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On Integral Representations Involving the Probability Generating Function for Inverse Moments of Positive Discrete Random Variables

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Abstract

I simplify and note extensions of results of Shibu et al. *Sankhyā*, *Ser. A* **85** (2023a,b) concerning integral representations involving the probability generating function for inverse moments of positive discrete random variables, both univariate and multivariate.

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1 Complements to Shibu et al. (2023a)

In a recent article in this journal, Shibu et al. (2023a) "derive exact expressions for inverse moments for any positive integer valued random variable" (X, with probability mass function [p.m.f.] $p_1, p_2, ...$) using "an integral representation for inverse moments involving the probability generating function" (p.g.f., $G(s) = \mathbb{E}(s^X) = \sum_{x=1}^{\infty} p_x s^x$). For the first inverse moment, $\mu_{-1} \equiv \mathbb{E}(1/X)$, they show that $\mu_{-1} = \lim_{t\to\infty}\{tH(t)\}$ where $H(t) \equiv \int_0^1 G(s^t)dt$. I would first like to note a different version of this result, namely,

$$\mu_{-1} = \int_0^1 s^{-1} G(s)\, ds. \tag{1}$$

To see this, note that the right-hand side equals

$$\sum_{x=1}^{\infty} p_x \int_0^1 s^{x-1}\, ds = \sum_{x=1}^{\infty} p_x/x = \mathbb{E}(1/X).$$

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Note also that, for $A > 0$, by a similar argument,

$$\mathbb{E}\{1/(X+A)\} = \int_0^1 s^{A-1} G(s)\, ds. \tag{2}$$

Example expressions given by Shibu et al. for μ_{-1} for geometric and negative binomial distributions are correct but both can be evaluated beyond the integral forms given. The first reduces to $p(-\log p)/(1-p)$. Using

$$\begin{aligned}\int_0^1 \frac{p^m s^{m-1}}{\{1-(1-p)s\}^m}\, ds &= \left(\frac{p}{1-p}\right)^m \int_p^1 \frac{(1-w)^{m-1}}{w^m}\, dw \\ &= \left(\frac{p}{1-p}\right)^m \sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^{m-1-j} \int_p^1 w^{-j-1}\, dw,\end{aligned}$$

the second becomes

$$\left(\frac{p}{1-p}\right)^m \left[(-1)^{m-1}(-\log p) + \sum_{j=1}^{m-1} \frac{1}{j} \binom{m-1}{j} (-1)^{m-1-j} \left\{(1/p)^j - 1\right\}\right];$$

here, integer m is both the index and the starting point of the negative binomial distribution. When $m = 1$, the latter reduces to the former since then the sum vanishes.

Shibu et al. (2023a) also provide an expression for higher order integer inverse moments which is a limiting value of an expression involving $H(t)$ and lower order integer inverse moments. To complement this with a corresponding direct integral expression, I note that Cressie et al. (1981) gave the formula

$$\mu_{-q} \equiv \mathbb{E}(1/X^q) = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} M(-t)\, dt, \tag{3}$$

$q > 0$, where $M(t)$ is the moment generating function given in terms of the p.g.f. by $M(t) = G(e^t)$. To see this, note that the right-hand side of (3) equals

$$\frac{1}{\Gamma(q)} \sum_{x=1}^\infty p_x \int_0^\infty t^{q-1} e^{-xt}\, dt = \frac{1}{\Gamma(q)} \sum_{x=1}^\infty p_x \Gamma(q)/x^q = \mathbb{E}(1/X^q).$$

In terms of the p.g.f., therefore, we have

$$\mu_{-q} = \frac{1}{\Gamma(q)} \int_0^\infty t^{q-1} G(e^{-t})\, dt = \frac{1}{\Gamma(q)} \int_0^1 s^{-1}(-\log s)^{q-1} G(s)\, ds. \tag{4}$$

Note that (4) reduces to (1) when $q = 1$.

