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COMMENTARY

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Who Puts the ‘Active’ into ‘Active Learning’?

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Abstract Learning is here considered to have taken place when someone has developed the habit, propensity, and disposition to attend productively to things not previously noticed, and in ways not previously experienced, to do with some specific and particular content. ‘Active learning’ sounds like a tautology, but was introduced as a contrast to the more passive activity of students sitting in lectures listening and transcribing mathematics written on a board or screen onto their own paper. The stance taken here is that effective and efficient learning involves active engagement in activity, but includes enculturation through being in the presence of a relative expert[1] who themselves is manifesting mathematical thinking, not simply passing on the records of the results of previous mathematical thought. Such ‘passivity’ does not necessarily require intention. Following Bennett[2] actions are here taken to involve three agents or impulses: initiating, responding, and reconciling or mediating. All three agents are thus active, but in different ways. Interactions intended to contribute to learning are considered to be actions, and so involve three agents: learner, teacher (in some manifestation), and mathematical content, all within a culture or ethos. Since there are six different ways in which the triple of agents can be assigned to the triple of impulses, six different modes are possible. Analysing these modes sheds light on different ways in which learning could be said to be ‘active’. Activity takes place within a mode of interaction. Again following Bennett, effective activity is here taken to require appropriate relationships among the gap between current state and intended goal, the resources available, and the tasks set.

[1]Vygotsky (1978) pointed out that ‘higher psychological processes’ are first encountered in others.

[2]Bennett (1993); see also Shantock Systematics Group (1975)

Résumé On considère qu’il y a apprentissage lorsque quelqu’un prend l’habitude, qu’il a la propension ainsi que la disposition à s’occuper de façon productive de choses qu’il n’avait pas remarquées auparavant, et d’une manière qu’il n’avait jamais expérimentée, et qui ont un rapport avec un contenu spécifique et particulier. L’expression « apprentissage actif» semble être une tautologie, mais elle a été

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instaurée pour distinguer ce processus de l'activité plus passive d'élèves assis dans des cours magistraux, écoutant et transcrivant sur leurs propres feuilles de papier des mathématiques écrites au tableau ou à l'écran. Notre estimons qu'un apprentissage efficace et performant implique un engagement actif dans l'activité, mais il inclut l'enculturation par la présence d'un expert relatif[1] qui manifeste lui-même la pensée mathématique, et ne se contente pas de rapporter des résultats d'une pensée mathématique antérieure. Une telle «passivité» ne requiert pas nécessairement une intention. À l'instar de Bennett[2], les actions sont ici considérées comme impliquant trois agents ou impulsions: l'initiation, la réponse et la réconciliation ou la médiation. Les trois agents sont donc actifs, mais de manière différente. Les interactions destinées à contribuer à l'apprentissage sont considérées comme des actions et elles supposent donc la présence de trois agents: l'élève, l'enseignant (sous une forme ou une autre) et le contenu mathématique, le tout dans le cadre d'une culture ou d'une éthique. Comme il existe six façons différentes d'affecter les trois agents à chacune des impulsions, six modes distincts sont possibles. L'analyse de ces modes met en lumière les différentes façons dont l'apprentissage peut être qualifié d'«actif». L'activité se déroule dans un mode d'interaction. Toujours selon Bennett, une activité efficace exige des relations appropriées entre l'écart qui existe entre l'état actuel des choses et l'objectif, les ressources disponibles et les tâches fixées.

[1]Vygotsky (1978) a souligné que les « processus psychologiques supérieurs » sont d'abord rencontrés chez les autres.

[2]Bennett (1993); voir aussi Shantock Systematics Group (1975)

Keywords Modes of interaction · Pedagogy · Active learning · Receptive learning · Pedagogic actions

Introduction

What is signalled by putting the adjective 'active' in front of learning? Isn't all learning, in a sense, active? And what is excluded by the adjective? Does inactive, passive, or receptive learning even make sense?

For generations, many mathematicians have implicitly and explicitly taken the stance that 'the aim of a mathematics teacher is to express the mathematical ideas clearly'. When learners do not appear to grasp what they have been told or shown, then either they are not studying effectively or the lecturer (text) has not been maximally clear. This aligns with Henri Poincaré's moan:

> That not everyone can invent is nowise mysterious. That not everyone can retain a demonstration once learned may also pass. But that not everyone can understand mathematical reasoning when explained appears very surprising when we think of it. (Poincaré, 2012, p. 383)

Poincaré had a strong sense of mathematics being entirely rational and logical, and so within the grasp of everyone. This resonates with the Meno dialogue of Plato in which the slave boy cannot but assent to each statement put to him.[1] For Plato, this was evidence of previous lives; for us, it raises questions about what constitutes learning. Sometimes learners develop an assentive rather than an assertive stance (Mason, 2009), acting as if assent is sufficient to produce learning, whereas encouragement to assert, to express their comprehension, is much more effective. Poincaré seems to accept assenting because he goes on to say that the fact that people.

