On large regular \((1, 1, k)\)-mixed graphs

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**A B S T R A C T**

An \((r, z, k)\)-mixed graph \(G\) has every vertex with undirected degree \(r\), directed in- and out-degree \(z\), and diameter \(k\). In this paper, we study the case \(r = z = 1\), proposing some new constructions of \((1, 1, k)\)-mixed graphs with a large number of vertices \(N\). Our study is based on computer techniques for small values of \(k\) and the use of graphs on alphabets for general \(k\). In the former case, the constructions are either Cayley or lift graphs. In the latter case, some infinite families of \((1, 1, k)\)-mixed graphs are proposed with diameter of the order of \(2 \log N\).

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1. Introduction

The relationship between vertices or nodes in interconnection networks can be undirected or directed depending on whether the communication between nodes is two-way or one-way. Mixed graphs arise in this case and in many other practical situations in which both kinds of connections are needed. Urban street networks are the most popular ones. Thus, a mixed graph \(G = (V, E, A)\) has a set \(V = V(G) = \{u_1, u_2, \ldots\}\) of vertices, a set \(E = E(G)\) of edges or unordered pairs of vertices \((u, v)\), for \(u, v \in V\), and a set \(A = A(G)\) of arcs, directed edges, or ordered pair of vertices \((uv)\). For a given vertex \(u\), its undirected degree \(r(u)\) is the number of edges incident to vertex \(u\). Moreover, its out-degree \(z^{-}(u)\) is the number of arcs emanating from \(u\), whereas its in-degree \(z^{-}(u)\) is the number of arcs going to \(u\). If \(z^{-}(u) = z^{-}(u) = z\) and \(r(u) = r\), for all \(u \in V\), then \(G\) is said to be a totally regular \((r, z, k)\)-mixed graph with whole degree \(d = r + z\).

The distance from vertex \(u\) to vertex \(v\) is denoted by \(\text{dist}(u, v)\). Notice that, when the out-degree \(z\) is not zero, the distance \(\text{dist}(u, v)\) is not necessarily equal to the distance \(\text{dist}(v, u)\). If the mixed graph \(G\) has diameter \(k\), its distance matrix \(A_i\), for \(i = 0, 1, \ldots, k\), has entries \((A_i)_{uv} = 1\) if \(\text{dist}(u, v) = i\), and \((A_i)_{uv} = 0\) otherwise. So, \(A_0 = I\) (the identity matrix) and \(A_1 = A\) (the adjacency matrix of \(G\)).

Mixed graphs were first considered in the context of the degree/diameter problem by Bosák [1]. The degree/diameter problem for mixed graphs reads as follows: Given three natural numbers \(r, z, k\), find the largest possible number of vertices \(N(r, z, k)\) in a mixed graph \(G\) with maximum undirected degree \(r\), maximum directed out-degree \(z\), and diameter \(k\).

For mixed graphs, an upper bound for \(N(r, z, k)\), known as a Moore(-like) bound \(M(r, z, k)\), was obtained by Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [2] (also by Dalfó, Fiol, and López [6] with an alternative computation).

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Theorem 1.1 (Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [2]). The Moore bound for an \((r, z)\)-mixed graph with diameter \(k\) is

\[
M(r, z, k) = A \frac{u_1^{k+1} - 1}{u_1 - 1} + B \frac{u_2^{k+1} - 1}{u_2 - 1},
\]

where

\[
u = (z + r)^2 + 2(z - r) + 1.
\]

This bound applies whether or not \(G\) is totally regular, but it is elementary to show that a Moore mixed graph must be totally regular. Thus, a Moore \((r, z, k)\)-mixed graph is a graph with diameter \(k\), maximum undirected degree \(r \geq 1\), maximum out-degree \(z \geq 1\), and order given by \(M(r, z, k)\). An example of a Moore \((3, 1, 2)\)-mixed graph is the Bosák graph [1], see Fig. 1.

Bosák [1] gave a necessary condition for the existence of a mixed Moore graph with diameter \(k = 2\). Such graphs have the property that, for any ordered pair \((u, v)\) of vertices, there is a unique walk of length at most 2 between them. In general, there are infinitely many pairs \((r, z)\) satisfying Bosák necessary condition for which the existence of a mixed Moore graph is not known yet. Nguyen, Miller, and Gimbert [15] proved the existence and unicity of some Moore mixed graphs of diameter 2. López, Miret, and Fernández, [11] proved that there is no Moore \((r, z, 2)\)-mixed graph when the pair \((r, z)\) equals \((3, 3), (3, 4), \) or \((7, 2)\).

For diameter \(k \geq 3\), it was proved that mixed Moore graphs do not exist, see Nguyen, Miller, and Gimbert [15]. In the case of total regularity, this result also follows from the improved bound in Dalfó, Fiol, and López [6], where it was shown that the order \(N\) of an \((r, z)\)-regular mixed graph \(G\) with diameter \(k \geq 3\) satisfies

\[
N \leq M(r, z, k) - r,
\]

where \(M(r, z, k)\) is given by (1). In general, a mixed graph with maximum undirected degree \(r\), maximum directed out-degree \(z\), diameter \(k\), and order \(N = M(r, z, k) - \delta\) is said to have defect \(\delta\). A mixed graph with defect one is called an almost mixed Moore graph. Thus, the result in (2) can be rephrased by saying that \(r\) is a lower bound for the defect of the mixed graph. In the case \(r = z = 1\), such a result was drastically improved by Tuite and Erskine [18] by showing that a lower bound \(\delta(k)\) for the defect of a \((1, 1)\)-regular mixed graph with diameter \(k \geq 1\) satisfies the recurrence

\[
\delta(k + 6) = \delta(k) + f_{k-1} + f_{k+4},
\]

where the initial values of \(\delta(k)\), for \(k = 1, \ldots, 6\), are, respectively, 0, 1, 1, 2, 3, 5, and \(f_k\) are the Fibonacci numbers starting from \(f_0 = f_1 = 1\), namely, 1, 1, 2, 3, 5, 8, 13, 21, \ldots Alternatively, starting from \(\delta(1) = 0\) and \(\delta(2) = 1\), we have \(\delta(k + 2) = \delta(k + 1) + \delta(k)\) if \(k + 2 \equiv 1, 2 \mod (\text{mod } 6)\), and \(\delta(k + 2) = \delta(k + 1) + \delta(k) + 1\), otherwise.

For more results on degree/diameter problem for graphs, digraphs, and mixed graphs, see the comprehensive survey by Miller and Širáň [14]. For more results on mixed graphs, see Buset, López, and Miret [3], Dalfó [4], Dalfó, Fiol, and López [5–7], Erskine [8], Jørgensen [10], López, Pérez-Rosés, and Pujolàs [12], Nguyen, Miller, and Gimbert [15], and Tuite and Erskine [17].

Fig. 1. The Bosák \((3, 1)\)-graph with diameter \(k = 2\) and \(N = 18\) vertices.
In this paper, we deal with \((1, 1, k)\)-mixed graphs, that is, mixed graphs with undirected degree \(r = 1\), directed out-degree \(z = 1\), and with diameter \(k\). Our study is based on computer techniques for small values of \(k\), and the use of graphs on alphabets for general \(k\). In the former case, the constructions are either Cayley or lift graphs. In the latter case, some infinite families of \((1, 1, k)\)-mixed graphs are proposed with \(N\) vertices and diameter \(k\) of the order of \(2 \log_2 N\). Most of the proposed constructions are closely related to line digraphs. Given a digraph \(G\), its line digraph \(LG\) has vertices representing the arcs of \(G\), and vertex \(x_1x_2\) is adjacent to vertex \(y_1y_2\) in \(LG\) if the arc \((x_1, x_2)\) is adjacent to the arc \((y_1, y_2)\) in \(G\), that is, if \(y_1 = x_2\). The \(k\)-iterated line digraph is defined recursively as \(L^kG = L^{k-1}(LG)\). Let \(K^+_d\) be the complete symmetric digraph with \(d\) vertices with loops, and \(K_{d+1}\) the complete symmetric digraph on \(d + 1\) vertices (in these complete graphs, each edge is seen as a digon or a pair of opposite arcs). Then, two well-known families of iterated line digraphs are the De Bruijn digraphs \(B(d, k) = L^k(K^+_d)\), and the Kautz digraphs \(K(d, k) = L^k(K_{d+1})\). Both \(B(d, k)\) and \(K(d, k)\) have diameter \(k\), but De Bruijn digraphs have \(d^k\) vertices, whereas Kautz digraphs have \(d^k + d^{k-1}\) vertices. See, for instance, Fiol, Yebra, and Alegre [9], and Miller and Širáň [14].

2. Some infinite families of \((1, 1, k)\)-mixed graphs

In this section, we propose some infinite families of \((1, 1, k)\)-mixed graphs with exponential order. All of them have vertices with out-degree \(z = 1\). When, moreover, all the vertices have in-degree \(1\), we refer to them as \((1, 1, k)\)-regular mixed graphs. If we denote by \(f(r, z, k)\) the order of a largest \((r, z, k)\)-mixed graph, which is upper bounded by the (exponential) Moore bound \(M(r, z, k)\), all the described graphs provide exponential lower bounds for \(f(1, 1, k)\).