Example 1 *For the geometric distribution (starting at 1) with p.g.f.* $G(s) = ps/\{1-(1-p)s\}$, *(4) shows that*

$$\mu_{-q} = \frac{p}{\Gamma(q)} \int_0^1 \frac{(-\log s)^{q-1}}{1-(1-p)s}\, ds = \frac{p}{\Gamma(q)} \int_0^\infty \frac{w^{q-1}}{e^w-(1-p)}\, dw. \tag{5}$$

While $\mu_{-1} = p(-\log p)/(1-p)$, *as noted above, (5) shows that for* $q = 2, 3, ...,$

$$\mu_{-q} = \frac{p\,\mathrm{Li}_q(1-p)}{1-p},$$

where $\mathrm{Li}_q(1-p)$ *is the polylogarithm function as given, for example, in NIST (2024, Section 25.12). This route gives the integral representation of the polylogarithm function; directly, minor manipulation gives* $p/(1-p)$ *times the series definition of the same function:*

$$\mu_{-q} = \sum_{x=1}^{\infty} \frac{1}{x^q}\, p(1-p)^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} \frac{(1-p)^x}{x^q}.$$

The second inverse moment, consequently, involves the dilogarithm function.

Note that the general results above apply to any positive discrete valued random variable, not just integer valued versions, via just a notational change to the indexing of x.

2 Related Expressions, Particularly Inverse Factorial Moments

"When studying discrete distributions, it is often advantageous to use the *factorial moments*" (Johnson et al., 2005, p.52). This is because many discrete distributions involve factorial terms, usually arising from combinatorial considerations. So, in a discrete, integer valued, context, it might be expected that inverse factorial moments can be easier to work with than inverse power moments. For $\nu_{-n} \equiv \mathbb{E}(1/\{X(X+1)\cdots(X+n-1)\} = \mathbb{E}\{(X-1)!/(X+n-1)!\}$, $n = 1, 2, ...,$ I earlier gave a closely related formula with an unfortunate typographical error (the power $n-1$ is missing in the theorem in Jones, 1987) which I here correct to

$$\nu_{-n} = \frac{1}{(n-1)!} \int_0^1 s^{-1}(1-s)^{n-1} G(s)\, ds. \tag{6}$$

To confirm this, observe that the right-hand side is

$$\begin{aligned}\frac{1}{(n-1)!}\sum_{x=1}^{\infty} p_x \int_0^1 s^{x-1}(1-s)^{n-1}\,ds &= \frac{1}{(n-1)!}\sum_{x=1}^{\infty} p_x B(x,n)\\ &= \frac{1}{(n-1)!}\sum_{x=1}^{\infty} p_x \frac{(x-1)!(n-1)!}{(x+n-1)!}\\ &= \mathbb{E}\{(X-1)!/(X+n-1)!\};\end{aligned}$$

equation (6) too reduces to (1) when $n = 1$. As shown in Jones (1987), (6) can also be proved by expanding $1/\{X(X+1)\cdots(X+n-1)\}$ in partial fractions and using (2). And there is also an integral representation for ν_{-n} rather like (3) for μ_{-q} but involving the factorial moment generating function $F_X(y) = \mathbb{E}\{(1+y)^X\} = G(1+y)$ rather than the moment generating function. This representation is

$$\nu_{-n} = \frac{1}{(n-1)!}\int_0^1 y^{n-1} F_{X-1}(-y)\,dy \tag{7}$$

(the corollary in Jones, 1987). By way of proof, the right-hand side of (7) is

$$\frac{1}{(n-1)!}\sum_{x=1}^{\infty} p_x \int_0^1 y^{n-1}(1-y)^{x-1}\,dy = \frac{1}{(n-1)!}\sum_{x=1}^{\infty} p_x \int_0^1 s^{x-1}(1-s)^{n-1}\,ds$$

and the argument is completed as above.

Example 2 *By using (6), the n'th inverse factorial moment of the geometric distribution is*

$$\nu_{-n} = \frac{p}{(n-1)!}\int_0^1 \frac{(1-s)^{n-1}}{1-(1-p)s}\,ds = \frac{p}{n!}\,{}_2F_1(1,1;n+1;1-p) \tag{8}$$

where ${}_2F_1(1,1;n+1;1-p)$ is a special case of the Gauss hypergeometric function (NIST, 2024, Section 15). Again, this approach directly yields an integral representation of the special function concerned; minor manipulation of the direct formula for the inverse factorial moment yields the usual series representation thereof.

More generally, other works concerning integral representations of inverse moments include Adell et al. (1996), Chao and Strawderman (1972) and Cressie and Borkent (1986).

