Parallel Product of Maps and Construction of Regular Maps

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Abstract

A map is a cellular embedding of a connected graph on a closed surface. An automorphism of a map is a permutation of its flags that preserves the cell structure of the map. The group of all automorphisms of a map acts freely on the flag set of the map. If this action is also transitive, and hence regular, the map itself is regular. Regularity of a map can be interpreted as the largest level of 'internal symmetry' a map exhibits.

A regular map of type (k, m), that is of face length m and valency k, can be identified with smooth quotients of extended $(2, k, m)$-triangle groups, which are generated by three involutions whose products have prescribed orders $2, m$ and k. Similarly, orientably-regular maps of the same type can be identified with smooth quotients of the ordinary $(2, k, m)$-triangle groups, constituting the 'even' subgroup of index 2 of the extended $(2, k, m)$-triangle group.

This Thesis considers construction of regular and orientably-regular maps with specified external symmetries, in particular, with specified invariance to rotational powers. Given a map M of type (k, m) and an integer j relatively prime to k, the operator of a j^{th} rotational power constructs a new map $M(j)$ from M by replacing all the local rotations by their j^{th} powers. If M is (orientably-) regular, then so is $M(j)$. If $M(j)$ is isomorphic to M, then j is an exponent of M. The collection of exponents of M forms a group isomorphic to a subgroup of the group of units mod k.

We consider the following:

1. Given k and a group U of units mod k, does there exist an (orientably-) regular map of valency k with exponent group U?

2. Given m and k, does there exist an (orientably-) regular map of type (k, m) with trivial exponent group?

Preface

Acknowledgements

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Declaration

I declare that this thesis represents my own work, unless otherwise stated. This work has been undertaken during my time as a registered student and has not been previously included in a thesis or dissertation submitted to this or any other institution. In the case of joint work, due reference has been made to the co-authors.

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Chapter 1

Introduction

The main objective of this thesis is the investigation of one aspect of a rich interplay area between graph theory, group theory and low-dimensional topology which is the theory of symmetries of maps, that is, graph embeddings on surfaces.

1.1 Background

Origins of this interplay may be traced centuries back to Kepler's work *Harmonice Mundi* [42] in which he came up with several constructions of spatial arrays (such as the Great Stellated Dodecahedron) which he thought could help explain certain planetary and stellar configurations and which generalised the shape of Platonic solids. Another impetus can be tracked to the second half of the 19th century, when interest in surface embeddings emerged through Heawood's suggestion [32] to generalise the Four Colour Problem which includes a lower bound for the number of colours needed in order to have a graph colouring on a surface which has a given genus. The inclusion of a highly symmetric toroidal embedding of a complete graph of order seven in [32] lead to a an instance of one of the two branches of investigation that brought symmetries of embeddings into consideration. The other was the work of Klein [43] by the discovery of a quartic complex curve of genus 3, now bearing his name, which was later realised to represent a highly symmetric embedding of a graph, with the smallest non-symmetric simple group as the automorphism group.

The platonic maps which can be seen in figure 2.5 and Klein's map are examples of regular maps, while Heawood's toroidal one is orientably-regular but chiral, not regular. At this stage we say very informally that, in what is now known as *topological graph theory*, maps are cellular embeddings of graphs on surfaces, and regular maps are those in which their automorphism groups are regular on flags (which, in non-degenerate cases, may be identified with mutually incident vertex-edge-face triples). The concept of an orientably-regular map is slightly weaker, where the embeddings are restricted to orientable surfaces and the automorphism group to orientation-preserving automorphisms, regular on darts (incident vertex-edge pairs). Precise definitions will follow in chapter 2; for now it is sufficient (and useful) to think of (orientably-) regular maps as those exhibiting the 'largest possible level of (orientation-preserving) symmetry' a map can have.

Platonic maps are the obvious example of regular maps on a sphere; there are infinitely many others but they are, in some sense, trivial (cycles, dipoles and semi-stars). Remarkably, apart from the tetrahedron, the remaining Platonic maps are double covers of regular maps on a projective plane, providing thus the first non-trivial examples of regular maps on a non-orientable surface. Infinitely many non-trivial regular and chiral maps on a torus may have been known to geometers in the late 19th century, as can be documented by the work of Dyck [30] on complex functions on a torus.

A new addition into the theory of maps came with Burnside's monograph [15] dated 1911, containing presentations of automorphism groups of all orientably-regular maps on a torus, but their complete and systematic classification had to wait until the work of Coxeter [27] and later the influential monograph of Coxeter and Moser [28] who also mentioned non-existence of a regular map on a Klein bottle. This was the state-of-the-art at the beginning of 20th century: identification of infinitely many regular maps on a sphere and a projective plane, infinitely many non-trivial regular and chiral maps on a torus, and no such map on a Klein bottle.

The four surfaces mentioned above are the only ones with non-negative Euler characteristic. In a sharp contrast with these, the Riemann-Hurwitz inequality (Chapter 2) implies that every surface of a negative Euler characteristic can support only a finite number of orientably-regular or regular maps. Without going into too many particulars about later development and referring to the survey [52] for details, let us just mention that by the late 1980's, classification of regular maps on orientable surfaces of positive genus were completed only up to genus 7, and for non-orientable surfaces only up to crosscap number 8, by the collective effort of a number of researchers [52].

We now pause for a moment to draw attention to another important feature of the study of regular maps. Apart from supporting surfaces, the early examples of Heawood and Klein point to two more aspects intimately connected with maps: the underlying graph (K_7 in Heawood's case) and the automorphism group ($\mathrm{PSL}(2,7)$ in Klein's case). Even the very definition of a regular map fits within what we started with, namely, the interplay between graph theory, group theory and the theory of surfaces. Thus, investigation into (orientably-) regular maps splits naturally into three broad directions: maps with a given supporting surface, with a given automorphism group, and with a given underlying graph.

So far we have reviewed the development in the study of regular maps on a given surface, until approximately late 1980's. How about the cases for a given automorphism group or given underlying graph? It appears that these have been somewhat neglected - with two considerable exceptions, however. The first was a classification of regular maps with automorphism group isomorphic to $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, implied by the group-theoretic work by Sah [48], and the second was an influential paper by James and Jones [33] on classification of orientably-regular embeddings of complete graphs. But one can note an increased amount of activity since 1990, in all the three directions of study of regular maps, on which we briefly report. We again leave a full historical account to the survey [52] and mention only the most influential results.

In classification of regular maps on a given surface there were two kinds of major breakthroughs after 2000: the first kind was computational, beginning with the Conder-Dobcsányi list [19] of regular maps on orientable surfaces of genus up to 15 and non-orientable up to crosscap number 30, which had gradually grown over years to the current computationally generated list of Conder [18] reaching as far as regular maps on orientable surfaces up to genus 301 (and non-orientable up to crosscap number 602). Parallel to the computational front, the two most influential results were the historically first classification results for regular maps on an infinite set of surfaces: for non-orientable ones this was the classification of regular maps on non-orientable surfaces of negative prime Euler characteristic by Breda, Nedela and Širáň [10], with an orientable counterpart (and for Euler characteristic $-2p$ for prime p) by Conder, Širáň and Tucker [25].

In the area of regular maps with a given underlying graph, the two milestones after dealing with complete graphs in [33], were a classification of orientably-regular embeddings of multidimensional binary cube graphs by Catalano, Conder, Du, Kwon, Nedela and Wilson [16], and a seminal classification of regular embeddings of complete bipartite graphs completed by Jones [34]. On classification of regular maps with given automorphism group the most important results for simple groups appear to be the ones by Jones and Silver [40] regarding Suzuki groups and by Jones [38] targeting Ree groups; an important counterpart for soluble groups is the one by Conder, Du, Nedela and Škoviera [20] for regular maps with nilpotent automorphism groups.

Our focus in the area of regular maps is in those which are, on some sense, 'more regular than others'. Let us explain what is meant by this seemingly contradictory description, resembling a quote from the novel 'Animal Farm' by George Orwell. Consider, for example, a map of a tetrahedron. This is a well known example of a regular map, and demonstration of its symmetries is often a good exercise in elementary group theory. But this map is also self-dual, that is, isomorphic to its dual (this is not the case for the remaining four Platonic maps). One may argue that self-duality gives the tetrahedral map (which is 'most symmetric' by automorphism) an extra 'level' of symmetry, which one may consider to be an 'external symmetry', one which does not come from automorphisms.

Another way of looking at this phenomenon is to say that the tetrahedral map is invariant under the duality operator (which, again, is not the case for the remaining Platonic maps, where a cube is the dual of an octahedron and a dodecahedron is the dual of an icosahedron).

So, what are the possible operators on maps worth looking at from the point of view of studying invariance of maps under them? It turns out that besides duality there are two more 'elementary operators' one can apply to maps: the Petrie duality and rotational powers, both of which are introduced and explained in detail in Chapter 2. A regular map invariant to all these types of operators is called *super-symmetric*. Constructions of super-symmetric maps of any even valency have been discovered by Archdeacon, Conder and Širáň [2] just about 10 years ago. Meanwhile, the problem of existence of such maps of odd valency is still open. Given a super-symmetric map, there is the question of the structure of its *external symmetry group* generated by its external symmetries. This can be surprisingly complicated even for super-symmetric maps of valency 8, as documented by the work of Conder, Kwon and Širáň [22].

1.2 Thesis Outline

The chapters in this Thesis will be split up as follows.

In Chapter 2 will take a look at the existing literature, setting the foundations for the results that will be discussed in subsequent chapters. We will start by introducing regular maps, explaining the importance of flags and how the automorphism group of a map is generated.

The building blocks of orientably-regular maps are darts (also known as arcs), where each dart is made up of two flags and an edge. Here the orientation-preserving automorphisms act semi-regularly on the darts and if the group is regular then we speak about orientably-regular. Furthermore, we will see how a map can be called reflexible if it emits orientation-reversing automorphisms, or chiral if not.

We then introduce *external symmetries*, which are not automorphisms since they destroy some of the structure but still return an object isomorphic to the original one. Particularly, we look at self-dualities and exponents.

For non-orientable regular maps, we take a look at known literature regarding maps of a given type over linear fractional groups. These results will allow us to prove results in subsequent chapters.

In Chapter 3 we look at more known literature that is used as the building blocks for our results. The first area of focus is the *parallel product* also known as the *join* which is one of the main methods used to prove our results. The second half of the chapter is dedicated to available knowledge regarding linear fractional groups over finite fields which will be used later to help construct regular maps with automorphism groups isomorphic to these groups.

The question posed in Chapter 4 relates to the existence of orientable and non-orientable regular maps with a given exponent group. In 2016, Conder and Širáň [24] proved that for every $k \geq 3$ and every group U of units mod k there is an infinite number of orientably-regular maps of valency k with exponent group equal to U.

After discussing known results and required preliminaries, a new proof of the main result in [24] about the existence of orientably-regular maps with a given exponent group is provided using the parallel product as found in Theorem 4.7. This result is then extended to Theorem 4.10 where the case of non-orientable regular maps is covered and we prove the existence of such maps with any given exponent group $U < U_k$ containing the unit -1 (which, along with 1, is an exponent in every non-orientable regular map). This is done through the use of linear fractional groups.

In [21], it was proved that there are infinitely many chiral orientably-regular maps of any given hyperbolic type. In Chapter 5 we look into extending this result to show the existence of orientably-regular maps of a given hyperbolic type that have no non-trivial exponent, that is, their only possible exponent is 1. Here we make use of maps with automorphism groups isomorphic to linear fractional groups over finite fields. This is once again achieved by introducing parallel products and details can be found in Theorem 5.2. For non-orientable regular maps the challenge here is to show the existence of non-orientable regular maps of any given hyperbolic type with no exponents other than ± 1 and our main results in this direction are Theorems 5.9 and 5.10.

In the penultimate chapter we look into a proof of part of the result discussed in the previous chapter using a different method. In [21], Conder et. al made use of coset diagrams to prove the existence of infinitely many orientably-regular but chiral maps of every given hyperbolic type (k, m). A similar approach is considered here. We start by constructing 10 families of 2-generated permutation groups based on the constructions in [21] with some modifications made as necessary. Here, every group G is in such a way that it is isomorphic to either the symmetric group S_n or the alternating group A_n. We then take a close look at their associated maps M which are shown to imply the existence of orientably-regular maps of any given hyperbolic type with no non-trivial exponents, that is, no other exponent apart from 1, as proved in Theorem 6.6.

In the concluding chapter we summarise the results discussed in the preceding chapters. We will then take a quick look at a few open questions and possible future work.

The results mentioned in Chapter 4 are a result of joint work with M. Conder, S. Pavlíková and J. Širáň, and have been published in [4]. Meanwhile, the paper containing the results in Chapter 5, which was joint work with M. Conder, O. Reade and J. Širáň, has been submitted for review [5].

Chapter 2

Regular and orientably-regular maps

Our main object of interest is 'symmetric maps'. Informally, a map is a cellular decomposition of a surface, and a symmetry of a map is a bijection between certain building blocks of the map that preserves the map structure, that is, vertices, edges and faces of the map. In this chapter we will explain the basic concepts regarding maps that are necessary for further development of the theory of maps and for presentation of our results in forthcoming chapters. In what is to follow, we will assume at least basic undergraduate level familiarity with the three fundamental concepts this Dissertation is built on - namely, graphs, groups and surfaces.

The first three subsections contain basic definitions of maps, their building blocks (flags) and map automorphisms. Later on, when dealing with orientably-regular and regular maps (those that are 'richest' in automorphisms), such maps are identified with their automorphism groups. This, from subsection 2.4 onwards, is essential for understanding our results, which are almost exclusively stated in terms of the associated group presentations 2.1.

2.1 Introducing maps

As it is customary in low-dimensional topology, by a *closed surface*, or, briefly, a *surface*, one means a connected, 2-dimensional real manifold, that is, a

Hausdorff topological space in which every point has a neighbourhood homeomorphic to an open disc; we make no preliminary assumptions about orientability or compactness. The following definitions are standard in map theory and more detail can by found in survey [52]. A *map* is then a cellular embedding of a connected graph on some surface; or, equivalently, a cellular decomposition of a surface. The 0-cells and 1-cells are respectively *vertices* and *edges* of both the map and its *underlying connected graph*, while the 2-cells of the cellular decomposition are *faces* of the map. Incidence between vertices, edges and faces of a map is understood in the usual way. Of course, cellularity of a map implies that its underlying graph is finite if and only if the *supporting* surface of the map is compact.

One can be more formal and introduce maps as follows. Let Γ be a connected graph, possibly with semi-edges (those which are continuous bijective images of a closed unit interval but with one end not considered to be a vertex of the graph), loops and parallel edges. Let us endow Γ with the usual CW-topology, turning it into a CW-complex. An *embedding* of Γ on some surface $\mathcal{S}$ is any continuous one-to-one function $\vartheta : \Gamma \to \mathcal{S}$ where the embedding is also a homeomorphism onto its image. The embedding ϑ is *cellular* if every connected component of $\mathcal{S} \backslash \vartheta(\Gamma)$ is homeomorphic to an open disc. If ϑ is a cellular embedding, the pair $(\mathcal{S}, \vartheta(\Gamma))$ is called a *map*. The graph Γ and the surface $\mathcal{S}$ are the *underlying connected graph* and the *supporting surface* of the map. In such a case, every component of $\mathcal{S} \backslash \vartheta(\Gamma)$ is called a *face* of the map. In what follows, however, we will use the earlier way of describing maps.

2.2 Flags - the building blocks of maps

In order to be able to speak about the structure of maps in detail we need to introduce their basic building blocks, which we do next. To begin with, for every map M on a supporting surface $\mathcal{S}$ we assign a new map M^b, constructed from M on the same surface as follows. Let e be an edge of M that is not a semi-edge, such that $e = uv$ is an edge joining vertices u and v (possibly, $u = v$, in which case e is a loop). Subdivide every edge by a new vertex w_e, so that e becomes a walk uw_ev. If e is a semi-edge at a vertex u, then we let w_e be a new vertex at the dangling end of e (so that the semi-edge turns into a new edge uw_e). If M^s is the map arising from M by these subdivisions, faces of M and M^s have the same interior (but not the same boundary walks). Next, for every face F of M^s one inserts exactly one new vertex w_F to (the interior of) F and, within every such F, an edge is drawn from w_F to every vertex v of the original map M lying on the boundary of F, and also to every new vertex w_e on the boundary of F. It is assumed that the addition of edges is done in such a way that no added pair of edges cross each other. The new cellular decomposition of $\mathcal{S}$ derived from M this way is the *barycentric subdivision* M^b of M; we reiterate that it is supported by the same surface $\mathcal{S}$. Every face of M^b is called a *flag* of the original map M. Hence every flag is incident with three vertices of the form v, w_e and w_F, where (v, e, F) is a mutually incident vertex-edge-face triple of M. The boundary of such a flag is formed by the three edges vw_e, w_ew_F and w_Fv; see Fig. 2.1.

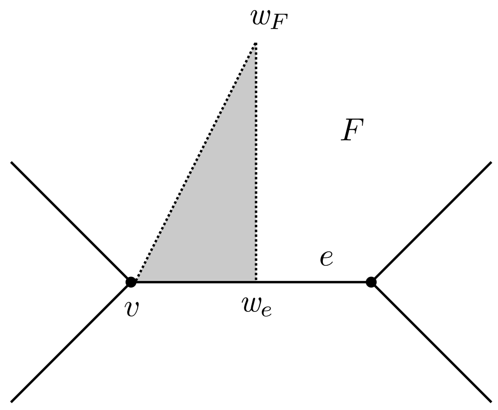

Figure 2.1: A flag of a map.

We can refer to a flag of M as a 'triangle' of the barycentric subdivision M^b. To help us visualise we can 'colour' the vertices v, w_e and w_F by labels 0, 1 and 2, respectively. This labelling corresponds to the dimension of the relevant element *associated* with the flag since a vertex, an edge, and a face have a dimension of 0, 1 and 2, respectively. We say that two distinct flags of the map M are *adjacent* if they have a common edge in M^b and we will say that the flags are adjacent *along an edge*, *across an edge*, and *through a corner*, if the edge they share in M^b is the one with vertices labelled $\{2, 1\}$, $\{1, 0\}$, and $\{0, 2\}$, respectively.

We assume familiarity with the well known construction of a *dual map* M^D to a map M, on the same supporting surface. Map duality reflects in the 3-coloured barycentric subdivision in an astonishingly simple way: the barycentric subdivisions M^b and $(M^D)^b$ are topologically the same and differ only in the interchange of labels 0 and 2 throughout. The new labelling means that every vertex of M^D corresponds to a face centre of M and vice versa, while for every edge e of M the dual edge e^D 'perpendicularly crosses' the original edge.

2.3 Monodromy groups and map automorphisms

Flags as building blocks of maps allow us to introduce two basic concepts that will enable us to study map symmetries, namely, the monodromy group and the automorphism group of a map. Here we introduce the first of these, deferring the second to the next section.

To address monodromy, consider a map M with 3-coloured barycentric subdivision as introduced in the previous section. From now on we will assume that the set of colours $\{0, 1, 2\}$ is the set of elements of the cyclic group C_3 of order 3. Let us define three involutory permutations ρ_j ($j \in C_3$) of the flag set $\mathcal{F}$ of M, permuting the flag set according to the following rule: for every $j \in C_3$ and for every 3-coloured flag $f \in \mathcal{F}$, the image $f\rho_j$ will be the unique flag of M adjacent to f along the side with corners coloured $\{j-1, j+1\}$ as can be seen in

Fig. 2.2. It follows that every orbit of every ρ_j is a cycle of length 2; it is also obvious that ρ_0 and ρ_2 commute. Moreover, for every vertex v of the map M the product $\rho_1\rho_2$ induces two orbits on the subset of flags incident to v, both of length equal to the valency of v. Similarly, for every face F of M the product $\rho_0\rho_1$ induces two orbits on the subset of flags incident to w_F, both of length equal to the length of the face boundary of F. The subgroup generated by the permutations ρ_j for $j \in C_3$, that is, the subgroup $\langle \rho_0, \rho_1, \rho_2 \rangle < \mathrm{Sym}(\mathcal{F})$ of the symmetric group on $\mathcal{F}$, is the *monodromy group* $\mathrm{Mon}(M)$ of the map M.

Observe that the (throughout assumed) connectedness of underlying graphs of our maps implies that for every map M the group $\mathrm{Mon}(M)$ is a *transitive* permutation group on the flag set of M.

The monodromy group of a map is an important combinatorial and group-theoretic structure. As it is clear, it can be associated with every map, and its relation to potential 'symmetries' is not obvious at first glance. What is obvious immediately is the following. Since for every flag of M the set $\{f\rho_0, f\rho_1, f\rho_2\}$ is the set of flags adjacent to f along edges labelled 12, 02 and 01, respectively, the permutations ρ_j for $j \in C_3$ can be regarded as 'building instructions' for assembling the map M from its flag set $\mathcal{F}$.

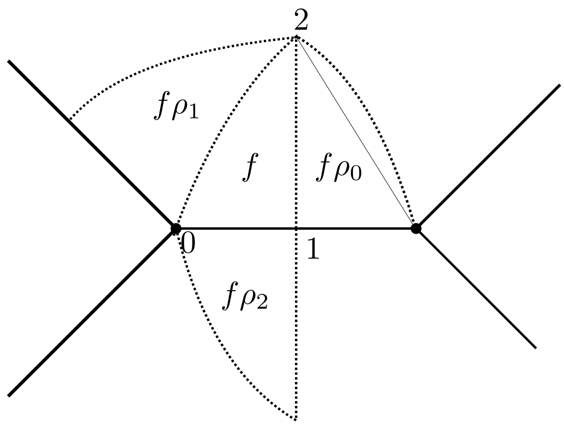

Figure 2.2: The flag f with involutory permutations, $\rho_j, j \in \{0, 1, 2\}$.

Branched coverings of surfaces have their natural counterpart in coverings of maps. In terms of monodromy groups they may be described as follows. Suppose M and M' are two maps on flag sets $\mathcal{F}$ and $\mathcal{F}'$ and with monodromy groups $\mathrm{Mon}(M) = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $\mathrm{Mon}(M') = \langle \rho_0', \rho_1', \rho_2' \rangle$ as subgroups of $\mathrm{Sym}(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F}')$, respectively. A *covering* M to M' (or M' by M) is any mapping $f : \mathcal{F} \to \mathcal{F}'$ such that $f(z\rho_i) = (f(z))\rho_i'$ for every flag $z \in \mathcal{F}$ and $i \in \{0, 1, 2\}$. Equivalently, f 'commutes' with the corresponding generators of monodromy groups. Topologically, f induces a covering of the supporting surface of M' by the supporting surface of M, which may be branched at corners of flags, that is, at vertices, face centres and edge midpoints but at no other points. Algebraically, existence of such a covering $f : M \to M'$ with the property that $f(z) = z'$ for some selected $z \in \mathcal{F}$ and $z' \in \mathcal{F}'$ translates into existence of a group epimorphism $\phi : \mathrm{Mon}(M) \to \mathrm{Mon}(M')$ such that $\phi(\rho_i) = \rho_i'$ for $i \in \{0, 1, 2\}$ and carrying the stabiliser of z in $\mathrm{Mon}(M)$ onto the stabiliser of z' in $\mathrm{Mon}(M')$.

2.4 The automorphism group of a map and regular maps

The monodromy group allows us to conveniently introduce the automorphism group of a map. This can be done in two ways. Given a map M with flag set $\mathcal{F}$, an *automorphism* of M is a permutation of $\mathcal{F}$ that preserves all three types of flag adjacency, that is, along edges marked 01, 02, and 12. Here, by preserves we mean that the adjacencies remain consistent after applying the automorphism. Preservation of all these types of adjacencies implies that every automorphism preserves vertices (sets of flags incident to a vertex of M), edges (sets of flags incident to an edge of M), and, similarly, faces of M. A composition of two automorphisms of M (as permutations of M) is obviously an automorphism of M again; the group of all such automorphisms of M thus forms a group, the *automorphism group* $\mathrm{Aut}(M)$ of M.

Preservation of adjacency of flags by an automorphism A of a map M is obviously equivalent to the statement that A, as a permutation of $\mathcal{F}$, *commutes* with each monodromy group generator ρ_j for $j \in C_3$ because an automorphism is automatically a covering (dealt with in subsection 2.3). This means that the group $\mathrm{Aut}(M)$ can be equivalently introduced as the *centraliser* of the monodromy group in the symmetric group on $\mathcal{F}$; in symbols, $\mathrm{Aut}(M) = \mathrm{Cent}_{\mathrm{Sym}(\mathcal{F})}\mathrm{Mon}(M)$. By Section 6 of [11], every automorphism of M is a semi-regular permutation of $\mathcal{F}$ (that is, the stabiliser of any point is the identity) and it then follows that the automorphism group acts freely on the flag set. In the special and extremal case when the group $\mathrm{Aut}(M)$ is also *transitive*, and hence *regular*, on the flag set of M, the map itself is called a *regular map*. It follows that, in a regular map M for every pair of flags f, f' of M there is *exactly one* automorphism $A \in \mathrm{Aut}(M)$ such that $A(f) = f'$.

Automorphisms of a map M, and the group $\mathrm{Aut}(M)$ in particular, are a formal way to express symmetries; this also implies that regular maps are the 'most symmetric' maps.

In order to get a deeper insight, let M be a regular map and let z be a fixed flag of M. By regularity, for every $j \in C_3$ there exists an automorphism r_j of M taking z onto the flag $r_j(z)$ adjacent to z along an edge, through a corner, and across an edge, respectively. Note that the only corner of z moved by r_j is the one whose label corresponds to a j-dimensional object of the map - a vertex, an edge, and a face for $j = 0$, 1, and 2, respectively. The situation is depicted in Fig. 2.3.

This above figure resembles the one we encountered when introducing the three monodromy generators ρ_j for $j \in C_3$ as seen in Fig. 2.2. Note, however, a substantial difference: while, say, the *monodromy generator* ρ_0 permutes throughout the entire map, only pairs of flags adjacent along edges coloured 02,

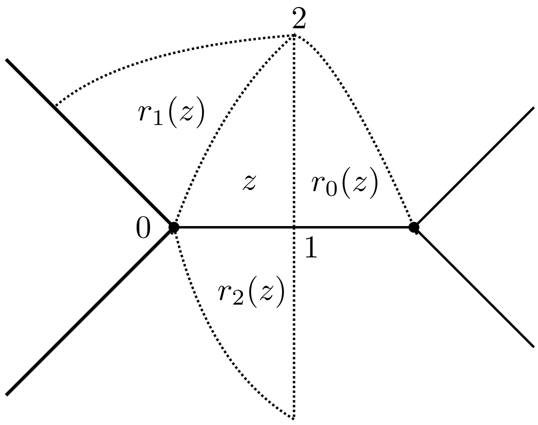

Figure 2.3: The flag z and its images $r_j(z)$.

the *automorphism* r_0 acting locally as a *'reflection in the axis'* 12 interchanges also the flags $r_1(z)$ and $r_0r_1(z)$, and so on; see Fig. 2.4. Yet, regular maps can be characterised also by the existence of an *isomorphism* between of $\mathrm{Mon}(M)$ and $\mathrm{Aut}(M)$ as permutation groups, which follows from [[37], page 238].

Proposition 2.1. *A map M is regular if and only if its monodromy group* $\mathrm{Mon}(M)$ *and automorphism group* $\mathrm{Aut}(M)$ *are isomorphic as permutation groups.*

Let the flag z be incident with a vertex v, an edge e and a face F of our regular map M. In view of Proposition 2.1 we may now mimic our earlier considerations about orders of compositions of generators of the monodromy group. First, notice that regularity of a map implies that all vertices of the map have the same valency, say, k, and all its faces are bounded by closed walks of the same length, say, m, in the underlying graph (for short, we will use the term *face length* for m). In such a case the map is said to be of *type* (k, m); this differs from the traditional but sometimes confusing notation $\{m, k\}$.

Returning to the group $\mathrm{Aut}(M) = \langle r_0, r_1, r_2 \rangle$ of a regular map M with a fixed flag z as in Fig. 2.4, the composition r_0r_1 fixes the corner labelled 2 and so automorphism r_0r_1 acts locally as a rotation about this corner, moving the flag z to the flag $r_0r_1(z)$ 'two steps away' from z. It follows that the order of r_0r_1 is equal to the face length of M. Similarly, r_1r_2 fixes the corner of z labelled 0 and so r_1r_2 acts locally as a rotation about this corner, and the order of r_1r_2 is the valency of M. The same reasoning shows that the automorphism r_2r_0 has order 2 and r_0 commutes with r_2. The action of the three rotations is indicated in Fig. 2.4.

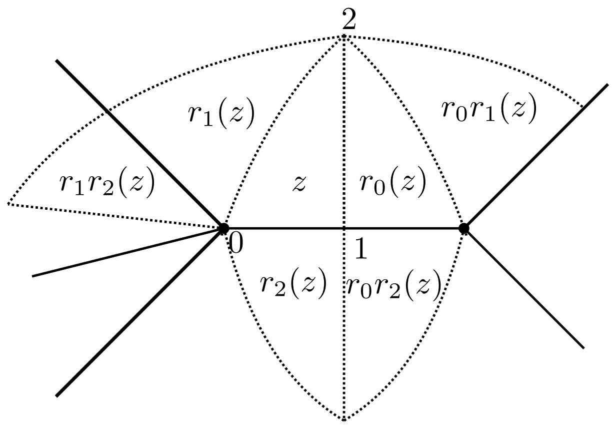

Figure 2.4: The images of the flag z under the rotations r_0r_1, r_1r_2 and r_0r_2.

For a regular map M of type (k, m), where k represents the valency and m the face length, our considerations imply that the automorphism group $G = \mathrm{Aut}(M)$ of the map admits the following presentation:

$$G = \langle r_0, r_1, r_2 \mid \ r_0^2, r_1^2, r_2^2, (r_0r_1)^m, (r_1r_2)^k, (r_2r_0)^2, \ldots \rangle \tag{2.1}$$

where dots indicate a possible presence of additional relators. We will assume throughout that in all such presentations the powers on generators and relators are their *true orders* which means we do not consider cases where, for example, the order of r^2r^0 is 1 implying that $r^0 = r^1$ or cases where any of the generators have order 1. Also, since we did not make any preliminary assumptions on finiteness here, we allow one or both entries in the type of a regular map to be ∞.

2.5 Examples of regular maps

The most famous examples of regular maps embedded on a sphere are (2-skeletons of) the Platonic solids. The five regular maps inherit the names of the solids - the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron; see Fig. 2.5.

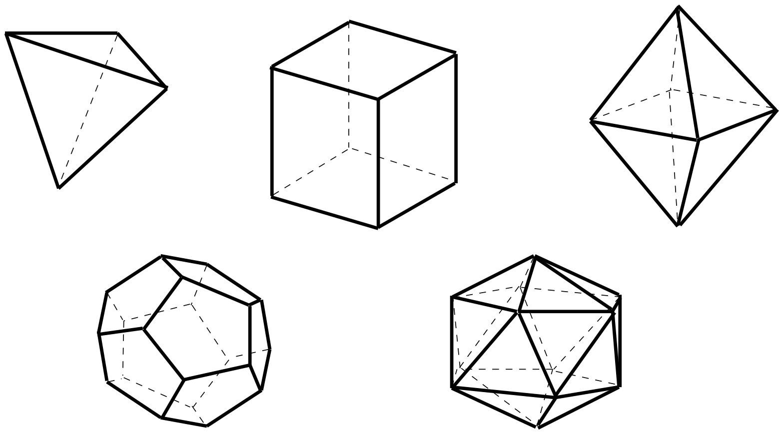

Figure 2.5: The five Platonic solids.