> should fail to understand the demonstrations [proofs] expounded to them, that they should remain blind when they are shown a light that seems to us to shine with a pure brilliance, it is this that is

[1] See for example Hamilton & Cairns (1961, pp. 353–384).

altogether miraculous. … We have here a problem that is not easy of solution, but yet must engage the attention of all who wish to devote themselves to education. (Poincaré, 1952)

Of course, anyone who has stood in front of learners and then marked their examination papers knows that this is, and probably always has been, a major source of frustration and amazement: when learners are examined, they do not seem fully to appreciate and comprehend the mathematics they have been 'taught' (i.e. been exposed to). As a teaching assistant in the 1960s, I worked for a professor who explicitly expected learners to display at least a 75% grasp of what he taught them. None of his assistants could get him to realise how unrealistic he was.

It is certainly possible to learn a great deal simply from being in the presence of an expert. For example, I learned to wire a house and to hang doors simply by watching my father and handing him the tools that he asked for. I learned to construct mathematical proofs by reading proofs and watching proofs being set out by lecturers. Indeed, there is a long history of apprentices learning this way, by being in the presence of expertise (see for example Lave, 1977). If any meaning is to be ascribed to passive or receptive learning, this would be it.

However, learning by merely being in the presence of expertise is haphazard: what is learned, and how thoroughly, depends greatly on the sensitivity, disposition, and commitment of learners. Learning can be enriched and made less haphazard by the nature and quality of interactions, pedagogic or otherwise, ranging from explanations offered by way of running commentary, to being questioned and urged to try for oneself. Whatever the practices and pedagogy, whatever the ethos, whatever the rubric in which practices are embedded, the actions enacted, and even some of the pedagogic practices themselves, are likely to be absorbed and internalised. In other words, a great deal more is learned than teachers think they are teaching.

For learners of mathematics, a culture of valuing only correct answers can alienate some while attracting others. A culture of valuing both the making of conjectures and the modifying of conjectures, exposing the creative aspects of mathematical thinking as well as the rigour of justifying conjectures, is likely to attract learners who value self-expression and creativity. Adopting the slogan 'active learning', which seems to have emerged in the 1990s, is presumably intended to stress the active engagement of each learner, and celebrates Piaget's (1970) notion of *genetic epistemology*, in which each learner is seen as an active agent, with learning arising from learner activity. Put another way, 'Mathematics is not a spectator sport' (Davis & Hersh, 1981). And yet, there is much to be gained simply by being in the (actual or virtual) presence of a relative expert.

Attending to the Mystery

In order to probe forms or ways of learning mathematics, some agreement is needed as to what is meant by mathematics and by learning.

Philosophers and educators have long debated what constitutes mathematics. For example, Pappus (fourth century BCE) describes a debate between Menaechmus (nephew of Plato) who took the view that mathematics was problem solving, and Speusippus (student of Eudoxus) who took the view that mathematics is about theory development (Bernard et al., 2014, p. 42). If, in common with Speusippus, mathematics is identified with Popper's third world of library (and now internet) collections of written mathematics (Popper, 1972), then a minimal pedagogy is to rehearse those texts in front of learners. If on the other hand mathematics is seen as an activity, even a way of being, then an appropriate pedagogy would engage learners in mathematical activity themselves, without falling prey to the false dichotomy of 'either you tell them, or they try to discover it for themselves'. Neither of these are helpful or effective as sole practices, and the variation in interpretations of what they mean in practice has been the source of endless debate.[2]

The fact that library-mathematical records are the product of activity suggests that learning mathematics is usefully taken to mean learning to engage in mathematical thinking, spurred and informed by what is written (and said) (Halmos, 1980). This of course merely passes the question on to *mathematical thinking*. Reflection on experience of both learning and teaching indicates that learning involves learners developing the use of their natural powers in ways which enable them to attend to details that are relevant, and to attend to these in relevant ways. By 'relevant ways' is meant in ways which have proved fruitful to others in the past, or in other words, as advocated by the curriculum (Mason, 2008).