Let us first give some basic properties of \((1, 1, k)\)-mixed graphs. It is readily seen that the Moore bound satisfies the Fibonacci-type recurrence

\[
M(1, 1, k) = M(1, 1, k-1) + M(1, 1, k-2) + 2,
\]

starting from \(M(1, 1, 0) = 1\) and \(M(1, 1, 1) = 3\). From this, or just applying (1), we obtain that the corresponding Moore bound is

\[
M(1, 1, k) = \left(1 - \frac{2}{\sqrt{5}}\right)^{k+1} + \left(1 + \frac{2}{\sqrt{5}}\right)^{k+1} - 2.
\]

The obtained values for \(k = 2, \ldots, 16\) are shown in Table 4. Then, for large values of \(k\), the Moore bound \(M(1, 1, k)\) is of the order of

\[
M(1, 1, k) \sim \left(1 + \frac{2}{\sqrt{5}}\right)^{k+1} \approx 1.8944 \cdot 1.6180^{k+1}.
\]

2.1. The mixed graphs \(E(n)\)

The first construction is the simplest one. Given \(n \geq 2\), the graph \(E(n)\) is defined as follows. As before, label the Fibonacci numbers so that \(f_0 = f_1 = 1\). Consider a Moore tree of radius \(n\) with base vertex \(u_0\). The set of vertices at distance \(i\) from \(u_0\) is called the vertices at level \(i\). There are \(f_{i-1}\) vertices at level \(i\). We can partition these vertices into two sets: \(V_i\) contains the \(f_i\) vertices at level \(i\) incident through an arc from level \(i - 1\), and \(W_i\) contains the \(f_{i-1}\) vertices at level \(i\) incident through an edge from level \(i - 1\). In what follows, we denote by \(v_1 \in V_1\) and \(w_1 \in W_1\) the two vertices in level 1; and by \(v_2, v_3 \in V_2\) and \(w_2 \in W_2\) the three vertices in level 2 (where, \(v_2\) and \(w_2\) are adjacent from \(v_1\), and \(v_3\) is adjacent from \(w_1\)). See Figs. 2 and 3. We must consider two cases to complete the graph, depending on whether \(f_n\) is even or odd.

- If \(f_n\) is even, then we add a matching among the vertices of \(V_n\) and add an arc from each vertex in level \(n\) to \(u_0\). In this case, the diameter is \(2n\) (except for \(n = 2\), in which case the diameter is 3). Note that the maximum distance occurs from the level-1 vertex \(v_1\) to a level-\(n\) vertex on the opposite edge, that is, in the subtree rooted at vertex \(w_1\).
- If \(f_n\) is odd, then we must modify the construction slightly. In this case, when we add a matching among the vertices of \(V_n\), there is one vertex \(v \in V_n\) missed by the matching. So, we must add another vertex \(v'\), join this vertex to \(v\) by an edge, and then add an arc from \(v'\) to the base vertex \(u_0\). All other vertices at level \(n\) have arcs directly to \(u_0\). In this case, the diameter is \(2n\) or \(2n + 1\) depending on the situation of vertex \(v\). If \(v\) is in the subtree rooted at \(v_1\), then the diameter is \(2n\), and a maximum path goes from the level-2 vertex \(v_3\) to \(v'\) (see an example of this case for \(n = 4\) in Fig. 2). Otherwise, if \(v\) belongs to the subtree rooted at \(w_1\), the diameter is \(2n + 1\), where the maximum distance occurs from the level-1 vertex \(v_1\) to \(v'\) (again for \(n = 4\), see an example in Fig. 3).

So, the graph \(E(n)\) has diameter \(2n\) or \(2n + 1\), and order \(M(1, 1, n)\) or \(M(1, 1, n) + 1\). This bound is very weak for small diameters, but at least it gives a first explicit construction that gives an exponential lower bound. In the following subsections, we show that we can do it better.
2.2. The mixed graphs $F(n)$

Given $n \geq 2$, the $(1, 1, k)$-mixed graph $F(n)$ has vertices labeled with $a : x_1 \ldots x_n$, where $a \in \{+1, -1\}$, with $x_i \in \mathbb{Z}_3$, and $x_{i+1} \neq x_i$ for $i = 1, \ldots, n - 1$. The adjacencies are as follows:

(i) $a|x_1x_2\ldots x_n \sim -a|x_1x_2\ldots x_n$ (edges);
(ii) $a|x_1x_2\ldots x_n \rightarrow a|x_2x_3\ldots x_n(x_n + a)$ (arcs).

Thus, $F(n)$ has $3 \cdot 2^n$ vertices and is an out-regular graph, but not in-regular since vertices $a|x_1x_2\ldots x_n$ and $a|x_1'x_2\ldots x_n$ are both adjacent to $a|x_2\ldots(x_n + a)$. Thus, this implies the presence of other vertices with in-degree 0.

The mixed graph $F(3)$ is shown in Fig. 4. It is easily checked that the mapping

$$a|x_1x_2\ldots x_n \mapsto -a|x_1\bar{x}_2\ldots\bar{x}_n,$$

where $\overline{0} = 1$, $\overline{1} = 0$, and $\overline{2} = 2$, is an automorphism of $F(n)$. This is because $x_n + a = \overline{x_n} - a$.

**Proposition 2.1.** The diameter of the mixed graph $F(n)$ is $k = 2n$.

**Proof.** Let us see that there is a path of length at most $2n$ from vertex $x = a|x_1\ldots x_n$ to vertex $y = b|y_1\ldots y_n$. Taking into account the automorphism in (5), we can assume that $a = +1$. Notice that, with at most two steps, depending on the values of $a$, $x_n$, and $y_1$ (at the beginning) or $a$, $y_1$, and $y_{i+1}$ (in the sequel), we can add a new digit of $y$. Thus, in principle, we would need at most $2n$ steps but, possibly, one last step to fix the first digit to the one of $y$ (for example, $b$). However,
in what follows, we show that the first two digits $y_1$ and $y_2$ can be ‘placed’, so reaching a vertex of the form $\alpha' \ldots y_1y_2$, with at most 3 steps.

(i) If $y_1 = x_n$, the first step is not necessary.

(ii) If $y_1 = x_n + 1$, go through the arc $+1|x_1 \ldots x_n \rightarrow +1|x_2 \ldots x_ny_1$.

(iii) If $y_1 = x_n - 1$ and $y_2 = y_1 - 1$, go through the edge and two arcs

$$+1|x_1 \ldots x_n \sim -1|x_1x_2 \ldots x_n \rightarrow -1|x_2 \ldots x_ny_1 \rightarrow -1|x_3 \ldots x_ny_1y_2.$$ 

(iv) If $y_1 = x_n - 1 (= x_n + 2)$ and $y_2 = y_1 + 1$, go through the three arcs

$$+1|x_1 \ldots x_n \rightarrow +1|x_2 \ldots x_nx_1 + 1 \rightarrow +1|x_3 \ldots x_n + 1x_n + 2$$

$$= +1|x_3 \ldots x_n + 1y_1 \rightarrow +1|x_4 \ldots x_ny_1y_2.$$ 

Thus, to reach $y$, we need at most $3 + 2(n - 2) + 1 = 2n$ steps. Finally, it is not difficult to find vertices that are at distance $2n$. For instance, for $n$ odd, go from $x = +1|0101 \ldots 0$ to $y = +1|2020 \ldots 2$; and, for $n$ even, go from $x = +1|1010 \ldots 0$ to $y = -1|2020 \ldots 0$. Thus, in the first case, if $n = 3$, we have the path

$$x = +1|010 \rightarrow +1|011 \rightarrow +1|012 \rightarrow +1|120$$

$$\sim -1|202 \sim +1|202 = y.$$ 

In the second case, if $n = 4$, we have the path

$$x = +1|1010 \rightarrow +1|0101 \rightarrow +1|0112 \rightarrow +1|0120$$

$$\sim -1|0120 \sim -1|202 \sim +1|1202 \sim +1|2020 \sim -1|2020 = y.$$ 

Notice that, in both cases, the first three steps are done according to (iv). □

More details about the proof of the diameter and the existence of paths of a given length are in Proposition 2.2. There, we deal with the mixed graph $F[n]$, which is introduced in the next subsection, which is proved to be isomorphic to $F(n)$.