Note that the cube and the octahedron are a pair of mutually dual maps, and so are the dodecahedron and the icosahedron. The dual of the tetrahedron is again a tetrahedron.

The Platonic solids can be represented as maps on a sphere in $\mathbb{R}^3$ centred at the origin in such a way that all their automorphisms are spatial rotations about the origin or compositions of such rotations with the antipodal reflection that maps every point onto the point centrally symmetric with respect to the origin. This does not apply to the tetrahedron.

The projective-planar quotients of the dodecahedron and the icosahedron by the antipodal reflection are represented in Fig. 2.6 where antipodal pairs on the dashed outer circle are assumed to be identified. The underlying graph of the map in thick and thin lines, respectively, is the Petersen graph and K_6, the complete graph on 6 vertices.

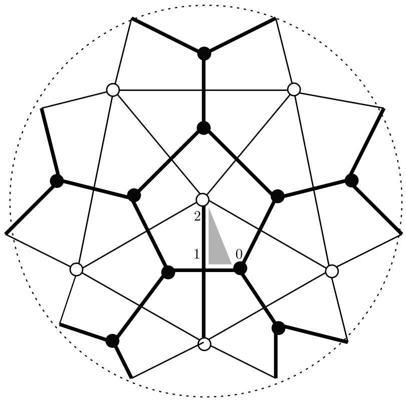

Figure 2.6: A pair of mutually dual regular maps on the projective plane.

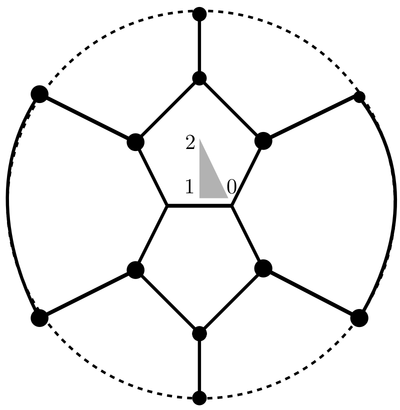

Figure 2.7: A different drawing of the Petersen graph on the projective plane.

Since the subgroup consisting of the antipodal reflection and the identity is normal in the automorphism group of the dodecahedron (and the icosahedron), it follows that the maps in Fig. 2.6 are regular [52]. It is also possible, however, that we may decide on regularity by producing a slightly different drawing, such as the one in Fig. 2.7 for the quotient of the dodecahedron.

In Fig. 2.6 one sees that the reflections of the picture in the axes through the segments $1, 2$ and $0, 2$ can be identified with the involutions r_0 and r_1 applied to the shaded flag, while from Fig. 2.7 it is obvious that the reflection in the horizontal axis can be identified with the involution r_2 applied to the same flag. Hence, the map in Fig. 2.7 is regular, and so is its dual in Fig. 2.6.

An example of a toroidal regular map is given in Fig. 2.8. The torus is obtained from the dashed rectangle in the diagram by identifying the pairs of sides labelled a into a single segment, followed by the identification of the two circles resulting from the pair of sides labelled b into a single circle, in both cases consistently with the indicated direction. The underlying graph of the map is $K_{3,3}$, the complete bipartite graph on 6 vertices.

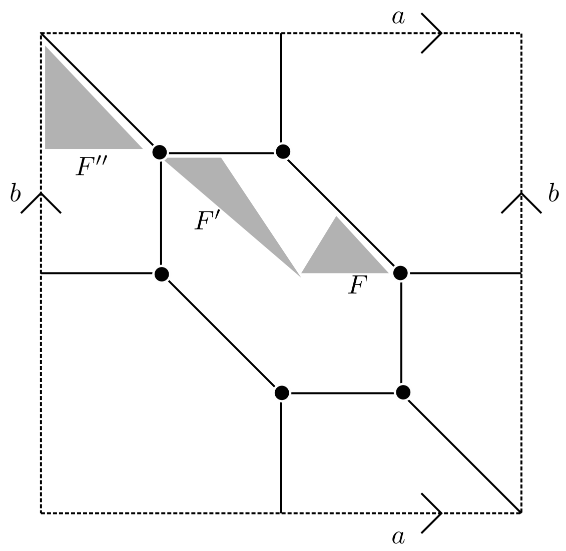

Figure 2.8: A regular map on the torus, with underlying graph $K_{3,3}$.

It follows from Fig. 2.8 that the reflection in the diagonal through the top right corner gives the involution r_0 on the flag F. We can produce the same diagram if the triangle F is drawn instead in positions F' and F'' where in this case the reflection through the diagonal starting from the top left corner induces the involutions r_1 and r_2. This implies that the map is regular.

2.6 Regular maps and groups

Given a regular map M with automorphism group $\text{Aut}(M) = G = \langle r_0, r_1, r_2 \rangle$ as in (2.1), regularity of G on the flag set $\mathcal{F}$ of M helps us determine the map M solely by means of the group G acting on itself by left multiplication. Consider a fixed flag $z \in \mathcal{F}$ and consider the bijection $G \to \mathcal{F}$ given by $g \mapsto g(z)$ for $g \in G$, which identifies flags in $\mathcal{F}$ with elements of G. This mapping will also induce a bijection between left cosets of the dihedral subgroups $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ and $\langle r_2, r_0\rangle$ of respective order $2m$, $2k$ and 4, and the faces, vertices and edges of M.

Conversely, take the group
$G = \langle r_0, r_1, r_2 \mid\ r_0^2, r_1^2, r_2^2, (r_0r_1)^m, (r_1r_2)^k, (r_2r_0)^2, \ldots\rangle$. Let $(G; r_0, r_1, r_2)$ be the corresponding regular map which can be constructed by letting the flag set $\mathcal{F}$ be equal to G and realising a flag $g \in G$ as a triangle with its corners labelled $0, 1$ and 2. If this triangle is labelled g, then for $j \in \{0, 1, 2\} = C_3$, two such flags $g, g' \in G$ will be adjacent (or j-adjacent), if $g' = gr_j$. We can observe that in such a reconstruction the edges, vertices and faces of the map $(G; r_0, r_1, r_2)$ will be formed by left cosets of the subgroups $\langle r_0, r_2\rangle$, $\langle r_1, r_2\rangle$ and $\langle r_0, r_1\rangle$, respectively.

We hence informally conclude that:

The study of regular maps is equivalent to the study of groups presented as in (2.1).

To distinguish between the left and right actions of the group G on the map $M = (G; r_0, r_1, r_2)$, where G is a group generated by r_0, r_1, r_2 and presented as in (2.1), note that our notation assumes that left multiplication by elements of G corresponds to automorphisms, while right multiplication by r_j for $j \in C_3$ determines adjacency of flags and so corresponds to the action of the monodromy group on M. The three reflection automorphisms in our fixed flag z but with localisation at the g-image of a fixed flag are given by left multiplication by the conjugates $gr_jg^{-1} \in G$ for $j \in C_3$.

With regard to representing isomorphism between regular maps we have the following observation from [11].

Proposition 2.2. *Two regular maps $M = (G; r_0, r_1, r_2)$ and $M' = (G'; r'_0, r'_1, r'_2)$ are isomorphic, that is, $M \simeq M'$, if and only if there is a group isomorphism*

$\theta : G \to G'$ such that $\theta(r_j) = r'_j$ for every $j \in C_3$.

A regular map $M = (G; r_0, r_1, r_2)$ on a supporting surface $\mathcal{S}$, with flag set $\mathcal{F}$ identified with $\mathrm{Aut}(M) = G$ as above, can also be represented by the Cayley graph $Cay(G; r_0, r_1, r_2)$ embedded on $\mathcal{S}$. Indeed, for any flag z identified with an element g, consider the element g to be a 'midpoint' of z. Then, right multiplication of g by r_j for $j \in C_3$ can be regarded as a definition of adjacency in the Cayley graph $Cay(G; r_0, r_1, r_2)$ between vertices g and gr_j (which means moving between g and g_j along an edge in the Cayley graph 'coloured' r_j); this edge may be thought of as intersecting the common side of the flags represented by g and gr_j. The embedded Cayley graph Γ on $\mathcal{S}$ has three types of faces which are those 'surrounding' the original face centres, vertices, and edge midpoints. These faces have length $2m$, $2k$, and 4, respectively, corresponding to the relators $(r_0r_1)^m$, $(r_1r_2)^k$, and $(r_2r_0)^2$; see Fig. 2.9. This embedded Cayley graph will be denoted by $\mathrm{Cay}(M)$ and called the *Cayley map associated with M*.

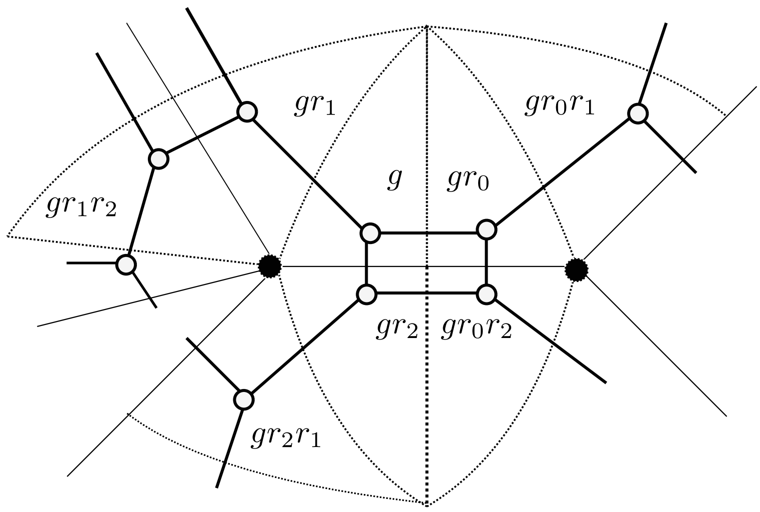

Figure 2.9: The associated Cayley map (in thick lines) of a regular map.

2.7 Supporting surfaces

A compact connected surface $\mathcal{S}$ is, up to homeomorphism, uniquely defined by its Euler characteristic $\chi = \chi(\mathcal{S})$ and by specifying whether it is orientable or not. If $\mathcal{S}$ is orientable then it is also uniquely determined by its *genus* g given by $\chi = 2 - 2g$. Topologically, the surface $\mathcal{S}$ can then be obtained from a sphere by a process known as 'adding g handles'. If $\mathcal{S}$ is non-orientable, it is determined by its *crosscap number* h given by $h = 2 - \chi$, and $\mathcal{S}$ can then be formed from a sphere by a similar process known as 'attaching h crosscaps'. One has $\chi(\mathcal{S}) \leq 2$ for every compact surface $\mathcal{S}$. The only compact surfaces of positive Euler characteristic are the sphere (with $\chi = 2$) and the projective plane (with $\chi = 1$), and the only compact surfaces of Euler characteristic 0 are the torus and the Klein bottle. However, we note that there is no regular map on the Klein bottle [28].

Let M be a finite map, meaning both the vertex set V and edge set E of its underlying graph are finite, and its set of faces F is finite as well. If $\mathcal{S}$ is the supporting surface of M, the well known Euler-Poincaré formula tells us that $\chi(\mathcal{S}) = |V| - |E| + |F|$. Suppose in addition that the map is regular, with a (finite) group $G = \mathrm{Aut}(M)$, and of type (k, m). Then, one has $|V| = |G|/(2k)$, $|E| = |G|/4$ and $|F| = |G|/(2m)$, and the Euler-Poincaré formula gives

$$\chi = \frac{|G|}{2k} - \frac{|G|}{4} + \frac{|G|}{2m} = -\mu(k, m) \cdot \frac{|G|}{2} \quad \text{where} \quad \mu(k, m) = \frac{1}{2} - \frac{1}{k} - \frac{1}{m} \ . \tag{2.2}$$

As we have alluded to, a group $G = \langle r_0, r_1, r_2 \rangle$ with presentation (2.1) as given in Section 2.4 gives rise to a uniquely determined regular map $M = (G; r_0, r_1, r_2)$. The Euler characteristic of its supporting surface $\mathcal{S}$ is given by (2.2), and its orientability or otherwise is determined as follows. For $G = \langle r_0, r_1, r_2 \rangle$, let G^+ be the subgroup of G generated by all products of the generators r_j ($j \in C_3$) of even length. The group G^+ is known also as the *even subgroup* of G; it has index 1 or 2 in G. Relation of this even subgroup to orientability of the corresponding map is well known; see, for example, Proposition 2.6 of [52], which we restate as follows.

Proposition 2.3. [[52], Proposition 2.6] *The supporting surface of a regular map* $(G; r_0, r_1, r_2)$ *is orientable if and only if the even subgroup* $G^+ < G$ *has index* 2 *in* G*; in such a case,* G^+ *is the group of all the orientation-preserving automorphisms of the map.*

If, say, $r = r_2 r_1$ and $s = r_1 r_0$ are automorphisms acting locally as rotations about the midpoint of a fixed vertex and about a fixed face incident with the face, then Proposition 2.3 can now be restated in a very useful way as follows:

Corollary 2.4. *The supporting surface of a regular map* $M = (G; r_0, r_1, r_2)$ *is non-orientable if and only if* $G = \langle r, s \rangle$*, where* $r = r_2 r_1$ *and* $s = r_1 r_0$*.*

2.8 Full triangle groups and tessellations

If there are no additional independent relators in the presentation (2.1), then we obtain the *full (or extended)* $(2, k, m)$*-triangle group*:

$$\Delta(2, k, m) = \langle R_0, R_1, R_2 \mid R_0^2, R_1^2, R_2^2, (R_0 R_1)^m, (R_1 R_2)^k, (R_2 R_0)^2 \rangle. \tag{2.3}$$

As mentioned in earlier sections, by our convention, powers of generators and relators in (2.3) are true orders. It is well known (cf. [11]) that $\Delta(2, k, m)$ is a finite group if and only if $1/k + 1/m > 1/2$. Further, by a somewhat deeper theory that has also been summarised in [11], the corresponding regular maps $\mathcal{U}_{k,m} = (\Delta(2, k, m); R_0, R_1, R_2)$, widely known as a *universal tessellations* of type (k, m), have *simply-connected* supporting surfaces. There are just two topologically inequivalent surfaces in the latter category: the sphere, and the plane. The supporting surface of $\mathcal{U}_{k,m}$ is a sphere if and only if $1/k + 1/m > 1/2$, and hence types (k, m) which satisfy this inequality are aptly referred to as *spherical*. The supporting surface of $\mathcal{U}_{k,m}$ is a plane if and only if $1/k + 1/m \leq 1/2$. More specifically, the types (k, m) for which $1/k + 1/m = 1/2$ are called *Euclidean* whilst those where $1/k + 1/m < 1/2$ are referred to as *hyperbolic*.

2.8.1 Spherical Types

The possible values of k and m which satisfy the condition that $1/k+1/m > 1/2$ are the following: $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$, $(5, 3)$, together with $(k, 2)$ and $(2, m)$ for all $k, m \geq 1$ and with (k, k) in the exceptional case when the underlying graph of the map is a semi-star of valency k consisting of just one vertex and k attached semi-edges (which are not considered edges). For the first five types the universal spherical tessellations $\mathcal{U}_{k,m}$ have a unique realisation in the form of maps coming from Platonic solids as is visible in Fig. 2.5 in Section 2.5. For the other universal spherical tessellations, their realisation is obtained through m-cycles and their duals, k-dipoles (and, of course, the exceptional semi-stars). Each spherical regular map $\mathcal{U}_{k,m}$ admits a geometric realisation on a unit sphere in $\mathbb{R}^3$ such that all flags are mutually congruent triangles with geodesic sides, and with angles of size π/k, π/m and $\pi/2$. By 'congruence' we are referring to the underlying Riemannian geometry of a sphere.

2.8.2 Euclidean Types

There exist only three possible pairs of values of k and m that satisfy the condition for Euclidean type and these are $(4, 4)$, $(6, 3)$ and $(3, 6)$. The corresponding tessellations $\mathcal{U}_{k,m}$ can be seen in Fig. 2.10 and are given by an infinite square grid, a honeycomb and its dual, respectively. In a Euclidean plane they each have a symmetric geometric realisation where all flags are pairwise congruent (with respect to Euclidean isometries) and formed by Euclidean right triangles with the remaining angles π/k and π/m.

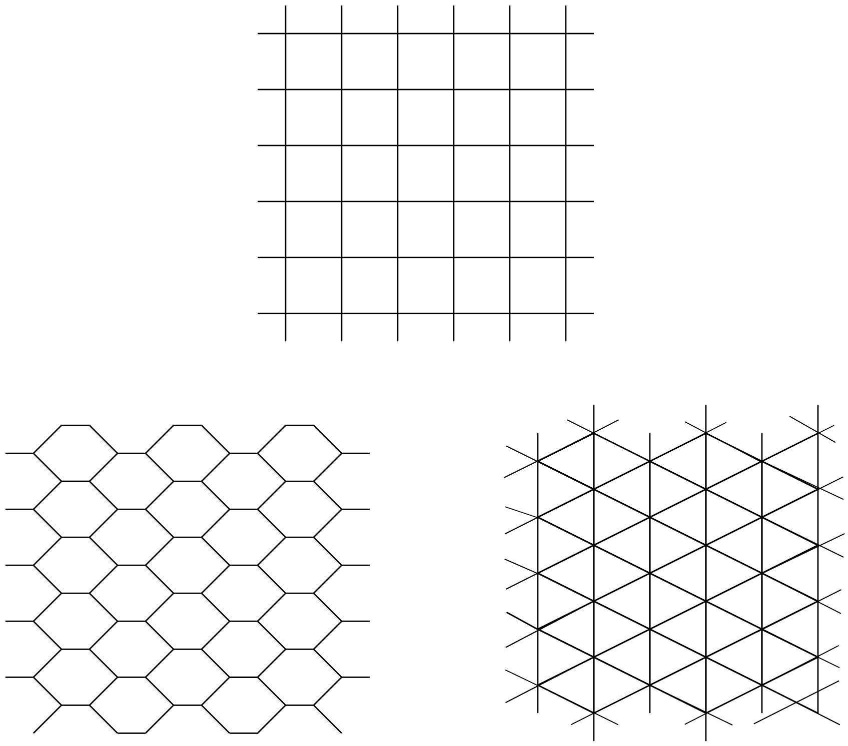

Figure 2.10: Fragments of the Euclidean tessellations $U_{4,4}$, $U_{3,6}$ and $U_{6,3}$.

2.8.3 Hyperbolic Types

The inequality $1/k + 1/m < 1/2$ has infinite pairs (k, m) that satisfy it and hence any pair where $k \leq m$ such that $k \geq 5$, or $k \geq 4$ and $m \geq 5$, or else $k \geq 3$ and $m \geq 7$, is hyperbolic. Given k and m as specified, it follows that (k, m) is hyperbolic if and only if the dual (m, k) is also hyperbolic. Suppose we attempt to draw the tessellation $\mathcal{U}_{k,m}$ on a plane for a hyperbolic pair (k, m). Then the resulting picture would contain m-sided polygons that get smaller and smaller the further they get from the 'centre' of the drawing. In this case, there still exists a geometry of a plane in which all flags can be realised by mutually congruent triangles with angles π/k, π/m and $\pi/2$ that sum to less than π which is the *hyperbolic geometry.* It is highly non-trivial to prove that for every hyperbolic pair (k, m) the map $\mathcal{U}_{k,m}$ can be realised in a *hyperbolic plane* in such a way that its flags are triangles with sides being hyperbolic straight line segments and angles having magnitude π/k, π/m and $\pi/2$, cf. [45]. The flags detailed here would automatically be congruent since a hyperbolic triangle is, up to congruence, uniquely determined by its angles. Fig. 2.11 represents a part of the hyperbolic tessellations $\mathcal{U}_{3,7}$ on the disc model of the hyperbolic plane, drawn with the help of the applet [41].

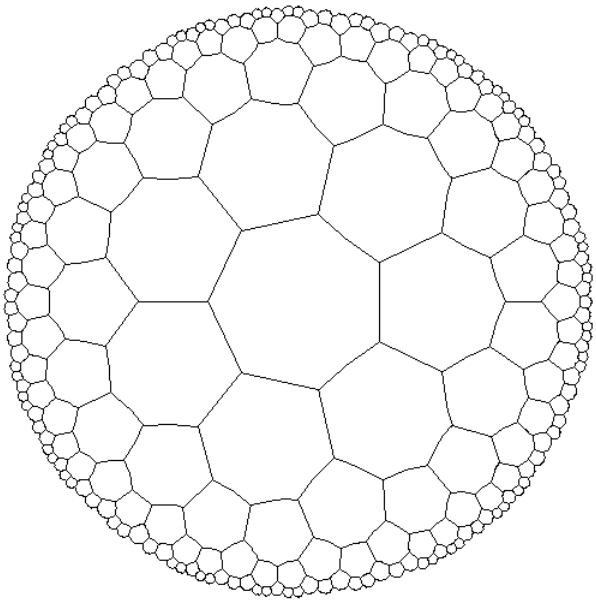

Figure 2.11: A fragment of the hyperbolic tessellation $\mathcal{U}_{3,7}$.

The comparison determining the type for a pair k and m, that is comparing $1/k + 1/m$ to $1/2$ is equivalent to comparison of the sum of angles $\pi/k + \pi/m + \pi/2$ in the flag triangles described above with the straight angle, π, which hence determines the underlying geometry.

Given that the presentation given in (2.1) is obtained by adding relators to (2.3) and the group G generated by $\langle r_0, r_1, r_2 \rangle$, there exists a group epimorphism $\Delta(2, k, m) \to G$ taking the ordered triple (R_0, R_1, R_2) onto (r_0, r_1, r_2). We take N to be the kernel of $\Delta(2, k, m)$; that is, N is a normal subgroup of $\Delta(2, k, m)$. By a non-trivial result on the full triangle groups, (see e.g. Theorem 2.9 of [45]) if $\Delta(2, k, m)$ is infinite, the only non-identity elements of finite order in

$\Delta(2, k, m)$ are elements conjugate to R_0, R_1, R_2, R_0R_1, R_1R_2, R_2R_0 and their powers. This implies that if $\Delta(2, k, m)$ is infinite, then the map subgroup N is torsion-free, which means that it does not contain any non-trivial elements of finite order, since $\mathcal{U}_{k,m}$ and M have the same type. This result is only for the readers knowledge and will not be used later on.

If $M = (G; r_0, r_1, r_2)$ is a regular map and N is the kernel of the above epimorphism $\Delta(2, k, m) \to G$, the subgroup N carries all the information about M and is called the *map subgroup* of the map M.

Further results summed up again in [11] lead to the following:

Proposition 2.5. [11] *The (isomorphism classes of) regular maps of a non-spherical type* (k, m) *are in a one-to-one correspondence with torsion-free normal subgroups of the full triangle group* $\Delta(2, k, m)$*; the maps are finite if and only if the subgroups are of finite index.*

Given that the universal tessellation $\mathcal{U}_{k,m}$ of a spherical, Euclidean or a hyperbolic type (k, m) is composed of congruent regular polygons in the appropriate geometry, the automorphism group $\Delta(2, k, m)$ of the regular map $\mathcal{U}_{k,m}$ can be identified with a discrete subgroup (A subgroup H of isometries of a sphere, a Euclidean plane or a hyperbolic plane is *discrete* if every point t of the surface has an open neighbourhood $O(t)$ such that for every $h \in H$ one has $h(t) = t$ or $h(O(t))$ is disjoint from $O(t)$) of the group of all (orientation-preserving and also reversing) isometries of the sphere, the Euclidean plane, or the hyperbolic plane. This results in a number of consequences of geometric and topological nature. We will proceed to discuss two of them.

Firstly, the three reflections R_0, R_1 and R_2 in the sides of a fixed flag can be written as matrices of order 2×2 or 3×3 which represent orientation-reversing isometries in the corresponding geometry. The matrices are defined over some finite extension $\mathbb{F}$ of $\mathbb{Q}$, the field of rational numbers. The extension $\mathbb{F}$ is dependent on the type (k, m) of the tessellation. Given that the matrices are representative of the distance-preserving transformations in the relevant geometry, then the groups $\Delta(2, k, m)$ can be embedded in the linear groups of the form $SL(3, \mathbb{F}) \rtimes \mathbb{Z}_2$ (for all types) or $PSL(2, \mathbb{F}) \rtimes \mathbb{Z}_2$ (for the hyperbolic types). We refer to [52] and [46] for details.

The second consequence referred to above is that the supporting surface $\mathcal{S}$ of *any* regular map M can be equipped with a geometry in which the faces of M appear as congruent regular polygons [45].

2.9 Orientably-regular maps

Consider a regular map on an *orientable* surface. Its automorphism group contains a (normal) subgroup of index 2 consisting of *orientation-preserving* automorphisms, acting on the flag set with two orbits. It is possible, however, to find maps on orientable surfaces that are *not* regular but their automorphism groups, containing only orientation-preserving automorphisms, still have two orbits on flags.

Given a map M on an orientable surface $\mathcal{S}$, automorphisms of M which preserve orientation are called *orientation-preserving.* By orientation we refer to a consistent choice of "clockwise" for our surface. The subgroup $Aut^+(M)$ of $Aut(M)$ is the *orientation-preserving automorphism group* of M. Consider the flag set $\mathcal{F}$ of the map M and take any flag $z \in \mathcal{F}$. Then there exists a unique flag z' wherein z and z' have the side labelled $0, 1$ in common which we shall call e. This side e together with the flags z and z' form what is known to be a *dart* due to its likeness to an arrow. Each dart is said to be *incident* to the (unique) vertex that is shared by the two flags z and z' whilst also being *incident* with (possibly two) faces present in z and z'. A representation of a dart can be found in Fig. 2.12. Hence, any edge of the underlying graph of M is part of two darts facing opposite directions. We refer to the *reverse* of a dart d by d^{-}. If $\mathcal{D}$ is the dart set of M, then it follows that $|\mathcal{D}| = |\mathcal{F}|/2$ provided that the sets are finite.

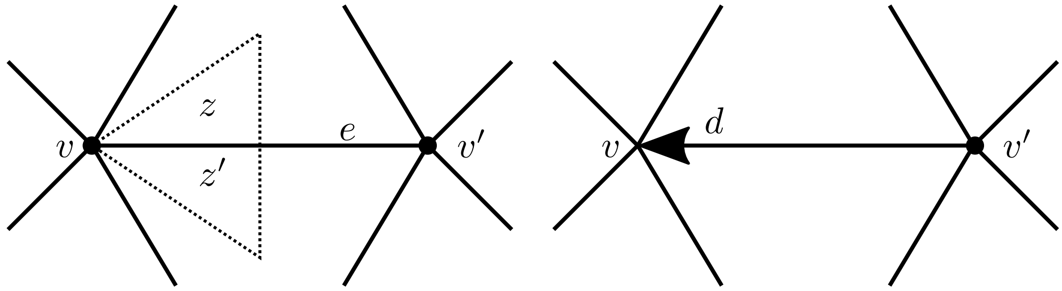

Figure 2.12: The flags z, z' adjacent across e (left) and the resulting dart d (right).

We note that the group $Aut^{+}(M)$ acts semi-regularly on the set $\mathcal{D}$ of darts of M. This means that for any two elements there is at most one element of the group taking the first element to the second. Hence, if the group $Aut^{+}(M)$ acts regularly on $\mathcal{D}$, that is, it is both transitive and semi-regular, we say that M is *orientably-regular.*

It so happens that an orientably-regular map M may or may not be regular. This depends on whether $\mathrm{Aut}^{+}(M)$ is a (normal) subgroup of $\mathrm{Aut}(M)$ of index 2 or $\mathrm{Aut}^{+}(M) = \mathrm{Aut}(M)$. The first case happens if and only if M has some automorphism that reverses the orientation of the supporting surface (an *orientation-reversing* automorphism, for short).

Maps that are both regular and orientably-regular which are regular maps on orientable surfaces are referred to as *reflexible.* Meanwhile, orientably-regular maps that are not regular, which are maps that do not contain any orientation-reversing automorphism, are called *chiral.* In Fig. 2.13 we visualise two examples of toroidal chiral maps whose underlying graphs are the complete graphs K_5 and K_7, respectively.

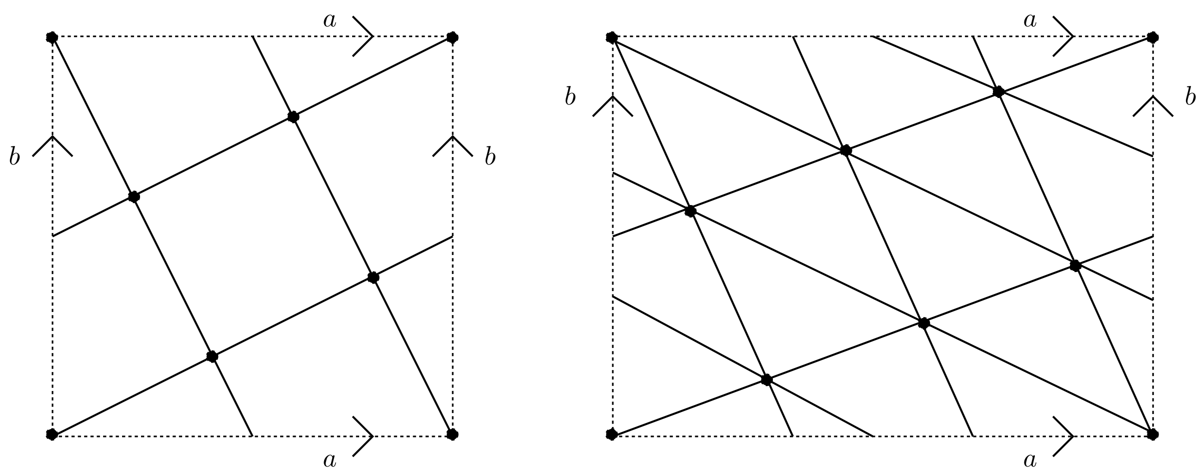

Figure 2.13: The toroidal chiral maps with underlying graphs K_5 and K_7.

To prove that the map on the left in Fig. 2.13 is chiral, consider the unique closed walk W of length 3 (a triangle in the underlying graph isomorphic to K_5) that starts at the bottom left corner and ends at the top left corner.

Suppose the surface has a clockwise orientation in the figure. If one considers the mirror image but still with clockwise orientation of the surface, and letting the image of d_1 in a supposed isomorphism of the two maps by d_1' (by regularity, d_1' can be any suitable dart in the figure), in continuing to construct the image of the walk W one would have to 'turn left' at the end-vertex of d_1' to enter the dart d_2' as image of d_2 and then 'continue straight' to reach the image d_3' of d_3, but one sees that $d_1'd_2'd_3'$ is not a closed walk, a contradiction. The group of all orientation-preserving automorphisms of the map is $\mathrm{AGL}(1,5)$, the 1-dimensional affine general linear group over the field of order 5 (isomorphic to a semi-direct product of cyclic groups of order 5 and 4).

To proceed we will consider presentations of the group $\mathrm{Aut}^+(M)$. For this, let d be any dart of M, let v be the vertex incident to it and let F be the face whose boundary contains the dart d^-. Given that the map is orientably-regular, then there exist an automorphism $s \in \mathrm{Aut}^+(M)$ taking d onto the next dart, $s(d)$, that is incident with v when moving around v along the established orientation.