A naïve approach might assume that whatever the teacher or text is saying is what learners are attending to. Even if this were the case, and a moment's reflection shows that it is rarely true, it also matters greatly whether teacher and learner are attending in the same way. For example, a teacher may be attending to relationships between specific details of an expression on a board or in a text, while the learners may be struggling to discern which detailed items are being referred to. As another example, when a teacher 'works through' a worked example (with or without inviting suggestions from learners), the teacher experiences the particular as an instance of a generality, a method, and as the application of expounded theorems. By contrast, the learners, not yet having any sense of generality, attend to the specific details of the worked example. When learners and teacher attend to different things, and even when they attend to the same thing but differently, communication between them is likely to be impoverished.[3]

Throughout this commentary, the term *teacher* includes, in addition to a more knowledgeable live person, a diagram, worksheet, or problem sheet, a text, and audio or video recording: in short, anything prepared by a teacher for use by learners.

Learning as (Inter)Action

I take pedagogy to consist of specific actions, usually but not always initiated by the teacher, which involves the learner, specific mathematical content, and the teacher, and which takes place within a developing ethos.

In mathematics, the importance of both culture and content might be captured in the slogans 'Being mathematical with and in front of learners' (Mason, 2008) and 'trying to do for learners only what they cannot yet do for themselves'. Even when sociological attention was diverted from the notions of near and far transfer of learning (Detterman & Sternberg, 1993; Lave & Wenger, 1991; Seeley-Brown et al., 1989; Watson & Winbourne, 2008) to acknowledge the situated aspect of learning (Lave, 1988) in order to explain why learners faced with a novel situation do not make use of what they have been recently taught, the importance of engaging in the practice as well as being in the presence of others was under-represented.

Participation is the key element (Halmos, 1994). However, a pedagogic action can be enacted by or with some of those present, while others remain indifferent, detached, and untouched. For example, mindlessly copying notes from a board or screen, mindlessly watching a video animation going by, or half-heartedly watching things happen around them do not constitute participating in the action. There is always a contract between teacher and learner,[4] usually implicit, occasionally more or less explicit. Its usual form is that learners assume that reasonable attempts at the tasks they are assigned will mean that the intended learning will take place. Some learners are satisfied with copying other people's solutions,

[2] See for example en.wikipedia.org/wiki/Math_wars#:~:text=Math%20wars%20is%20the%20debate,and%20subsequent%20development%20and%20widespread.

[3] For details about forms of attention, see Mason (2008).

[4] Known as the *contrat didactique* (Brousseau, 1997).

showing that they have an impoverished sense of even the most minimal of contracts. The teacher is expected to design and assign tasks which will produce the desired learning. Associated with such a contract is an implicit tension (Brousseau, 1997; Mason, 2002, p. 17): the more precisely the teacher indicates the behaviour expected, the easier it is for the learners to display that behaviour without generating it for and from themselves.

Structurally, it can be helpful to distinguish six different modes of interaction.

Exposition

Traditional mathematics teaching has been described as 'the sage on the stage' (Davis & Hersh, 1981; King, 1993) in which a teacher expounds to an audience of learners (or in the case of seminars, colleagues). But even here there can be significant variations in quality. We have all attended lectures in which we, 'ever more came out by the same door wherein [we] went', none the wiser,[5] in contrast to lectures in which we have felt drawn into the lived experience of the presenter.

In quality exposition, the teacher initiates, the learner responds, and mathematics is what mediates between the two, holding them together. Sometimes the presence of the learner(s) (both imminent and actual) enables the presenter to make contact with the content in a fresh manner. Perhaps a fresh perspective, an insight as to possible epistemological obstacles (Bachelard, 1938), or a different (re)presentation becomes available and is enacted. Both what the presenter is attending to and how they are attending to it becomes available to those in the audience who engage.

Successful exposition requires the expositor to be trying to draw the audience into their world, but it also requires the audience to be responsive, to allow themselves to be drawn into that world. This means that the content being communicated has to be suitable both for the expositor and for the audience: reading out a text one does not oneself comprehend is unlikely to draw much of the audience in!

Traditional mathematics teaching follows exposition with *worked examples* followed by *exercises*, and then *assessment*, leading to the slogans 'reverse dentistry: fill-em and drill-em' and 'I do, We do, You do' as descriptions of the underlying pedagogy.

Worked Examples and Exercises

Even the most ancient of pedagogic texts, consisting of problems and solutions, tend to end with a statement such as 'thus is it done', or 'Do thou likewise' (Gillings, 1972/1982). The author is attending to the general while instantiating it in the particular. Somehow the learner is expected to detect what is general in the solution, what can be varied, and what must remain invariant, constituting the 'method'. Several instances can be helpful (Watson & Mason, 2006), and this is the subject of uses of the *principle of variation*, specifically under the heading of *procedural variation* (Jacques, 2023; Huang & Lee, 2016; Gu et al., 2017).