2.2.1. A numeric construction

An alternative presentation $F[n]$ of $F(n)$ is as follows: Given $n \geq 1$, let $N' = 3 \cdot 2^{n-1}$ so that the number of vertices of $F[n]$ is $2N'$. The vertices of $F[n]$ are labeled as $\alpha | i$, where $\alpha \in \{1, 2\}$, and $i \in \mathbb{Z}_{N'}$. Let $T = 2$ and $Z = 1$. Then, the adjacencies of $F[n]$ defining the same mixed graph as those in (i) and (ii) are:

$$\alpha | i \sim \overline{\alpha} | i \text{ (edges)};$$

$$\alpha | i \rightarrow \alpha | -2i + \alpha \text{ (mod} N') \text{ (arcs).}$$
To show that both constructions give the same mixed graph, $F[n] \cong F(n)$, define first the mapping $\pi$ from the two digits $x_1x_2$ to $\mathbb{Z}_6$ as follows

$$\pi(01) = 0, \quad \pi(10) = 1, \quad \pi(12) = 2, \quad \pi(21) = 3, \quad \pi(20) = 4, \quad \pi(02) = 5.$$ 

Then, it is easy to check that, for $n = 2$, the mapping $\psi$ from the vertices of $F(2)$ to the vertices of $F[2]$ defined as

$$\psi(a|x_1x_2) = \alpha(a)|\pi(x_1x_2),$$

where $\alpha(a) = \frac{a+3}{2}$, is an isomorphism from $F(2)$ to $F[2]$. (Note that $\alpha(−1) = 1$ and $\alpha(+1) = 2$). From this, we can use induction. First, let us assume that $\psi'$ is an isomorphism from $F(n−1)$ to $F[n−1]$ of the form

$$\psi'(a|x_1x_2 \ldots x_{n−1}) = \alpha(a)|\pi'(x_1x_2 \ldots x_{n−1}),$$

where the linear mapping $\alpha$ is defined as above, and $\pi'$ is a mapping from the sequences $x_1x_2 \ldots x_{n−1}$ to the elements of $\mathbb{Z}_{N'}$, with $N' = 3 \cdot 2^{n−2}$. Then, we claim that the mapping $\psi$ from the vertices of $F(n)$ to the vertices of $F[n]$ defined as

$$\psi(a|x_1x_2 \ldots x_n) = \alpha(a)|\pi(x_1x_2 \ldots x_n) + \alpha(x_n - x_{n−1}) \pmod{N'},$$

where $N' = 3 \cdot 2^{n−1}$ and the value of $x_n - x_{n−1}$ is taken in $\{-1, 0, 1\}$ mod $3$, is an isomorphism from $F(n)$ to $F[n]$. Indeed, since $\psi'$ is an isomorphism from $F(n−1)$ to $F[n−1]$, we have that $\psi' \Gamma = \Gamma \psi'$ and $\psi' \Gamma^+ = \Gamma^+ \psi'$, where $\Gamma$ and $\Gamma^+$ denote undirected and directed adjacency, respectively. Thus, from

$$\psi'(a|x_1x_2 \ldots x_{n−1}) = \psi'(-a|x_1x_2 \ldots x_{n−1}) = \alpha(-a)|\pi'(x_1x_2 \ldots x_{n−1}),$$

$$\Gamma^+\psi'(a|x_1x_2 \ldots x_{n−1}) = \Gamma^+(\alpha(a)|\pi'(x_1x_2 \ldots x_{n−1})) = \alpha(a)|2\cdot\pi'(x_1x_2 \ldots x_{n−1}) + \alpha(a),$$

we conclude that $\alpha(-a) = \overline{\alpha(a)}$ for every $a \in \{+1, -1\}$ (as it is immediate to check), and

$$\pi'(x_2 \ldots x_{n−1}(x_n + a)) = -2 \cdot \pi'(x_1 \ldots x_{n−1}) + \alpha(a).$$

Now, we can assume that $a = +1$ (because of the automorphism (5)), and let $a' = x_n - x_{n−1}$. Then, since clearly $\psi' \Gamma = \Gamma^+ \psi'$, edges map to edges, we focus on proving that the same holds for the arcs, that is, $\psi^+ \Gamma = \Gamma^+ \psi^+$. With this aim, we need to prove the following two calculations, where we use (8), give the same result:

$$\psi^+\Gamma^+(+1|x_1x_2 \ldots x_n) = \psi(+1|x_2 \ldots x_n x_n + 1)$$

$$= 2| - 2 \cdot \pi'(x_2 \ldots x_n) + 2,$$

$$\Gamma^+\psi^+\Gamma^+(+1|x_1x_2 \ldots x_n) = \Gamma^+(2| - 2 \cdot \pi'(x_1 \ldots x_{n−1}) + \alpha(a'))$$

$$= 4 \cdot 2 \cdot \pi'(x_1 \ldots x_{n−1}) - 2\alpha(a') + 2.$$ 

The required equality follows since, from (9) with $a'$ instead of $a$, we have

$$-2 \cdot \pi'(x_2 \ldots x_n) = 2 \cdot \pi'(x_2 \ldots x_{n−1}(x_n + a')) = -2[ -2 \cdot \pi'(x_1 \ldots x_{n−1}) + \alpha(a')]$$

$$= 4 \cdot \pi'(x_1 \ldots x_{n−1}) - 2\alpha(a').$$

In Fig. 4, every vertex has been labeled according to both presentations.

Using this presentation, we extend (and again prove) Proposition 2.1.

**Proposition 2.2.** The diameter of $F[n]$ is $k = 2n$. More precisely, there is a path of length $n$ or $n−1$ between any pair of edges $\alpha | i − \alpha | i'$ and $\alpha' | i' − \alpha | i''$. Moreover, there is a path of length between $n−1$ and $2n$ between any pair of vertices.

**Proof.** Let us consider a tree rooted at a pair of vertices of an edge, $u_1 = 1i$ and $u_2 = 2i$, and suppose the $n = 2v + 1$ is odd (the case of even $n$ is similar). Then,

- The vertices at distances $1, 2$ of $u_1$ or $u_2$ are $\alpha | -2i + 1, \alpha | -2i + 2$ with $\alpha = 1, 2$.
- The vertices at distances $3, 4$ of $u_1$ or $u_2$ are $\alpha | 4i, \alpha | 4i - 1, \alpha | 4i - 2$ and $\alpha | 4i - 3$ with $\alpha = 1, 2$.
- The vertices at distances $5, 6$ of $u_1$ or $u_2$ are $\alpha | -8i + 1, \alpha | -8i + 2, ..., \alpha | -8i + 8$ with $\alpha = 1, 2$.

... 

- The vertices at distances $2n − 3, 2n − 2$ of $u_1$ or $u_2$ are $\alpha | 2n−1 + v$ with $v = 0, \ldots, -2n−1 + 1$ and $\alpha = 1, 2$.
- The vertices at distances $2n − 1, 2n$ of $u_1$ or $u_2$ are $\alpha | -2n + v$ with $v = 1, 2, \ldots, 2n$ and $\alpha = 1, 2$. 

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Fig. 5. The paths of length at most 6 in $F[3]$ from the vertices of the edge $\{1|i, 2|i\}$.

See Fig. 5 for the case of $F[3]$, which has 24 vertices. Note that, from the pair of vertices $1|i$ and $2|i$, the 3rd and 4th columns contain all the ‘consecutive’ vertices of $F(3)$ from $\alpha|4i - 3$ to $\alpha|4i + 8$, with $\alpha = 1, 2$. More precisely, from vertex $2|i$ (we can fix $\alpha$ because of the automorphism), we reach all of such vertices with at most 6 steps, except $2|4i + 1$ (in boldface, on the top of the 4th column), which would require the 7 adjacencies $'\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow'$. But this vertex is reached following the path $'\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$' (in boldface, in the 5th column). In general, using the notation $f(\alpha|i) = \alpha| - 2i + \alpha$ and $g(\alpha|i) = \overline{\alpha|i}$, we have the following:

- If $n$ is even, then the exception vertex is $g(fg)^n(2|i)(\text{mod } N) = 1|2i + 2$ (2n steps).
- If $n$ is odd, then the exception vertex is $g(fg)^n(2|i)(\text{mod } N) = 2|2n + 1$ (2n steps). $\square$

2.3. The mixed graphs $F^*(n)$

A variation of the mixed graphs $F(n)$ allows us to obtain $(1, 1, k)$-regular mixed graphs that we denote $F^*(n)$. Given $n \geq 2$, the $(1, 1, k)$-regular mixed graph $F^*(n)$ has vertices labeled as those of $F(n)$. That is, $a|x_1 \ldots x_n$, where $a \in \{+1, -1\}$ and $x_i \in \mathbb{Z}_3$. Now, the adjacencies are as follows:

$$a|x_1x_2\ldots x_n \sim -a|x_1x_2\ldots x_n \quad \text{(edges)};$$

$$a|x_1x_2\ldots x_n \rightarrow a|x_2x_3\ldots x_n(x_n + a(x_2 - x_1)) \quad \text{(arcs)}.$$
where, when computed modulo 3, we take $x_3 - x_1 \in [+1, -1]$. Hence, the vertices $a|x_1x_2 \ldots x_n$ and $a|x'_1x_2 \ldots x_n$, with $x'_1 \neq x_1$, are adjacent to different vertices of the form $a|x_2 \ldots (x_n \pm 1)$.

For example, the mixed graph $F^*(3)$ is shown in Fig. 6.