This automorphism s acts locally as a rotation about the midpoint of F and the direction of rotation is the same as the orientation of M. Similarly, we can consider an automorphism $r \in \mathrm{Aut}^+(M)$ taking the dart $s(d)$ onto d^- where r acts locally as a rotation about the vertex v consistently with the orientation of M. Now given that $rs(d) = d^-$, the automorphism rs acts locally as a rotation of the dart d about the midpoint of the corresponding edge, and by regularity we know that $(rs)^2$ is the identity automorphism, as can be seen in Fig. 2.14.

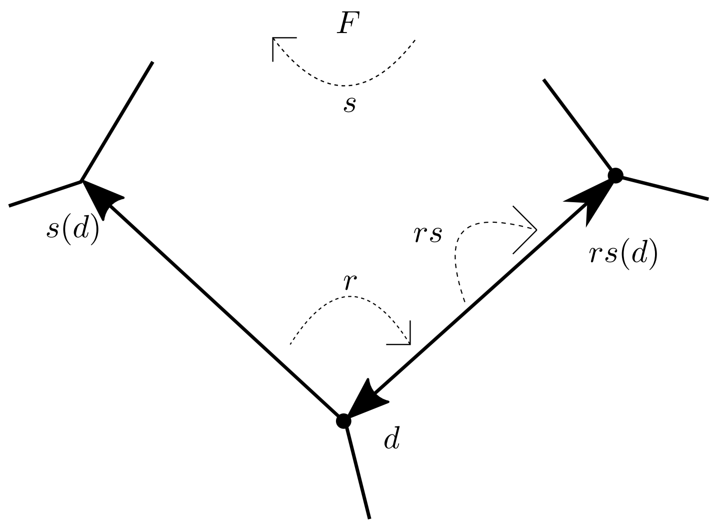

Figure 2.14: The action of s, r and rs on a dart d.

By a previous argument found at the very end of Section 2.7 the rotations r and s generate the group $Aut^+(M)$. If M is of type (k, m), the orders of r and s are equal to k and m, respectively. This gives the following presentation of the orientation-preserving automorphism group of an orientably-regular map M:

$$\mathrm{Aut}^+(M) = \langle r, s \mid r^k, s^m, (rs)^2, \ \ldots \rangle \tag{2.4}$$

where, the powers represent the true orders of the generators and dots indicate the possibility of having additional relators independent from those displayed. Letting $x = rs$ and $y = r^{-1}$ and observing that s and rsr^{-1} have the same order, m, we may rewrite (2.4) as follows

$$\mathrm{Aut}^+(M) = \langle x, y \mid x^2, y^k, (xy)^m, \ldots \rangle. \tag{2.5}$$

Conversely, for any group H generated by two elements r, s or x, y with a presentation as in (2.4) or (2.5), the corresponding orientably-regular map can be constructed as discussed in Section 2.6, except here individual flags are replaced with pairs of flags adjacent across edges. The maps are denoted by $(H; r, s)$ or $(H; x, y)$, respectively,

This results in the following:

Proposition 2.6. *Orientably-regular maps $(H; r, s)$ and $(H'; r', s')$ are isomorphic if and only if there is a group isomorphism from H onto H' taking r to r' and s to s'. Similarly, in the representation* (2.5)*, orientably-regular maps $(H; x, y)$ and $(H; x', y')$ are isomorphic if and only if there is a group isomorphism from H onto H' taking x to x' and y to y'.*

The lack of an orientation-reversing automorphism in an orientably-regular but chiral map also has its algebraic counterpart that follows from Proposition 2.6:

Corollary 2.7. *An orientably-regular map $(H; r, s)$ with H presented as in* (2.4) *is chiral if and only if there is no automorphism of the group H inverting both r and s. Similarly, an orientably-regular map $(H; x, y)$ with H presented as in* (2.5) *is chiral if and only if there is no automorphism of the group H preserving x and inverting y.*

For an orientably-regular map $M = (H; x, y)$ with H as in (2.5), the map $M_{\text{mirr}} = (H; x, y^{-1})$ is often called the *chiral mate* or the *mirror image* of M; by Corollary 2.7 the maps M and M_{mirr} are not isomorphic if and only if M is chiral.

We wrap up this section by making observations analogous to those when relating (fully) regular maps to full triangle groups $\Delta(2,k,m)$ with presentation (2.3). Letting $R = R_2R_1$ and $S = R_1R_0$ in (2.3), the two new elements generate a subgroup $\Delta^+(2,k,m)$ of index 2 in $\Delta(2,k,m)$. The group

$$\Delta^+(2,k,m) = \langle R, S \mid R^k,\ S^m,\ (RS)^2\rangle \tag{2.6}$$

is often called the *ordinary* $(2,k,m)$-triangle group, or just *triangle group* if no confusion is likely. Since $\Delta^+(2,k,m)$ contains only orientation-preserving automorphisms of the universal tessellation $\mathcal{U}_{k,m}$, it is sometimes referred to as the orientation-preserving part of the full triangle group.

Every orientably-regular map M of type (k,m) can thus be considered to be a normal quotient of the universal tessellation $\mathcal{U}_{k,m}$ formed by a normal subgroup N of the ordinary triangle group $\Delta^+(2,k,m)$ as in (2.6), that is, one may write $M = \mathcal{U}_{k,m}/N$, or $M = (H; r, s)$ for $H = \Delta^+(2,k,m)/N$ and $r = RN$, $s = SN$. Making the substitution $X = RS = S^{-1}R^{-1}$ and $Y = R^{-1}$ as above, one obtains the presentation

$$\Delta^+(2,k,m) = \langle X, Y \mid X^2,\ Y^k,\ (XY)^m\rangle \tag{2.7}$$

which is another frequently used form to represent the ordinary $(2,k,m)$-triangle group, quotients of which may be identified with orientably-regular maps with automorphism groups as in (2.5).

If now $M = (G; x, y)$ is an orientably-regular map in the presentation (2.5), as in the case of regular maps there is a natural epimorphism $\Delta^+(2,k,m) \to G$ taking the pair (X,Y) to the pair (x,y); the kernel of this epimorphism is again called the *map subgroup* of M, with no confusion likely.

As it follows from Corollary 2.4, a group $H = \langle r, s\rangle$ with presentation as in (2.4) may also define a *non-orientable* regular map M' of type (k, m), which happens if and only if H contains an involution inverting both r and s; in the notation of Corollary 2.4 an involution with this property is r_1. Equivalently, a group $H = \langle x, y\rangle$ presented as in (2.5) defines a non-orientable regular map M' provided that H contains an involution inverting both x and y, or, equivalently, commuting with x and inverting y; in this case (and again in the notation of Corollary 2.4) the inverting involution would be r_2. In either case, such a non-orientable regular map M' is orientably doubly-covered by the (orientably-regular) map $M = (H; r, s) = (H; x, y)$. Since we will use this observation for presentation (2.5) later in Chapter 4, we state it separately; for a proof we refer to the Theorem in Section 2 on page 368 of [26].

Proposition 2.8. [[26], Main Theorem] *Let $M = (G; x, y)$ be an orientably-regular map, with $G = \langle x, y \mid x^2, y^k, (xy)^m, \ldots\rangle$. This presentation of G determines, at the same time, a non-orientable regular map M' if and only if the group G contains an involution commuting with x and inverting y; in such a case M is a smooth orientable double cover of M'.*

2.10 External Symmetries

Along with map automorphisms, a map may admit ‘external symmetries’ that are not automorphisms but endow the map with a higher ‘level of symmetry’. The usual way of looking at ‘external symmetries’ is to introduce the ‘elementary’ ones first, which are self-dualities and the ones represented by the so-called ‘exponents’; we will give a description of these in what follows. General external symmetries are then obtained as arbitrary compositions of the elementary ones. They form a group under composition, called the *external symmetry group* of a map. We note that the determination of the external symmetry group of a map may be non-trivial, cf [22].

2.10.1 Duality and self-duality

As mentioned in earlier sections, the dual of a regular map $(G; r_0, r_1, r_2)$ is formed by swapping the labels 0 and 2 of each flag which is equivalent to interchanging the roles of the involutions r_0 and r_2. As a result, the corresponding *dual* regular map can therefore be identified by $(G; s_0, s_1, s_2)$ where $s_0 = r_2$, $s_2 = r_0$ and $s_1 = r_1$. Let D be the mapping which allocates the regular map $M = (G; r_0, r_1, r_2)$ its dual (regular) map $M^D = (G; r_2, r_1, r_0)$; this is the *duality operator*. In the cases where M is isomorphic to its dual M^D, we can say that M is *self-dual*. This, together with Proposition 2.2, imply that a regular map $(G; r_0, r_1, r_2)$ is self-dual if and only if there is an automorphism of the group $G = \langle r_0, r_1, r_2 \rangle$ fixing r_1 and interchanging r_0 with r_2.

Now if $M = (G; r_0, r_1, r_2)$ is orientably-regular, then so is its dual. Let G^+ be the even subgroup of G and let M be represented by $(G^+; r, s)$ where $r = r_2r_1$ and $s = r_1r_0$. Then the interchange of r_0 with r_2 corresponds to the interchange of r with s^{-1} and s with r^{-1}. This will give rise to the dual map whilst also causing a change in the orientation of the supporting surface. If we would like to retain the orientation we can use conjugation by r_1 to invert both r and s.

2.10.2 Petrie duality and self-Petrie-duality

We now look at the Petrie dual of a regular map. Such a map is obtained from $(G; r_0, r_1, r_2)$ by specifying that two flags $g, g' \in G$ are 0-adjacent in the new map if $g' = gr_0r_2$, leaving the j-adjacency rule intact for $j = 1, 2$ but changing it for $j = 0$. This means that $_1$ and r_2 remain the same as previously stated whilst r_0 is now interchanged with r_0r_2. The resulting face boundary walks are then (pairs of) orbits of the cyclic group $\langle (r_0r_2)r_1 \rangle$. Topologically, replacing r_0 with r_2 leads to the face boundary walks of the newly formed map being obtained by walking along edges on the sides of the supporting surface and following the map corners of the *original* map, but then switching the side every time the midpoint of the edge is reached. As seen in Fig. 2.15, the new face boundary walk will be in form of a 'zigzag' or 'left-right' walks, known as *Petrie walks.*

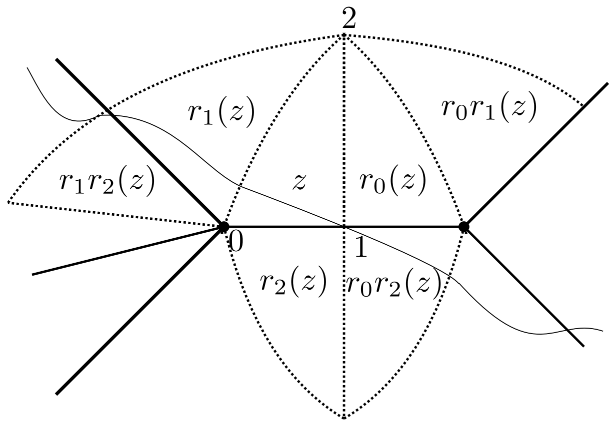

Figure 2.15: A part of a Petrie walk, illustrated by the thin line.

When considering the Petrie dual of a map, this resultant map preserves the underlying graph but not the supporting surfaces in general. So much so, that if we start off with a map on a non-orientable surface, the resultant map might be a regular map on an orientable surface; or vice-versa. Formally, a regular map $(G; s_0, s_1, s_2)$ is the *Petrie dual* of $(G; r_0, r_1, r_2)$ if $s_0 = r_0r_2$, $s_1 = r_1$ and $s_2 = r_2$. The operator P by which a regular map $M = (G; r_0, r_1, r_2)$ is assigned its (regular) Petrie dual $M^P = (G; r_0r_2, r_1, r_2)$ is called the *Petrie duality operator*. Furthermore, M is said to be *self-Petrie-dual* if $M \simeq M^P$. By using Proposition 2.2 we conclude that a regular map $(G; r_0, r_1, r_2)$ is self-Petrie-dual if and only if there is an automorphism of the group G fixing both r_1 and r_2 and interchanging r_0 with r_0r_2.

2.10.3 Rotational powers and exponents

Taking the alternative notation (2.5) for orientably-regular maps, consider an orientably-regular map $M = (H; x, y)$ of type (k, m). Suppose we want to keep the underlying graph and the orientation-preserving automorphism group the same whilst altering the map; by 'keeping' we mean that one is not allowed to replace the graph and the group by isomorphic copies. By keeping the graph, the left cosets of the cyclic groups $\langle x\rangle$ and $\langle y\rangle$ remain unchanged as they represent vertices and edges. However, the left cosets of $\langle xy\rangle$ have no restrictions as they represent the new faces on the resulting surface.

Therefore there is only one way to define a new map of the form $(H; x', y')$ with these restrictions, up to conjugation. Let y' be some other generator of $\langle y\rangle$ and let $x' = x$ as this element has to be fixed. The only choice then is to let $y' = y^e$ for some e relatively prime to k (the order of y). Then we define the e^{th} *rotational power* $M^{(e)}$ of our orientably-regular map M to be the orientably-regular map $M^{(e)} = (H; x, y^e)$. This construction has earlier been known under the name *hole operator* [28], or Wilson operator [56].

Without loss of generality let C_k^* represent the set of generators of $\langle y\rangle \simeq C_k$. Here C_k^* is the group of units of the ring of integers mod k. It is possible that, while M and $M^{(e)}$ have the same underlying graphs and the same orientation-preserving automorphism group, the supporting surfaces might be different. If k is infinite, the only possible choice for a generator for all k, distinct from 1 is when $e = -1$. In general, (for arbitrary k), supporting surfaces for M and M^{-1} are the same, and the two maps are non-isomorphic if and only if they form a chiral pair.

Now given an orientably-regular map $M = (H; x, y)$ of type (k, m) with k finite, then an $e \in C_k^*$ is an *exponent* of M if $M^{(e)} \simeq M$. Using Proposition 2.6 for generating sets of the form $\{x, y\}$, we see that an element $e \in C_k^*$ is an exponent of an orientably-regular map $(H; x, y)$ of valency k if and only if there is a group automorphism of H which fixes x and maps y onto y^e.

Let us now consider non-orientable surfaces. We similarly want to find a replacement of the three generating involutions r_0, r_1, r_2 whilst keeping the underlying graph and the automorphism group the same. In order to not change the left cosets of $r_2 = \{1, r_2\}$ or the left multiples of the coset $r_0\langle r_2\rangle = \{r_0, r_0r_2\}$, we fix r_2 and either fix r_0 or interchange r_0 with r_0r_2. But if r_0 and r_2 are left unchanged, the only way to keep the left cosets of $\langle r_1, r_2\rangle = \langle r_1r_2, r_2\rangle$ representing vertices intact, is to replace the element r_1r_2 by another generator of the cyclic group $\langle r_1r_2\rangle$.

Hence, to change the regular map $(G; r_0, r_1, r_2)$ to a regular map $(G; r_0, r_1', r_2)$ in the way just described, the only possibility is to let $r_1'r_2 = (r_1r_2)^e$ for some e such that $(r_1r_2)^e$ generates the cyclic part of the subgroup $\langle r_1, r_2\rangle$. Then we obtain the involution $r_1' = (r_1r_2)^e r_2$. Now suppose $M = (G; r_0, r_1, r_2)$ is a regular map of valency k and $e \in C_k^*$. Then we can define the e^{th} *rotational power* of M as the regular map $M^{(e)} = (G; r_0, (r_1r_2)^e r_2, r_2)$. Using Proposition 2.2 we can conclude that $M \simeq M^{(e)}$ if and only if there is a group automorphism G fixing r_0 and r_2 while taking r_1 to $(r_1r_2)^e r_2$. We mention that a non-orientable map automatically has the values 1 and -1 as exponents.

2.11 The Riemann-Hurwitz Formula and Hurwitz bounds

In this section we are looking at a few results that are of general interest given their importance in the theory of maps and their covers but will not be used in this thesis.

The Riemann-Hurwitz formula, aptly named after Bernhard Riemann and Adolf Hurwitz, describes the relationship between the Euler characteristics of two surfaces. It is often used to find the genus of a complicated Riemann surface that can be mapped to a simpler surface such as the sphere.

Consider a Riemann surface $\mathcal{S}$ with Euler characteristic $\chi(\mathcal{S}) = 2 - 2g$, where g

is the genus of this surface. Let $\mathcal{S}'$ be some other Riemann surface such that there exists a complex analytic map $\pi : \mathcal{S}' \to \mathcal{S}$, that is surjective and of degree N.

Then we obtain the formula $\chi(\mathcal{S}') = N.\chi(\mathcal{S})$ since each simplex of $\mathcal{S}$ should be covered by N simplices in $\mathcal{S}'$. Now, the map π is said to be *ramified* at a point P in $\mathcal{S}'$ if there exist analytic coordinates near P and $\pi(P)$ such that $\pi(z) = z^n$, for $n > 1$, where n is referred to as the *ramification index* at P which can also be denoted by e_P.

Now, when calculating the Euler characteristic for $\mathcal{S}'$, there is a loss of $e_P - 1$ copies of P above $\pi(P)$. This gives the celebrated Riemann-Hurwitz formula in the form,

$$\chi(\mathcal{S}') = N.\chi(\mathcal{S}) - \sum_{P \in \mathcal{S}'} (e_P - 1)$$

which if we substitute the Euler characteristic as above we can rewrite as:

$$2g(\mathcal{S}') - 2 = N.(2g(\mathcal{S}) - 2) + \sum_{P \in \mathcal{S}'} (e_P - 1)$$

We wrap up by exhibiting two important bounds. In Section 2.7 we saw that, for a finite regular map M of type (k, m) with $\mathrm{Aut}(M) = G$, the Euler characteristic χ of the supporting surface of M and the type are tied with the order of G by the formula (2.2). If one further assumes that the type of M is hyperbolic, that is, such that $1/k + 1/m < 1/2$, then the quantity $\mu(k, m) = 1/2 - 1/k - 1/m$ is positive. In such a situation the minimum of $\mu(k, m)$, taken over all hyperbolic pairs (k, m), is equal to $1/42$, attained for exactly two pairs, namely, $(k, m) = (3, 7)$ or $(7, 3)$. Combined with (2.2) this gives

$$\chi = -\mu(k, m)|G|/2 \leq -|G|/84, \quad \text{or, equivalently,} \quad |G| = |\,\mathrm{Aut}(M)| \leq -84\chi$$

for every regular map of a hyperbolic type supported by a surface of a given Euler characteristic $\chi < 0$. For an orientably-regular map M of a hyperbolic type on an orientable surface of Euler characteristic $\chi < 0$ a similar reasoning shows that $|\,\mathrm{Aut}^+(M)| \leq -42\chi$. The two estimates of the order of the automorphism groups of (orientably-) regular maps are known as *Hurwitz bounds*. They have the following obvious but important consequence:

Corollary 2.9. *Every surface of a negative Euler characteristic supports only a finite number of pairwise non-isomorphic regular maps, and every such orientable surface supports only a finite number of regular or orientably-regular maps.*

Chapter 3

Parallel products of maps, and maps from linear fractional groups

In this chapter we will review two construction methods of regular maps, which will play essential roles in subsequent chapters. The first is building new regular maps from old with the help of the so-called *parallel product of maps*, or, more precisely, by parallel products of their automorphism groups. The second method uses the available very detailed knowledge of *linear fractional groups* over finite fields for direct construction of regular maps with automorphism groups isomorphic to these groups.

We emphasise that there have been a number of other constructions of both kinds (new from old as well as directly through constructions) developed over decades of the study of regular maps; for an overview we refer to the survey paper [52]. Here we just mention two other examples. A very prolific construction of new regular maps from old is the lifting construction (also known as covering spaces construction); this is not a surprise as coverings are at the very heart of the theory of regular maps and we have encountered them in the previous chapter. Direct constructions are mostly based on a choice of suitable pairs of generators in a known group; the most studied here are matrix and permutation groups. The latter will play a role in Chapter 5 devoted to constructions of orientably-regular maps with a trivial exponent group.

The main tools based on parallel products of groups and maps in this brief review are Lemma 3.4 together with Propositions 3.5 and 3.7, all taken from [37] and addressing the situation when the quotient groups featuring in parallel products are simple; these will be used later in Chapters 4 and 5. As regards linear fractional groups, the main results in our brief survey that are used later in the same Chapters 4 and 5 as tools are Theorems 3.7 and 3.10 from [51], together with Theorem 3.11 and Proposition 3.12 from [23].

3.1 Parallel product of groups and maps

Parallel products were introduced by Wilson in [57] where he defined their existence across various types of groups and maps. In our introduction to parallel products we will also take advantage of their description in [37].

Let Γ be an arbitrary group, which we shall often refer to as the *parent group* here. For any $\ell \geq 1$ let $\{N_j \mid j \in J = \{1, 2, \ldots, \ell\}\}$ be a family of normal subgroups of Γ, and let $G_j = \Gamma/N_j$ be the corresponding quotient groups. The *parallel product* of the groups G_j for $j \in J$, denoted $||_{j\in J} G_j$, is defined as the quotient group $\Gamma/\bigcap_{j\in J} N_j$. Obviously, the parallel product as introduced here depends on the parent group, which does not explicitly appear in the notation but its choice will always be clear from the context.

Recalling a well known group-theoretic fact that an intersection of a finite number of normal subgroups of finite index in a parent group is again of finite index, it follows that a parallel product of a finite number of finite subgroups is again a finite group.

A more specific and useful description of a parallel product of groups in the context of regular and orientably-regular maps is obtained when one assumes that Γ is a finitely generated group with a specified generating set, say,

$\Gamma = \langle X_1,\, X_2,\, \ldots,\, X_n \rangle$. Then, for any fixed $j \in J$ if one lets $x_{i,j} = X_i N_j$ ($1 \le i \le n$) denote the corresponding generators of the quotient G_j, the parallel product $||_{\,j\in J} G_j$ is isomorphic to the subgroup of the direct product $\prod_{j\in J} G_j = G_1 \times G_2 \times \ldots \times G_\ell$ generated by the n elements $\overline{x}_i = (x_{i,1},\, x_{i,2},\, \ldots,\, x_{i,\ell})$ for $i \in \{1, 2, \ldots, n\}$. For the latter we will simply use the notation $\overline{x}_i = \prod_{j\in J} x_{i,j}$ customary in direct product of groups.

Moreover, in the notation introduced above, one can prove equivalence of the two ways of introducing parallel product as follows. Let $W(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ be a word determining an element of the group $G = ||_{j\in J} G_j$. Fix now some $j \in J$ and define W_j to be the word obtained from W by replacing every occurrence of the generator $\overline{x}_i$ for $i \in \{1, 2, \ldots, n\}$ by the generator $x_{i,j}$, so that the word W_j represents an element of the group G_j. Having done this for every $j \in J$, one has a correspondence of the form $W \mapsto (W_1, W_2, \ldots, W_\ell)$ sending elements of G to elements of the subgroup G_* of the product $\prod_{j\in J} G_j$ generated by $\overline{x}_i$ for $i \in \{1, 2, \ldots, n\}$. To demonstrate that this correspondence well-defines an isomorphism between G and G_* it is sufficient to show that a word W as above evaluates to identity in G if and only if every W_j for $j \in J$ evaluates to identity in G_j, but this is obvious from the way how W and W_j are related. This proof is just an extension of the argument appearing in [[57], Section 1, page 539] to an arbitrary finite number of groups.

The way parallel products of groups can be transferred to orientably-regular and fully regular maps should now be obvious. Take, for example, the full triangle group $\Delta(2, k, m) = \langle R_0, R_1, R_2 \mid R_0^2, R_1^2, R_2^2, (R_0R_1)^m, (R_1R_2)^k, (R_2R_0)^2 \rangle$ given by (2.3) as our parent group Γ; we may assume that one or both of k and m are infinite (in which case the corresponding relators are vacuous). Suppose now one has a collection of regular maps $M_j = (G_j;\, r_{0,j},\, r_{1,j}\, r_{2,j})$ for $j \in J$, of types (k_j, m_j), where $k_j \mid k$ and $m_j \mid m$ for every $j \in J$ (again, the divisibility conditions are automatically satisfied if one or both k, m are infinite). Recall that here every group G_j is a quotient of $\Gamma = \Delta(2,k,m)$ by some normal subgroup N_j for $j \in J$. Then, the *parallel product* $M = ||_{j\in J} M_j$ of these regular maps is the (regular) map $M = (G; \overline{r}_0, \overline{r}_1, \overline{r}_2)$, where $G = ||_{j\in J} G_j$ and $\overline{r}_i = \prod_{j\in J} r_{i,j}$. Equivalently, we may simply say that the map M is uniquely determined by the intersection $N = \cap_{j\in J} N_j$ as the quotient $\Delta(2,k,m)/N$.

In an analogous way, one may take a collection of normal subgroups K_j $(j \in J)$ of the ordinary triangle group $\Delta^+(2,k,m) = \langle X, Y \mid X^2,\, Y^k,\, (XY)^m, \rangle$ from (2.7) and form parallel products of orientably-regular maps $M_j^+ = (H_j; x_j, y_j)$ determined by the groups $H_j = \Delta^+(2,k,m)/K_j$ and generators $x_j = XK_j$ and $y_j = YK_j$. The parallel product $M = ||_{j\in J} M_j$ will be an orientably-regular map of the form $M = (H; \overline{x}, \overline{y})$, where $H \cong \Delta^+(2,k,m)/\cap_{j\in J} K_j$ embeds in $\prod_{j\in J} H_j$ with $\overline{x} = \prod_{j\in J} x_j$ and $\overline{y} = \prod_{j\in J} y_j$.

We have described parallel products of maps that are regular or orientably-regular, but the concept (and the construction) may be introduced for arbitrary maps, as done in [57] as well. We will illustrate this on parallel products of two maps, with an obvious way to extend it iteratively. Let M and M' be an arbitrary pair of two maps, with flag sets $\mathcal{F}$ and $\mathcal{F}'$ and monodromy groups $\mathrm{Mon}(M) = \langle \rho_0,\, \rho_1,\, \rho_2 \rangle$ and $\mathrm{Mon}(M') = \langle \rho_0',\, \rho_1',\, \rho_2' \rangle$. Let $z \in \mathcal{F}$ and $z' \in \mathcal{F}'$ be fixed flags, and for a word w in terms of ρ_i $(i = 0,1,2)$ let w' be the naturally corresponding word in terms of ρ_i' $(i = 0,1,2)$. Let $\mathcal{F} \,||\, \mathcal{F}'$ be the set of all pairs $(zw, z'w')$ where w ranges over all words as described, and for $i \in \{0,1,2\}$ let $(zw, z'w')\overline{\rho}_i = (zw\rho_i, z'w'\rho_i')$. Then the parallel product of M and M' is the map $M \,||\, M'$ with flag set $\mathcal{F} \,||\, \mathcal{F}'$ and with monodromy group $\langle \overline{\rho}_0, \overline{\rho}_1, \overline{\rho}_2 \rangle$. It may be shown (cf. [57], Section 4, p. 541) that, due to transitivity of monodromy groups, the isomorphism type of the map $M \,||\, M'$ resulting from this construction does not depend on the choice of the fixed flags in M and M'.

Let us consider an important example of this more general construction. Let M

be an arbitrary *non-orientable* map (with no assumption on symmetry), that is, a map supported by a non-orientable surface, with flag set $\mathcal{F}$. As a second map we take a 'trivial' map $M' = \Theta$, made up of a vertex and a single semi-edge on the sphere, with flag set $\mathcal{F}' = \{z_1, z_2\}$ consisting of just two flags, and with monodromy group $\mathrm{Mon}(M') = \langle \rho_0', \rho_1', \rho_2' \rangle$ containing only one non-identity permutation, the transposition $\rho_1' = \rho_2'$ interchanging z_1 with z_2 (note that $\rho_0' = id$). If $z \in \mathcal{F}$ and $z_1 \in \mathcal{F}'$ are fixed, the flag set $\mathcal{F} \,||\, \mathcal{F}'$ partitions into two sets, those containing flags of the form (zw, z_1) and flags of the form (zw, z_2), depending on whether w' is trivial or not. Hence, we end up with two orbits, transposed by $\rho_1' = \rho_2'$. It follows that for an arbitrary non-orientable map M the parallel product $M \,||\, \Theta$ is the smooth orientable double cover of M.

In [57], Wilson proved the following result (the proof in [57] is for general maps but for regular maps it follows automatically from the correspondence between regular maps and normal subgroups of triangle groups).

Theorem 3.1. [[57], Theorem 1] *A map N is a covering of M if and only if $N = N \,||\, M$.*

Corollary 3.2. [57] *A map M is orientable if and only if $M = M \,||\, \Theta$.*

From this corollary we can deduce that if either M_1 or M_2 is orientable, then the parallel product $M_1 \,||\, M_2$ is orientable. The converse to this statement is false, however, as shown by examples in [57].

With regard to regular and orientably-regular maps, which are the main object of investigation in this Thesis, Wilson in [57] stated and proved several facts. Proofs of these can easily be obtained by an appropriate modification of the argument presented in the context of construction of a double cover by means of a parallel product of a map with the trivial map Θ, or using representations of these maps as quotients of triangle parent groups by normal subgroups and their intersections; here we give just a summary statement. We recall that if

$M = (H; x, y)$ is a chiral (that is, an orientably-regular but irreflexible) map with $H = \mathrm{Aut}^+(M)$ presented as in (2.5), then its mirror image is the (orientably-regular but chiral) map $M_{\mathrm{chir}} = (H; x, y^{-1})$.

Proposition 3.3. *Let M_1 and M_2 be finite maps.*

(a) If M_1 and M_2 are orientably-regular, then so is $M = M_1 \,||\, M_2$.

(b) If M_1 and M_2 are fully regular, then so is $M = M_1 \,||\, M_2$.

(c) M_2 is a cover of M_1 if and only if M_2 is isomorphic to $M_1 \,||\, M_2$.

(d) If M_1 is an orientably-regular but chiral map and $M_2 = M_{\mathrm{chir}}$ is its mirror image, then $M_1 \,||\, M_2$ is a fully regular orientable map, universal in the sense that any reflexible orientable map that covers M is also a cover of $M_1 \,||\, M_2$.

In connection with part (c) of Proposition 3.3 let us mention still another way of introducing parallel products of maps. Namely, in [9] it is pointed out that, given a parallel product of a collection $\mathcal{M} = \{M_j \mid j \in J\}$ of maps (with no restriction on types and not even on finiteness), the parallel product $||\,\mathcal{M}$ of members of this family is isomorphic to the smallest map (with respect to the partial order induced by map coverings) which covers each of the maps in the family; the latter is known also as the *join* of the family $\mathcal{M}$.

Hence, the join is the map $\vee\mathcal{M}$, uniquely defined by taking the intersection $N^+(\vee\mathcal{M}) = \cap_{j\in J} N_j^+$ as the map subgroup in the orientably-regular case and in the parent group $\Delta^+(2, k, \infty)$. For the non-orientable case we instead take the intersection $N(\vee\mathcal{M}) = \cap_{j\in J} N_j$ in the parent group $\Delta(2, k, \infty)$. Joins and parallel products of families of maps are thus equivalent concepts.

Useful observations related to parallel products of maps have been proved in [37]; we begin with an auxiliary lemma inspired by [37] but not contained therein.

