and the uppermost t order statistics from a sample of size $n+s+t$, $s,t = 0,1,\ldots$. Here, s and t denote different quantities than they did earlier in the article, but no confusion should be caused. For $r = 1, 2, \ldots$, the trimmed L-moments turn out to be defined by

$$\lambda_r^{s,t} \equiv k_r^{s,t} \int_0^1 F^{-1}(u)\,u^s(1-u)^t J_{r-1}^{*(t,s)}(u)\,du \tag{20}$$

where

$$k_r^{s,t} = \frac{(r-1)!(r+s+t)!}{r(r+s-1)!(r+t-1)!} > 0$$

and $J_{r-1}^{*(t,s)}(u)$ is the shifted Jacobi polynomial given by

$$J_{r-1}^{*(t,s)}(u) = \sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1+t}{k} \binom{r-1+s}{k+s} u^k (1-u)^{r-1-k}, \qquad 0<u<1$$

(Hosking, 2007). Of course, $\lambda_r = \lambda_r^{0,0}$ and the shifted Jacobi polynomial reduces to the shifted Legendre polynomial when $s = t = 0$.

Table 2
Values of the right-hand side of (22) for $\sigma = 1$, $s = 0, \ldots, 3$ and $t = 0, \ldots, s$.

$t \downarrow$ $s \rightarrow$	0	1	2	3
0	$\sqrt{\frac{1}{3}} \simeq 0.577350$	$\sqrt{\frac{3}{10}} \simeq 0.547723$	$\sqrt{\frac{12}{35}} \simeq 0.585540$	$\sqrt{\frac{25}{63}} \simeq 0.629941$
1		$\sqrt{\frac{6}{35}} \simeq 0.414039$	$\sqrt{\frac{10}{63}} \simeq 0.398410$	$\sqrt{\frac{25}{154}} \simeq 0.402911$
2			$\sqrt{\frac{10}{77}} \simeq 0.360375$	$\sqrt{\frac{35}{286}} \simeq 0.349825$
3				$\sqrt{\frac{140}{1287}} \simeq 0.329818$

Similarly to the shifted Legendre polynomials, orthogonality of the shifted Jacobi polynomials with respect to $u^s(1-u)^t$ implies that, for $r = 2, 3, \ldots$, a constant c can be subtracted from $F^{-1}(u)$ without changing the value of $\lambda_r^{s,t}$. Thus, from (20), we get the new bounds

$$|\lambda_r^{s,t}| \le k_r^{s,t} A_p(c)^{1/p} \left\{ \int_0^1 u^{qs}(1-u)^{qt} \, |J_{r-1}^{*(t,s)}(u)|^q \, du \right\}^{1/q} \equiv k_r^{s,t} A_p(c)^{1/p} J_{r,s,t,q}, \tag{21}$$

say.

An explicitly calculable special case is that of bounding the trimmed second L-moment by a multiple of σ, that is, $r = p = q = 2$. Now, $k_2^{s,t} = \binom{s+t+2}{s+1}/2$ and $J_1^{*(t,s)}(u) = (s+t+2)u - (s+1)$, so

$$\begin{aligned}
J_{2,s,t,2}^2 &= \int_0^1 u^{2s}(1-u)^{2t}\{(s+t+2)u - (s+1)\}^2 \, du \\
&= (s+t+2)^2 B(2s+3, 2t+1) - 2(s+1)(s+t+2)B(2s+2, 2t+1) + (s+1)^2 B(2s+1, 2t+1) \\
&= \frac{\Gamma(2s+1)\Gamma(2t+1)}{\Gamma(2(s+t+2))} \big\{ 2(s+t+2)^2(s+1)(2s+1) - 2(s+1)(s+t+2)(2s+1)(2s+2t+3) \\
&\qquad + 2(s+1)^2(2s+2t+3)(s+t+1) \big\} \\
&= \frac{2(s+1)(2s)!(2t)!}{(2(s+t)+3)!} \big\{ (s+t+2)^2(2s+1) - (s+t+2)(2s+1)(2s+2t+3) + (s+1)(2s+2t+3)(s+t+1) \big\} \\
&= \frac{2(s+1)(t+1)(s+t+1)(2s)!(2t)!}{(2(s+t)+3)!}.
\end{aligned}$$