A detailed study of the role of worked examples (Chi & Bassok, 1989; Chi et al., 1989, 1994) has revealed that what really matters is not so much what to do at each step of a procedure, but how you know what to do at each step. What brings the next mathematical action into presence, to be enacted? This is part of the self-explanation, personal narrative, or inner incantation available to the learner in the midst of carrying out a task such as resolving an exercise problem, for themself.

Exercise seems to be a misunderstood action. There is little or no use in a teacher advocating, even demanding, the completion of a set of exercises if the action does not emerge as or resonate with a desire

[5] Stanza XXVII of Fitzgerald's translation of Omar Khayyam's Rubaiyat.

from within the learner. The rise in the use of personal trainers for motivation is evidence of this, where an initial but unsustainable desire for exercise is outsourced so as to be repeated, reinforced, and amplified. But this requires and depends upon subordination of will. In fact, for effective exercise, the desire to exercise is experienced as an arising within the learner, but coming, as it were, from the mathematical content itself. It usually arises as a desire to seek facility, competence, even mastery. The expertise of the teacher has been employed to create suitable exercises, engaging with which will efficiently and effectively lead the learner to mastery. This entails careful consideration of the sequence of tasks, what is varied, and what remains the same. It really involves treating a set of related exercises as a single mathematical object (Watson & Mason, 2006). As an action, true exercising is initiated by the mathematical content with the learner responding, mediated by the exercises chosen or constructed by the teacher.

It can be so tempting to construct a sequence of exercises which are intended to reproduce the stages in a pleasurable personal exploration which ended with a sense of 'and I could teach that'. When the experience of an expert, their awareness, is turned into a sequence of instructions to learners as to what to do at each stage, the learners are not likely to experience any of the creativity, the exploratory and constructive nature of the original exploration. *Transposition didactique* (Chevellard, 1985) is a useful label for the transformation that takes place when expert awareness is transformed into instructions for learners.

Again, to be effective, the learner needs to treat the exercise as a learning opportunity. They have to do more than simply try to resolve each task in turn. Rather, they need to look for what is varying and what invariant. Again, it is a matter of what is being attended to, and in what way. Learners need to construct their own narrative as to what the exercise sequence is revealing, and to imagine themselves carrying out the procedure again in the future. As Poincaré (2012, p. 44) pointed out, and many have discovered, it is not the specific objects but the relationships between them that mathematics studies.[6]

A further consideration concerns learner expectations, and the difference between an outer task and an inner task (Tahta, 1981; Mason & Johnston-Wilder, 2004, 2006; Mason & Davis, 1989). If the learner knows in advance that the result of working on a certain set of tasks will be that they will encounter a specified mathematical theme or a particular mathematical concept, they are likely to be on the alert for it, to look out for it. Consequently, they may not experience the arising and gradual getting-a-sense-of the concept or procedure which is eventually brought to articulation. It seems that the latter is likely to be more deeply felt, more accessible in future, than superficially encountering some ideas or constructs.

One of the extraordinary features of historical pedagogy is that although mathematicians depend on having access to an extensive space of examples of mathematical objects, it is only relatively recently that discussion of examples appears in the literature (Michener, 1978; Watson & Mason, 2005; Sangwin, 2005, 2006) about how learners can be supported in constructing and stimulated to construct their own examples of mathematical objects meeting specified constraints, yet this is one of the most powerful ways of probing learner comprehension. Care is of course needed: absence of evidence of comprehension or facility cannot be taken as evidence of absence.[7] For the learner, whenever a statement is encountered which is not completely familiar, it helps to have accessible a space of examples, together with various construction techniques, on which one can instantiate the generalities and so appreciate their import.

Examining

As an action, submitting oneself to examination is ideally initiated by the learner, wishing to check whether their personal criteria, their sense of appreciating and comprehending, match the expectations of the expert, the examiner. Thus ideally, learners initiate this when they are ready, as happens in some European countries. The mathematics mediates and the teacher-assessor responds. So not simply the mathematical content but

[6] But 'poetry is concerned with connections between things' (Fry, 2005, p. 124).

[7] Origins unknown, often referred as an archaeologists dictum. See en.wikipedia.org/wiki/Evidence_of_absence.

the sense of comprehending is what brings the learner into contact with the examiner. Unfortunately the fact of large classes has institutionalised examinations so that the learner usually knows when the examination will take place even before the course begins. The result requires learning to take place within a pre-specified time frame, when it may be quite different for different learners.

Interlude

It is perfectly possible for learners to be subjected to indifferent exposition, recommended exercise, and enforced examination, without any sense of creativity or opportunity to take initiative. When learners are attracted to engage more fully, there are some opportunities to experience a richer side of mathematics. But even more is possible.