Lemma 3.4. *Let K and L be normal subgroups of a group G. If $G = KL$, then $G/(K \cap L) \cong G/K \times G/L$. In particular, this holds when K and L do not contain each other and G/K or G/L is simple.*

Proof. First, let $J = K \cap L$, which is normal in G, and set $\overline{H} = H/J$ for every subgroup H of G containing J. Then $\overline{G} = \overline{KL} = \overline{K}\,\overline{L}$, with $\overline{K} \cap \overline{L} = \overline{J}$ being trivial, and therefore $\overline{G} \cong \overline{K} \times \overline{L}$.

Also by the Second Isomorphism Theorem, $\overline{K} = K/(K \cap L) \cong KL/L = G/L$ and $\overline{L} = L/(K \cap L) \cong KL/K = G/K$, so $G/(K \cap L) \cong G/L \times G/K$.

In particular, if G/K or G/L is simple, then K or L is a maximal normal subgroup of G, and hence if K and L do not contain each other, then $G = KL$ and the conclusion holds. □

This simple Lemma has the following consequence which we will make use of in Chapter 5.

Proposition 3.5. *Let M and N be non-isomorphic maps of the same hyperbolic type (k, m), both orientably-regular, or both fully regular and non-orientable, but not a cover of each other, and let $M \,||\, N$ be their parallel product. If M, N are orientably-regular and at least one of the groups* $\mathrm{Aut}^+(M)$ *and* $\mathrm{Aut}^+(N)$ *is simple, then* $\mathrm{Aut}^+(M \,||\, N) \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N)$. *Similarly, if M and N are fully regular and non-orientable but not covering each other, and at least one of* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(N)$ *is simple, then* $\mathrm{Aut}(M \,||\, N) \cong \mathrm{Aut}(M) \times \mathrm{Aut}(N)$.

Proof. As before, let $\Delta^+(2, k, m)$ be the ordinary $(2, k, m)$-triangle group with presentation $\langle\, X, Y \mid X^2, Y^k, (XY)^m \,\rangle$, and let K and L be the *map subgroups* of $\Delta = \Delta^+(2, k, m)$ for the maps M and N respectively, so that $\mathrm{Aut}^+(M) \cong \Delta/K$ and $\mathrm{Aut}^+(N) \cong \Delta/L$. By the above description of parallel products, we have $\mathrm{Aut}^+(M \,||\, N) \cong \Delta/(K \cap L)$, with K and L not containing each other because

M and N are not a cover of each other. Then since at least one of Δ/K and Δ/L is assumed to be simple, Lemma 3.4 implies that

$$\mathrm{Aut}^+(M \,||\, N) \cong \Delta/(K \cap L) \cong \Delta/K \times \Delta/L \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N) \ .$$

The argument for fully regular non-orientable maps M and N is analogous, with Δ the full $(2, k, m)$-triangle group $\Delta(2, k, m)$ presented e.g. as in (2.3). □

One more fact that we will use here comes from a combination of a restricted version of Corollary 3.12 in [6] and a special case of Theorem 3.1 from [7] on automorphism groups of direct products.

Proposition 3.6. *If two finite groups G and H, each with trivial centre, have no common non-trivial direct factor, then* $\mathrm{Aut}(G \times H) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(H)$*. If G is a finite simple group, then* $\mathrm{Aut}(G \times G) \cong (\mathrm{Aut}(G) \times \mathrm{Aut}(G)) \rtimes C_2$*, where the C_2 factor interchanges the two copies of* $\mathrm{Aut}(G)$*.*

Referring to observations made in [37] on parallel products of groups, the most useful for us concern families containing an arbitrary finite number of groups, and hence parallel products of an arbitrary number of maps. As one would expect, in general it may not be easy to determine the automorphism group of the join $\vee\mathcal{M}$ of a family of (orientably-) regular maps M_j in terms of their automorphism groups G_j for $j \in J$. But Jones in [37] has shown that the task is manageable when the G_j are simple and non-abelian, which is a consequence of the following result (for a proof of which we refer to [37]).

Proposition 3.7. [[37], Proposition 7.1] *Let $N_1, N_2, \dots N_\ell$ be distinct normal subgroups of a parent group Γ, with each $G_j := \Gamma/N_j$ non-abelian and simple. If $K = N_1 \cap \dots \cap N_\ell$ then $\Gamma/K \cong G_1 \times \dots \times G_\ell$.*

It follows that if $\mathcal{M} = \{M_j \mid j \in J\}$ is a family of, say, regular maps with simple automorphism groups G_j, then their join $\vee\mathcal{M}$, or, equivalently, their parallel product $|| \mathcal{M}$, has automorphism group isomorphic to the direct product $G_1 \times G_2 \times \ldots \times G_\ell$. We will develop this into a tool for proving further results in Chapter 5.

3.2 Regular maps of given type from linear fractional groups

In the previous section we presented an important method for construction of new (orientably-) regular maps from old, namely, the 'parallel product method'. In this section we will examine aspects of perhaps the most prolific *direct* method of construction of regular maps of a given hyperbolic type, on orientable as well as non-orientable surfaces, which is regular maps obtained from suitable generating matrices of fractional linear groups over finite fields, that is, groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ for prime powers q.

These two families of groups have been intimately known to geometers, complex analysts and group theorists of the late 19th century, and a detailed classification of their subgroups was available through Dickson's monograph [29] as early as 1901. Generation of these groups by means of a triple of generators of given orders with product equal to identity, however, was thoroughly studied in connection with epimorphic images of triangle groups only in the 1960's in two seminal papers, by Macbeath [44] and Sah [48], the second one containing more explicit information about generating elements. The importance of the two papers for construction of regular maps was recognised somewhat later, and the work of Sah [48] was put under scrutiny from the point of view of construction of regular maps by Conder, Potočnik and Širáň in [23], giving very detailed information about ways to construct orientable as well as non-orientable regular maps of a given type with automorphism groups isomorphic to one of the groups $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$.

We will not go into extensive details which can be found in [23]. Instead, we extract a few results from other available resources which contain constructions relevant for our purposes, based on the material in [23]. The first one concerns constructions of non-orientable regular maps of a given hyperbolic type with linear fractional automorphism groups. It also illustrates a common approach to constructions within fractional linear groups over finite fields: one begins by considering matrices of dimension 2 and determinant 1 over an algebraically closed field of a given prime characteristic p, that is, with the group $\mathrm{SL}(2, K)$ for such a field K, and then gradually works their way towards the groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ for q a power of p. The approach of Širáň [51] presented here is restricted to primes sufficiently large for the purpose of building suitable regular maps.

Theorem 3.8. [[51], Theorem 1] *Let (k, m) be a hyperbolic pair and let H be a finite group with presentation $\langle r, s \mid r^k = s^m = (rs)^2 = \ldots = 1\rangle$. Assume that H is a subgroup of $\mathrm{PSL}(2, K)$, where K is an algebraically closed field of characteristic p larger than k and m. Let $e = e(k, m)$ be the smallest positive integer j such that $n|(p^j - \epsilon_n)$ for each $n \in \{2k, 2m, 4\}$ and some $\epsilon_n \in \{1, -1\}$. Then:*

1. *there exist primitive $(2k)^{th}$ and $(2m)^{th}$ roots of unity ξ and η in the $GF(p^{2e})$, and an injective homomorphism φ from H into a subgroup of $\mathrm{PSL}(2, k)$ conjugate to $\mathrm{PSL}(2, p^e)$ such that*

$$\varphi(r) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} \qquad \varphi(s) = \beta \begin{bmatrix} (\eta^{-1} - \eta)\xi^{-1} & -D \\ 1 & (\eta - \eta^{-1})\xi \end{bmatrix},$$

   where $D = \xi^2 + \xi^{-2} + \eta^2 + \eta^{-2}$ and $\beta = (\xi - \xi^{-1})^{-1}$;

2. *if e is even, $e = 2f$, then the injection φ is an isomorphism $H \cong \mathrm{PGL}(2, p^f)$ if and only if $D \neq 0$ and either*

(a) there is an even entry $n \in \{k, m\}$ and an $\epsilon \in \{1, -1\}$ such that n divides $(p^f - \epsilon)$ but $2n$ does not, while the other entry divides $(p^f - \epsilon')/2$ for some $\epsilon' \in \{1, -1\}$; or

(b) both k and m are even and for any $n \in \{k, m\}$ there exists an ϵ'_n such that n is a divisor of $(p^f - \epsilon'_n)$ but $2n$ is not;

3. *the injection φ is an isomorphism $H \cong \mathrm{PSL}(2, p^e)$ if and only if $D \neq 0$ and either e is odd, or the pair (k, m) together with an even e do not satisfy any of the above conditions (a) and (b);*

4. *the regular map $M = (H; r, s)$ has type (k, m) and is reflexible in any case; and*

5. *if $D \neq 0$, the supporting surface of M is non-orientable if and only if either $\varphi(H) = \mathrm{PSL}(2, p^e)$ and $-D$ is a square in $\mathrm{GF}(p^e)$, or else if $e = 2f$ and $\varphi(H) = \mathrm{PGL}(2, p^f)$.*

Theorem 3.8 is proved by following almost verbatim the proof of Proposition 2.2 of [23] and then applying Proposition 4.6 of [23] to determine the smallest power of p referred to in Theorem 3.8; this is why we omit the details.

This theorem lead to the main result in [51], by ensuring that D is non-zero. If we let ζ be a primitive element of $\mathrm{GF}(p^e)$, then

(a) all $(2k)^{th}$ primitive roots of unity have the form $\xi_i = \zeta^{d(i)}$ for $d(i) = i(p^e - 1)/(2k)$, where $gcd(i, 2k) = 1$ and $-k < i < k$, and

(b) all $(2m)^{th}$ primitive roots of unity have the form $\eta_j = \zeta^{d(j)}$ for $d(j) = j(p^e - 1)/(2k)$, where $gcd(j, 2m) = 1$ and $-m < j < m$.

Such i and j are called *admissible*, and we let $D = D(i, j) = \xi_i^2 + \xi_i^{-2} + \eta_j^2 + \eta_j^{-2}$.

Lemma 3.9. [[51], Lemma 1] *For any hyperbolic pair (k, m) with $k \leq m$ and any prime $p > k$ we have $D(i, j) = 0$ for some admissible i and j if and only if* *either k is even, $m = k$ and $\pm 2i \pm 2j \equiv k \mod 4k$ for some choice of signs, or else k is odd, $m = 2k$ and $\pm 2i \pm j \equiv k \mod 4k$ for some choice of signs. In this situation, the group from Theorem 3.8 is cyclic, of order k and $2k$, respectively.*

The above helps us understand when $D(i, j)$ is non-zero. It follows then that if either both k and m are odd or both are even or if they have different parity such that $k \neq 2m$ and $m \neq 2k$, then $D(i, j)$ is non-zero for all admissible i and j. The next lemma tells us about the existence of such admissible i and j.

Lemma 3.10. [[51], Lemma 2] *For any hyperbolic pair (k, m) and any prime p larger than k and m there exist admissible i and j such that $D(i, j) \neq 0$.*

The main construction result of Širáň in [51] was then the following theorem.

Theorem 3.11. [[51], Theorem 2] *Let (k, m) be a hyperbolic pair with at least one even entry. Then, there is an infinite set P of primes such that for every $p \in P$ there exists a finite, non-orientable, regular maps of type (k, m) with automorphism group isomorphic to* $\mathrm{PGL}(2, p)$ *for suitable primes p.*

Proof. Let t be the least common multiple of k and m, and observe that for any chosen $\varepsilon \in \{+1, -1\}$ the numbers 2ℓ and $\ell + \varepsilon$ are relatively prime because ℓ is even. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes in the arithmetic sequence with first term $\ell + \varepsilon$ and difference 2ℓ. Let P be the set of such primes and let $p \in P$. By Lemma 3.10 there is a pair of admissible positive integers (i, j) such that $D(i, j) \neq 0$. Let now 2^a and 2^b be the largest powers of 2 dividing, respectively, k and m. By the assumption that one of k, m is even, taking duality into the account we may without loss of generality assume that $a \geq b$ and $a \geq 1$.

If $a > b$, then k and $2m$ divide $(p - \varepsilon)$ but $2k$ does not, which shows that condition (a) of part (2) of Theorem 3.8 holds. Alternatively, if $a = b$, for any

$n \in \{k, m\}$, the entry n is a divisor of $p - \varepsilon$ while $2n$ is not, implying that condition (b) of part (2) of Theorem 3.8 holds. Since $D(i, j) \neq 0$ in both cases, we conclude from (2) and (5) of 3.8 that $\mathrm{PGL}(2, p)$ is the automorphism group of a non-orientable regular map of type (k, m) if one of the entries is even. □

This result proved the existence of an infinite number of non-orientable regular maps of a given hyperbolic type with automorphism group isomorphic to a linear fractional group over a finite field, for approximately three quarters of types. The case when both entries are odd was established later in the work of Jones, Mačaj and Širáň [39] by more advanced algebraic methods, but we will not need it here; all we will make use of later in Chapter 5, is Theorem 3.11.

In Chapter 5 we will make use of a restatement of Theorem 3.8, obtained from the original statement by letting $y = \varphi(r)$ and $x = \varphi((rs)^{-1})$, so that x is an involution and $xy = \varphi(s^{-1})$ has order m; we also replace K by a field of characteristic p dividing nether k nor m and containing primitive $(2k)^{\mathrm{th}}$ and $(2m)^{\mathrm{th}}$ roots of unity for a hyperbolic pair (k, m). Existence of such primitive roots modulo a suitable power of a prime (coprime to k and m) may be shown as follows. In the infinite set of powers an odd prime p there are two giving the same residue mod $2km$; say $p^j \equiv p^i \bmod 2km$ for some positive integers i, j such that $i < j$. Since p is relatively prime to $2km$, it follows that $p^{j-i} \equiv 1 \bmod 2km$, so that $p^{j-i} - 1$ is divisible by both $2k$ and $2m$. This implies that the finite field of order $q' = p^{j-i}$ contains primitive $(2k)^{\mathrm{th}}$ and $(2m)^{\mathrm{th}}$ roots of unity.

The indicated restatement of Theorem 3.8 is now as follows.

Theorem 3.12. *Let (k, m) be a hyperbolic pair and let $p \neq 2$ be a prime dividing nether k nor m. Let q' be a power of p such that the $\mathrm{GF}(q')$ contains a primitive $(2k)^{\mathrm{th}}$ root of unity ξ and a primitive $(2m)^{\mathrm{th}}$ root of unity η. Let e be the smallest positive integer such that both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$ are in $\mathrm{GF}(p^e)$. Let $y = \pm\mathrm{diag}(\xi, \xi^{-1}) \in \mathrm{PSL}(2, q')$ be a diagonal matrix with ξ and ξ^{-1} in the* *main diagonal. Then, the same group contains an involution x such that the product xy has trace $\pm(\eta+\eta^{-1}$ and hence has order m. If $D(\xi,\eta)\neq 0$, then the group $\langle x, y \mid x^2, y^k, (xy)^m, \ldots\rangle$ is conjugate either to* $\mathrm{PSL}(2,p^e)$ *or to* $\mathrm{PGL}(2,p^{e/2})$ *(for e even in the latter case).*

Our last construction of regular maps with linear fractional automorphism groups over finite fields will take place in groups based on fields of characteristic 2. In such a case, for any $n\geq 1$ one has the isomorphisms $\mathrm{PSL}(2,2^n)\cong\mathrm{PGL}(2,2^n)\cong\mathrm{SL}(2,2^n)$. Moreover, here the group $\mathrm{SL}(2,2^n)$ has the striking property that its only elements of even order are involutions, which follows from the classification of subgroups of special 2-dimensional linear groups [29]. As it turns out (just as in [48, 23]), the construction of maps $M(\xi,\eta)$ given in Theorems 3.8 and 3.12 carries over to characteristic 2, except that in that case the elements ξ and η are k^{th} and m^{th} primitive roots in some (possibly larger) field of characteristic 2, with q again being the smallest power of 2 containing both $\xi{+}\xi^{-1}$ and $\eta{+}\eta^{-1}$.

To develop a few more details which will be needed later in Chapter 5, let (k,m) be a hyperbolic type but this time *with both entries odd*, and let ξ and η be primitive k^{th} and m^{th} roots of unity in some field $\mathrm{GF}(q')$ of characteristic 2, such that $\xi\neq\eta$, η^{-1}, and then define $\delta=\xi+\xi^{-1}+\eta+\eta^{-1}$; note that this δ is a square root of the quantity D referred to earlier. Furthermore, let $q=2^n$ be the smallest power of 2 such that the field $\mathrm{GF}(2^n)$ contains both $\xi+\xi^{-1}$ and $\eta+\eta^{-1}$. This condition is known ([48], proof of Proposition 1.1, or [23], Lemma 4.2 and Proposition 4.3) to be equivalent to n being the smallest positive integer such that k divides one of 2^n-1 and 2^n+1, and also m divides one of 2^n-1 and 2^n+1; which follows e.g. from Lemma 4.2 and Proposition 4.3 of [23]. This condition is usually abbreviated by stating that k and m divide $2^n\pm 1$.

For powers of 2 we are in a similar situation as with Theorem 3.7. Namely, if one follows (and again, almost verbatim) the proof of Proposition 2.2 of [23] but this time for $p = 2$ and odd k and m (with $l = 2$ in the notation of [23]), a subsequent application of Proposition 4.6 of [23] to determine the smallest power of 2 alluded to above, and making a similar substitution as done before the statement of Theorem 3.12 yields the following result.

Theorem 3.13. *Let (k, m) be a hyperbolic pair with both entries odd, and let q' be a power of 2 such that $\mathrm{GF}(q')$ contains a primitive k^{th} root of unity ξ and a primitive m^{th} root of unity η, and suppose that $\delta = \xi + \xi^{-1} + \eta + \eta^{-1} \neq 0$. Further, let n be the smallest positive integer such that both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$ are in $\mathrm{GF}(2^n)$. Then, letting*

$$x = x(\xi, \eta) = \frac{1}{\xi+\xi^{-1}} \begin{bmatrix} \eta+\eta^{-1} & \delta^2\xi \\ \xi^{-1} & \eta+\eta^{-1} \end{bmatrix}, \quad y = y(\xi, \eta) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}.$$

the group $H = \langle x, y \mid x^2, y^k, (xy)^m, \ldots \rangle$ is conjugate to $\mathrm{SL}(2, 2^n)$ and isomorphic to the automorphism group of a non-orientable regular map $M = (H; x, y)$.

Note here that $\eta+\eta^{-1} \neq 0$ because a finite field of characteristic 2 has no element of multiplicative order 2, and the condition $\delta \neq 0$ is, by Lemma 2.4 of [23], equivalent to $\xi \neq \eta, \eta^{-1}$. Moreover, the map $M = (H; x, y)$ is non-orientable since by Proposition 3.2 of [23] and our Proposition 2.8 in Section 2.9, the group H contains an involution conjugating x and y to their inverses.

Remark. To wrap up, we prove that, in the situation of Theorem 3.12, the involution of H inverting x and y is unique. Indeed, suppose that c and c' were two distinct involutions commuting with x and inverting y. Then the product cc' would commute with both x and y and hence would be a central element of H. But the group $H \cong \mathrm{SL}(2, 2^n)$ has trivial centre (it is simple for $n \geq 2$, which is the case here as (k, m) is a hyperbolic pair), which implies that the involutions c and c' must be identical, a contradiction. We note that the same proof works in the more general setting for any regular map with automorphism group isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$ for an arbitrary prime power q.

Chapter 4

Orientable and Non-Orientable maps with given exponent group

4.1 Preliminaries

This chapter and the one that follows, present results on exponents of regular and orientably-regular maps, which have been published [4] or submitted for publication [5] and have been co-authored by the author of this Dissertation. Notation regarding orientably-regular maps in these papers is, to a large extent, unified and follows the one that has been introduced in Section 2.9 in the presentation (2.5). That is, if M is an orientably-regular map with $\mathrm{Aut}^+(M) = G$, say, of type (k, m), then we will assume that $G = \langle x, y \mid x^2, y^k, (xy)^m, \ldots \rangle$, and we let $M = (G; x, y)$ as before. By the theory outlined in Sections 2.7 and 2.9, and by Proposition 2.8 in particular, the group G with the same presentation may also be the automorphism group of a non-orientable regular map M' provided that G contains an involution, say, c, inverting both x and y (or, equivalently, commuting with x and inverting y). Such a map M' is then smoothly double-covered by the orientably-regular map M. For M' we will then reserve the notation $M' = (G; x, y, c)$.

We continue by recalling from Section 2.10 what an exponent of a map is. If $M = (G; x, y)$ is an orientably-regular map, then j is an exponent of M if and only if there exists an automorphism that fixes x and maps y to y^j. In the case where M is a non-orientable regular map presented in the form $(G; x, y, c)$, then j is an exponent if and only if there is an automorphism of G that fixes both x

and c whilst taking y to y^j. We refer to such an automorphism for both orientable and non-orientable maps as the *j-rotational automorphism*.

Clearly the product of any two exponents of an orientably-regular or fully regular M of valency k is also an exponent of M. The set of all exponents of M forms a subgroup called the *exponent group* of M which is denoted by $\text{Exp}(M)$. We also recall that this type of mapping was studied by Wilson [56] and was referred to as a *hole operator*.

For an orientably-regular map of valency k, we can have the case where the exponent group is trivial or is solely made up of 1 and -1, or the other extreme where the map admits the full exponent group C_k^*, the group of units in the ring C_k of order k.

There are infinitely many finite orientably-regular maps of valency k with trivial exponent group [3]. This was proved using a method that prohibited the creation of new automorphisms in lifted maps, but with no control over face lengths. In [53], Širáň et al., proved that for every valency k and face length m such that $1/k + 1/m \leq 1/2$, there exist infinitely many finite orientably-regular and reflexible maps of type (k, m) that admit no exponents other than 1 and -1. This was done using residual finiteness of triangle groups.

Furthermore, [24] used residual finiteness of triangle groups to show that for every integer $k \geq 3$, there exist infinitely many finite orientably-regular maps with exponent group equal to C_k^*. In this method, there was once again no restriction over the face length. These types of maps were called *kaleidoscopic* maps in [2] where a covering construction was given for a kaleidoscopic k-valent regular map which is invariant under duality and Petrie duality for every even k. A different construction was considered in [37] for an infinite set of odd numbers of the form $2^{2n} - 1$.

In [24], arbitrary subgroups of the group of units modulo k were considered to show that for every $k \geq 3$ and every given subgroup U of C_k^*, there exist infinitely many finite orientably-regular maps of valency k with exponent group equal to U.

The result was formally stated as follows.

Theorem 4.1. [[24], Theorem 1] *For every $k \geq 3$ and every subgroup U of C_k^*, there are infinitely many finite orientably-regular maps of valency k with exponent group equal to U.*

The aim of this chapter is twofold. First, we give an alternative proof of Theorem 4.1 using parallel product of maps. While the original proof of this result in [24] was based on a clever but rather ad-hoc method, our approach appears to be a bit more systematic. What is more, this approach allows for an adaptation to prove a non-orientable analogue of Theorem 4.1, which is Theorem 4.10, representing (besides the new proof mentioned above) another main result of this Chapter.

While the new proof of Theorem 4.1 emerged as a result of our joint explorations (under the guidance of my supervisor and by correspondence with the remaining two co-authors of the paper [4] during my study of parallel products and especially the paper [37], the essential ingredients for the proof of Theorem 4.10 were developed as part of my independent exploration of generating matrices for groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ (Theorems 4.8 and 4.9).

4.2 Orientably-regular maps

Let J be a finite linearly-ordered set, and let $\mathcal{M} = (M_j : j \in J)$ be a correspondingly ordered family such that one of the following holds.

a) Every map M_j is orientably-regular and has the form $M_j = (G_j; x_j, y_j)$ where the group G_j has presentation of the form $\langle x, y \mid x^2 = y^k = (xy)^m = \ldots = 1\rangle$ with map subgroup $N_j^+ = N^+(M_j)$ in the parent group $\Gamma = \Delta^+(2, k, \infty)$; or

b) Every M_j is fully regular and has the form $M_j = (G_j; x_j, y_j, c_j)$ where G_j has a presentation $\langle x, y, c \mid x^2 = y^k = (xy)^m = c^2 = (xc^2) = (cy)^2 = \ldots = 1\rangle$ with map subgroup $N_j = N(M_j)$ in the parent group $\Gamma = \Delta(2, k, \infty)$.

Note that in the second case above, the parent group $\Delta(2, k, \infty)$ is assumed to have presentation $\langle X, Y, C \mid X^2, Y^k, C^2, (XC)^2, (YC)^2\rangle$.

Consider the *join*, or, equivalently, the *parallel product* of the family of maps $\mathcal{M}$, introduced in Section 3.1. We recall that this join is the map $\vee\mathcal{M}$, uniquely defined by taking the intersection $N^+(\vee\mathcal{M}) = \cap_{j\in J} N_j^+$ as the map subgroup in the orientably-regular case and in the parent group $\Delta^+(2, k, \infty)$. For the non-orientable case we instead take the intersection $N(\vee\mathcal{M}) = \cap_{j\in J} N_j$ in the parent group $\Delta(2, k, \infty)$.

The map subgroup $N^+ = N^+(\vee\mathcal{M})$ is a normal subgroup of finite index in $\Delta^+(2, k, \infty)$ and hence $\vee(M)$ is a finite orientably-regular map with valency k. We can also describe $\vee\mathcal{M}$ as the map $(G; x, y)$ or $(G; x, y, c)$, where x and y are the images of X and Y in the quotient $G = \Delta^+(2, k, \infty)/N^+(\vee\mathcal{M})$. Furthermore, $\vee\mathcal{M}$ is a regular cover of each of the maps M_j since $gN^+ \mapsto gN_j$ for $g \in \Delta^+(2, k, \infty)$.

Similarly for the non-orientable case, $N = N(\vee\mathcal{M})$ is a normal subgroup of finite index in $\Delta(2, k, \infty)$, making $\vee\mathcal{M}$ a k-valent fully regular map. As above, $\vee\mathcal{M}$ is the map $(G; x, y, c)$ where x, y and c are the images of X, Y and C in the quotient $G = \Delta(2, k, \infty)/N(\vee\mathcal{M})$, and hence, $\vee\mathcal{M}$ is a regular cover given that $gN \mapsto gN_j$ for $g \in \Delta(2, k, \infty)$.

By Proposition 3.7, in the case when the groups G_j for $j \in J$ are simple and non-abelian the group $G = \Gamma/N(\vee\mathcal{M})$ for Γ equal to $\Delta^+(2,k,\infty)$ or $\Delta(2,k,\infty)$ is determined relatively easily: one simply has $G \cong G_1 \times \ldots \times G_r$. But we will state a slightly more general (but still obvious) consequence of Proposition 3.7 here.

Theorem 4.2. *Let Γ be the free product of cyclic groups of (not necessarily distinct) orders m and n with representation $\Gamma = \langle R, S \mid R^m = S^n = 1\rangle$ and let $(\Gamma_j : j \in J)$ be a non-empty finite ordered family of epimorphic images of Γ, given by pairwise distinct normal subgroups K_j of Γ such that $\Gamma/K_j \cong \Gamma_j$ for all $j \in J$ and with intersection $K = \cap_{j\in J} K_j$. If each Γ_j is a simple non-abelian group, then the mapping*

$$\theta : \Gamma/K \to \prod_{j\in J} \Gamma_j \cong \prod_{j\in J} (\Gamma/K_j) \quad \text{given by} \quad \theta : Kz \mapsto (K_j z)_{j\in J} \tag{4.1}$$

is a group isomorphism taking the images of R and S in Γ/K to the product of the images of R and S in the groups $\Gamma_j \cong \Gamma/K_j$.

This theorem tells us that under the given assumptions, the group homeomorphism θ defined by (4.1) is *surjective*, which is equivalent to the statement that the θ-images of R and S form a generating pair for the whole product $\prod_{j\in J}(\Gamma/K)$, rather than for a proper subgroup; we refer to [37] and [36] for more details.

We now apply Theorem 4.2 in the special case where $(m,n) = (2,k), (R,S) = (X,Y), \Gamma = \Delta^+(2,k,\infty) = \langle X, Y \mid X^2 = Y^k = 1\rangle$ and $\Gamma_j = G_j$ for all $j \in J$, coming from the family $M_j = (G_j; x_j, y_j)$ of orientable-regular maps considered earlier.

Theorem 4.3. *Let $\mathcal{M} = (M_j : j \in J)$ be a finite ordered family of orientably-regular maps $M_j = (G_j; x_j, y_j)$ where the groups G_j are non-abelian and simple, but not necessarily distinct. If the map subgroups $N^+(M_j)$ are* *pairwise distinct (as j runs through J), then the direct product $\overline{G} = \prod_{j \in J} G_j$ is generated by $\overline{x} = (x_j)_{j \in J}$ and $\overline{y} = (y_j)_{j \in J}$, and the corresponding orientably-regular map $(\overline{G}; \overline{x}, \overline{y})$ is isomorphic to the join $\vee \mathcal{M}$.*

We are now almost ready to present our first main result in this chapter, which is a construction of orientably-regular maps of a given valency and with a given exponent group. The one extra bit we will need is a way to determine the automorphism group of a direct product of some number of isomorphic copies of a group. For this, we refer to a deep study of automorphism groups of direct products of groups by Bidwell [7] and state here just a consequence we need. For brevity, we let K^n denote a direct product of n copies of K.

Theorem 4.4. [[7], Theorem 3.1] *Let H be a non-abelian simple group, and let $G = H^n$. Then $\mathrm{Aut}(G) = (\mathrm{Aut}(H))^n \rtimes S_n$ with the symmetric group S_n of degree n acting on $(\mathrm{Aut}(H))^n$ by permuting the n constituents of the direct product.*

All the ingredients required to prove our main result are now complete.

Theorem 4.5. *Let $M = (G; x, y)$ be an orientably-regular map with valency $k \geq 3$ and let U be a subgroup of the group U_k of units modulo k. For every $j \in U$, let $M^{(j)} = (G; x, y^j)$ be the j^{th} rotational power of M, and let $\mathcal{M}$ be the family $(M^{(j)} : j \in U)$, linearly-ordered by elements of U. If G is a non-abelian simple group and M admits only the trivial exponent 1, then the exponent group of the orientably-regular map $\vee \mathcal{M}$ is equal to U.*

Proof. Recall from subsection 2.10.3, that given two maps M and $M^{(j)}$ embedded on different surfaces, even if the orders of xy and xy^j are the same, the maps themselves need not be isomorphic. The unit j mod k is called an exponent of the map M when M and $M^{(j)}$ are isomorphic.
Here, $\mathrm{Exp}(M)$, the group of exponents of M, is trivial. Hence, the maps $M^{(j)}$ are pairwise non-isomorphic. This in turn implies that there is no automorphism

of the group G that takes (x, y^j) to (x, y^l) for two different $j, l \in U$. This also means that the map subgroups $N^+(M^{(j)})$ for $j \in U$ are pairwise distinct. Then since G is simple, using Theorem 4.3 to the family $\mathcal{M} = (M^{(j)} : j \in U)$ which allows us to show that the group $\overline{G} = \prod_{j \in U} G_j$ of orientation-preserving automorphisms of the join $\overline{M} = \vee_{j \in U} M^{(j)}$ is a direct product of $|U|$ copies of G, and is generated by the two elements $\overline{x} = (x)_{j \in U}$ and $\overline{y} = (y^j)_{j \in U}$.