2.3.1. An alternative presentation

To study some properties of $F^*(n)$, it is useful to work with the following equivalent presentation: The vertices are now labeled as $a|b : a_1 \ldots a_{n-1}$, where $a_i \in [+1, -1]$ for $i = 1, \ldots, n-1$, and $b \in \mathbb{Z}_3$. Then, the adjacencies (12) and (13) become

$\begin{align*}
    a|b : a_1a_2 \ldots a_{n-1} & \sim -a|b : a_1a_2 \ldots a_{n-1} \quad \text{(edges)}; \\
    a|b : a_1a_2 \ldots a_{n-1} & \rightarrow a|b + a_1 : a_2a_3 \ldots a_{n-1} : aa_1 \quad \text{(arcs)}.
\end{align*}$

Hence, the vertices

$\begin{align*}
    a|b : a_1a_2 \ldots a_{n-1},
    \quad b \in \mathbb{Z}_3,
\end{align*}$

where $\phi + 1 = 0$, $\phi(0) = 0$, $\phi(1) = 2$, and $\overline{a_i} = -a_i$ for $i = 1, \ldots, n-1$. To prove that $\Phi$ is an automorphism of $F^*(n)$, observe that the vertex in (16) is adjacent, through an edge, to

$\begin{align*}
    a|\phi(b) + \overline{a_1} \overline{a_2} \ldots \overline{a_{n-1}}a\overline{a_1} = \Phi(a|b + a_1 : a_2a_3 \ldots a_{n-1} : aa_1),
\end{align*}$

and, through an arc, to

$\begin{align*}
    a|\phi(b) + \overline{a_1} \overline{a_2} \ldots \overline{a_{n-1}}\overline{a_1} = \Phi(a|b + a_1 : a_2a_3 \ldots a_{n-1} : aa_1),
\end{align*}$

where the last equality holds since $\phi(b + a_1) = \phi(b) + \overline{a_1}$, and $aa_1 = \overline{a_1}$ for every $b \in \mathbb{Z}_3$ and $a, a_1 \in [+1, -1]$. Similarly, we can prove that $\Psi$ is also an automorphism of $F^*(n)$. Clearly, $\Phi$ is involutive, and $\Psi$ has order three. Moreover, $(\Phi\Psi)^3 = \text{id}$ (the identity). Then, the automorphism group $\text{Aut}(F^*(n))$ must contain the subgroup $\langle \Phi, \Psi \rangle = D_3$. It is easy to see that the graph $F^*(n)$ has exactly three digons between pairs of vertices of the form $\overline{a} = xyxy \ldots xy$ and $\overline{a} = x$$\overline{a} = x$$\overline{a}$ when $n$ is even, or $\overline{a} = x$$\overline{a} = x$$\overline{a}$ when $n$ is odd; see again Fig. 6. Thus, any automorphism of $F^*(n)$ must interchange these digons; hence, the automorphism group has at most $3! = 6$ elements. Consequently, $\text{Aut}(F^*(n)) \cong D_3 \cong S_3$, as claimed. □

Before giving the diameter of $F^*(n)$, we show that, for every vertex $u$, there is only a possible vertex $v$ at distance $2n + 1$ from $u$. Suppose first that $n$ is odd (the case of even $n$ is similar). It is clear that, excepting possibly one case, from vertex $u = a|b : a_1a_2 \ldots a_{n-1}$ to vertex $v = a'|b' : y_1y_2 \ldots y_{n-1}$, there is a path with at most $2n$ steps of the form $- \rightarrow - \rightarrow \ldots \rightarrow$, where ‘$\rightarrow$’ stands for ‘$\overline{\rightarrow}$’ (edge) or ‘$\overline{\rightarrow}$’ (nothing), and ‘$\rightarrow$’ represents an arc. The exception occurs when all the edges of the path are necessary. That is:
At first glance, the proof of this result seems to be involved, although we managed to prove the following.

The adjacencies through arcs are:

\[ G \]

So, the number of vertices of \( G \) is \( 2^{n+1} - 4 \).

The adjacencies (with arithmetic modulo 2) through edges are:

(i) For any \( n \): \( 100 \ldots 0 \sim 111 \ldots 1 \);
(ii) For odd \( n \): \( 0101 \ldots 0 \sim 1010 \ldots 1 \);
(iii) For even \( n \): \( 0101 \ldots 01 \sim 01010 \ldots 10 \);
(iv) For the other vertices, \( x_0 x_1 \ldots x_n \sim (x_0 + 1)x_1 \ldots x_n \).

The adjacencies through arcs are:

(v) \( x_0 x_1 \ldots x_n \rightarrow x_0 x_2 \ldots x_n (x_1 + x_0) \).

The graph \( G \) is an in- and out-regular mixed graph with \( r = z = 1 \). Its only nontrivial automorphism is the one that sends \( x_0 x = x_0 x_1 x_2 x_3 \) to \( x_0 x = x_0 x_1 x_2 x \), where \( x_2 = x_i + 1 \) for \( i = 1, 2, 3, \ldots \). In Fig. 8, we show the mixed graph \( G(3) \).

Looking at the computer-generated results for \( n \leq 12 \), we are led to conjecture that the diameter of \( G(n) \) is \( k = 2n - 1 \).
Proposition 2.4. The diameter of $G(n)$ is at most $2n$.

Proof. Consider the digraph $G^+(n)$ defined by considering all $2^{n+1}$ vertices of the form $0|x_1\ldots x_n$ and $1|x_1\ldots x_n$, with $x_i \in \mathbb{Z}_2$, with undirected adjacencies as in (iv), and directed adjacencies as in (v). Then, $G^+(n)$ has the self-loops at vertices $0|00\ldots 0$ and $0|11\ldots 1$ and one digon (or two opposite arcs) between $0|0101\ldots 01$ and $0|1010\ldots 10$ for even $n$, and $0|0101\ldots 0$ and $0|1010\ldots 1$ for odd $n$. In fact, if every edge of $G^+(n)$ is ‘contracted’ to a vertex, what remains is the De Bruijn digraph $B(2, n)$, with $2^n$ vertices and diameter $n$. Moreover, notice that $G(n)$ is obtained by removing the above four vertices and adding the edges in (i), (ii), and (iii). By way of examples, Fig. 9 shows the graph $G^+(2)$, whereas Fig. 10 shows the mixed graph $G^+(3)$ ‘hanging’ from a vertex with eccentricity $2n = 6$

Consequently, since the diameter of $G(n)$ is upper bounded by the diameter of $G^+(n)$, we concentrate on proving that the diameter of $G^+(n)$ is $2n$ for $n > 1$ ($G(1)$ has diameter 3). The proof is constructive because we show a walk of length at most $2n$ between any pair of vertices. To this end, we take the following steps:

1. There is a walk of length at most $2n$ from vertex $x_0|x = x_0|x_1x_2\ldots x_n$ to vertex $(x_n + y_n)y_1y_2\ldots y_n$. Indeed, as $x_i + x_i = 0$ for any value of $x_i$, we get

$$x_0|x_1x_2x_3\ldots x_n \sim (x_1 + y_1)|x_1x_2x_3\ldots x_n \rightarrow (x_1 + y_1)|x_2x_3\ldots x_ny_1$$
4. For the cases, the strategy is to put first the last digit of destiny. Namely, if \( n \) is even, we make the walk consisting of exactly 2\( n \) steps (vertices at maximum distance) if \( |x \cap y| = 0 \) and none of the situations in (b) holds. Assuming first that \( x_0 = 0 \) (the case \( x_0 = 1 \) is similar), the latter occurs when \( x_1 + y_1 = 1 \Rightarrow y_1 = \overline{x_1} \), \( x_2 + y_2 = 0 \Rightarrow y_2 = x_2 \), \( x_3 + y_3 = 1 \Rightarrow y_3 = \overline{x_3} \), and so on. Consequently, starting from \( 0|x = 0|x_1x_2x_3 \ldots x_n \), we only need to test the destiny vertices of the form \( 1|y = 1|\overline{x_1}x_2\overline{x_3} \ldots x_n \) (\( n \) even), and \( 0|z = 0|\overline{x_1}x_2\overline{x_3} \ldots \overline{x_n} \) (\( n \) odd), with the additional constraints \( |x \cap y| = |x \cap z| = 0 \).

Thus, if we can save some steps, one last step \( (x_n + y_n)|y \sim (\overline{x_n} + y_n)|y \) assures a walk of length at most 2\( n \) from \( x_0|x \) to \( y_0|y \) for any \( y_0 \in \{0, 1\} \).

In the ‘worst case’, the walk in (18) consists of exactly 2\( n \) steps (vertices at maximum distance) if \( |x \cap y| = 0 \) and none of the situations in (b) holds. Assuming first that \( x_0 = 0 \) (the case \( x_0 = 1 \) is similar), the latter occurs when \( x_1 + y_1 = 1 \Rightarrow y_1 = \overline{x_1} \), \( x_2 + y_2 = 0 \Rightarrow y_2 = x_2 \), \( x_3 + y_3 = 1 \Rightarrow y_3 = \overline{x_3} \), and so on. Consequently, starting from \( 0|x = 0|x_1x_2x_3 \ldots x_n \), we only need to test the destiny vertices of the form \( 1|y = 1|\overline{x_1}x_2\overline{x_3} \ldots x_n \) (\( n \) even), and \( 0|z = 0|\overline{x_1}x_2\overline{x_3} \ldots \overline{x_n} \) (\( n \) odd), with the additional constraints \( |x \cap y| = |x \cap z| = 0 \).