Interpreting pedagogy as a collection of actions to be enacted, each action seen as consisting of the interplay between the three impulses stemming from the teacher, the learner(s), and the mathematical content, suggests that there are three further possible actions (six ways to assign teacher, student, and mathematical content to the three roles of initiating, mediating, and responding). Furthermore, it is likely that encountering a mixture of all six possible actions might enrich learner experience beyond what is available from the three traditional interactions.

Exploration and Investigation

Exploration arises when the learner initiates further activity, guided by suggestions from the teacher, and it provides fodder for constructive discussion with and comment by the teacher. For example, John Wallis (1616–1703) was challenged by Pierre de Fermat (1601–1665) to solve various number theory problems. Wallis developed a 'method of investigation' consisting of systematic consideration of particular examples from which he recognised, articulated and conjectured generalistions which he then justified. Fermat decried this approach, preferring his analytic 'method of descent'.

As long as learners find themselves acted upon rather than taking initiative, they are likely to form the impression that mathematics is a collection of tools designed to answer pre-determined 'problems', all fully worked out. This contrasts with the mathematician's experience of mathematics as a constructive enterprise, taking creative initiative, including posing their own problems.

A pedagogy based sometimes on the order 'You do, We do' with the occasional 'I do' is likely to enrich learners' experience of mathematical thinking, and to promote a disposition to engage and explore.

Expressing

There are times in thinking mathematically, trying to detect or comprehend some relationship(s), when the urge to talk to someone else becomes irresistible. The phenomenon of finding myself going into a colleague's office, expressing something, then saying 'thank you' and leaving, is not uncommon. At the Open University, I used to recommend to learners studying alone at home to talk to their goldfish if they couldn't find any other audience (Chi & Bassok 1989; Chi et al., 1989, 1994; Hodds et al., 2014). There is something about the urge to express and the presence of an audience which helps things become clearer. Uttering a conjecture to someone else is much more effective than holding onto it until it has been thoroughly checked. Here the mathematical content initiates and the audience responds, mediated by the learner's articulation.

Listening (Explaining)

Teaching by listening sounds like an oxymoron, but in fact it underpins some of the most effective teaching (Davis, 1996). Through listening, the teacher attempts to enter the world of the learner, so it is the mathematical content that mediates between them, but it is still an action initiated by the teacher, in contrast to the action of expressing. The moment the teacher experiences 'Oh, that is where they are going wrong', the action is liable to turn into exposition, when the teacher experiences the learner's confusion as 'the problem' and launches into their own articulation (Zaslavsky, 2010).

In other writing (Mason, 2002, 1979), I described this action as *explaining*, even though the common meaning of explaining is 'making plain or clear'. By listening, and encouraging articulation, clarity can emerge in a completely different way to exposition. For me, teaching by listening is indeed making clear, making clear such details as what the learner is attending to, and how, that may be hindering their further comprehension.

Summary

The six modes of interaction described above can conveniently by labelled as the *Six Ex's*: expounding, exercising, examining, exploring, expressing, and explaining. Each action arises from one of the three components, teacher, learner, and content taking each of the roles of initiating, mediating, or responding (Mason, 1979). The involvement of the teacher, or some artifact taking the teacher's role, is what makes these pedagogic actions. But in many cases, over time, the learner can internalise the action, turning it into a mathematical action which enriches their mathematical thinking.

Initiating an action is not sufficient for it to proceed fruitfully. There can be significant differences between the activity intended by the author, by the teacher, and as experienced by the learner, so it is important to distinguish between tasks as set, and consequent activity (Tahta & Brookes, 1966; Christiansen & Walther, 1986). Learners need to participate in the action, willingly and appropriately. Effective pedagogic actions generate activity in which there is an alignment between the resources available, the tasks undertaken, and the intended learner development (Bennett, 1993; see also Shantock Systematics Group, 1975).

Furthermore, the quality of a pedagogic action will depend considerably on the ethos or milieu (Brousseau, 1997). The more everyone treats every utterance as a conjecture to be tested in their experience, the more participants are encouraged to modify their conjectures, and of course to treat each other with respect as seekers rather than know-it-alls, the more likely it is that these pedagogic actions will proceed productively.

To depend on only some of these actions is to impoverish learner experience, and possibly to infuse them with a mistaken or incomplete appreciation of what mathematics is about, and what it is like to think mathematically.

Who then puts the 'active' into 'active learning'? The teacher, the learners, and even the content have roles to play, in order to exploit the different opportunities being offered to experience mathematical thinking, and hence to learn mathematics, in different modes.

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