We now show that U is the exponent group of the orientably-regular map $\overline{M} = (\overline{G}; \overline{x}, \overline{y})$. It is easy to see that $\text{Exp}(M)$ contains U, because multiplication by any element of U induces a permutation of the constituents $M^{(j)} = (G; x, y^j)$ of $\overline{M}$ and an automorphism group of the group $\overline{G}$ permuting its $|U|$ direct factors.

Conversely, let d be an exponent of $\overline{M}$, and let ψ be an automorphism of $\overline{G}$ that fixes $\overline{x}$ and takes $\overline{y} = (y^j)_{j \in U}$ to $\overline{y}^d = (y^{jd})_{j \in U}$. Then since $\overline{G}$ is a direct product of $|U|$ copies of the same simple group G, then, by Theorem 4.4, the group $\text{Aut}(\overline{M})$ is isomorphic to $(\text{Aut}(G))^n \rtimes S_n$ where $n = |U|$. It follows that we can multiply ψ by a suitable element of $\text{Aut}(\overline{M})$ to obtain an automorphism τ of $\overline{G}$ that preserves G_1, which is the first copy of G in the expansion $\overline{G} = \prod_{j \in U} G_j$, and then τ induces an automorphism of G_1 ($\cong G$) taking (x, y) to (x, y^{ld}) for some $l \in U$. But once again we can use the fact that $\text{Exp}(M)$ is trivial, to conclude that $ld = 1$ in U, and hence $d = l^{-1} \in U$. Thus $\text{Exp}(\overline{M}) = U$, completing the proof. $\square$

There are many orientably-regular maps of a given valency $k \geq 3$ with a non-abelian simple automorphism group and a trivial exponent group. A number of such constructions can be found in [21] for chiral orientably-regular maps of given valency $k \geq 4$ with automorphism group isomorphic to some alternating group; in forthcoming chapters we will show that these maps have trivial exponent groups.

To find other such maps we can use a result from [13].

Theorem 4.6. [[13], Theorem 6.3] *For every integer $m \geq 7$, all but finitely many of the alternating groups A_n can be generated by two elements x and y such that x, y and xy have orders 2, 3 and m respectively, and such that there exists no automorphism of $\langle x, y \rangle = A_n$ taking x and y to x^{-1} and y^{-1}, respectively.*

This shows that for every $m \geq 7$ there are infinitely many chiral orientably-regular maps with type $(3, m)$ and with automorphism group isomorphic to an alternating group.

Note also that finding examples for $k = 3, 4, 6$ is trivial, as, in each of these cases, the only possible exponents are ± 1.

We now note a few explicit examples with valency $k \geq 5$. Let G be the alternating group A_{2k} and define two permutations x and y in G by $x = (k-2, k-1)(k, k+1)(2k-3, 2k-2)(2k-1, 2k)$ and $y = (1, 2, \ldots, k)(k+1, k+2, \ldots, 2k)$. This can be represented by the coset diagram in Fig 4.1.

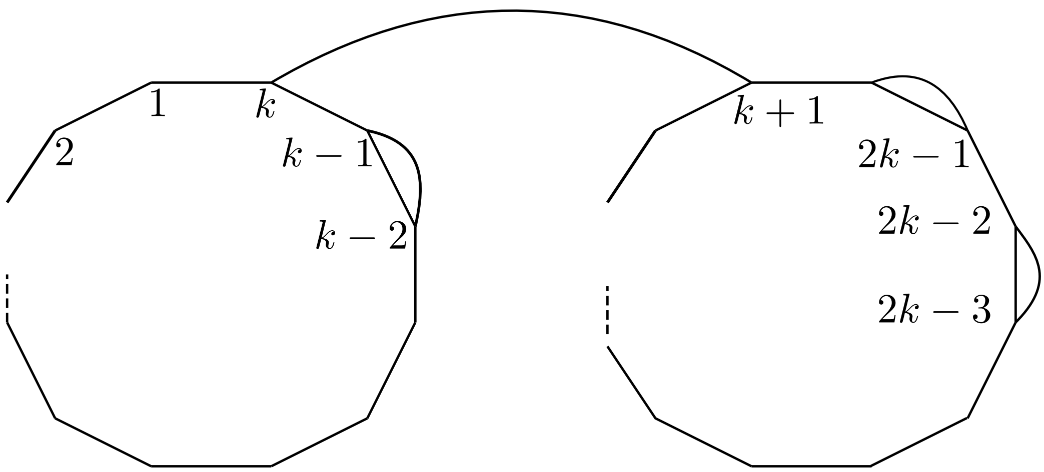

Figure 4.1: Coset diagram for G defined by x and y

It was shown in [21] that $xy = (1, 2, \ldots, k-2, k, k+2, \ldots, 2k-3, 2k-1, k+1)$ which is a single cycle of length $2k-3$ fixing points $k-1, 2k-2$ and $2k$. It's conjugate $y^{-2}(xy)y^2$ of xy gives another cycle of length $2k-3$ fixing $1, k+2$ and $2k$, implying that the stabiliser of the point $2k$ (which contains $\langle xy, y^{-2}(xy)y^2\rangle$ as a subgroup) is transitive on the set of the remaining $2k-1$ points. It follows that G is 2-transitive and primitive, Since xy is a single cycle fixing exactly 3 points, the group G is isomorphic to the symmetric or alternating group of degree $2k$ by a classical theorem of Jordan, see [35]. But as both x and y are even permutations we conclude that $G \cong A_{2k}$

Assume that the corresponding map M is reflexible, that is, a map that is both regular and orientably regular. Then the mirror symmetry would interchange the fixed points of xy with the fixed points of xy^{-1} which would induce a reflection of each of the k-cycles of y. This would mean that the mirror symmetry maps the transposition $(k, k+1)$ to $(k-3, 2k-4)$ which is not possible. Hence M is chiral. In subsequent chapters we will also prove that the exponent group of M is trivial.

A combination of these examples with Theorem 4.5 yields the following theorem, proved in [24] using a different approach and in a stronger form (furnishing there infinitely many k-valent orientably-regular maps with given exponent group, for every $k \geq 3$). Our approach presents an alternative proof using parallel products, as stated earlier in this chapter.

Theorem 4.7. *For every integer $k \geq 3$ and for every subgroup U of units modulo k, there exists an orientably-regular map of valency k with exponent group equal to U.*

4.3 Non-Orientable regular maps

We now turn our attention towards proving the same result for non-orientable regular maps of arbitrary valency greater than two, having the smallest possible exponent group $\{\pm 1\}$ which we will refer to as *almost trivial* exponent group. To do this, we will take what we have shown for orientably-regular maps and adjust it, through the use of linear fractional groups, to apply to non-orientable regular maps.

Recall from Section 4.2 that a non-orientably-regular map M of type (k, m) is given by $M = (G; x, y, c)$ and its automorphism group is represented by

$$\mathrm{Aut}(M) = \langle x, y, c \mid x^2 = y^k = (xy)^m = c^2 = (xc)^2 = (cy)^2 = \ldots = 1 \rangle. \tag{4.2}$$

We will look at a series of results that leads us to our main result in this section about the existence of infinitely many non-orientable regular maps with almost trivial exponent group.

The first result will show the existence of non-orientable regular maps of a given type with an almost trivial exponent group for the group $G = \mathrm{PGL}(2, p)$ and a generator which lies in the subgroup $\mathrm{PGL}(2, p) \backslash \mathrm{PSL}(2, p)$, for some prime p.

Theorem 4.8. *For every integer $k \geq 4$ and every prime p congruent to $2k + 1$ modulo $4k$, there exists a non-orientable regular map $M_{k,p} = (G; x, y, c)$ of type $(k, 2k)$ for the group $G = \mathrm{PGL}(2, p)$ with $\mathrm{Exp}(M_{k,p}) = \{\pm 1\}$, such that the generator y of order k lies in the subgroup $\mathrm{PSL}(2, p)$ of G while the the involution x is not contained in the subgroup $\mathrm{PSL}(2, p)$ of G.*

Proof. Let p be a prime number such that $p \equiv 2k + 1 \bmod 4k$, where $k \geq 4$ is the valency. Applying Theorem 3.10 for the hyperbolic pair (k, m) with $m = 2k$ using the matrices from Theorem 3.7, it follows that the group $G = \mathrm{PGL}(2, p)$ is generated by two elements y and z of orders k and $2k$, with product $y^{-1}z$ of order 2. (This is a consequence of Theorems 1 and 2 of [51]).

By Theorem 3.7 the element y is contained in the subgroup $K \cong \mathrm{PSL}(2, p)$ of index 2 in G. But by our assumption on p we have $(p - 1)/2 \equiv k \bmod 2k$.

Hence, no element of K has order $2k$ and so $z \in G\backslash K$. Consequently, the involution $x = zy^{-1}$ is not contained in K.

This implies that the involution $x = y^{-1}z$ lies in $G\backslash K$, and obviously $\langle x, y\rangle = G$.

By Proposition 3.2 of [23], in our group $G = \langle x, y\rangle \cong \mathrm{PGL}(2, p)$ there is an inner automorphism conjugating x and y to their inverses, that is, there is an element $c \in G$ such that $(cx)^2 = (cy)^2 = 1$. It follows that $G = \langle x, y\rangle = \langle x, y, c\rangle$, and so by theorem 4.8 presented earlier, we find that $M_{k,p} = (G; x, y, c)$ is a non-orientable regular map of type $(k, 2k)$.

Finally, we show that this map has exponent group $\{\pm 1\}$. Let j be a unit mod k for which $M_{k,p}$ is isomorphic to $M_{k,p}^{(j)} = (G; x, y^j, c)$, so that there exists an automorphism of G taking y to y^j whilst fixing both x and c. The automorphism group of $G = \mathrm{PGL}(2, p)$ is isomorphic to G, acting on itself by conjugation, and therefore y and y^j must be conjugate in G and hence must be represented by 2×2 matrices having the same trace, up to multiplication by ± 1. Then

$$y = \pm \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix}, \qquad y^j = \pm \begin{bmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{bmatrix}.$$

Hence, $\mathrm{Tr}(y) = \xi + \xi^{-1}$ and $\mathrm{Tr}(y^j) = \pm(\xi^j + \xi^{-j})$. Then, $\xi + \xi^{-1} = \pm(\xi^j + \xi^{-j})$, which was shown in [23] to occur if and only if $j \in \{\pm 1\}$ which is equivalent to $(\xi^{j+1} \pm 1)(\xi^{j-1} \pm 1) = 0$. Now, a simple calculation (as in the proof of Lemma 4.4 of [23]) using the fact that the order of ξ is $2k$ implies that the last equation is equivalent to $j \equiv \pm 1 \bmod k$. This proves that the exponent group of M is almost trivial. □

By Dirichlet's theorem on primes in arithmetic progression, for every positive relatively prime integers a, b there exist infinitely many primes congruent to a mod b. Letting, for a fixed k, $a = 2k + 1$ and $b = 4k$, by Dirichlet's theorem there are infinitely many primes p satisfying $p \equiv 2k + 1 \bmod 4k$. Hence for each such k we have an infinite number of non-orientable regular maps $M_{k,p}$ of valency k with properties guaranteed by Theorem 4.8, and in particular, with exponent group $\{\pm 1\}$. We can now extend this to an arbitrary subgroup $U \le U_k$ containing $\{\pm 1\}$.

For any such U, let V be a set of coset representatives for its subgroup $\{1, -1\}$, that is, chosen so that $|V| = |U|/2$ and $V \cup (-V) = U$, and consider the join $\vee\mathcal{M}_{k,p,V}$ of the family $\mathcal{M}_{k,p,V} = (M_{k,p}^{(j)} \colon j \in V)$ of j^{th} rotational powers of the map $M_{k,p} = (G; x, y, c)$ constructed in Theorem 4.8 for the given prime p.

By the theory outlined in Section 4.2, we know that $\vee\mathcal{M}_{k,p,V}$ is a regular map of valency k. Specifically, $\vee\mathcal{M}_{k,p,V} = (\overline{G}; \overline{x}, \overline{y}, \overline{c})$, where $\overline{G} = \prod_{j\in V} G_j$ with $G_j = \langle x, y^j, c\rangle$ for all $j \in V$, and $\overline{x}$ and $\overline{c}$ are the $|V|$-tuples $(x, x, \ldots, x)$ and $(c, c, \ldots, c)$ while $\overline{y} = (y^j)_{j\in V}$. Also it is easy to see that $\vee\mathcal{M}_{k,p,V}$ is non-orientable, with $\overline{c} \in \langle \overline{x}, \overline{y}\rangle = \overline{G}$.

We proceed by determining the structure of $\overline{G}$, noting that G is no longer simple in the current context.

Theorem 4.9. *The map $\vee\mathcal{M}_{k,p,V}$ is non-orientable, with automorphism group $\overline{G}$ isomorphic to a semi-direct product $(\prod_{j\in V} K_j) \rtimes \langle \overline{x}\rangle$, where $K_j = \langle y^j, xy^jx\rangle \cong \mathrm{PSL}(2,p)$ for all $j \in V$.*

Proof. First, we note that by non-orientability, $G_j = \langle x, y^j\rangle = G \cong \mathrm{PGL}(2,p)$ for every $j \in V$. Let $K_j = \langle y^j, xy^jx\rangle$ be a subgroup of G_j. Now, for every $j \in V$ the subgroup K_j has index 2 in G_j. Also, K_j is isomorphic to $\mathrm{PSL}(2,p)$, and it is the image of the index 2 subgroup Γ of $\Delta^+(2,k,\infty) = \langle\, X, Y \mid X^2 = Y^k = 1\,\rangle$ generated by Y and XYX, under an epimorphism $f_j \colon \Gamma \to G_j$ such that it takes Y to y^j and XYX to xy^jx. Note also that Γ is isomorphic to $C_k * C_k$.

By Theorem 4.8, we know that $\mathrm{Exp}(M_{k,p}) = \{\pm 1\}$, and then it follows from the way in which the index-set V was introduced above that the rotational powers

$M_{k,p}^{(j)}$ and $M_{k,p}^{(\ell)}$ are not isomorphic whenever j and ℓ are distinct members of V. Accordingly, the epimorphisms $F_j : \Delta^+(2, k, \infty) \to G_j$ given by $(X, Y) \mapsto (x, y^j)$ for each $j \in V$ have pairwise distinct kernels, as do their restrictions $f_j \colon \Gamma \to K_j$. (Indeed $\ker F_j = \ker f_j$ for all $j \in V$.) This means we can apply Theorem 4.2 to $\Gamma \cong C_k * C_k$, and find that with $\overline{x} = (x, x, \ldots, x)$ and $\overline{y} = (y^j)_{j \in V}$ as above, the elements $\overline{y}$ and $\overline{x}\,\overline{y}\,\overline{x}$ make up a generating pair for $\overline{K} = \prod_{j \in V} K_j$.

Also conjugation by the involution $\overline{x}$ swaps the two generators $\overline{y}$ and $\overline{x}\,\overline{y}\,\overline{x}$ of $\overline{K}$, and as x lies in $\mathrm{PGL}(2, p) \backslash \mathrm{PSL}(2, p)$, it follows that $\overline{x} \notin \overline{K}$, and so $\overline{K} = \prod_{j \in V} K_j$ has index 2 in $\langle \overline{x}, \overline{y} \rangle = \langle \overline{x}, \overline{y}, \overline{c} \rangle = \overline{G}$, making $\overline{G}$ isomorphic to a semi-direct product $(\prod_{j \in V} K_j) \rtimes \langle \overline{x} \rangle$. □

We will now proceed to prove the main result in this section, about the existence of non-orientable regular maps with valency k and exponent group U.

Theorem 4.10. *For every integer $k \geq 3$, and every subgroup U of the group of units modulo k containing -1, there exist infinitely many non-orientable regular maps with valency k and exponent group equal to U.*

Proof. As above, let V be a set of coset representatives for $\{1, -1\}$ in U with $1 \in V$, and let p be any prime chosen from the infinite set of primes congruent to $2k + 1 \bmod 4k$. Also let $M_{k,p} = (G; x, y, c)$ be a non-orientable regular map for $G = \langle x, y \rangle \cong \mathrm{PGL}(2, p)$ with exponent group $\{\pm 1\}$, where $y \in K \cong \mathrm{PSL}(2, p)$ and $x \in G \backslash K$, as given by Theorem 4.8, and let $\overline{M} = \vee \mathcal{M}_{k,p,V} = (\overline{G}; \overline{x}, \overline{y}, \overline{c})$ be the non-orientable regular map of valency k constructed from the family $\mathcal{M}_{k,p,V} = (M_{k,p}^{(j)} \colon j \in V)$ before the statement of Theorem 4.9.

Our aim was to show that the exponent group of $\overline{M}$ is equal to U. For this we let $[u]$ be the representative in V of any given element $u \in U$, namely either u or $-u$.

It is easy to see that $\mathrm{Exp}(\overline{M})$ contains U, because multiplication by any element of U induces a permutation of the constituents $M_{k,p}^{(j)} = (G; x, y^j, c)$ of $\overline{M}$. Indeed if $\ell \in U$ then multiplication by ℓ takes $M_{k,p}^{(j)} = (G; x, y^j, c)$ to $M_{k,p}^{([j\ell])} = (G; x, y^{[j\ell]}, c)$ for all $j \in V$, as the fact that $\mathrm{Exp}(M_{k,p}) = \{1, -1\}$ implies that each of $(G; x, y^{j\ell}, c)$ and $(G; x, y^{-j\ell}, c)$ is isomorphic to $M_{k,p}^{([j\ell])}$ because of the automorphism of G taking (x, y, c) to (x, y^{-1}, c).

Conversely, let d be any exponent of $\overline{M}$, and let ψ be an automorphism of $\overline{G}$ that fixes $\overline{x}$ and takes $\overline{y} = (y^j)_{j\in V}$ to $\overline{y}^d = (y^{[jd]})_{j\in V}$. Then the restriction of ψ to $\overline{K}$ is an automorphism of $\overline{K}$ taking $\overline{y}$ to $\overline{y}^{[d]}$, and $\overline{x}\,\overline{y}\,\overline{x}$ to $\overline{x}\,\overline{y}^{[d]}\,\overline{x}$. Next, as $\overline{K} = \prod_{j\in V} K_j$ is a direct product of $|V|$ copies of the simple group $K \cong \mathrm{PSL}(2, p)$, we may use [7, Theorem 3.1] again to conclude that $\mathrm{Aut}(\overline{K})$ is isomorphic to the wreath product $\mathrm{Aut}(K) \wr \mathrm{Sym}(n)$ where $n = |V| = |U|/2$. Hence we can multiply $\psi \restriction_{\overline{K}}$ by a suitable element of $\mathrm{Aut}(\overline{K})$ to obtain an automorphism τ of $\overline{K}$ that preserves K_1 (the first copy of K in the expansion $\overline{K} = \prod_{j\in V} K_j$), and then τ induces an automorphism of K_1 ($\cong K$) taking (x, y) to $(x, xy^{\ell d}x)$ for some $\ell \in U$. But now since $\mathrm{Exp}(M_{k,p}) = \{1, -1\}$, we conclude that $\ell d = \pm 1$ in U, and hence that $d = \pm \ell^{-1} \in U$. Thus $\mathrm{Exp}(\overline{M}) = U$, completing the proof. □

4.4 Further Observations

Let us sum up what has been said so far in this chapter regarding proving the existence of orientably-regular or regular maps with given valency and with given exponent group U. When dealing with orientable case, we first look into constructing corresponding maps with trivial exponent group. Then, we form rotational powers of these maps using the elements of the given exponent group U. We then proceed to take the parallel product of all of these maps. Similarly, when we consider the non-orientable case, we take the same approach, however we produce maps that have an 'almost-trivial' exponent group $\{\pm 1\}$. In both cases, the required properties happen to be controllable if the automorphism groups of the constituent maps are either simple or 'nearly simple'.

We can now go a step forward, using a method known as the 'Macbeath trick', as can be seen in the following theorem. Here we can construct regular covers in order to obtain infinite families of orientably-regular and non-orientable regular maps with given valency k and given exponent group, using the maps that were established in Theorems 4.5 and 4.10. In this result, note that the *characteristic* of a map is the Euler characteristic of its carrier surface.

Theorem 4.11. *Let M be an orientably-regular or a non-orientable regular map with characteristic $\chi < 0$. Then for any prime p not dividing $|\operatorname{Aut}(M)|$ there is (respectively) an orientably-regular or a non-orientable regular map $M^{[p]}$ covering M, with covering group a non-trivial elementary abelian p-group, such that $M^{[p]}$ has the same exponent group as M.*

Proof. Consider first the non-orientable case, which is harder; at the end of the proof we comment on the modifications needed to obtain a proof for the orientable case.

So, let M be a non-orientable regular map of a hyperbolic type (k, m), with $\operatorname{Aut}(M) = \langle x, y, c \mid x^2, y^k, c^2, (xc)^2, (yc)^2, (xy)^m, \ldots$, presented as in (4.2). Let $\Delta = \Delta(2, k, \infty) = \langle X, Y, C \mid X^2, Y^k, C^2, (XC)^2, (YC)^2 \rangle$ be the corresponding parent group from section 4.2. Consider the natural epimorphism ψ from Δ onto $\operatorname{Aut}(M)$, given by by $\psi(X) = x$, $\psi(Y) = y$ and $\psi(C) = c$. In this situation we will need a lemma a formal statement of which appears to be folklore (and we could not find an explicit reference, except in-passing mentions in the survey [52], for example.

Lemma 4.12. *The kernel of the epimorphism $\psi:\ \Delta(2, k, \infty) \to \operatorname{Aut}(M)$ is isomorphic to the fundamental group of the punctured surface $\mathcal{S}$ obtained from* *the carrier surface of the map M by removing exactly one point from the interior of each face of M.*

Proof. This is a consequence of a general result in algebraic topology; see, for example, the last statement in subsection 19B on page 38 of [1], by which any topological space with a universal cover is the quotient of its universal cover by its fundamental group, acting as a group of covering transformations. In our case, the universal cover of the punctured surface $\mathcal{S}$ obtained from the carrier surface of M is the universal tessellation $\mathcal{U}_{k,m,\infty}$ of the hyperbolic plane, with each vertex of valency k but with each face of infinite length. Topologically, the fundamental group acts as a covering transformation group of the covering $\mathcal{U}_{k,\infty} \to \mathcal{S}$ with branch points of infinite order in the faces of $\mathcal{U}_{k,\infty}$. Geometrically, the action of the fundamental group can be visualised as if the covering 'wraps' each face of $\mathcal{U}_{k,\infty}$ around its 'centre' an infinite number of times. A condensed purely algebraic translation of this argument can be found in [50]. □

Since M is of hyperbolic type and non-orientable, for the Euler characteristic $0 > \chi$ of its carrier surface one has $\chi = 2 - h$, where $h \geq 3$ is the non-orientable genus of the surface (also known as its crosscap number). By Lemma 4.12, the kernel $K = \ker\psi$ is isomorphic to the fundamental group of the punctured surface $\mathcal{S}$. Algebraic topology has developed a standard way to determine a presentation for K [1]. One begins by taking an oriented $2h$-sided fundamental polygon for $\mathcal{S}$ with sides marked consecutively $a_1a_1a_2a_2\ldots a_ha_h$. If the map M has f faces, one introduces further f loops inside the polygon, marked (say) $x_1, x_2, \ldots, x_f$, in such a way that each puncture lies inside exactly one loop. The group K is then determined as a single-relator group by a presentation of the form

$$K = \langle a_1, a_2, \ldots, a_h, x_1, x_2 \ldots, x_f \mid a_1^2 a_2^2 \ldots a_h^2 x_1 x_2 \ldots x_f \rangle \tag{4.3}$$

Eliminating, say, x_f from the equation $a_1^2 a_2^2 \ldots a_h^2 x_1 x_2 \ldots x_f = 1$ shows that K is, at the same time, generated by the $h + f - 1$ generators $a_1, \ldots, a_h, x_1, \ldots, x_{f-1}$ satisfying no relations at all; hence K is isomorphic to a free group of rank $h + f - 1$. It follows that its abelianisation K/K' is a free Abelian group of rank $h + f - 1$, isomorphic to $\mathbb{Z}^{h+f-1}$.

Next, let p be any (odd) prime that does not divide $|A| = |\operatorname{Aut}(M)|$, and let $K^{[p]}$ be the subgroup of K generated by all the pth powers of elements in K, and let $L = K'K^{[p]}$. Then clearly L is a characteristic subgroup of K and hence a normal subgroup of Δ, with K/L isomorphic to $(C_p)^{f+h-1}$, and then since $\Delta/K \cong (\Delta/L)/(K/L)$, the quotient $A^{[p]} = \Delta/L$ is an extension of $K/L \cong (C_p)^{f+h-1}$ by $\Delta/K \cong A$. Moreover, by choice of p the order of $K/L \cong A$ is coprime to the order of $K/L \cong (C_p)^{f+h-1}$, and hence by the Schur-Zassenhaus theorem we conclude that $A^{[p]} = \Delta/L$ is isomorphic to a semi-direct product $(C_p)^{f+h-1} \rtimes A$.

The group $A^{[p]}$ is the automorphism group of a fully regular map $M^{[p]}$ of valency k, with face length either m or mp (and we need not be any more specific about that). The subgroup $K/L \cong (C_p)^{f+h-1}$ is a normal Sylow p-subgroup of $A^{[p]}$ and hence is characteristic in $A^{[p]}$. Topologically, the natural epimorphism $\psi :\ A^{[p]} \cong \Delta/L \to \Delta/K \cong A$ with kernel K/L makes $M^{[p]}$ a (p^{f+h-1})-fold cover of M, which is possibly branched at faces, but as p is odd the carrier surface of $M^{[p]}$ must be non-orientable. Applying the Riemann-Hurwitz formula to the covering ψ one sees that $M^{[p]}$ has characteristic $p^{f+h-1}\chi = p^{f+h-1}(2-h)$, irrespective of possible branch points at face centres.

It remains for us to show that the k-valent non-orientable regular maps M and $M^{[p]}$ have the same exponent group. To state this in an equivalent form but in a different language, for j coprime to k we say that, for a group G presented in the form $G = \langle x, y, c \mid x^2, y^k, c^2, (cx)^2, (cy)^2, \ldots$, an automorphism of G is j-rotational of it preserves x and c and maps y to y^j. So, to prove that M and $M^{[p]}$ have the same exponent group amounts to showing that for every j coprime with K, or equivalently, that for every $j \in U_k$, the group $\mathrm{Aut}(M) \cong \Delta/K$ admits a j-rotational automorphism if and only if $\mathrm{Aut}(M^{[p]}) \cong \Delta/L$ does.

The 'if' part of this is easy: because K/L is characteristic in Δ/L, every j-rotational automorphism of Δ/L preserves K/L and hence induces a j-rotational automorphism $(\Delta/L)/(K/L) \cong \Delta/K$.

Conversely, if Δ/K admits a j-rotational automorphism, then this lifts to the unique j-rotational automorphism α of the infinite full triangle group $\Delta = \Delta(2, k, \infty)$, and then α preserves K and hence also preserves its characteristic subgroup L, so α induces a j-rotational automorphism of Δ/L. This completes the proof in the non-orientable case.

The proof for the orientable case is almost verbatim identical, with the following exceptions. For the parent group one replaces Δ by $\Delta^+ = \Delta^+(2, k, \infty)$ obtained from (2.7) by omitting the relator $(XY)^m$. Since the map M is now orientable, for the Euler characteristic χ of its supporting surface one has $\chi = 2 - 2g$, where $g \geq 2$ is the (orientable) genus of the surface. For the presentation of the kernel K of the epimorphism $\psi : \Delta^+ \to \mathrm{Aut}^+(M)$ one starts with a $4g$-sided fundamental polygon for an orientable surface of genus g of the form $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ and adds f loops $x_1, ..., x_f$ corresponding to punctures as before, so that K has now a presentation of the form

$$K = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, x_1, x_2, \ldots, x_f \mid [a_1, b_1] \ldots [a_g, b_g] x_1 x_2 \ldots x_f \rangle \quad (4.4)$$

By similar arguments as before, K is isomorphic to a free group of rank $2g + f - 1$. (This, together with a derivation of the presentation for K, is also explained in a very accessible form in [17].) The remaining parts of the proof for orientable maps is (modulo making the above changes) identical with the proof for the non-orientable case. This completes the proof. □

Chapter 5

Regular maps of a given hyperbolic type with no non-trivial exponents

5.1 Preliminaries

In this chapter we will focus on regular maps of a given hyperbolic type that are on the other end of spectrum compared to those considered in Chapter 4 when it comes to exponents. Recall that the main results of Chapter 4 imply, for example, that for every valency $k \geq 3$ there exist both orientably-regular as well as non-orientable regular maps of valency k with the 'largest possible exponent group', of order $\varphi(k)$. We also remarked that such a result does not extend to maps of arbitrary given type. Here we will consider orientably-regular and regular maps with the 'smallest possible' exponent group trivial in the first case and $\{\pm 1\} \cong C_2$ in the second case (because a non-orientable regular maps automatically admits the exponent -1.) This time, however, we aimed for results valid for an arbitrary hyperbolic type (and not just for a given valency). For orientably- regular maps the corresponding main result is Theorems 5.2. In the non-orientable case, existence of regular maps with exponent group $\{\pm 1\}$ was shown in Theorem 5.9 for hyperbolic types (k, m) with at least one even entry, and in Theorem 5.10 for hyperbolic types (k, m) with both entries odd and not relatively prime.

Proofs of both these main results of Chapter 5 are based on my explorations of generating matrices for groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ as in Chapter 4 during my PhD study under the guidance of my supervisor; in the non-orientable case this was done with emphasis on q a power of 2.

5.2 Orientably-regular maps of arbitrary hyperbolic type with no non-trivial exponent

In this section we will prove the existence of orientably-regular maps of arbitrary hyperbolic type (k,m), with no non-trivial exponent. We will start by showing a construction of such maps with no exponents other than ± 1. The existence of such maps was proved in [53] with the help of residual finiteness of triangle groups.

Theorem 5.1. *For any given hyperbolic pair (k,m), there exist infinitely many primes p and infinitely many finite reflexible orientably-regular maps M of type (k,m) with $\mathrm{Aut}^+(M)$ isomorphic to $\mathrm{PSL}(2,p)$, such that the only exponents of M are ± 1.*

Proof. Let $p = p(k,m)$ be an odd prime congruent to 1 modulo each of $2k$ and $2m$. Existence of an infinite number of such primes is a consequence of Dirichlet's theorem, which gives infinitely many primes congruent to 1 mod $2\mathrm{lcm}(k,m)$. Our choice of p implies that the field $\mathrm{GF}(p)$ contains a $(2k)^{th}$ and $(2m)^{th}$ primitive roots of unity. By Theorem 3.12, there exist elements $x, y \in \mathrm{PSL}(2,p)$ such that x has trace 0, while y and xy respectively have traces $\pm(\xi+\xi^{-1})$ and $\pm(\eta+\eta^{-1})$ for some $(2k)^{\mathrm{th}}$ and $(2m)^{\mathrm{th}}$ primitive roots ξ and η mod p , such that $G = \langle x, y\rangle = \mathrm{PSL}(2,p)$ and the map $M = (G;, x, y) = \mathrm{Map}(\xi,\eta)$ is orientably-regular, of type (k,m). The fact that the only exponents of M are ± 1 follows in exactly the same way as in the proof of Theorem 4.8. □

We are now ready to prove our main theorem on orientably-regular maps.