Thus, if we can save some steps, one last step \( (x_n + y_n)|y \sim (\overline{x_n} + y_n)|y \) assures a walk of length at most 2\( n \) from \( x_0|x \) to \( y_0|y \) for any \( y_0 \in \{0, 1\} \).

This thus leads to the step pattern \( \sim \rightarrow \sim \rightarrow \ldots \sim \rightarrow \ldots \rightarrow \ldots \sim \rightarrow \overline{x_n} \sim \rightarrow \overline{x_n} \ldots \sim \rightarrow \overline{x_n} \ldots \sim \rightarrow \overline{x_n} \) uniquely determines the destiny vertex.

2. Clearly, some of the steps in (18) are not necessary if some of the following situations occur:

(a) The ‘intersection’ of the sequences \( x = x_1x_2 \ldots x_n \) and \( y = y_1y_2 \ldots y_n \) (that is, the maximum length of the last subsequence of \( x \) that coincides with a first subsequence of \( y \)), denoted \( |x \cap y| \), is greater than zero. (For instance, for \( x = 0 \ldots 01 \) and \( y = 100 \ldots 0 \), we get \( |x \cap y| = 2 \).) In this case, the first \( \ell = |x \cap y| \) step pairs ‘\( \sim \rightarrow \)’ of the walk in (18) are useless and can be avoided. Then, we say that we save 2\( \ell \) steps.

(b) Some of the following equalities hold: \( x_0 = x_1+y_1 \) or \( x_i+y_i = x_{i+1}+y_{i+1} \) for some \( i = 1, \ldots, n-1 \). In this case, some steps ‘\( \sim \)’ are absent. More precisely, if either both equalities \( x_i = y_i \) and \( x_{i+1} = y_{i+1} \) (or both inequalities \( x_i \neq y_i \) and \( x_{i+1} \neq y_{i+1} \)) hold, then the step ‘\( \sim \)’ through an edge leading to \( x_{i+1}+y_{i+1} \big|x_{i+1} \ldots x_n y_1 \ldots y_i \ldots \) is absent. So, we save 1 step.

Thus, if we can save some steps, one last step \( (x_n + y_n)|y \sim (\overline{x_n} + y_n)|y \) assures a walk of length at most 2\( n \) from \( x_0|x \) to \( y_0|y \) for any \( y_0 \in \{0, 1\} \).

This thus leads to the step pattern \( \sim \rightarrow \sim \rightarrow \ldots \sim \rightarrow \ldots \rightarrow \ldots \sim \rightarrow \overline{x_n} \sim \rightarrow \overline{x_n} \ldots \sim \rightarrow \overline{x_n} \ldots \sim \rightarrow \overline{x_n} \) uniquely determines the destiny vertex.
There is a walk of length 2

\begin{align*}
&\sim (x_2 + x_1) | x_2 x_3 \ldots x_n x_1 \rightarrow (x_2 + x_1) | x_3 x_4 \ldots x_n x_1 x_1 \\
&\sim (x_3 + x_2) | x_3 x_4 \ldots x_n x_1 x_1 \rightarrow (x_3 + x_2) | x_4 x_5 \ldots x_n x_1 x_1 x_2 \\
&\sim (x_4 + x_3) | x_4 x_5 \ldots x_n x_1 x_1 x_2 \rightarrow (x_4 + x_3) | x_5 \ldots x_n x_1 x_1 x_2 x_3 \\
&\vdots \\
&\sim (x_n + x_{n-1}) | x_n x_1 x_1 x_2 \ldots x_{n-1} x_{n-2} \\
&\rightarrow (x_n + x_{n-1}) | x_1 x_1 x_2 \ldots x_{n-2} x_{n-1} \\
&\sim (x_1 + x_n) | x_1 x_1 x_2 x_3 \ldots x_n \\
&\sim 1 | (x_1 + x_n) | x_1 x_1 x_2 x_3 x_4 x_5 \ldots x_n 
\end{align*}

This walk can have 2n + 2 steps whenever all steps ‘\sim’ through edges are present. This is the case when \(x_2 + x_1 \neq 0\), \(x_3 + x_2 \neq x_2 + x_1\), \(x_4 + x_3 \neq x_3 + x_2\), \ldots, \(x_1 + x_n \neq x_n + x_{n-1}\), and \(x_1 + x_n \neq 1\). In turn, this implies the \(n + 1\) equalities

\begin{align*}
x_1 &= x_3, \quad x_3 = x_5, \quad \ldots, \quad x_{n-3} = x_{n-1}, \quad x_{n-1} = x_1, \\
x_1 &= x_2, \quad x_2 = x_4, \quad x_4 = x_6, \quad \ldots, \quad x_{n-2} = x_n, \quad x_n = x_1. 
\end{align*}

Note that these sequences of equalities form two cycles (with odd and even subscripts) rooted at \(x_1\). Thus, the number of inequalities, if any, must be at least 2. In this case, at least 2 steps ‘\sim’ are absent in \((19)\), and we have a walk of length at most 2n between the vertices considered.

Otherwise, if all the equalities \((20)-(21)\) hold, the initial vertex must be 0000 \([9]\) 00 (the first digit \(x_1\) can be fixed to 0 since the mixed graph has an automorphism that sends \(x_0 x_1 x_2 \ldots x_n\) to \(x_0 x_1 x_2 \ldots x_n\), and the destiny vertex is 01010 \([9]\) 10. The same reasoning for \(n\) odd leads that, in the worst case (walk in \((19)\) of length 2n + 2), the initial vertex is 00000 \ldots 0 and the final vertex 01010 \ldots 1. In such cases, we have a particular walk of the desired length.

5. There is a walk of length 2n from 0000 \ldots 0 to 11010 \ldots 10 (\(n\) even) or to 01010 \ldots 01 (\(n\) odd) by using the following step pattern

\begin{align*}
\sim \rightarrow \rightarrow \sim \rightarrow \sim \rightarrow \ldots \ldots \sim \rightarrow \rightarrow .
\end{align*}

For instance, for \(n = 6\), we get

\begin{align*}
000000 &\sim 1000000 \rightarrow 1000001 \rightarrow 1000011 \\
&\sim 0000011 \rightarrow 0000110 \\
&\sim 1000110 \rightarrow 1001101 \\
&\sim 0001101 \rightarrow 0011010 \\
&\sim 1011010 \rightarrow 1101101 \rightarrow 1101010,
\end{align*}

and, for \(n = 7\),

\begin{align*}
00000000 &\sim 10000000 \rightarrow 10000001 \rightarrow 10000011 \\
&\sim 00000011 \rightarrow 00000110 \\
&\sim 10000110 \rightarrow 10001101 \\
&\sim 00001101 \rightarrow 00011010 \\
&\sim 10011010 \rightarrow 10110101 \\
&\sim 01011010 \rightarrow 10110101 \rightarrow 01010101.
\end{align*}

6. The case \(x = 1\) is similar, and we only mention the main facts. Now, the ‘worst case’ (2n steps) in the walk in \((18)\) (2n steps) occurs when, starting from 1\(|x| = 1|x_1 x_2 x_3 \ldots x_n\), we want to reach the destiny vertices of the form 0\(|y| = 0|x_1 x_2 x_3 x_4 \ldots x_n\) (\(n\) even), or 1\(|z| = 1|x_1 x_2 x_3 x_4 \ldots x_n\) (\(n\) odd), with the additional constraints \(|x \cap y| = |x \cap z| = 0\). Now, following the same strategy as in step 4 above, it turns out that for the case of 2n + 2 steps, the following conditions must hold (assuming \(n\) odd, the even case is similar):

\begin{align*}
x_1 &= x_2, \quad x_2 = x_3, \quad \ldots, \quad x_{n-1} = x_n, \quad x_n \neq x_1, 
\end{align*}

which are clearly incompatible, and at least 2 steps must be another inequality (the last one in \((22)\) is forced since the final vertex has \(x_0 = 1\)). Again, at least 2 steps ‘\sim’ are absent in \((19)\), and we have a walk of length at most 2n between the vertices considered. For example, for \(n = 5\), and assuming that \(x_4 \neq x_5\) and \(x_1 = 0\), the walk of 10 steps from 100001 to 101011 is:

\begin{align*}
100001 &\rightarrow 100011 \\
&\sim 000011 \rightarrow 0000110
\end{align*}
Proof. For any fixed functions


\[ \begin{align*}
\sim 100110 &\rightarrow 101101 \\
\sim 01101 &\rightarrow 010101 \\
\rightarrow 010101 &\rightarrow 010111 \sim 101011. 
\end{align*} \]

This completes the proof. \( \square \)

In fact, we implicitly proved the following.

**Lemma 2.5.** For every \( n > 1 \), the mixed graph \( G^+(n) \) satisfies the following.

(i) The vertices \( 000 \ldots 0 \) and \( 111 \ldots 1 \) have maximum eccentricity \( 2n \).