3 Complement to Shibu et al. (2023b)

Shibu et al. (2023b) generalize the univariate results of Shibu et al. (2023a) to the multivariate case. Let $X_1, ..., X_k$ follow a positive integer valued multivariate distribution with p.m.f. $p_{x_1 \dots x_k}$, p.g.f. $G(s_1, ..., s_k) = \mathbb{E}(s_1^{X_1} \cdots s_k^{X_k})$ and inverse moments $\mu_{-q_1,...,-q_k} = \mathbb{E}(X_1^{q_1} \cdots X_k^{q_k})$. Then, Shibu et al. (2023b) show that

$$\mu_{-1,...,-1} = \lim_{t_1 \to \infty, ..., t_k \to \infty} t_1 \cdots t_k \int_0^1 \cdots \int_0^1 G(s_1^{t_1}, \dots, s_k^{t_k}) \, ds_1 \cdots ds_k$$

together with a complicated expression for general $\mu_{-q_1,...,-q_k}$ involving limits of multiple summations each involving lower order inverse moments. Formulas (1) and (4) generalise straightforwardly to the multivariate case, however. We have

$$\mu_{-q_1,...,-q_k} = \frac{1}{\Gamma(q_1) \cdots \Gamma(q_k)} \int_0^1 \cdots \int_0^1 \frac{(-\log s_1)^{q_1 - 1}}{s_1} \cdots \frac{(-\log s_k)^{q_k - 1}}{s_k} G(s_1, ..., s_k) \, ds_1 \cdots ds_k. \tag{9}$$

The proof is no more difficult than in the univariate case: the right-hand side of (9) is

$$\begin{aligned}
&\frac{1}{\Gamma(q_1) \cdots \Gamma(q_k)} \int_0^\infty \cdots \int_0^\infty t_1^{q_1 - 1} \cdots t_k^{q_k - 1} \, G(e^{-t_1}, ..., e^{-t_k}) \, dt_1 \cdots dt_k \\
&= \frac{1}{\Gamma(q_1) \cdots \Gamma(q_k)} \sum_{x_1=1}^\infty \cdots \sum_{x_k=1}^\infty p_{x_1 \dots x_k} \int_0^\infty \cdots \int_0^\infty t_1^{q_1 - 1} \cdots t_k^{q_k - 1} \, e^{-x_1 t_1} \cdots e^{-x_k t_k} \, dt_1 \cdots dt_k \\
&= \frac{1}{\Gamma(q_1) \cdots \Gamma(q_k)} \sum_{x_1=1}^\infty \cdots \sum_{x_k=1}^\infty p_{x_1 \dots x_k} \left(\int_0^\infty t_1^{q_1 - 1} \, e^{-x_1 t_1} \, dt_1 \right) \cdots \left(\int_0^\infty t_k^{q_k - 1} \, e^{-x_k t_k} \, dt_k \right) \\
&= \frac{1}{\Gamma(q_1) \cdots \Gamma(q_k)} \sum_{x_1=1}^\infty \cdots \sum_{x_k=1}^\infty p_{x_1 \dots x_k} \frac{\Gamma(q_1)}{x_1^{q_1}} \cdots \frac{\Gamma(q_k)}{x_k^{q_k}} \\
&= \sum_{x_1=1}^\infty \cdots \sum_{x_k=1}^\infty \frac{p_{x_1 \dots x_k}}{x_1^{q_1} \cdots x_k^{q_k}} = \mu_{-q_1,...,-q_k}.
\end{aligned}$$

Example 3 *Example 2 of Shibu et al. (2023b) concerns inverse moments of the Barbiero (2019) bivariate geometric distribution which has p.g.f. of the form*

$$G(s_1, s_2) = \sum_{j=1}^4 \frac{a_j s_1 s_2}{(1 - b_j s_1)(1 - c_j s_2)}$$

for $a_j, b_j, c_j,\ j = 1, ..., 4$*, being certain simple functions of parameters* $0 < \theta_1, \theta_2 < 1$ *and* $-1 \le \alpha \le 1$*. Since in the bivariate case*

$$\mu_{-1,-1} = \int_0^1 \int_0^1 \frac{G(s_1, s_2)}{s_1 s_2} \, ds_1 ds_2,$$

it is easy to see that

$$\mu_{-1,-1} = \sum_{j=1}^{4} \frac{a_j}{b_j c_j} \log(1 - b_j) \log(1 - c_j).$$

This is simpler than the formula given by Shibu et al. (2023b).

Declarations

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