Theorem 5.2. *For any given hyperbolic pair (k, m), there exists an orientably-regular map of type (k, m) with no exponent except* 1.

Proof. Let M be a map of type (k, m) as constructed in the proof of Theorem 5.1, with $\mathrm{Aut}^+(M)$ isomorphic to the simple group $\mathrm{PSL}(2, p)$ for some sufficiently large p. Also let N be an orientably-regular but chiral map of the same type (k, m), as constructed in [21], with $\mathrm{Aut}^+(N)$ isomorphic to the symmetric group S_n or the alternating group A_n, for some $n > 6$. Now consider the parallel product $M \,||\, N$ of these two maps. By Proposition 3.5, $\mathrm{Aut}^+(M \,||\, N) \cong \mathrm{Aut}^+(M) \times \mathrm{Aut}^+(N)$. Thus, if $G = \mathrm{Aut}^+(M) = \langle\, x, y \mid x^2, y^k, (xy)^m, \dots \rangle$, $H = \mathrm{Aut}^+(N) = \langle\, u, v \mid u^2, v^k, (uv)^m, \dots \rangle$, then the group $\mathrm{Aut}^+(M \,||\, N) \cong G \times H$ is generated by the elements (x, u) and (y, v), of orders 2 and k, with product (xy, uv) of order m.

Next, suppose that some unit e mod k is an exponent of $M \,||\, N$. This implies existence of an automorphism of $G \times H$ fixing (x, u) and mapping (y, v) to $(y, v)^e = (y^e, v^e)$. Then since G is a simple group and one obviously may assume that M and N do not cover one another, Proposition 3.6 tells us that such an automorphism is formed by a pair of automorphisms of G and H, taking $(x, y) \mapsto (x, y^e)$ and $(u, v) \mapsto (u, v^e)$ respectively. It follows that e is an exponent of both M and N. By Theorem 5.1, however, we find that $e \in \{1, -1\}$, while on the other hand, chirality of N implies that $e \neq -1$. We conclude that $e = 1$, and hence that the orientably-regular map $M \,||\, N$ of type (k, m) has no non-trivial exponent. □

We note that in the proof above, the map M was chosen in such a way that the exponents were restricted to being 1 and -1. Choosing the map N to be chiral, further restricted the possible exponents by eliminating -1 and leaving just 1 as a possibility. This shows how the parallel product construction can be very helpful in constructing orientably-regular maps with a particular exponent group.

5.3 Non-orientable regular maps on linear fractional groups over fields of characteristic two

We begin by recalling Proposition 3.13 from Section 3.2, which will play an important role in subsequent considerations. It says that for a hyperbolic pair (k, m) with both entries odd, when one lets ξ and η be k^{th} and m^{th} primitive roots of unity modulo a suitable power of 2, with $\delta = \xi + \xi^{-1} + \eta + \eta^{-1}$, the matrices x and y given by

$$x = x(\xi, \eta) = \frac{1}{\xi+\xi^{-1}} \begin{bmatrix} \eta+\eta^{-1} & \delta^2\xi \\ \xi^{-1} & \eta+\eta^{-1} \end{bmatrix} , \quad y = y(\xi, \eta) = \begin{bmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{bmatrix} . \tag{5.1}$$

generate a group G with presentation $\langle\, x, y \mid x^2, y^k, (xy)^m, \dots \rangle$, conjugate to $\mathrm{SL}(2, 2^n)$, where n is the smallest positive integer such that $k \mid 2^n \pm 1$ and $m \mid 2^n \pm 1$. Further, by Theorem 3.13, the group $G = \langle x, y\rangle \cong \mathrm{SL}(2, 2^n)$ determines a non-orientable regular map $M = (G; x, y)$ of type (k, m); by Proposition 2.8 in Section 2.9 this means that there is an involution $c \in \langle x, y\rangle$ inverting both x and y. To link this map $M = (G; x, y)$ with ξ and η as above we will from now on write $M = (G; x, y) = \mathrm{Map}(\xi, \eta)$.

We will now investigate possible exponents of the map $M = \mathrm{Map}(\xi, \eta)$ of 'odd' type (k, m) introduced above, with $\mathrm{Aut}(M) \cong \langle x, y\rangle \cong \mathrm{SL}(2, 2^n)$ from (5.1). This requires a closer look at automorphisms of the group $\langle x, y\rangle$, which we do next.

For $q = 2^n$, the automorphism group of $\mathrm{SL}(2,q)$ is known to be a split extension of $\mathrm{SL}(2,q)$ by the cyclic group of Galois automorphisms of $\mathrm{GF}(q)$ [55]. If both k and m divide $q-1 = 2^n - 1$, then the group $G = \langle x, y\rangle$ for x, y given by (5.1) coincides with the unique subgroup J of $\mathrm{SL}(2,q^2)$ isomorphic to $\mathrm{SL}(2,q)$, which consists of unimodular matrices *defined over* $\mathrm{GF}(q)$, and then all automorphisms of G come from the split extension described above. But if one of k, m divides $2^n + 1$, then $G = \langle x, y\rangle$ is a conjugate of J contained in $\mathrm{SL}(2,q^2)$ but distinct from J. In this situation, let $h \in \mathrm{SL}(2,q^2)$ be such that $hGh^{-1} = J$ and let $J \mapsto \alpha J^\theta \alpha^{-1}$ be an automorphism of J for some $\alpha \in \mathrm{SL}(2,q)$ and a Galois automorphism θ induced by $z \mapsto z^{2^i}$ for some $i \in \{0, 1, \ldots, n-1\}$ and all $z \in \mathrm{GF}(q)$. Note that θ induces also an automorphism of $\mathrm{SL}(2,q^2)$ when applied to elements $z \in \mathrm{GF}(q^2) > \mathrm{GF}(q)$. Now substituting $hGh^{-1} = J$ in $J \cong \alpha J^\theta \alpha^{-1}$ gives $hGh^{-1} \cong \alpha(hGh^{-1})^\theta \alpha^{-1}$, which implies that $G \cong \beta G^\theta \beta^{-1}$ for $\beta = h^{-1}\alpha h^\theta$. So, even in the case where k divides $2^n + 1$, all automorphisms of $G = \langle x, y\rangle$ have the form $u \mapsto \beta u^\theta \beta^{-1}$ for suitable $\beta \in \mathrm{SL}(2,q^2)$ and some Galois automorphism θ of the field $\mathrm{GF}(q)$.

Hence in both cases $e \neq \pm 1$ is an exponent of $\mathrm{Map}(\xi, \eta)$ if and only if there is an element $\alpha \in \mathrm{SL}(2,q')$ and a Galois automorphism $\theta:\ z \mapsto z^{2^i}$ of $\mathrm{GF}(q)$ for some $i \in \{1, \ldots, n-1\}$ such that

$$\alpha x^\theta = x\alpha \quad \text{and} \quad \alpha y^\theta = y^e \alpha\ , \tag{5.2}$$

where x and y are given by (5.1). (We may exclude $i = 0$ from consideration because then α would conjugate y to y^e, and we would find $e = \pm 1$ by the trace argument used before.)

The second equation of (5.2) gives two kinds of solutions, one of the form

$\alpha = \mathrm{diag}(d^{-1}, d)$ for some non-zero $d \in GF(q')$ and some Galois automorphism given by $\theta(z) = z^{2^i}$ such that $2^i \equiv e \bmod k$, and the other in the form of an off-diagonal matrix and with Galois automorphism $\theta(z) = z^{-2^i}$ with $e \equiv -2^i \bmod k$. We may disregard the latter, since it arises as a composition of the former with the (unique) automorphism of $\mathrm{SL}(2, 2^n)$ inverting both x and y; uniqueness of this automorphism follows from the Remark made after the statement of Theorem 3.13 and here it is induced by conjugation by the off-diagonal matrix with entries $\delta\xi$ and $(\delta\xi)^{-1}$. Note also that e is an exponent of $\mathrm{Map}(\xi, \eta)$ if and only if $-e$ is. Continuing to apply $\alpha = \mathrm{diag}(d^{-1}, d)$ and $\theta(z) = z^{2^i}$ and comparing entries in the products appearing in the first equation in (5.2) one finds that $\alpha x^\theta = x\alpha$ is equivalent to the following three equations:

$$\left[\frac{\eta+\eta^{-1}}{\xi+\xi^{-1}}\right]^{2^i} = \frac{\eta+\eta^{-1}}{\xi+\xi^{-1}}\ , \quad \left[\frac{\delta^2\xi}{\xi+\xi^{-1}}\right]^{2^i} = \frac{d^2\delta^2\xi}{\xi+\xi^{-1}}\ , \quad \text{and} \quad \left[\frac{\xi^{-1}}{\xi+\xi^{-1}}\right]^{2^i} = \frac{d^{-2}\xi^{-1}}{\xi+\xi^{-1}}\ . \tag{5.3}$$

A further calculation (details of which we omit) reveals that the first equation of (5.3) implies equivalence of the second and third equations of (5.3), and taking into account the facts that $2^i \equiv e \bmod k$ and every non-zero element of $\mathrm{GF}(q')$ has a unique square root, the third equation finally gives $d = (\xi^e{+}1)/(\xi{+}1)$.

Of importance here is the first equation of (5.3), which states that $\rho^{2^i} = \rho$ for the ratio $\rho = \rho(\xi, \eta) = (\eta{+}\eta^{-1})/(\xi{+}\xi^{-1})$. Note also that $\rho \neq 1$ because of the condition $\xi \neq \eta,\ \eta^{-1}$. The order $o(\rho)$ of ρ is then a divisor of $2^i{-}1$, but obviously $o(\rho)$ is also a divisor of $2^n{-}1$, and hence a divisor of $\gcd(2^i{-}1, 2^n{-}1) = 2^j{-}1$, where $j = \gcd(i, n)$. Note here that j divides n by the well known fact that $2^j{-}1$ divides $2^n{-}1$ if and only if j divides n, and our assumption on the range of i implies that $j \neq n$. But from this point on, working backwards and letting $e \geq 1$ be a positive integer smaller than k such that $e \equiv 2^j \bmod k$ for j as above, we find that e is also an exponent of $\mathrm{Map}(\xi, \eta)$. As $\rho^{2^j} = \rho$, the ratio

$\rho(\xi, \eta)$ is an element of the proper subfield $\mathrm{GF}(2^j)$ of $\mathrm{GF}(2^n)$.

It follows that the order of the *smallest* subfield containing $\rho(\xi, \eta)$ is an exponent of $\mathrm{Map}(\xi, \eta)$, and, conversely, the smallest power of 2 which (mod k) is an exponent of $\mathrm{Map}(\xi, \eta)$ is the order of a subfield containing $\rho(\xi, \eta)$. This smallest power is then a generator of a (cyclic) subgroup of the exponent group of $\mathrm{Map}(\xi, \eta)$, namely the subgroup induced by involving the Galois automorphism as above. More precisely, let 2^ℓ be the order of the smallest subfield containing $\rho(\xi, \eta)$. Since the order of ξ divides $2^n \pm 1$, there is a smallest positive integer t such that $2^{t\ell} \equiv 1 \bmod k$, and then $t\ell$ is a divisor of n if k divides $2^n - 1$, and a divisor of $2n$ if k divides $2^n + 1$. Then if e satisfies $2^\ell \equiv e \bmod k$ and $1 \leq e \leq k-1$, the units $e, e^2, \ldots, e^{t-1}$ form a cyclic group of exponents of $\mathrm{Map}(\xi, \eta)$, of order t, induced by automorphisms of $\langle x, y\rangle$ defined with the help of the Galois action; and moreover, there are no other exponents of $\mathrm{Map}(\xi, \eta)$ of this kind.

It remains to clarify the role of $e = -1 \equiv k-1 \bmod k$ which, as we know, is always an exponent of $\mathrm{Map}(\xi, \eta)$ arising from conjugation inverting x and y. Can the same exponent be obtained also by an automorphism of $\langle x, y\rangle$ of the form $u \mapsto \alpha u^\theta \alpha^{-1}$ for some $\alpha \in \langle x, y\rangle$ and some non-trivial Galois automorphism θ of the field $\mathrm{GF}(2^n)$, as considered above?

By the first equation of (5.3) and its consequences, θ would need to have the form $z \mapsto z^{2^i}$ for some i such that $2^i \equiv -1 \bmod k$. Note, however, that if $2^i \equiv -1 = e \bmod k$ so that k divides $2^i + 1$, then the first equation of (5.3) reduces to $(\eta + \eta^{-1})^{2^i} = \eta + \eta^{-1}$, which is equivalent to $(\eta^{2^i} + \eta)(1 + \eta^{-(2^i+1)}) = 0$ and hence to m dividing one of $2^i \pm 1$. But we have assumed at the very beginning that the smallest positive i such that each of k and m divides $2^i \pm 1$ is n. For $i = n$, however, the Galois automorphism $z \mapsto z^{2^n}$ appearing in (5.3) is trivial (giving $d = 1$). It follows that $e = -1$ never arises as an exponent from involving Galois conjugation $z \mapsto z^{2^i}$ for $i \in \{1, 2, \ldots, n-1\}$. It follows that the exponent -1 is not among those identified in the multiplicative group $\{e, e^2, \ldots, e^{t-1}\} \cong C_t$ but commutes with each of these, implying that the exponent group of $\mathrm{Map}(\xi, \eta)$ is isomorphic to $C_t \times C_2$.

Collecting the above arguments yields a proof of the following statement.

Proposition 5.3. *Let (k, m) be a hyperbolic type with odd entries, let ξ and η be primitive k^{th} and m^{th} roots in some field of characteristic 2 such that $\xi \neq \eta,\ \eta^{-1}$, and let $\mathrm{Map}(\xi, \eta)$ be the corresponding non-orientable regular map, with $q = 2^n$ being the smallest power of 2 such that $\mathrm{GF}(2^n)$ contains both $\xi+\xi^{-1}$ and $\eta+\eta^{-1}$. Also let ℓ be a positive divisor of n such that 2^ℓ is the smallest order of a subfield of $\mathrm{GF}(2^n)$ containing the ratio $\rho(\xi, \eta) = (\eta+\eta^{-1})/(\xi+\xi^{-1})$, and let t be the smallest positive integer such that $2^{t\ell} \equiv 1 \bmod k$. Then every exponent of $\mathrm{Map}(\xi, \eta)$ has the form $2^{j\ell}$ or $-2^{j\ell}$ mod k for some $j \in \{0, 1, \ldots, t-1\}$, and the exponent group of $\mathrm{Map}(\xi, \eta)$ is isomorphic to the direct product $C_t \times C_2$.* □

With the above notation, we have the following obvious consequence of Proposition 5.3.

Corollary 5.4. *A non-orientable regular map $M = \mathrm{Map}(\xi, \eta)$ with $\mathrm{Aut}(M) \cong \mathrm{SL}(2, 2^n)$ has no exponents distinct from ± 1 if and only if the ratio $\rho(\xi, \eta)$ is contained in no proper subfield of $\mathrm{GF}(2^n)$.* □

5.4 Non-orientable regular maps of arbitrary hyperbolic type with almost trivial exponent group

We begin by proving four consequences of Proposition 5.3 and Corollary 5.4 as a preparation towards constructions of non-orientable regular maps with no exponents except ± 1 with help of parallel products of maps.

Proposition 5.5. *For every odd integer $k \geq 5$, there exists a non-orientable regular map of type (k, k) having no exponent distinct from ± 1.*

Proof. Let n be the smallest positive integer such that k divides $2^n \pm 1$, let ξ be a corresponding primitive k^{th} root of 1 in $\mathrm{GF}(2^n)$ or $\mathrm{GF}(2^{2n})$, and let $\eta = \xi^2$. Note that $\xi \neq \eta$, η^{-1} since $k \geq 5$. From $\eta + \eta^{-1} = (\xi + \xi^{-1})^2$ it follows that $\mathrm{GF}(2^n)$ is the smallest field of characteristic 2 containing $\xi + \xi^{-1}$ (and of course also $\eta + \eta^{-1}$). As the ratio $\rho(\xi, \eta)$ is now simply equal to $\xi + \xi^{-1}$, the non-orientable regular map $\mathrm{Map}(\xi, \eta)$ constructed in Proposition 5.3 has no non-trivial exponents, by Corollary 5.4. □

Proposition 5.6. *For every odd integer $k \geq 7$, there exists a non-orientable regular map of type $(k, 3)$ such that neither the map nor its dual have any exponents distinct from ± 1.*

Proof. Let η be a primitive 3^{rd} root of unity in some finite field of characteristic 2. Then η is a root $\eta^2 + \eta + 1$, and so $\eta + \eta^{-1} = 1$. Next let n be the smallest positive integer such that k divides $2^n \pm 1$. Then n is also the smallest positive integer for which $\mathrm{GF}(2^n)$ contains $\xi + \xi^{-1}$ (and $1 = \eta + \eta^{-1}$) for every choice ξ of a primitive k^{th} root of unity. Now if 2^ℓ were a non-trivial exponent of $M(\xi, \eta)$ induced by the Galois action, with the smallest positive ℓ such that $1 \leq \ell < n$, then by (5.3) and Proposition 5.3, the smallest subfield containing $\xi + \xi^{-1}$ would be $\mathrm{GF}(2^\ell)$. But by minimality of n this would imply that $\ell = n$, a contradiction. Thus $M(\xi, \eta)$ has no non-trivial exponents, as does its dual (because it has valency 3). □

Proposition 5.7. *For every odd integer $k \geq 7$, there exists a non-orientable regular map of type $(k, 5)$ such that neither the map nor its dual has an exponent distinct from ± 1.*

Proof. Let η be a primitive 5^{th} root of unity in some finite field of characteristic

2, and let $\nu = \eta + \eta^{-1}$. Then since η satisfies the equation $\eta^4 + \eta^3 + \eta^2 + \eta + 1 = 0$, it follows that $\nu^2 + \nu + 1 = 0$, so that ν has multiplicative order 3, and $\nu^4 = \nu$. Next let ξ be a primitive k^{th} root of unity, again in some finite field of characteristic 2, and also let n be the the smallest integer greater than 1 such that both 5 and k divide $2^n \pm 1$. Observing that 5 divides $2^j \pm 1$ if and only if j is even or $j = 1$, we see that n must be even.

Now suppose that $M(\xi, \eta)$ has a non-trivial exponent. Then by Proposition 5.3 we may assume that this exponent has the form 2^ℓ for some proper divisor ℓ of n. If ℓ is even, then from $\nu^4 = \nu$ it follows that $\nu^{2^\ell} = \nu$, and then the leftmost part of (5.3) implies that $\xi + \xi^{-1}$ is contained in GF(2^ℓ), but then $\ell = n$ by the properties of n, a contradiction. Hence ℓ must be odd, and then the leftmost part of (5.3) reduces to

$$(\xi + \xi^{-1})^{2^\ell - 1} = \nu \ . \tag{5.4}$$

Moreover, since the order of ν (namely 3) does not divide $2^\ell - 1$ for odd ℓ, it follows from (5.4) that the order of $\xi + \xi^{-1}$ is a divisor of $2^{2\ell} - 1$. But now 2ℓ cannot be a proper divisor of n, as this would contradict minimality of n with respect to 5 and k dividing $2^n \pm 1$. We conclude that for η and ξ as above, if the map $M(\xi, \eta)$ has a non-trivial exponent, then it has the form 2^ℓ for $\ell = n/2$, with $n/2$ odd.

Let us now mimic the above considerations for our chosen η but with ξ replaced by ξ^2, assuming that 2^ℓ is a non-trivial exponent of the map $M(\xi^2, \eta)$ for some proper divisor ℓ of n. (Note that the minimal n here is the *same* as above.) It can be checked that (5.4) will then have the form

$$(\xi^2 + \xi^{-2})^{2^\ell - 1} = \nu \tag{5.5}$$

and the conclusion that $\ell = n/2$ remains the same for the map $M(\xi^2, \eta)$. But then comparison of (5.4) and (5.5) gives $(\xi + \xi^{-1})^{2^\ell - 1} = 1$, which contradicts (5.4). It follows that one of the maps $M(\xi, \eta)$ and $M(\xi^2, \eta)$ has only trivial exponents, proving the existence of non-orientable regular maps of hyperbolic type $(k, 5)$ with only trivial exponents.

For the dual, observe that if a map of hyperbolic type $(5, k)$ with automorphism group isomorphic to $\mathrm{SL}(2, 2^n)$ for some n admitted a non-trivial exponent e, then e would be 2 or 3 ($\equiv -2$) mod 5, and the exponent group would be cyclic of order 4, which contradicts the final assertion of Proposition 5.3. □

Proposition 5.8. *For every hyperbolic pair (k, m) with distinct odd entries $k, m \geq 7$ and such that $\gcd(k, m) \in \{3, 5\}$ there exists a non-orientable regular map of type (k, m) with no exponent distinct from ± 1.*

Proof. Let k and m be as in the statement, with $\gcd(k, m) = d \in \{3, 5\}$; we will assume that one of the two values of d is fixed in what follows. Propositions 5.6 and 5.7 then guarantee existence of non-orientable regular maps $M_1 = (G_1; x_1, y_1)$ of type (k, d) and $M_2 = (G_2; x_2, y_2)$ of type (d, m) for suitable groups $G_i \cong \mathrm{SL}(2, 2^{n_i})$ for $i \in \{1, 2\}$, both having ± 1 as the only exponents. Our assumptions also imply that the two maps do not cover each other but note that the two groups may be abstractly isomorphic, e.g. for $k = 39$ and $m = 105$, with $n_1 = n_2 = 12$ and $d = 3$. For the parallel product $M = M_1 \,||\, M_2$ our earlier Proposition 3.5 implies that $\mathrm{Aut}(M) \cong G_1 \times G_2$ and so $M = (G; x, y)$ with $G \cong G_1 \times G_2$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Further, by our assumptions, for $i \in \{1, 2\}$ the order of y, that is, the valency of M, is equal to $\mathrm{lcm}(k, d) = k$ and the order of $xy = (x_1 y_1, x_2 y_2)$, the face length of M, is equal to $\mathrm{lcm}(d, m) = m$. It follows that the resulting (non-orientable) map M is of type (k, m).

Now, if e is an exponent of M, then the assignment $x \mapsto x$ and $y \mapsto y^e$ extends to an automorphism of G. If G_1 and G_2 are not isomorphic, Proposition 3.6 tells us that $\mathrm{Aut}(G) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$. In this case, for $i \in \{1, 2\}$ the above assignment would give rise to automorphisms of G_i fixing x_i and sending y_i to y_i^e. In other words, e would be an exponent of both M_1 and M_2 and hence $e = \pm 1$. If $G_1 \simeq G_2$, Proposition 3.6 implies that the only other possibility for the assignment $x \mapsto x$ and $y \mapsto y^e$ to extend to an automorphism of G is to exchange y_1 with y_2^e and y_2 with y_1^e, which is impossible by orders of these elements. □

We are now ready to address existence of non-orientable regular maps of a given hyperbolic type $\{m, k\}$ having no exponent other than ± 1. We begin with the case when at least one of k and m is even, using a direct construction.

Theorem 5.9. *Let (k, m) be a hyperbolic pair with at least one even entry. Then, there exists a non-orientable regular map of type $\{m, k\}$ with no exponent distinct from ± 1.*

Proof. Suppose that at least one of k, m is even. Then by Theorem 2 of [51] and its proof, there is an infinite set of odd primes p congruent to 1 mod both $2k$ and $2m$, such that for any $2k^{th}$ and $2m^{th}$ primitive roots ξ and η mod p the map $M = \mathrm{Map}(\xi, \eta)$ is regular and non-orientable, and has type (k, m), and $\mathrm{Aut}(M) \cong \mathrm{PGL}(2, p) \cong \langle x, y \rangle$, where $y = \pm\mathrm{diag}(\xi, \xi^{-1})$. The fact that the only exponents of M are ± 1 follows almost verbatim from the second part of the proof of our Theorem 5.1 in Section 5.2. □

If both entries of a hyperbolic pair (k, m) are odd, we can only offer a partial result for k and m that are not relatively prime. This will be done by a combination of Propositions 3.5 and 3.6 from Section 5.1, supported by the material developed in Section 5.3 together with Propositions 5.5 and 5.8, extending the tricks used in the corresponding proofs.

Theorem 5.10. *Let (k, m) be a hyperbolic pair with odd and non-coprime entries. Then, there exists a non-orientable regular map of type (k, m) with ± 1 as the only exponent.*

Proof. Let both entries k and m in our hyperbolic pair be odd and let $d = \gcd(k, m) > 1$. The situations when $k = m \geq 5$ or $d = 3$ are treated in Propositions 5.5 and 5.8 and so we will assume that for the odd highest common factor d is such that $d \geq 5$. Also, Proposition 5.7 allows us to assume that both $k, m \geq 7$ in what follows.

Let ξ and η be primitive k^{th} and m^{th} roots of unity in a finite field of characteristic 2, and let $q = 2^n$ be the smallest power of 2 such that the field $F = \mathrm{GF}(q)$ contains both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$. Since $(z + z^{-1})^2 = z^2 + z^{-2}$ in $\mathrm{GF}(q)$ and this field is closed under taking square roots (which are unique), it follows that the same field F is also the smallest containing both $\xi + \xi^{-1}$ and $\eta^2 + \eta^{-2}$. Our assumption that $k \neq m$ implies that both η and η^2 are distinct from ξ and ξ^{-1}.

It follows that $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^2)$ are non-orientable regular maps, with the same type (k, m), and with both having automorphism group isomorphic to $G = \mathrm{SL}(2, q)$. We next show that M and N are not isomorphic (and hence not a cover of each other, albeit having the same automorphism group). To demonstrate this, observe that a map isomorphism from M to N would have to be induced by an automorphism of $G = \mathrm{SL}(2, q)$, that is, by composition of conjugation by some element of G with a Galois automorphism of G, with the consequence that the pairs of traces $(\xi + \xi^{-1}, \eta + \eta^{-1})$ and $(\xi + \xi^{-1}, \eta^2 + \eta^{-2})$ corresponding to M and N would have to be related by the same Galois automorphism. We show that the latter implies $d = 3$, contrary to our assumption.

Suppose that a Galois automorphism of F, of the form $z \mapsto z^{2^\ell}$ for some $\ell \in \{1, ..., n-1\}$ and every $z \in F$, fixes $\xi + \xi^{-1}$ but sends $\eta + \eta^{-1}$ to $\eta^2 + \eta^{-2}$. Observe first that, in general, for any given non-zero element $u \in F$ of multiplicative order $\mathrm{ord}(u)$ and every integers i, j, one has $u^i + u^{-i} = u^j + u^{-j}$ if and only if $(u^{i+j} + 1)(u^{i-j} + 1) = 0$, which is equivalent to $\mathrm{ord}(u)$ dividing one of $i + j$, $i - j$, commonly written in the form $\mathrm{ord}(u) \mid i \pm j$. Using this, with $\mathrm{ord}(\xi) = k$ and $\mathrm{ord}(\eta) = m$ the condition $\xi^i + \xi^{-i} = \xi + \xi^{-1}$ for $i = 2^\ell$ and $j = 1$ translates to $k \mid 2^\ell \pm 1$, and the condition $\eta^i + \eta^{-i} = \eta^2 + \eta^{-2}$ for the same $i = 2^\ell$ with $j = 2$ similarly translates to $m \mid 2^\ell \pm 2$. But $d = \gcd(k, m)$ shares the divisibility properties of both k and m, that is, $d \mid 2^\ell \pm 1$ and $d \mid 2^\ell \pm 2$, which implies that $d \in \{1, 3\}$, a contradiction. It follows that M is not isomorphic to N if $d \geq 5$.

Our next aim is to show that M and N cannot have the same exponent $e \neq \pm 1$. So suppose the contrary, and let $e \neq \pm 1$ be a common exponent of both M and N. By Proposition 5.3 we may assume that $e = 2^i$, where i is the smallest *proper* divisor of n such that both $\rho(\xi, \eta)$ and $\rho(\xi, \eta^2)$ are contained in $\mathrm{GF}(2^i)$. Then $\rho(\xi, \eta^2)/\rho(\xi, \eta) = \eta + \eta^{-1}$ is an element of $\mathrm{GF}(2^i)$, as is $(\eta + \eta^{-1})/\rho(\xi, \eta) = \xi + \xi^{-1}$. But this means that both $\xi + \xi^{-1}$ and $\eta + \eta^{-1}$ are contained in a proper subfield of $\mathrm{GF}(2^n)$, which is a contradiction to the minimality of $q = 2^n$. It follows that the only common exponents of M and N are ± 1.

Next, let $M = (G; x, y)$ be a representation of $M = \mathrm{Map}(\xi, \eta)$ in the form $G = \langle\, x, y \mid x^2, y^k, (xy)^m, \ldots \rangle$, with x and y given by (5.1). For the map $N = \mathrm{Map}(\xi, \eta^2)$ we may use the same automorphism y, but with a modification x' of x obtained by replacing η with η^2 in (5.1). Then we may represent N in the form $N = (G; x', y)$ for the same group G but with presentation $G = \langle\, x', y \mid (x')^2, y^k, (x'y)^m, \ldots \rangle$. The maps M and N are distinct and their automorphism groups are both isomorphic to the simple group $G \cong \mathrm{SL}(2, q)$, and so it follows from Proposition 3.5 that the automorphism group of the parallel product $M \,||\, N$ is isomorphic to $G \times G$. Furthermore, by Proposition 3.6, the automorphism group of $\mathrm{Aut}(M \,||\, N)$ is isomorphic to $(\mathrm{Aut}(M) \times \mathrm{Aut}(N)) \rtimes C_2$, with the C_2-part inducing a transposition of the two factors.

Now suppose that this parallel product has an exponent $e \neq \pm 1$. Then just as in the proof of Theorem 5.2, there exists an automorphism γ of the group $\mathrm{Aut}(M \,||\, N)$ fixing the pair (x, x') and sending the pair (y, y) to $(y, y)^e = (y^e, y^e)$. By the final observation in the previous paragraph, γ is induced either by isomorphisms $M \to M^e$ and $N \to N^e$ such that $(x, y) \mapsto (x, y^e)$ and $(x', y) \mapsto (x', y^e)$, or by isomorphisms $M \to N^e$ and $N \to M^e$ such that $(x, y) \mapsto (x', y^e)$ and $(x', y) \mapsto (x, y^e)$.

In the first case M and N would have the same exponent $e \neq \pm 1$, a possibility that has been excluded. In the second case, composing the two isomorphisms $M \to N^e$ and $N \to M^e$ in both ways, that is, $M \to N^e \to (M^e)^e$ and $N \to M^e \to (N^e)^e$, implies that M and N have the same exponent e^2, and so $e^2 = \pm 1$. But here our assumed exponent e is induced by a Galois automorphism, and then by Proposition 5.3 the only possibility is that n is even and $e = 2^{n/2}$. Under the isomorphism $M \to N^e$ for $e = n/2$ the trace $\eta + \eta^{-1}$ of xy is mapped onto the trace $\eta^2 + \eta^{-2}$ of $x'y^e$, so that $\eta^{n/2} + \eta^{-n/2} = \eta^2 + \eta^{-2}$. By our previous trace calculation the latter implies that $m \mid 2^{n/2} \pm 2$, so that m also divides $(2^{n/2} + 2)(2^{n/2} - 2) = 2^n - 4$. But at the same time $m \mid 2^n \pm 1$ and combining the two divisibility condition gives $m \mid 4 \pm 1$, contrary to our assumption that $m \geq 7$.