(ii) The vertices \( 100 \ldots 0 \) and \( 111 \ldots 1 \) have eccentricity \( 2n - 1 \).

(iii) If \( n \geq 5 \), the vertices \( 100 \ldots 01 \) and \( 111 \ldots 10 \) have eccentricity \( 2n - 2 \).

**Proof.** (i) and (ii) follow from the previous reasoning. To prove (iii), we only need to check the distance from \( 100 \ldots 01 \) to \( 000 \ldots 0 \). A shortest path between these two vertices is \( 100 \ldots 01 \sim 000 \ldots 01 \rightarrow 000 \ldots 010 \rightarrow \cdots \rightarrow 010 \ldots 00 \sim 1010 \ldots 00 \rightarrow 100 \ldots 00 \sim 000 \ldots 00 \) of length \( n + 3 \leq 2n - 2 \) if \( n \geq 5 \). \( \square \)

Let \( \Psi_0 \) and \( \Psi_1 \) be the functions that map a vertex \( x \) to its adjacent vertex from an edge or an arc, respectively. That is,

\[ \Psi_0(x_0|x_1x_2 \ldots x_n) = \overline{x_0}x_1x_2 \ldots x_n, \]

\[ \Psi_1(x_0|x_1x_2 \ldots x_n) = x_0x_1x_2 \ldots (x_0 + x_1). \]

Let \( \Phi = (\phi_1, \phi_2, \ldots, \phi_n) \) be the function that maps every \( x_i \) to either \( x_i \) or \( \overline{x_i} \), for \( i = 1, 2, \ldots, n \).

**Lemma 2.6.** For any fixed functions \( \Psi_j \) and \( \Phi \), and first digit \( x_0 = 0, 1 \), we have

\[ \Psi_j(x_0|\Phi(x)) = \Phi(\Psi_j(x_0|x)), \]

where \( \Phi \) only acts on the digits \( x_1, x_2, \ldots, x_n \).

**Proof.**

\[ \Psi_0(x_0|\Phi(x)) = \Psi_0(x_0|\phi_1(x_1)\phi_2(x_2) \ldots \phi_n(x_n)) = \overline{x_0}\phi_1(x_1)\phi_2(x_2) \ldots \phi_n(x_n) \]

\[ = \Phi(\Psi_0(x_0|x)). \]

\[ \Psi_1(x_0|\Phi(x)) = \Psi_1(x_0|\phi_1(x_1)\phi_2(x_2) \ldots \phi_n(x_n)) = x_0|\phi_2(x_2) \ldots \phi_n(x_n)(x_0\phi_1(x_1)) \]

\[ = \Phi(\Psi_1(x_0|x)). \]

Another property of the mixed graph \( G^+(n) \) for \( n > 1 \) is that from every pair of (not necessarily distinct) vertices \( u \) and \( v \), there is at least a walk of length \( 2n \) from \( u \) to \( v \). For instance, for \( n = 2 \), fixing as before \( x_1 = 0 \) and setting \( y = x_0 + x_2 \), we have the following walks of length \( 4 \) from \( x_0|0x_2 \) to every vertex of \( G^+(2) \).

\[ x_0|0x_2 \sim \overline{x_0}|0x_2 \sim x_0|0x_2 \rightarrow x_0|x_2x_0 \rightarrow x_0|x_2x_0 \rightarrow x_0|0x_0(x_0 + x_2) = x_0|x_0y \]

\[ \rightarrow x_0|x_2x_0 \sim \overline{x_0}|x_2x_0 \rightarrow \overline{x_0}|x_0(x_0 + x_2) \sim x_0|0x_0(x_0 + x_2) = x_0|0x_0y \]

\[ \sim \overline{x_0}|0x_2 \rightarrow \overline{x_0}|x_2x_0 \rightarrow \overline{x_0}|x_0(x_0 + x_2) \sim x_0|0x_0(x_0 + x_2) = x_0|0x_0y \]

\[ \rightarrow x_0|x_2x_0 \rightarrow x_0|x_2x_0 \rightarrow x_0|x_2x_0 \rightarrow \overline{x_0}|(x_2 + x_0) \rightarrow \overline{x_0}|(x_2 + x_0) = x_0|0y \]

\[ \rightarrow x_0|x_2x_0 \rightarrow x_0|x_2x_0 \rightarrow x_0|x_2x_0 \rightarrow \overline{x_0}|(x_2 + x_0) \rightarrow \overline{x_0}|(x_2 + x_0) = x_0|0y \]

\[ \sim \overline{x_0}|0x_2 \rightarrow \overline{x_0}|x_2x_0 \rightarrow \overline{x_0}|x_0(x_0 + x_2) \rightarrow \overline{x_0}|(x_2 + x_0) = x_0|0y \]

\[ \rightarrow x_0|x_2x_0 \sim \overline{x_0}|x_2x_0 \overline{x_0}|x_0(x_0 + x_2) \overline{x_0}|(x_2 + x_0) = x_0|0y \]

\[ \rightarrow x_0|0x_2 \sim \overline{x_0}|0x_2 \overline{x_0}|x_0(x_0 + x_2) \overline{x_0}|(x_2 + x_0) = x_0|0y. \]

Working with the adjacency matrix \( A \) of \( G^+(2) \) (indexed according to Fig. 9), the above property is apparent when we look at the power \( A^4 \).

\[ \begin{pmatrix}
 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 \end{pmatrix} \]

\[ \begin{pmatrix}
 5 & 3 & 3 & 1 & 1 & 1 & 1 & 1 \\
 3 & 3 & 1 & 3 & 1 & 1 & 3 & 1 \\
 1 & 1 & 3 & 3 & 1 & 3 & 3 & 1 \\
 1 & 1 & 1 & 5 & 1 & 3 & 3 & 1 \\
 1 & 3 & 3 & 1 & 5 & 1 & 1 & 1 \\
 3 & 1 & 3 & 3 & 1 & 3 & 1 & 1 \\
 1 & 3 & 1 & 1 & 3 & 1 & 3 & 3 \\
 1 & 1 & 1 & 1 & 1 & 3 & 3 & 5 \\
 \end{pmatrix} \]
2.6. The n-line mixed graphs

Let $G = (V, A)$ be a 2-regular digraph with a given 1-factorization, that is, containing two arc-disjoint spanning 1-regular digraphs $H_1$ and $H_2$. Assuming that the arcs of $H_1$ have color blue and the arcs of $H_2$ have color red, we can also think about a (proper) arc-coloring $\gamma$ of $G$. Then, if $xy$ represents an arc of $G$, we denote its color as $\gamma(xy)$.

Given an integer $n \geq 3$, the vertices of the $n$-line mixed graph $H(n) = H_n(G)$ are the set of $n$-walks in $G$, $x_1x_2 \ldots x_{n-1}x_n$, with $x_i \in V$ and $x_i x_{i+1} \in A$, for $i = 1, \ldots, n-1$. The adjacencies of $H(n)$ are as follows:

$$x_1 x_2 \ldots x_{n-1} x_n \sim y_1 y_2 \ldots y_{n-1} y_n \quad \text{(edges)},$$

where $\gamma(y_1 y_2) \neq \gamma(x_1 x_2)$; and

$$x_1 x_2 \ldots x_{n-1} x_n \rightarrow x_2 x_3 \ldots x_{n-1} x_n y_{n+1} \quad \text{(arcs)},$$

where $\gamma(x_n y_{n+1}) = \text{red}$ if $\gamma(x_1 x_2) = \gamma(x_{n-1} x_n)$, and $\gamma(x_n y_{n+1}) = \text{blue}$ if $\gamma(x_1 x_2) \neq \gamma(x_{n-1} x_n)$. The reason for the name of $H_n(G)$ is because when we contract all its edges, so identifying the vertices in (23), the resulting digraph is the $(n-1)$-iterated line digraph $L^{n-1}(G)$ of $G$, see Fiol, Yebra, and Alegre [9]. Indeed, under such an operation, each pair of vertices in (23) becomes a vertex that can be represented by the sequence $x_1 x_2 \ldots x_n$, which, according to (24), is adjacent to the two vertices $x_3 \ldots x_{n} y_{n+1}$ with $y_{n+1} \in \Gamma^+(x_n)$ in $G$.

In the following result, we describe other basic properties of $H_n(G)$.

**Proposition 2.7.** Let $G = (V, A)$ be a digraph with $r$ vertices and diameter $s$, having a 1-factorization. For a given $n \geq 3$, the following holds.

(i) The mixed graph $H_n = H_n(G)$ has $N = r \cdot 2^{n-1}$ vertices, and it is totally $(1,1)$-regular with no digons.

(ii) The diameter of $H_n$ satisfies $k \leq 2s + 3$.

**Proof.** (i) Every vertex $x_1 \ldots x_n$ of $H_{n}$ corresponds to a walk of $G$ with first vertex $x_1$, which gives $r$ possibilities and, since $G$ is 2-regular, for every other $x_i, i = 2, \ldots, n$, we have 2 possible options. This provides the value of $N$.