It follows that

$$\lambda_2^{s,t} \le \sigma \frac{(s+t+2)!}{2(s+1)!(t+1)!} \sqrt{\frac{2(s+1)(t+1)(s+t+1)(2s)!(2t)!}{(2(s+t)+3)!}}. \tag{22}$$

This upper bound is symmetric in s and t and reduces to $\sigma/\sqrt{3}$ when $s = t = 0$, both of which are as they should be. When $s = t$, the bound can be shown to decrease in s, tending slowly to 0. But when t is fixed, the bound first decreases and then increases. Values of the bound for $\sigma = 1$, $s = 0, \ldots, 3$ and $t = 0, \ldots, s$ are given in Table 2.

The first trimmed L-moment,

$$\lambda_1^{s,t} \equiv \frac{1}{B(s+1, t+1)} \int_0^1 F^{-1}(u) u^s (1-u)^t \, du,$$

also has an interesting upper bound. We have

$$\begin{aligned}
\lambda_1^{s,t} &\le \frac{1}{B(s+1,t+1)} \int_0^1 |F^{-1}(u)| u^s(1-u)^t \, du \le \frac{1}{B(s+1,t+1)} \left\{ \int_0^1 |F^{-1}(u)|^p \, du \right\}^{1/p} \left\{ \int_0^1 u^{qs}(1-u)^{qt} \, du \right\}^{1/q} \\
&= \left\{ \mathbb{E}(|X|^p) \right\}^{1/p} \frac{\{B(qs+1, qt+1)\}^{1/q}}{B(s+1,t+1)}.
\end{aligned} \tag{23}$$

When $s = t = 0$, this reduces to $\mathbb{E}(X) \le \mathbb{E}(|X|^p)^{1/p}$, $p > 1$, as is known via the Lyapunov inequality. As $p \to \infty$, $q \to 1$, we get $\lambda_1^{s,t} \le \sup(|X|)$; when $p \to 1$, $q \to \infty$, we get

$$\lambda_1^{s,t} \le \frac{\mathbb{E}(|X|)}{B(s+1,t+1)} \frac{s^s t^t}{(s+t)^{s+t}}.$$

Recall that probability weighted moments (Greenwood et al., 1979) are defined by $M_{w,s,t} \equiv \mathbb{E}[X^w\{F(X)\}^s\{1-F(X)\}^t]$. Since $\lambda_1^{s,t}$ is proportional to a special case of probability weighted moments – with just a powering up of $F^{-1}(u)$ required – similar bounds, changing only in the absolute moment term, are available for them, namely

$$M_{w,s,t} \le \left\{ \mathbb{E}(|X|^{wp}) \right\}^{1/p} \{B(qs+1, qt+1)\}^{1/q}. \tag{24}$$

No such bound appears to be previously available in the literature.

7. Concluding remarks

All the quantities bounded in this article are L-functionals in the sense of Hössjer and Karlsson (2023), that is, quantities of the form $\int_0^1 F^{-1}(u)g(u)du$. Hölder's inequality immediately allows these to be upper bounded by $\left\{\mathbb{E}(|X|^p)\right\}^{1/p} \{\int_0^1 |g(u)|^q du\}^{1/q}$, for $p > 1$, $q = p/(p-1)$. The bound for probability weighted moments, (24), is of this type, as are bounds for expectations involving certain order statistics given in Arnold (1985). L-moments – including Gini's mean difference – and their trimmed counterparts are L-functionals with the additional property that $\int_0^1 g(u)du = 0$. It is this that has allowed us to tighten bounds by replacing raw absolute moments of X by centred versions thereof, and to attempt to capitalise optimally on this observation.

Data availability

No data was used for the research described in the article.

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