Thus, for distinct odd $k, m \geq 7$ such that $\gcd(k, m) \geq 7$ the parallel product $M \,||\, N$ is a non-orientable regular map of type (k, m) with no exponents except ± 1. This completes the proof. □

5.5 Remarks

The theorems presented in this chapter can also be seen as a demonstration of the usefulness of parallel products of maps in constructing new maps with given properties from suitable smaller ones. There are, however, limitations to this approach, and one has to be careful because some seemingly straightforward ideas might not work.

We illustrate this with reference to Theorem 2 of [53], which states that for every hyperbolic type (n, m) there exists a reflexible orientably-regular map of type (n, m), with exponent group $\{1, -1\}$. The proof uses residual finiteness of the triangle group, and hence has a non-constructive flavour. In an attempt to give a constructive proof using parallel products of maps, suppose instead that one takes an orientably-regular but chiral map M of type (n, m) with trivial exponent group, and constructs the parallel product of M with its mirror image. In that case the resulting map is reflexible, but its exponent group can be larger than $\{-1, 1\}$.

For example, if M is the dual of the chiral map C46.6 from the list of chiral maps in [18], then M has type $(10, 25)$ and trivial exponent group, but the parallel product of M and its mirror image turns out to have exponent group $\{1, 3, 7, 9\}$, which is the entire group of units mod 10 (the valency of M).

Another limitation of our approach emerges by analysing the proof of Theorem 5.10. The key ingredient there was formation of a parallel product of the maps $M = \mathrm{Map}(\xi, \eta)$ and $N = \mathrm{Map}(\xi, \eta^2)$, and for this operation to make sense we needed the two maps to be non-isomorphic. Unfortunately, there are infinitely many counterexamples for hyperbolic pairs with coprime entries, even in a very strong sense that the Galois automorphism $z \mapsto z^2$ of $\mathrm{GF}(2^n)$ giving the power of 2 at η cannot be replaced by any other automorphism $z \mapsto z^{2^i}$ for $i < n$.

We believe that the assumption of non-coprimality of entries of hyperbolic types in Theorem 5.10 may be removed, but this remains beyond methods developed in this thesis.

Chapter 6

Construction of regular maps of a given hyperbolic type with no non-trivial exponents

In this chapter we will again look at the existence of orientably-regular maps with no non-trivial exponents. However, we will do this by building on the coset diagrams presented by Conder, Hucíková, Nedela and Širáň in [21], used for construction of orientably-regular but chiral maps (that is, without the exponent -1 of arbitrary hyperbolic type). Here, however, our goal is to exclude every exponent other than 1, and this requires to modify the diagrams of [21] appropriately.

The work on this approach was done with roughly equal share with my colleague PhD student Olivia Reade (under the guidance of Jozef Širáň who was at that time the supervisor of both of us) at several workshop-like sessions in Milton Keynes and Malta.

6.1 Families of 2-generated permutation groups

We begin by outlining the methods of [21] of construction of chiral maps of given hyperbolic type. Since existence of chiral orientably-regular maps of valency 3 was established in [13], and because of the fact that a map is chiral if and only if its dual is (a consequence of Corollary 2.7 in subsection 2.9),

considerations in [21] were restricted to types (k, m) such that $4 \leq k \leq m$. For every such type, a pair of permutations x, y of the set $\{1, 2, \ldots, n\}$ for suitable $n \geq 7$ was constructed in [21], using coset diagrams to visualise the action of the two permutations and their product. The group $G = \langle x, y \rangle$ is then proved in [21] to be isomorphic to the symmetric group S_n or the alternating group A_n of degree n, using a remarkable extension [35] of the classical Jordan's theorem on primitive permutation groups containing a cycle. For every such group G it was then shown that it admits no automorphism fixing x and inverting y.
Consequently, the resulting regular map $M = (G; x, y)$ of type (k, m) is chiral, and so is its dual $M^D = (G; x, xy)$, of type (m, k).

Our aim is to show that for every hyperbolic type (k, m) there exists an orientably-regular map $M = (G; x, y)$ of that type with no non-trivial exponent, which means that there is *no* integer e mod k for which there would be an automorphism of G fixing x and taking y onto y^e. To accomplish this, it is worth trying to check if the permutations xy from [21] and xy^e have a different cycle structure, and a different number of fixed points in particular. As it turns out (and as we show), this *is* indeed the case, and hence in this way one obtains existence of orientably-regular maps of any given hyperbolic type (k, m) for $k \leq m$.

The restriction $k \leq m$ from chirality considerations is, however, not sufficient to exclude non-trivial exponents for maps of hyperbolic type in which the valency is larger than the face length. But even in this case, it is still worth trying to apply the above strategy to the dual maps $M^D = (G; x, xy)$ to those constructed in [21]. This time, to exclude exponents f such that $2 \leq f \leq m - 2$ in the dual, that is, excluding existence of an automorphism of G fixing x and taking xy onto $(xy)^f$ for any f as above, one could try to check if the permutations $x(xy) = y$ and $x(xy)^f$ have a different number of fixed points. This turns out to work in many cases of the permutation groups G constructed in [21], but there are infinitely many exceptions.

To simultaneously handle the hyperbolic types (k, m) with $k \leq m$ and their dual types by comparing the number of fixed points of the pair of permutations xy, xy^e $(2 \leq e \leq k)$ and y, $x(xy)^f$ $(2 \leq f \leq m-2)$ we will introduce suitable modifications of some of the coset diagrams of [21] that define the permutations x and y. In this section, we will recall constructions 3.1, 3.2 and 3.4 from [21] and introduce a suitable alteration of the latter together with modifications of the constructions 3.3, 3.5, 3.6 and 3.7 of [21]. The resulting 9 families of maps $F0$ – $F8$ will turn out to be sufficient to cover types (k, m) for $\min(k, m) \geq 4$. An extra coset diagram is introduced to define one more family, $F9$, to handle exponents of maps of type (k, m) for $k \geq 7$.

In the diagrams, the permutation y will consist of k-cycles represented by k-sided polygons (to be read anticlockwise), with fixed points of y represented by heavy dots. Transpositions of the involution x will be represented by additional edges.

The family F0. This family is identical to the one obtained by construction 3.1 of [21], as can be seen in Figure 6.1. We also list the corresponding permutations x and y on the set $\{1, 2, \ldots, n\}$, with parameters as follows.

F0: $k \geq 6$, $k{+}2 \leq m \leq 2k{-}4$, and $n = m{+}1 = k{+}r$, where $3 \leq r \leq k{-}3$;

$x = (1, k{+}1)(2, k{+}2) \ldots (r, k{+}r)(k{-}1, k)$, fixing the $k{-}r{-}2$ points $r{+}1, \ldots, k{-}2$;

$y = (1, 2, \ldots, k)$, fixing the r points $k{+}1, k{+}2, \ldots, k{+}r$;

$xy = (1, k{+}1, 2, k{+}2, \ldots, r, k{+}r, r{+}1, \ldots, k{-}2, k{-}1)$, an m-cycle fixing k only.

The family F1. This family is identical with the one coming from the

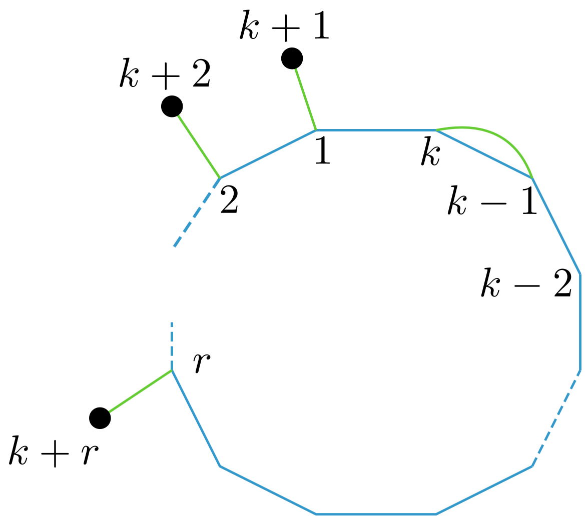

Figure 6.1: The Family F0

construction 3.2 of [21], and we again list the corresponding permutations x and y on $\{1, 2, \ldots, n\}$. The coset diagram can be seen in Figure 6.2.

F1: $k \geq 8$, $k{+}1 \leq m \leq 2k{-}7$, and $n = m{+}2 = k + r$, where $3 \leq r \leq k{-}5$;

$x = (1, k{+}1)(2, k{+}2)\ldots(r, k{+}r)(k{-}3, k{-}2)(k{-}1, k)$, fixing $r{+}1, \ldots, k{-}5, k{-}4$;

$y = (1, 2, \ldots, k)$, fixing the r points $k{+}1, k{+}2, \ldots, k{+}r$;

$xy = (1, k{+}1, 2, k{+}2, \ldots, r, k{+}r, r{+}1, \ldots, k{-}3, k{-}1)$, an m-cycle fixing $k{-}2$ and k.

The family F2. This family is the same as the one obtained from the construction 3.4 of [21], except that we will require $r \geq 2$ (instead of the original bound $r \geq 0$) in later arguments.

F2: $k \geq 5$, $2k{-}1 \leq m \leq 4k{-}11$, $n = m{+}3 = 2k{+}r{+}s$, $2{\leq}r{\leq}k{-}3$, $0{\leq}s{\leq}k{-}5$, $s{\neq}r{-}1$;

$x = (1, 2k{+}1)\ldots(r, 2k{+}r)(k{-}2, k{-}1)(k, k{+}1)(2k{-}s{-}3, n)$
$\ldots(2k{-}5, 2k{+}r{+}2)(2k{-}4, 2k{+}r{+}1)(2k{-}3, 2k{-}2)(2k{-}1, 2k)$, fixing the following $2k{-}8{-}r{-}s$ points: $r{+}1, \ldots, k{-}3$ and $k{+}2, \ldots, 2k{-}s{-}4$;

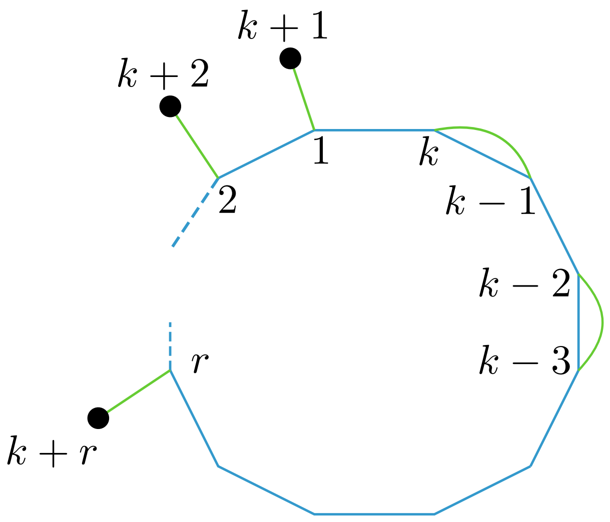

Figure 6.2: The Family F1

$y = (1, 2, \ldots, k)(k+1, k+2, \ldots, 2k)$, fixing the remaining $r+s$ points,

$xy = (1, 2k{+}1, \ldots r, 2k{+}r, r{+}1, \ldots, k{-}3, k{-}2, k, k+$
$2, k{+}3, \ldots, 2k{-}s{-}3, 2k{+}r{+}s,$
$2k{-}s{-}2, 2k{+}r{+}s{-}1, \ldots, 2k{-}3, 2k{-}1, k+1)$, an m-cycle fixing $k{-}1$, $2k{-}2$ and $2k$.

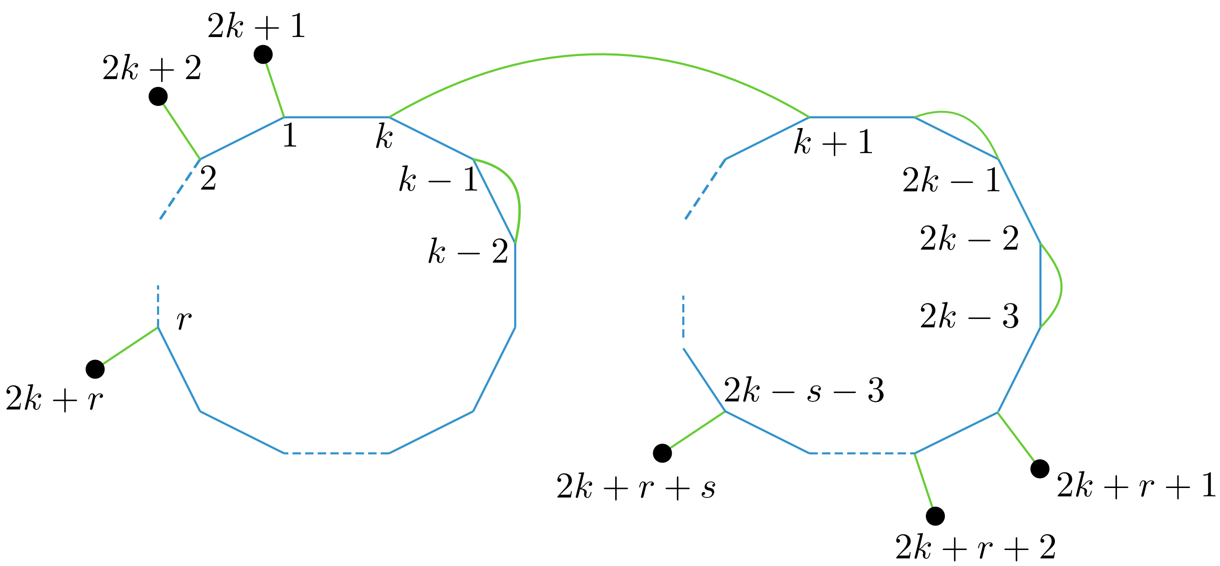

Figure 6.3: The Family F2

For each of these three families it was proved in [21] that the permutation group $G = \langle x, y\rangle$ of degree n is primitive, and since y is a k-cycle fixing the remaining

$n-k \geq 3$ points in F0 and F1 and xy is an m-cycle fixing 3 points in the family F2, by Jones' extension of the Jordan Theorem [35] the group G is isomorphic to S_n or A_n (note that $n \geq 9$ in F0, $n \geq 11$ in F1 and $n \geq 10$ in F2). In addition, in [21] all the three families have been shown to contain only chiral maps.

The family F3. This is a modification of the family 3.3 of [21] that enables us to work with a slightly more extended range of values of k and m.

F3: $k \geq 9$, $k-2 \leq m \leq 2k-9$, $n = m+3 = k+r$, where $1 \leq r \leq k-6$;

$x = (1,2)(3,4)(5,k+1)(6,7)(k-r+2,k+r)\dots(k,k+2)$, fixing $8,\dots,k-r+1$;

$y = (1,2,\dots,k)$, fixing the points $k+1,\dots,k+r$;

$xy = (1,3,5,k+1,6,8,\dots,k-r+2,k+r,k-r+3,k+r-1,\dots,k,k+2)$, an m-cycle fixing 2, 4 and 7.

A diagram representation of the permutations x and y for this family is in Fig. 6.4.

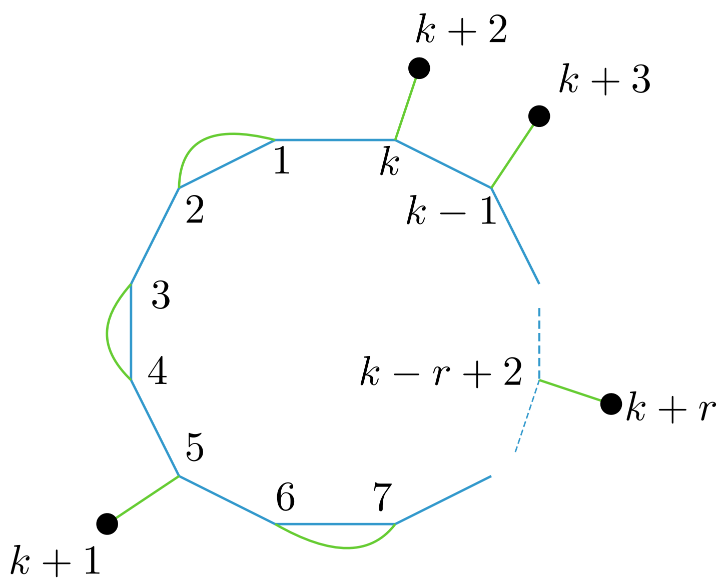

Figure 6.4: The Family F3

The family F4. This is a modification of the family 3.4 of [21], with 5 fixed points of xy.

F4: $k \geq 7$, $2k{-}3 \leq m \leq 4k{-}17$, $n = m{+}5 = 2k{+}2{+}r{+}s$, with $0 \leq r, s \leq k{-}7$, $s \neq r{-}3$;

$x = (1,2)(3,4)(5,2k{+}1)(6,7)(8,2k+2)\ldots(6{+}r,2k{+}r)$
$(k,k{+}1)(k{+}2,n{-}1)(k{+}3,n)(2k{-}s{-}3,2k{+}r{+}s)\ldots$
$(2k{-}4,2k{+}r{+}1)(2k{-}2,2k{-}3)(2k,2k{-}1)$, fixing the points
5 if $r = 0$, $6{+}r{+}1,\ldots,k{-}1$ if $r \geq 1$, and $k{+}4,\ldots,2k{-}s{-}4$;

$y = (1,2,\ldots,k)(k{+}1,\ldots,2k)$, fixing $2k{+}1,\ldots,2k{+}r$,
$2k{+}r{+}1,\ldots 2k{+}r{+}s, n-1, n$;

$xy =$
$(1,3,5,2k{+}1,6,8,2k{+}2,\ldots,6{+}r,2k{+}r,\ldots,k{-}1,k,k{+}2,n{-}1,k{+}3,n,\ldots,2k{-}s{-}3,$
$2k{+}r{+}s,\ldots,2k{-}4,2k{+}r{+}1,2k{-}3,2k{-}1,k{+}1)$, an m-cycle fixing 2, 4, 7, $2k{-}2$, $2k$.

A diagram representation of the permutations x and y for this family is in Fig. 6.5.

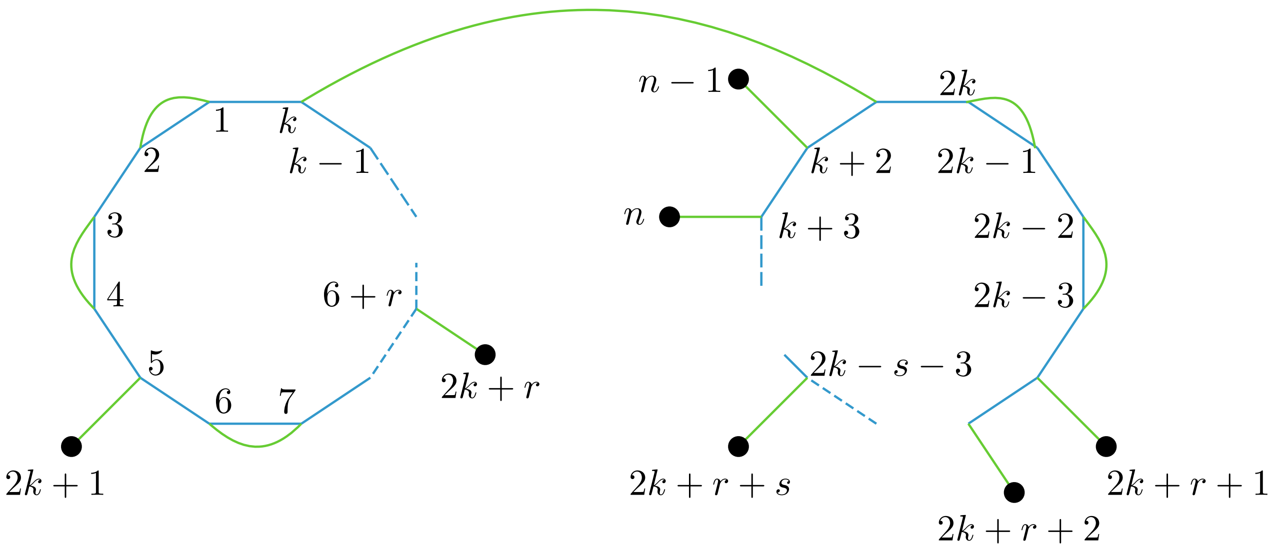

Figure 6.5: The Family F4

The family F5. This is a modification of the family 3.5 of [21] with $k \geq 8$, in which the number of k-cycles in y is c for an arbitrary $c \geq 3$.

F5: $k \geq 8$, $c \geq 3$, $3k-2 \leq m = ck{+}r-2$, $0 \leq r \leq k{-}1$, $n = m{+}5 = ck{+}r{+}3$;

$x =$

$(1,2)(3,4)(5,n{-}2)(6,7)(k,k{+}1)(k{+}2,k{+}3)(k{+}4,k{+}5),(k{+}6,n{-}1),(k{+}7,n)$

$(2k,2k{+}1)\ldots((c{-}1)k,(c{-}1)k{+}1)(ck{-}r{+}1,ck{+}r)(ck{-}r{+}2,ck{+}r{-}1)\ldots(ck,ck{+}1)$,

fixing $(c{-}1)k{+}2$, $\ldots$, $ck-r$, and for $3 \leq j \leq c{-}2$ also the points

$jk{+}2,\ldots,(j{+}1)k{-}1$;

$y = (1,2,\ldots,k)\ldots((c{-}1)k{+}1,\ldots,ck)$, fixing the remaining $r{+}3$ points;

$xy =$

$(1,3,5,n{-}2,6,8,\ldots,k,k{+}2,k{+}4,k{+}6,n{-}1,k{+}7,n,k{+}8,\ldots,2k,2k{+}2,\ldots,$

$3k,\ldots,(c{-}1)k,(c{-}1)k+2,\ldots,ck{-}r{+}1,ck{+}r,ck{-}r{+}2,ck{+}r{-}1,\ldots,$

$(c{-}1)k{+}1,(c{-}2)k{+}1,\ldots,2k{+}1,k{+}1)$, an m-cycle fixing the points 2, 4, 7, $k{+}3$ and $k{+}5$.

A diagram representation of the permutations x and y for this family is in Fig. 6.6.

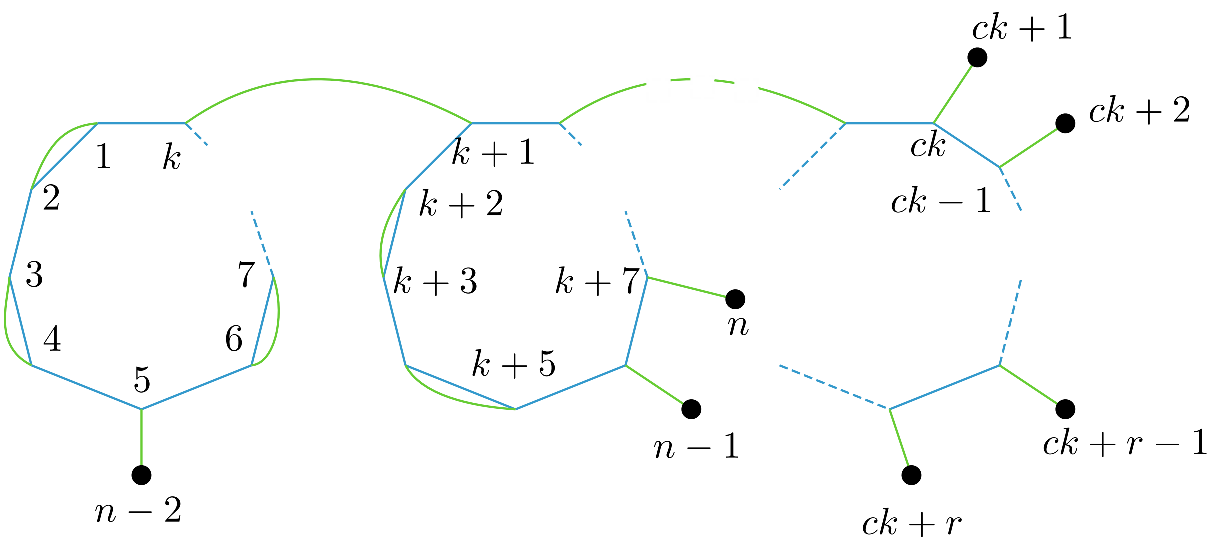

Figure 6.6: The Family F5

The family F6 for types $(6,m)$ will depend on the parity of m. Suppose first that m is odd and $m \geq 13$, so that $m = 13 + 2j$ for $j \geq 0$, and $n = |P| = m + 3$. The point set P and the permutations x and y of P will be as follows:

F6: $P = \{1, 2, \ldots, 15, 16\} \cup (\cup_{i=1}^{j} \{u_i, v_i\})$, with an empty union if $j = 0$;

$x =$

$(1, 13)(2, 3)(4, 5)(6, 7)(8, 9)(10, 14)(11, 15)(12, u_1)(v_1, u_2) \ldots (v_{j-1}, u_j)(v_j, 16)$

if $j \geq 1$ and $x = (1, 13)(2, 3)(4, 5)(6, 7)(8, 9)(10, 14)(11, 15)(12, 16)$ if $j = 0$;

$y = (1, \ldots, 6)(7, \ldots, 12)(u_1, v_1) \ldots (u_j, v_j)$, fixing the points $13, \ldots, 16$;

$xy = (1, 13, 2, 4, 6, 8, 10, 14, 11, 15, 12, v_1, v_2 \ldots, v_j, 16, u_j, u_{j-1} \ldots, u_1, 7)$ for $j \geq 1$,

and $xy = (1, 13, 2, 4, 6, 8, 10, 14, 11, 15, 12, 16, 7)$ for $j = 0$, where in both cases it is fixing 3, 5 and 9.

A diagram representation of the permutations x and y for this family is in Fig. 6.7. For even $m \geq 12$, that is, for $m = 12 + 2j$, the family is modified by deleting the point 16 from P and reducing the three permutations accordingly. By further deleting the point 15 for $j = 0$ from the diagram in Fig. 6.7, one obtains generators x and y for $m = 11$, that is, a map of type $(6, 11)$. Using a more drastic reduction by using just the point set $\{1, 2, \ldots, 6\} \cup \{1', 2', 4'\}$ and the permutations $x = (1, 1')(2, 2')(4, 4')$ and $y = (1, 2, \ldots, 6)$, with xy a cycle of length 9, gives a diagram for the type $(6, 9)$, that is, with $m = 9$.

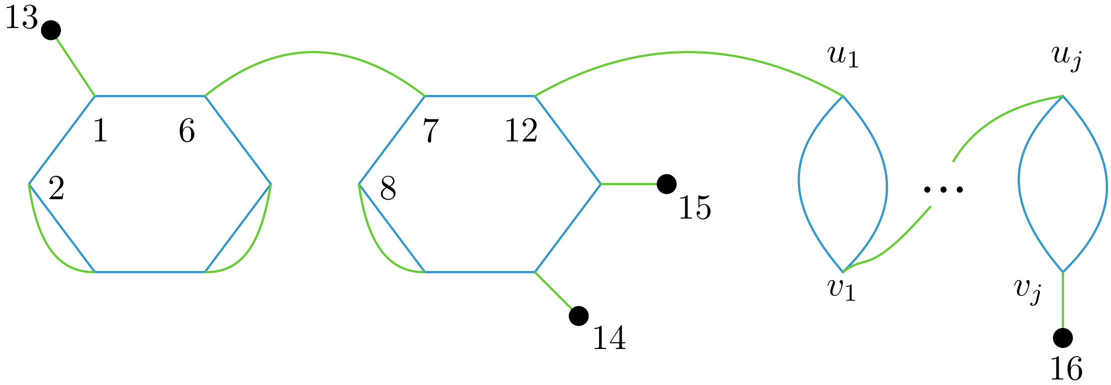

Figure 6.7: The Family F6

The family F7 for types $(5, m)$ will depend on residue class of m mod 5. Consider first the case when $m \equiv 3 \bmod 5$ and $m \geq 18$, that is, $m = 18 + 5j$ for $j \geq 0$. The point set P with $|P| = m + 5$ and the permutations x and y of P will now have the following form:

F7: $P = \{1, 2, \ldots, 19, 20, 9', 14', 17'\} \cup (\cup_{i=1}^{j}\{s_i, t_i, u_i, v_i, w_i\})$, $j \geq 0$;

$x = (1, 2)(3, 4)(5, 6)(7, 8)(9, 9')(10, 11)(12, 13)(14, 14')(15, s_1)(w_1, s_2)\ldots$ $(w_{j-1}, s_j)(w_j, 16)(17, 17')(19, 20)$, fixing 18 and each point in the set $\cup_{i=1}^{j}\{t_i, u_i, v_i\}$;

$y = (1, \ldots, 5)(6, \ldots, 10)(11, \ldots, 15)\ldots(s_i, t_i, u_i, v_i, w_i)\ldots(16, \ldots, 20)$, fixing $9'$, $14'$, $17'$;

$xy =$
$(1, 3, 5, 7, 9, 9', 10, 12, 14, 14', 15, .., t_i, u_i, v_i, w_i, .., 17, 17', 18, 19, 16, s_j, .., s_1, 11, 6)$
if $j \geq 1$, and $xy = (1, 3, 5, 7, 9, 9', 10, 12, 14, 14', 15, 17, 17', 18, 19, 16, 11, 6)$
if $j = 0$,
in both cases fixing 2, 4, 8, 13 and 20.

A diagram representation of the above permutations x and y for this family is in Fig. 6.8. For $m \equiv 4 \bmod 5$, of the form $m = 19 + 5j$, $j \geq 0$, one replaces x by x' which is achieved by eliminating the permutation $(19, 20)$ from x, leaving y intact and changing xy accordingly. For $m \equiv 0, 1, 2 \bmod 5$, with $m = 20 + 5j + \delta$ for $j \geq 0$ and $\delta \in \{0, 1, 2\}$ one replaces the ground set P from the description of Fig. 6.8 by P_δ and the involution x by x_δ, respectively, by letting $P_0 = P \cup \{18'\}$ and $x_0 = x'(18, 18')$, $P_1 = P_0 \cup \{19'\}$ and $x_1 = x_0(19, 19')$, and finally $P_2 = P_1 \cup \{20'\}$ and $x_2 = x_1(20, 20')$, again leaving y unchanged.