To show total (1,1) regularity, it is enough to prove that $H_{n}$ is 1-in-regular. Indeed, any vertex adjacent to $x_1 x_2 \ldots x_n$ with $\gamma(y_1 y_2 x_n) = \text{blue}$ (respectively, $\gamma(x_1 x_2 y_n) = \text{red}$) must be of the form $x_1 y_3 \ldots x_{n-2} x_{n-1}$ with $\gamma(x_1 x_2) \neq \gamma(x_{n-2} x_{n-1})$ (respectively, with $\gamma(x_1 y_2) = \gamma(x_{n-1} x_n)$). But, in both cases, there is only one possible choice for vertex $y$.

With respect to the absence of digons, notice that a vertex $u = x_1 x_2 \ldots x_{n-1} x_n$ belongs to a digon if, after two steps, we come back to $u$, which means that $x_1 x_2 \ldots x_{n-1} x_n = x_1 x_2 \ldots x_{n-2} y_1 x_n y_2$ and, hence, $x_1 = x_3 = \ldots$ and $x_2 = x_4 = \ldots$. In other words, vertex $u$ must be of the form $x y x y \ldots x y$ $(n$ even) or $x y x y \ldots x y$ $(n$ odd), and $G$ itself must have a digon between vertices $x$ and $y$. Assuming that $n$ is even and $\gamma(x y) = \text{blue}$ (the other cases are similar), the digon should be $u = x y x y \ldots x y \rightarrow v = y x y x \ldots y x \rightarrow u$.

But the last adjacency is not possible since both the first and last arcs of $v$ would have color $\gamma(y x) = \text{red}$ and, hence, so should be the color of $x y$, a contradiction.

(ii) Given both vertices $x_1 x_2 \ldots x_{n-1} x_n$ and $y_1 y_2 \ldots y_{n-1} y_n$, let us consider a shortest path in $G$ of length at most $s$ from $x_n$ to $y_2$. Then, using both types of adjacencies, we can go from $x_1 x_2 \ldots x_{n-1} x_n$ to a vertex of the form $z_1 \ldots z_{s+2} x_2$. From this vertex, we now reach the vertex $y_1 y_2 \ldots y_n$ in at most $2(n-2)$ steps. Finally, if necessary, we can change $y$ by $y_1$. In total, we use $k \leq 2s + 2(n-2) + 1 = 2(s+n) - 3$ steps, as claimed.

For example, if $G$ is the complete symmetric digraph $K_3$ (edges seen as digons) with vertices in $\mathbb{Z}_3$, blue arcs $i \rightarrow i + 1$ and red arcs $i \rightarrow i - 1$ for $i = 0, 1, 2$, the adjacencies of $H_n(K_3)$, with $3 \cdot 2^{n-1}$ vertices, are

$$x_1 x_2 \ldots x_{n-1} x_n \sim y_1 y_2 \ldots y_{n-1} y_n, \quad y_1 \neq x_1, x_2,$$

$$x_1 x_2 \ldots x_{n-1} x_n \rightarrow x_3 x_4 \ldots x_{n-1} x_n, \quad y_{n+1} = x_n - (x_2 - x_1)(x_n - x_{n-1}).$$

Thus, the $(1,1)$-regular mixed graphs $H_3(K_3)$ and $H_4(K_3)$, with diameter $k = 5$ and $k = 6$, respectively, are shown in Fig. 11. In this case, when we contract all the edges of $H_5(K_3)$, we obtain the $(n-1)$-iterated line digraph of $K_3$, which, as commented in the Introduction, is isomorphic to the Kautz digraph $K(2, n-1)$.

3. A first computational approach: The $(1, 1, k)$-mixed graphs with diameter at most 6

The Moore bound $M(1,1,k)$ coincides with the number of binary words of length $\ell \leq k$ without consecutive zeros. In this sense, the corresponding Moore tree can be rooted to a vertex labeled with the empty word. Every vertex labeled with a word $\omega$ (of length $\ell$) with the last symbol different from 0 is joined by an edge to a vertex labeled $\omega 0$ (of length $\ell + 1$), for all $0 \leq \ell \leq k - 1$. Moreover, the arcs are defined by $\omega \rightarrow \omega 1$ (see an example in Fig. 12).

This new description of the Moore tree is very useful for performing an exhaustive computational search of the largest mixed graphs for some small values of the diameter $k$. Let $a(k)$ be the number of vertices at distance $\ell$ from the root in the Moore tree. Using the above-mentioned equation, it is easy to see that $a(k)$ satisfies the recurrence equation

$$a(\ell) = a(\ell - 1) + a(\ell - 2),$$

(25)
with initial conditions $a(0) = 1$ and $a(1) = 2$. Indeed, $a(\ell)$ is the number of words of length $\ell$ (whose symbols are in the alphabet $\Sigma = \{0, 1\}$) without consecutive zeros. The words of length $\ell$ non-ending with 0 are constructed by a word of length $\ell - 1$ by adding 0. This gives $a(\ell - 1)$. Moreover, the words of length $\ell$ ending with 0 are constructed by adding 1. This gives $a(\ell - 2) = b(\ell)$, where $b(\ell)$ is the number of vertices at distance $\ell$ from the root joined by an edge to a vertex at distance $\ell - 1$. So $b(\ell)$ satisfies the same recurrence relation as $a(\ell)$ but with initial conditions $b(0) = 0$ and $b(1) = 1$. Finally, let $c(\ell) = a(\ell) - b(\ell) = a(\ell - 1)$, that is, the number of vertices at distance $\ell$ from the root pointed by an arc from a vertex at distance $\ell - 1$. Again, $c(\ell)$ satisfies the same type of recurrence relation but with initial conditions $c(0) = 1$ and $c(1) = 1$. Thus, $a(\ell)$, $b(\ell)$, and $c(\ell)$ are all Fibonacci-like numbers. For instance, $a(\ell)$ equals the following closed formula

$$a(\ell) = \frac{5 + 3\sqrt{5}}{10} \left( 1 + \frac{\sqrt{5}}{2} \right)^\ell + \frac{5 - 3\sqrt{5}}{10} \left( 1 - \frac{\sqrt{5}}{2} \right)^\ell.$$ 

Note that the sequence obtained from $a(\ell)$ corresponds to the Fibonacci numbers starting with $a(0) = 1$ and $a(1) = 2$ (see the sequence A000045 in [16]). Similar formulas for $b(\ell)$ and $c(\ell)$ can be obtained.

Now, we can perform an algorithmic exhaustive search to find all the largest $(1, 1, k)$-mixed graphs with order close to the Moore bound. For instance, in the case of almost mixed Moore graphs (with diameter $k$ and order $M(1, 1, k) - 1$), the number of different cases of mixed graphs to analyze is bounded by $N'(k)$, where $N'(k)$ is computed next.

1. We remove a vertex in the Moore tree at distance $k$ from the empty word. Notice there are $a(k)$ different choices for this vertex.
2. Now, we count the number $N_1$ of possibilities to complete the undirected part of the mixed graph. We recall that the number of perfect matchings in a complete graph of even order $n$ is $(n - 1)!!$. This number $N_1$ depends on the vertex removed in the previous step. If the removed vertex has a label ending with 0, that is, it is a vertex hanging

Fig. 11. The mixed graphs $H_3(K_3)$ and $H_4(K_3)$.

Fig. 12. The Moore tree with parameters $r = z = 1$ and depth 4 labeled with binary words without consecutive zeros.
there is no mixed graph with parameters $s_1, s_2, \ldots, s_r$ and presentation $(r, s_1, s_2, \ldots, s_r)$ (when $s$ ends in 1), where $s' \neq s$ is any word of length 4.

It remains to add the corresponding edges and arcs in the pruned Moore tree. The computational exhaustive search shows there is no $(1, 1)$-mixed Moore graph of diameter 4 and order 16. Now, the maximum order becomes $n = 14$ for a mixed graph with parameters $r = z = 1$ and $k = 4$. There are many more possibilities to prune the Moore tree, so we decide to implement a direct method to perform an exhaustive search in this case: taking the perfect matching with a set of vertices $V = \{0, 1, \ldots, 13\}$ and where $i \sim i + 1$ for all even $i$, we add the three arcs $(0, 2), (1, 5)$ and $(5, 7)$. Looking at vertex 0 as the root of the Moore tree, the existence of these three arcs in the mixed graph is given because $\delta = 5$ in this case. Now, we proceed with the exhaustive search by adding the remaining arcs in the graph. There are 11! possibilities, but excluding avoided permutations (those permutations with elements of order at most 2 or including edges of the perfect matching) significantly reduces the number of cases to analyze. We have the following result after computing the diameter of all these mixed graphs and keeping those non-isomorphic mixed graphs with diameter $k = 4$.

**Proposition 3.2.** The maximum order for a $(1, 1)$-mixed regular graph of diameter $k = 4$ is 14. There are 27 such mixed graphs (see Table 2), and only one of them is a Cayley graph. Namely, that of the dihedral group $D_n$ with generators $r$ and $s$, and presentation $(r, s) = (rs)^2 = 1$, also obtained as the line digraph of $C_7$, see the mixed graph at the top left in Fig. 13.