Some of the smaller values of m can be covered by reducing the diagram in Fig. 6.8 to the three left-hand-side pentagons. For example, if
$P = \{1, 2, \ldots, 15\} \cup \{9', 14'\}$ and

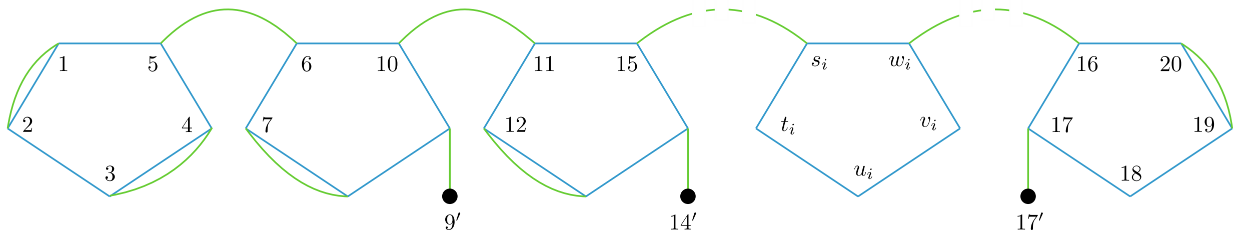

Figure 6.8: The Family F7

$x = (1,2)(3,4)(5,6)(7,8)(9,9')(10,11)(12,13)(14,14')$, for $m = 13 + i$ such that $i \in \{0,1,2,3,4\}$ one may form new ground sets $P_{(i)}$ and involutions $x_{(i)}$ on these sets by letting $P_{(0)} = P$ and $x_{(0)} = x$, $P_{(1)} = P \cup \{15'\}$ with $x_{(1)} = x(15,15')$, $P_{(2)} = P_{(1)}$ with $x_{(2)} = x_{(1)}(12,13)$, $P_{(3)} = P_{(2)} \cup \{13'\}$ with $x_{(3)} = x_{(2)}(13,13')$, and $P_{(4)} = P_{(3)} \cup \{12'\}$ with $x_{(4)} = x_{(3)}(12,12')$, in all cases using $y = (1,\ldots,5)(6,\ldots,10)(11,\ldots,15)$; in the corresponding permutation groups $\langle x_{(i)}, y\rangle$ the order of $x_{(i)}y$ is $m = 13+i$, $0 \le i \le 4$. A more extreme modification of the diagram in Fig. 6.8 is obtained by keeping just the two leftmost pentagons, shrinking the point set to $\{1,2,\ldots,10\} \cup \{9',10'\}$ and defining $x = (1,2)(3,4)(5,6)(7,8)(9,9')(10,10')$ and $y = (1,\ldots,5)(6,\ldots,10)$; this gives a map of type $(5,9)$.

The family F8 for types $(4,m)$ will depend on the parity of m. Suppose first that m is odd and $m \ge 17$, so that $m = 17 + 2j$ for $j \ge 0$. The point set P with $|P| = m + 3$ and the permutations x and y of P will now be as follows:

F8: $P = \{1,2,\ldots,13,14,14',15,15',16,16'\} \cup (\cup_{i=1}^{j}\{u_i,v_i\})$, with empty union if $j = 0$;

$x =$

$(1,1')(2,3)\ldots(12,13)(14,14')(15,15')(16,u_1)(v_1,u_2)\ldots(v_{j-1},u_j)(v_j,16')$ if $j \ge 1$;

$x = (1,1')(2,3)\ldots(12,13)(14,14')(15,15')(16,16')$ if $j = 0$, with no fixed points;

$y = (1, .., 4)(5, .., 8) \dots (13, .., 16) \dots (u_i, v_i) \dots$, fixing $1'$, $14'$, $15'$ and $16'$;

$xy = (1, 1', 2, 4, .., 12, 14, 14', 15, 15', 16, v_1, v_2.., v_j, 16', u_j, .., u_2, u_1, 13, 9, 5)$ if $j \geq 1$,

and $xy = (1, 1', 2, 4, .., 12, 14, 14', 15, 15', 16, 16', 13, 9, 5)$ if $j = 0$, fixing 3, 7 and 11.

A diagram representation of the permutations x and y for this family is in Fig. 6.9. For even $m \geq 16$, the family is modified by removing the point labeled $16'$ from P and the transposition $(v_j, 16')$ or $(16, 16')$ from x, leaving y intact.

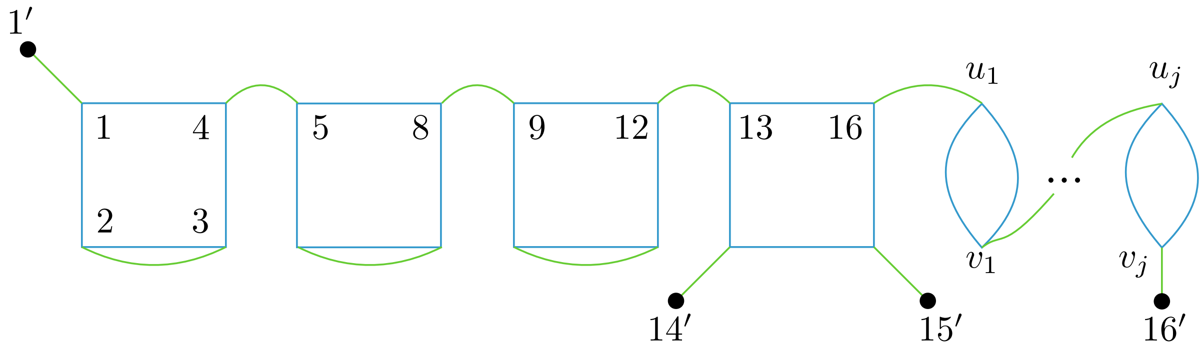

Figure 6.9: The Family F8

The family F9 for types $(k, 3)$ will depend on residue class of k mod 3. To begin with, suppose $k \equiv 2 \bmod 3$ and $k \geq 17$, so that $k = 17 + 3j$ for $j \geq 0$. The point set P with $|P| = k + 3$ and the permutations x and y of P will have the following form:

F9: $P = \{1, 2, \dots, 14, a, a', b, b', c, c'\} \cup (\cup_{i=1}^{j} \{u_i, v_i, w_i\})$, with an empty union if $j = 0$;

$x =$
$(a, a')(b, b')(c, c')(1, 6)(2, 3)(4, 5)(7, 12)(8, 9)(10, 11)(13, 14)(u_1, v_1) \dots (u_j, v_j)$, fixing the points $w_1, \dots, w_j$;

$y = (1, 2, a, 3, 4, b, 5, 6, u_1 \dots, u_j, 7, 8, 9, 10, c, 11, .., 14, v_j, w_j, \dots, v_1, w_1)$, a $(17 + 3j)$-cycle fixing the points a', b', c';

$xy =$

$(2, 4, 6)(8, 10, 12)(3, a, a')(5, b, b')(11, c, c')(1, u_1, w_1)(v_1, u_2, w_2) \dots (v_j, 7, 13),$

with the product following $(11, c, c')$ reducing to $(1, 7, 13)$ if $j = 0$, fixing 9 and 14.

A diagram representation of the permutations x and y is in Fig. 6.10. For $k \equiv 1 \mod 3$ and $k \geq 13$ the family is modified by deleting the four points 7, 12, 13 and 14 from P, removing the transpositions $(7, 12)$ and $(13, 14)$ from x, and adjusting the non-trivial cycle of y by simply removing the four deleted points from their positions. A further modification for $k \equiv 0 \mod 3$ and $k \geq 12$ is obtained by merging the points 8 and 9 into a single point in both the ground set and in y (the merged point becomes fixed by x).

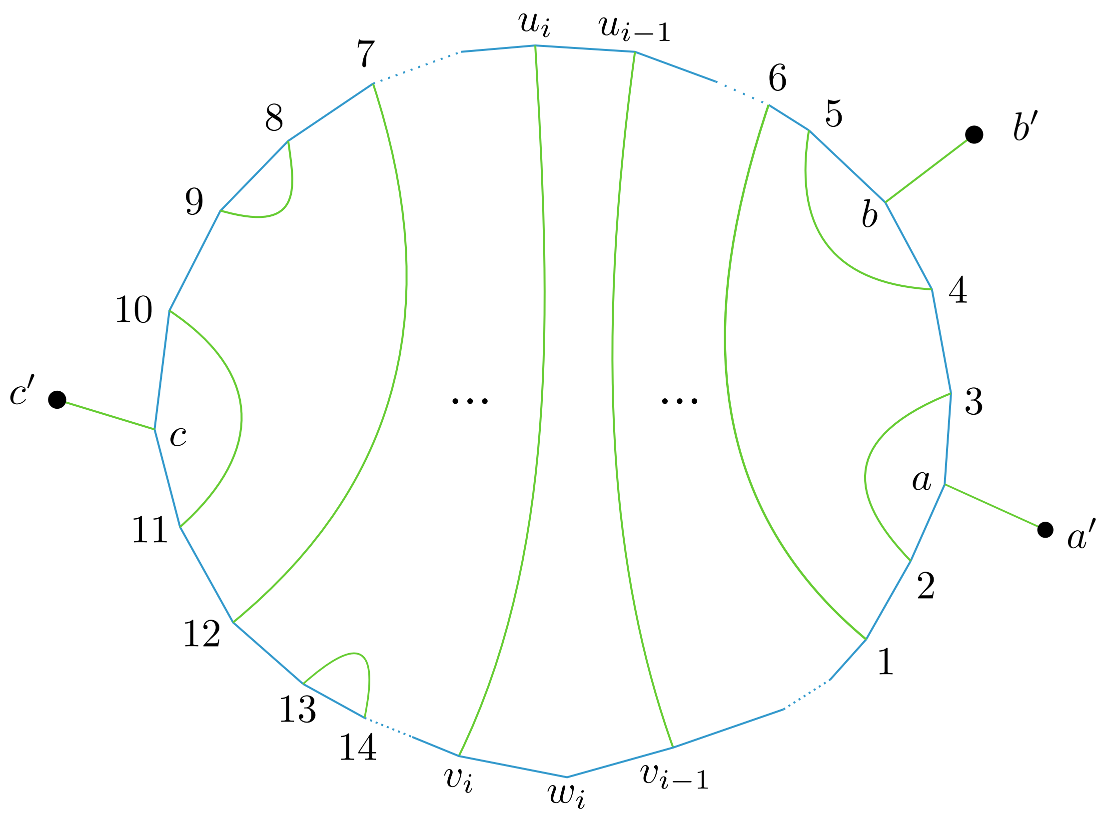

Figure 6.10: The Family F9

This last family, F9, differs from all the previous ones by having k larger than m and also by the smallest possible value of m (which is 3 for a hyperbolic type of a map), which will require a different approach to proving some of its properties.

For $k = 6$ and 5 we introduced modifications giving maps of types (k, m) for certain values of $m > k$ but 'close' to k. One could produce more modifications of a similar kind but we restricted ourselves only to those providing suitable ingredients for the forthcoming Proposition 6.4 that allows for an easy control over exponents in the duals of the maps defined by modified diagrams.

6.2 Orientably-regular maps of given type with no non-trivial exponents

In a series of steps we will show that the orientably-regular maps $M = (G; x, y)$ arising from the groups $G = \langle x, y \rangle$ in the families F0 – F9 have no non-trivial exponents. We begin by investigating the group G as a *permutation* group.

Connectedness of the diagrams in Figs. 6.4 – 6.10 implies that the corresponding permutation groups $\langle x, y \rangle$ are transitive on the given point sets. Next, observe that in the diagrams for the families F3, F4 and F5, the conjugate $y^{-3}(xy)y^3$ of xy fixes the point 7 (which is also fixed by xy) but moves all the remaining fixed points of xy. Similarly, the permutation $y^{-2}(xy)y^2$ fixes the point 5 in all the maps from F6 except the one for $m = 9$, and the point 4 in all maps in F7, while this permutation fixes no other point fixed by xy. For the same kind of feature in the remaining two families, in F8 one may use the conjugate $(xy^{-1}x)y(xyx)$ of y, which fixes the point $15'$ but no other point fixed by y, and in F9 the permutation $(xy^{-3}x)y(xy^3x)$ fixes b' but no other point fixed by y. It follows that in each of the above cases the stabilizer of a point in $G = \langle x, y \rangle$ is transitive on the remaining points, implying that G is doubly transitive, and hence primitive, permutation group in all the families F3 – F9. One may check that the same conclusion is valid for the exceptional group of F6 for $m = 9$.

Further, in all the families F3 – F8 except for the modification of F6 for $m = 9$, the permutation xy is a single cycle with 3, 4, or 5 fixed points, and in F9 and in the above modification of F6 the permutation y is a single cycle with 3 fixed points. From the Jones' generalization [35] of Jordan's theorem, it follows that in each of these cases the group $G = \langle x, y\rangle$ is isomorphic to the symmetric or the alternating group of degree equal to the number of points in the corresponding diagram. Taking into account the previously mentioned analysis of the families F0 – F2 in [21], we obtain:

Proposition 6.1. *The permutation groups $G = \langle x, y\rangle$ for all the ten families* F0 – F9 *are isomorphic to the symmetric or the alternating group of the corresponding degree.* □

We now turn to excluding non-trivial exponents of the orientably-regular maps $M = (G; x, y)$, beginning with -1. This has already been done for the first three families in [21]. To eliminate the exponent -1 (i.e., to establish chirality) in the remaining 7 families we use the following observation implicitly contained in [21], which enables one to reduce the question to the existence of a certain reflective symmetry of the corresponding coset diagrams in Figs 6.4 – 6.10.

Lemma 6.2. *Let $M = (G; x, y)$ be an orientably-regular map and let* $\mathrm{Diag}(M)$ *be the corresponding coset diagram with $n \geq 7$ points. Assume that G is isomorphic to the symmetric or the alternating group of degree n. Then, M admits the exponent -1 if and only if there is a reflection of the diagram* $\mathrm{Diag}(M)$ *that maps every y-cycle onto a y-cycle with a reverse orientation.*

Proof. Suppose that -1 is an exponent of M, that is, there is an automorphism of G fixing x and taking y onto y^{-1}. Since $G \cong S_n$ or A_n, this happens if and only if there is a *permutation* α of the set S of the n points (vertices) of $\mathrm{Diag}(M)$

such that $\alpha x \alpha^{-1} = x$ and $\alpha y \alpha^{-1} = y^{-1}$. Such a permutation α has the property that α^2 fixes both x and y and hence fixes pointwise the entire group $G \cong S_n$ or A_n, so that $\alpha^2 = id$ and hence $\alpha = \alpha^{-1}$. Further, the equations tying x and y with α are equivalent to $j\alpha x = jx\alpha$ and $j\alpha y = jy^{-1}\alpha$ for every $j \in S$. But this means that α induces a reflection (i.e., an orientation-reversing involutory automorphism) of the *diagram* Diag(M), sending every y-cycle to another y-cycle but with reverse orientation (and setwise preserves the x-edges). □

Corollary 6.3. *All the orientably-regular maps in the families* F0 – F9 *together with their dual maps are chiral.*

Proof. Since families F0 – F2 are identical to the ones in [21], we can use the results from that paper that proved existence of chiral maps. For families F3 – F9 we use Proposition 6.1 which confirms that the relevant groups are isomorphic to S_n or A_n, through which we can then use Lemma 6.2 to say that M admits the exponent -1 if and only if there is a reflection of the diagram Diag(M) that maps every y-cycle onto a y-cycle with a reverse orientation. Chirality of these maps follows since none of the diagrams in Figs 6.4 – 6.10 admit a reflection inverting orientation of all the y-cycles. The dual maps to the ones determined by Figs 6.4 – 6.10 are also chiral, because of the fact that a (not necessarily regular) map is isomorphic to its mirror image if and only if its dual is. □

Elimination of the remaining non-trivial exponents in the maps $M = (G; x, y)$ of type (k, m) and their duals $M^D = (G; x, xy)$ of type (m, k) will be based on the built-in features of their construction; the family F9 will be dealt with separately because of its different character.

Proposition 6.4. *For every orientably-regular map* $M = (G; x, y)$ *of type* (k, m) *from any of the families* F0 – F8, *the group* $G = \langle x, y \rangle$ *has the following two properties:*

(P1) *for each* $e \in \{2, 3, \ldots, k-2\}$ *the permutation* xy^e *has no fixed points, and*

(P2) *for each* $f \in \{2, 3, \ldots, m-2\}$ *the permutation* $x(xy)^f$ *has fewer fixed points than* y.

Proof. We will refer to the description of the families F0 – F8, including the cases that depend on residue classes mod 2 or 5, and we will refer to the accompanying diagrams in Figs. 6.1 – 6.9. Edges corresponding to transpositions of x and to the cycles of y will simply be referred to as x-edges and y-edges, respectively.

Observe that every x-edge in the above diagrams is either pendant, that is incident to a vertex of valency 1, or joining a pair of points on distinct y-cycles (a *bridging* edge, for short). Existence of a fixed point of the permutation xy^e for some $e \in \{2, 3, \ldots, k-2\}$ would imply existence of an x-edge which would be a chord joining a pair of non-adjacent points on a y-cycle. Since no such chord in Figs. 6.1 – 6.9 exists, the property (P1) follows.

For (P2) it is useful to realise that a point u is fixed by $x(xy)^f$ if and only if $ux = u(xy)^f$, which means that the end-points u and ux of an x-edge are f steps apart from each other on the xy-cycle. But in all the families F0 – F8, the permutation xy is a single cycle with 3, 4 or 5 fixed points, and these fixed points are situated on 'digons' formed by pairs of points joined both by an x-edge and a y-edge. Recalling that every x-edge in Figs. 6.4 – 6.9 is either pendant or bridging, it follows that the only fixed points of the permutation $x(xy)^f$ for some $f \in \{2, 3, \ldots, m-2\}$ are end-points of bridging x-edges.

This means that (P2) is automatically satisfied for the families F0, F1, F3 and the modification of F6 for $m = 9$, as there are no bridging x-edges and y has at least one fixed point in each.

There is one bridging x-edge in the diagram in Fig. 6.5 describing the family F4 (and, with a small modification, the family F2). The extra conditions $s \neq r-1$ and $s \neq r-3$ in the descriptions of the families F2 and F4 guarantees that the endpoints of the single bridging x-edge are fixed points of $x(xy)^f$ for two *distinct* values of $f \in \{2, 3, \ldots, m-2\}$. This property of endpoints of the single bridging x-edge is valid also for the modifications of the construction F6 for $m = 11$ and F7 for $m = 9$. Since in both families y has at least two fixed points, (P2) is satisfied for F2 and F4. Note that (P2) holds also for the family F6 for $m \in \{12, 13\}$, in which case there is just one bridging x-edge but y has three fixed points.

To check the property (P2) for the families F5 – F8 with at least 3 bridging x-edges, observe that the distribution of the bridging x-edges in Fig. 6.6 – 6.9 implies that for every $f \in \{2, 3, \ldots, m-2\}$, there are at most two end-points of some bridging x-edges fixed by $x(xy)^f$ for that particular f. At the same time, the corresponding permutation y has at least 3 fixed points. This proves validity of (P2) in these cases, and hence for all the families F0 – F8. □

Absence of non-trivial exponents in the maps from all our ten families is now established as follows.

Proposition 6.5. *Let M be an arbitrary map from any of the families* F0 – F9. *Then, neither M nor its dual M^D has a non-trivial exponent.*

Proof. By Corollary 6.3 neither of the maps in the statement has exponent -1. For maps of type (k, m) we can therefore restrict ourselves to considering powers $e \in \{2, 3, \ldots, k-2\}$, and in their dual maps of type (m, k) to powers $f \in \{2, 3, \ldots, m-2\}$.

Let $M = (G; x, y)$ be an orientably-regular map from one of the families F0 – F8, of type (k, m) and valency k, and let $M^D = (G; x, xy)$ be its dual map, of type (m, k) and valency m. For some $e \in \{2, 3, \ldots, k-2\}$ to be an exponent of M, and for some $f \in \{2, 3, \ldots, m-2\}$ to be an exponent of M^D, there would have to be an automorphism of the group G fixing x and, respectively, taking y to y^e for M, and taking xy to $(xy)^f$ for M^D. But by properties (P1) and (P2) of Proposition 6.4, no such automorphisms exist, because of a different number of fixes points of xy and xy^e in the first case, and $x(xy) = y$ and $x(xy)^f$ in the second case.

The dual map to any of the maps in the family F9 has valency 3 and hence cannot have any non-trivial exponent, so that it remains to consider the primal maps $M = (G; x, y)$ from F9, of type $(k, 3)$ and valency $k \geq 12$ but $k \neq 14$ (by inequalities following from the description of this family). Suppose that for some $e \in \{2, 3, \ldots, k-2\}$ there is an automorphism of G fixing x and taking y to y^e. Since the order of xy here is 3, this would mean that the order of xy^e is 3 as well. In particular, referring to the diagram in Fig. 6.10 and the notation therein, this would mean that $a(xy^e)^3 = a$ and $b(xy^e)^3 = b$. Observe that both a and b are fixed by the permutation $xy^e x$, so that the above equations simplify to $ay^e xy^e = a$ and $by^e xy^e = b$.

Taking into account that $b = ay^3$, the last pair of equations mean that, on the y-cycle of length k in Fig. 6.10, there would have to be a closed walk from the point a of length $2e + 1$ consisting of e consecutive y-edges in the counterclockwise direction along the y-cycle, followed by an x-chord (not by a pendant x-edge) and continuing by further e consecutive y-edges in the counterclockwise direction along the y-cycle, finally terminating at the same point, a. But a similar closed walk, of the same length and with the same sequence of y- and x-edges, would have to exist, starting and ending at the point b, situated just three y-edges apart from a. By inspecting the diagram in 6.10 one concludes that neither of the two x-edges in the closed walks can join a pair of adjacent points on the y-cycle, or a pair of points on the y-cycle at distance two. It follows that the x-edges in the two closed walks would have to be chords of the y-cycle crossing each other (in the obvious sense), which contradicts the situation in the diagram. This completes the proof. □

Observe that the chiral orientably-regular maps in the families F6, F7, F8, and the dual maps to those in the family F9, automatically have no non-trivial exponents; this applies also to chiral maps of valency 10. The reason is that the only units mod $k \in \{6, 4, 3\}$ are ± 1, and for $k \in \{5, 10\}$ it follows from the fact that the square of each unit mod k distinct from ± 1 is equal to -1. It follows that, regarding these four families, Proposition 6.5 renders a non-trivial statement only for the dual maps to those in F6 – F8 and for the primal maps in F9.

We are now in position to prove the main result.

Theorem 6.6. *For every hyperbolic type (k, m) there exists an orientably-regular map M with automorphism group isomorphic to the symmetric or the alternating group of degree n for some n, such that M has only the trivial exponent.*

Proof. We will successively examine the constructions in the previous sections with emphasis on the range of the parameters k and m they cover, assuming first that $k \leq m$. For brevity we will refer just to 'maps', meaning 'orientably-regular' throughout.

For $k \geq 9$, the family F3 gives maps of type (k, m) for all m in the range $k \leq m \leq 2k - 9$, and the families F1 and F0 supply maps of type (k, m) for m such that $k + 1 \leq m \leq 2k - 7$ (even for $k \geq 8$) and $k + 2 \leq m \leq 2k - 4$ (even for $k \geq 6$), respectively. Still assuming that $k \geq 9$, from the family F4 one has maps of type (k, m) with $2k - 3 \leq m \leq 4k - 17$, and the family F2 furnishes maps of type (k, m) for all m in the range $2k - 1 \leq m \leq 4k - 11$ (in both cases even for $k \geq 7$), and as $2k-1 \leq 4k-16$ for $k \geq 8$ the two families cover the interval $2k-3 \leq m \leq 4k-11$. Finally, the family F5 provides maps of type (k,m) for every $m \geq 3k-2$ and $k \geq 8$, and since $3k-2 \leq 4k-10$ for $k \geq 8$, the six families cover maps of all types (k,m) such that $9 \leq k \leq m$. If $k=8$, the chain of the above arguments applies except for the type $(8,8)$, so that they extend to all types (k,m) such that $8 \leq k \leq m$ save the case $k=m=8$.

If $k \geq 7$, the arguments using the union of types covered by the families F3, F1 and F0 for the range $k \leq m \leq 2k-4$ apply except for $k=7$ and $m \in \{7,8\}$. Further, still for $k \geq 7$, the union of the families F4 and F2 covers the interval $2k-3 \leq m \leq 4k-11$ except for $k=7$ and $m=12$, and the union of the families F2 and F5 takes care of every $m \geq 3k-2$ except for $k=7$ and $m=18$. The family F6 covers all types $(6,m)$ such that $m \geq 11$ (and also for $m=9$), the family F7 takes care of the types $(5,m)$ for every $m \geq 13$ (including $m=9$), and the family F8 covers the types $(4,m)$ for every $m \geq 16$. Finally, the outlier family F9 implies existence of orientably-regular maps of type $(k,3)$ for $k \geq 12$, $k \neq 14$.

By Propositions 6.1 and 6.5, all these maps, together with their duals, have no non-trivial exponents and their automorphism groups are isomorphic to symmetric or alternating groups.

The remark on no non-trivial exponents in chiral maps of valency $k \in \{3,4,5,6,10\}$, made after the proof of Proposition 6.5, implies that if a type (k,m) for one of these values of k is *not* covered by the above analysis, it can be *excluded* from further consideration.

This leaves the following 27 types to be examined further: $(8,8)$, $(7,7)$, $(7,m)$ for $m \in \{8,12,18\}$ and their dual types, $(7,6)$, $(8,6)$, $(k,5)$ for $k \in \{7,8,11,12\}$, $(k,4)$ for $k \in \{7,8,9,11,12,13,14,15\}$, and $(k,3)$ for $k \in \{7,8,9,11,14\}$. Existence of orientably-regular maps of corresponding types has been checked computationally through the use of [8] and [58], with a list of generators x and y included in the subsequent table. This completes the proof. □

Table of groups $G = \langle x, y \rangle$ completing the proof of Theorem 6.6:

Type	**Group**	x	y
$(7,3)$	A_{15}	$(1,4)(5,13)(6,11)(7,14)(8,10)(9,15)$	$(1,2,\ldots 7)(8,\ldots 14)$
$(7,4)$	S_8	$(1,5)(2,3)(4,8)$	$(1,2,\ldots 7)$
$(7,5)$	A_{10}	$(1,4)(2,9)(3,10)(7,8)$	$(1,2,\ldots 7)$
$(7,6)$	S_8	$(1,2)(3,4)(7,8)$	$(1,2,\ldots 7)$
$(7,7)$	A_8	$(1,2)(7,8)$	$(1,2,\ldots 7)$
$(7,8)$	S_9	$(1,2)(6,9)(7,8)$	$(1,2,\ldots 7)$
$(7,11)$	A_{11}	$(1,8)(2,9)(3,10)(5,11)$	$(1,2,\ldots 7)$
$(7,12)$	S_8	$(1,2)(3,8)(4,7)$	$(1,2,\ldots 7)$
$(7,18)$	S_{18}	$(1,14)(10,18)(11,17)(12,16)(13,15)$	$(1,2,\ldots 7)(8,\ldots,14)$
$(7,19)$	A_{19}	$(1,14)(10,18)(9,19)(11,17)(12,16)(13,15)$	$(1,2,\ldots 7)(8,\ldots,14)$
$(8,3)$	S_{18}	$(1,10)(3,9)(4,11)(5,8)(12,16)(13,17)(15,18)$	$(1,2,\ldots 8)(9,\ldots 16)(17,18)$
$(8,4)$	S_9	$(4,8)(5,6)(7,9)$	$(1,2,\ldots 8)$
$(8,5)$	S_{11}	$(1,6)(2,3)(4,11)(7,10)(8,9)$	$(1,2,\ldots 8)$
$(8,6)$	A_9	$(2,3)(4,5)(6,7)(3,9)$	$(1,2,\ldots 8)$
$(8,8)$	S_9	$(1,2)(8,9)$	$(1,2,\ldots 8)$
$(9,3)$	A_{10}	$(1,3)(2,10)(5,9)(6,7)$	$(1,2,\ldots 9)$
$(9,4)$	A_9	$(1,6)(2,5)$	$(1,2,\ldots 9)$
$(11,3)$	A_{13}	$(1,3)(2,12)(4,5)(6,11)(8,10)(9,13)$	$(1,2,\ldots 11)$
$(11,4)$	A_{11}	$(1,6)(2,3)(8,9)(10,11)$	$(1,2,\ldots 11)$
$(11,5)$	A_{12}	$(1,6)(2,3)(4,13)(7,12)(8,9)(10,11)$	$(1,2,\ldots 11)$
$(12,4)$	S_{12}	$(1,6)(2,3)(7,8)(9,10)(11,12)$	$(1,2,\ldots 12)$
$(12,5)$	S_{13}	$(1,6)(2,3)(4,13)(9,10)(11,12)$	$(1,2,\ldots 12)$
$(13,4)$	S_{13}	$(1,6)(2,3)(7,8)(10,11)(12,13)$	$(1,2,\ldots 13)$
$(14,3)$	S_{15}	$(1,3)(2,15)(5,14)(6,7)(8,13)(9,10)(11,12)$	$(1,2,\ldots 14)$
$(14,4)$	A_{14}	$(1,6)(2,3)(8,12)(9,15)(10,11)(13,14)$	$(1,2,\ldots 14)$
$(15,4)$	A_{16}	$(1,5)(2,3)(4,16)(8,13)(11,12)(14,15)$	$(1,2,\ldots 15)$

Chapter 7

Conclusion

7.1 Looking ahead

There are numerous challenges in the theory of regular and orientably-regular maps. Within the world of maps in general, which are arbitrary cellular decompositions of surfaces, maps with the highest level of 'internal' symmetry (that is, rich in automorphisms) are rare. When it comes to understanding of these objects, despite enormous progress made in the last five decades as summed up in the survey [52], we still know relatively little when it comes to their classification on a given surface, or with a given underlying graph, or else with a given automorphism group.

In this Thesis we have focused on even more rare objects - regular and orientably-regular maps exhibiting also a high degree of 'external symmetry', that is, invariance with respect to map operators. By an analysis of [52], the operators that preserve the underlying graph of a map comprise Petrie duality and rotational powers. Together with duality (which preserves the supporting surface but not the underlying graph in general), one arrives at the three basic operators that preserve the automorphism group of a regular map. The 'most symmetric map' one can think of in this sense - a super-symmetric map - is then a regular map invariant with respect to taking the dual, the Petrie dual and all the admissible rotational powers.

Existence of super-symmetric maps of every even valency was established just about 10 years ago in [2]. An 'odd' counterpart of this result, existence of super-symmetric maps of every odd valency at least 7 (valencies 3 and 5 are trivially out of question here) is still open, and the best we have here is application of an idea of [37] to form a parallel product of all pairwise non-isomorphic copies of regular maps with automorphism group isomorphic to $\mathrm{SL}(2, 2^n)$, which leads to a construction of super-symmetric maps of valency $2^{2n} - 1$ for every $n \geq 2$. Even the smallest open case of valency 7 appears to be a big challenge.

A related problem here is the determination of the group of *external symmetries* generated by its self-duality, self-Petrie-duality and exponents, for a super-symmetric map. The only published result in this area appears to be [22], where it is shown that even for as small a valency as 8, there is an infinite sequence of super-symmetric maps (of increasing size) with the property that the orders of their external symmetry group grows beyond any limit. A deeper look into the structure of external symmetry groups of regular maps was undertaken also in [47], with findings still to be published.

Various relaxations of super-symmetry have been studied as well. If only dualities are considered, the best result appears to be that of [31] where existence of self-dual and self-Petrie-dual regular maps was established for any odd valency at least 5 by methods from the theory of finite rings (the even valency counterpart follows from the aforementioned result of [2]). On the other hand, if only exponents are considered, the best currently available results are those of [24] and [4] on regular and orientably-regular maps of a given valency with a specified group of exponents.

It is a striking fact that all results on regular maps with given exponent group are confined only to maps with prescribed valency and not to a given type. Strictly speaking, an extension to a given type is not possible in general, due to results of [54] where it was shown that for every $k \equiv \pm 1 \bmod 6$, no orientably-regular map of type $(k, 3)$ can have more than $\varphi(k)/2$ exponents, where φ is the Euler totient function. This result, however, appears to be the only one available to demonstrate impossibility of the extension mentioned above. The situation regarding regular and orientably-regular maps of a given hyperbolic type (k, m) for $m \geq 4$ with a given exponent group is completely open.

In my opinion, the two open problems, of existence of super-symmetric maps of odd valency on the one hand, and of existence of regular maps of a given hyperbolic type with face length at least four and with a given exponent group on the other hand, are the most important challenges in the study of external symmetries of regular maps.

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