### Table 1

<table>
<thead>
<tr>
<th>$N(k)$</th>
<th>396</th>
<th>889,980</th>
<th>0</th>
<th>2 - $10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Table 3 Classification of the largest mixed graphs for $r = z = 1$ and diameter $k = 4$ according to their spectra. The third row gives the spectrum of the 5 cospectral graphs in Fig. 13.

<table>
<thead>
<tr>
<th>Number of graphs</th>
<th>Spectra</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>${2^1, 0^6, a_{1j}^1, a_{1j}^2}$ for $j = 1, 2, 3, 4$ and $s = 1, 2, 3$</td>
</tr>
<tr>
<td>6</td>
<td>${2^1, -1^1, 1^1, 0^3, a_{1j}^1, a_{1j}^2}$ for $j = 1, 2, 3$</td>
</tr>
<tr>
<td>5</td>
<td>${2^1, 0^7, a_{1j}^2}$ for $j = 1, 2, 3$</td>
</tr>
<tr>
<td>4</td>
<td>${2^1, 0^1, a_{1j}^1, a_{1j}^3}$ for $j = 1, 2, 3, 4$ and $k = 1, \ldots, 6$</td>
</tr>
<tr>
<td>2</td>
<td>${2^1, 1^1, 0^3, a_{1j}^1}$ for $j = 1, \ldots, 9$</td>
</tr>
<tr>
<td>1</td>
<td>${2^1, 0^1, a_{1j}^1, a_{1j}^3}$ for $j = 1, \ldots, 6$</td>
</tr>
</tbody>
</table>

The spectra of all 27 mixed graphs with the largest order can be described with the help of the (complex) roots $a_{ij}$ of the irreducible polynomials $p_i(x) \in \mathbb{Q}(x)$ given below (see Table 3):

- $p_1(x) = x^4 + x^3 - 2x^2 - x + 2$,
- $p_2(x) = x^3 + x^2 - 2x - 1$,
- $p_3(x) = x^3 - x + 1$,
- $p_4(x) = x^3 + 2x^2 - x - 3$,
- $p_5(x) = x^4 + x^3 - x^2 - x + 1$,
- $p_6(x) = x^6 + x^5 - 3x^4 - x^3 + 5x^2 - 4$,
- $p_7(x) = x^6 + 3x^8 - 6x^6 + 2x^5 + 11x^4 - 3x^3 - 9x^2 + 3x + 3$,
- $p_8(x) = x^6 + x^5 - 3x^4 - 2x^3 + 5x^2 + 2x - 3$,
- $p_9(x) = x^6 + x^5 - x^4 + 3x^2 - 1$.

4. A second computational approach: Cayley or lift $(1, 1, k)$-mixed graphs with small diameter

To obtain the results of this section, we followed a different strategy. We mainly concentrate our search on looking at large $(1, 1, k)$-mixed graphs that are either Cayley or lift graphs. Let us first recall these two classes of graphs.

Given a finite group $\Omega$ with generating set $S \subseteq \Omega$, the Cayley graph $\text{Cay}(\Omega, S)$ has vertices representing the elements of $\Omega$, and arcs from $\omega$ to $\omega s$ for every $\omega \in \Omega$ and $s \in S$. Notice that if $s, s^{-1} \in S$, then we have an edge (as two opposite arcs)
between \( \omega \) and \( \omega s \). Thus, if \( S = S_1 \cup S_2 \) where \( S_1 = S_1^{-1} \) and \( S_2 \cap S_2^{-1} = \emptyset \), the Cayley graph \( \text{Cay}(\Omega, S) \) is an \((r, z)\)-mixed graph with undirected degree \( r = |S_1| \) and directed degree \( z = |S_2| \).

Given a digraph \( G \), or base graph, and a finite group \( \Omega \) with generating set \( S \), a voltage assignment \( \alpha \) is a mapping \( \alpha : E \to S \), that is, a labeling of the arcs with the elements of \( S \). Then, the lift digraph \( G^\alpha \) has vertex set \( V(G^\alpha) = V \times \Omega \) and arc set \( E(G^\alpha) = E \times S \), where there is an arc from vertex \((v, g)\) to vertex \((w, g\alpha(uv))\) if and only if \( uv \in E \). In particular, the Cayley digraph \( \text{Cay}(\Omega, S) \) with \( S = \{g_1, \ldots, g_r\} \) can be seen as the lifted digraph \( G^\alpha \), where \( G = K^r_1 \) (a singleton with \( V = \{u\} \) and \( E = \{e_1, \ldots, e_r\} \) are \( r \) loops) and voltage assignment \( \alpha(e_i) = g_i \) for \( i = 1, \ldots, r \). An example of a lift digraph is shown in Fig. 14.

The results obtained by computer search are shown in Table 4, see next section. In what follows, we comment upon some of the cases. Notice that for diameter \( k = 2, 3, 4 \), the known \((1, 1, k)\)-mixed graphs have the maximum possible order. The mixed graph of diameter \( k = 2 \) is the Kautz digraph \( K(2, 2) \). The graph with \( k = 3 \) is isomorphic to the line digraph of the cycle \( C_5 \). Some of the maximal graphs with diameter \( k = 4 \) were already shown in Fig. 13.

Two maximal graphs of diameter \( k = 5 \) are shown in Fig. 15.

The graph of order 72 listed in the table for \( k = 8 \) is a lift graph using the dihedral group \( D_{18} \) of order 18. This group consists of the 18 symmetries of the nonagon. To describe our graph, we consider a regular nonagon whose vertices are labeled 0 to 8 in clockwise order. Label the elements of \( D_{18} \) as follows. There are nine counter-clockwise rotations, each through an angle \( 2\pi k/9 \) and denoted \( \text{Rot}(k) \), for \( 0 \leq k < 9 \). Finally, there are the nine reflections \( \text{Ref}(k) \) about the line through vertex \( k \) and the midpoint of the opposite side. This notation is used to specify the voltages on the edges and arcs of the base graph shown in Fig. 16.

The graph of order 544 for \( k = 13 \) is a lift of the base graph shown in Fig. 17 with voltages in the group \( \mathbb{Z}_{17} : \mathbb{Z}_8 \).

The remaining graphs are partially identified as notes following Table 4. Where a graph is identified as a lift using a voltage group of order half the order of the graph, the base graph is an undirected edge together with a directed loop at
Fig. 16. The base graph of the \((1, 1, 8)\)-mixed graph of order 72.

Fig. 17. The base graph of the \((1, 1, 13)\)-mixed graph of order 544.

each vertex. A complete description of such larger graphs, especially those that use unfamiliar groups, would take a lot of pages. The interested reader can address the third author to request more information.

5. Table of large \((1, 1, k)\)-mixed graphs

A summary of the results for a \((1, 1)\)-regular mixed graphs with diameter \(k\) at most 16 is shown in Table 4, where the lower bounds come from the mentioned constructions. Moreover, the upper bounds follow by Proposition 3.2 \((k = 4)\), a computer exploration \((k = 5)\), and the numbers \(M(1, 1, k) − \delta(k)\) with \(\delta(k)\) given in (3) and adjusted even parity (since \(r = 1\), the graph contains a perfect matching and, so, it must have even order), see Tuite and Erskine [18].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgments

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Table 4
Bounds for mixed graphs with \((r, z, k) = (1, 1, k)\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Moore (M(1, 1, k))</th>
<th>Notes</th>
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<tr>
<td>2</td>
<td>6</td>
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<td>3</td>
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<td></td>
</tr>
<tr>
<td>7</td>
<td>54</td>
<td>78</td>
<td>87</td>
<td>Cayley(^a)</td>
</tr>
<tr>
<td>8</td>
<td>72</td>
<td>126</td>
<td>142</td>
<td>Lift(^b)</td>
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<td>112</td>
<td>206</td>
<td>231</td>
<td>Lift(^c)</td>
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<td>336</td>
<td>882</td>
<td>985</td>
<td>Lift(^f)</td>
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<tr>
<td>13</td>
<td>544</td>
<td>1428</td>
<td>1595</td>
<td>Lift(^g)</td>
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<tr>
<td>14</td>
<td>800</td>
<td>2312</td>
<td>2582</td>
<td>Cayley(^h)</td>
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<td>1024</td>
<td>3744</td>
<td>4179</td>
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<td>16</td>
<td>1600</td>
<td>6058</td>
<td>6763</td>
<td>Lift(^j)</td>
</tr>
</tbody>
</table>

\(^a\) Cayley graph on SmallGroup(54,6); \(Z_9 : Z_6\).
\(^b\) Lift group is the dihedral group of order 18.
\(^c\) Lift group is \(AGL(1, 8) = (Z_3^2) : Z_7\).
\(^d\) Cayley graph on SmallGroup(144,182).
\(^e\) Lift group is \(A_5 \times Z_2\).
\(^f\) Cayley graph on \(PSL(2, 7) : Z_2\).
\(^g\) Lift group is \(Z_{17} : Z_8\).
\(^h\) Cayley graph on SmallGroup(800,1191).
\(^i\) Lift group is SmallGroup(512,1727).
\(^j\) Lift group is SmallGroup(800,1191).

References