Received: 2 February 2024 | Accepted: 4 March 2024

DOI: 10.1111/stan.12337

SHORT NOTE

WILEY

Duals of convolution thinned relationships

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In a recent article, J. Peyhardi gives a number of novel results related to quasi Pólya thinning which encompass a number of important mixture relationships between univariate discrete distributions. In this note, I explore the duals of the general results on convolution thinning given in Peyhardi's Theorem 1 in order to obtain new relationships and to gain new insights into old relationships. Some consequences—for integer-valued autoregressive processes—and analogues—in the continuous case—are noted.

KEYWORDS

discrete distribution, integer-valued autoregressive process, mixture, quasi Pólya thinning

1 | INTRODUCTION

Peyhardi (2023) gives a number of novel results related to the quasi Pólya thinning operator of Janardan and Raja Rao (1982). Inter alia, these encompass a number of important (discrete) mixture relationships between univariate discrete distributions. As one simple, known, example, binomial thinning of a Poisson distribution leads to a Poisson distribution with an altered parameter. In general, for discrete random variables K and N say, these mixture relationships can be interpreted as the combination of the conditional distribution of $K|N = n$ with the marginal distributions of N and K. As such, they each have a dual relationship made up of the conditional distribution of $N|K = k$ and the marginal distributions of K and N. In the simple example just given, this dual relationship turns out to be the convolution property of Poisson distributions. In this note, I explore the duals of the general results on convolution thinning given in Theorem 1 of Peyhardi (2023) in order to obtain new relationships and to gain new insights into old relationships.

To these ends, Section 2 provides the necessary definitions and notation. The main results of this note are in Section 3 while some consequences—for integer-valued autoregressive processes—and analogues—in the continuous case—are briefly considered in the closing Section 4.

2 | DEFINITIONS AND NOTATION

This section provides a précis of the required definitions and notation of Peyhardi (2023) which I follow in this note. Let K and N be discrete random variables with support $\mathbb{N}$ or an upper truncated version thereof. Let a_θ be a parametric sequence $\{a_\theta(n)\}_{n\in\mathbb{N},\ \theta\in\mathbb{N} \text{ or } \mathbb{R}^+}$ such that $a_\theta * a_\gamma = a_{\theta+\gamma}$ where $*$ denotes convolution. Denote by $\mathcal{C}_n(\theta,\gamma)$, $\mathcal{APS}(\theta, g(\alpha))$ and $\mathcal{IC}(r;\theta,\gamma)$ the convolution distribution, the additive modified power series distribution and the inverse convolution distribution, respectively. The convolution distribution has probability mass function (pmf)

$$P(K=k) = \frac{a_\theta(k)a_\gamma(n-k)}{a_{\theta+\gamma}(n)}, \qquad k = 0, \dots, n.$$

The power series distribution has pmf

$$P(K=k) = \frac{a_\theta(k)\{g(\alpha)\}^k}{h_\theta(\alpha)},$$

where $0 < \alpha < R$ and $g(\alpha)$ and $h_\theta(\alpha)$ are functions such that the sum over k of the numerator terms equals the denominator. And the inverse convolution distribution has pmf

$$P(K=k) = \frac{r}{k+r}\,\frac{a_\theta(k)a_\gamma(r)}{a_{\theta+\gamma}(k+r)},$$

$r = 0, \dots, K_\gamma$. For a thorough study of choice of a_θ and listings of myriad special case distributions and families thereof, see Peyhardi (2023), including its supplementary material. To set notation, the special case distributions explicitly considered in this note are the binomial, $\mathcal{B}_n(p)$, Poisson, $\mathcal{P}(\lambda)$, negative binomial, $\mathcal{NB}(r,p)$, beta binomial, $\beta\mathcal{B}(\theta,\gamma)$ and beta negative binomial, $\beta\mathcal{NB}(r;\theta,\gamma)$. Also, if $\mathcal{L}_n(\theta)$, say, is a generic distribution then its mixture with the distribution $\mathcal{L}$ is written $\mathcal{L}_n(\theta) \underset{n}{\wedge} \mathcal{L}$ or $\mathcal{L}_n(\theta) \underset{\theta}{\wedge} \mathcal{L}$ depending on which parameter is mixed over, while $\sim$ denotes "is distributed as" and $\overset{d}{=}$ denotes "has the same distribution as."

Theorem 1 of Peyhardi (2023) on the closure of the convolution thinning operation is central to what follows. Its three component results are repeated here:

$$\mathcal{C}_n(\theta,\gamma) \underset{n}{\wedge} \mathcal{C}_m(\theta+\gamma,\lambda) = \mathcal{C}_m(\theta,\gamma+\lambda); \tag{1}$$

$$\mathcal{C}_n(\theta,\gamma) \underset{n}{\wedge} \mathcal{APS}(\theta+\gamma, g(\alpha)) = \mathcal{APS}(\theta, g(\alpha)); \tag{2}$$

$$\mathcal{C}_n(\theta,\gamma) \underset{n}{\wedge} \mathcal{IC}(r;\theta+\gamma,\lambda) = \mathcal{IC}(r;\theta,\lambda). \tag{3}$$

3 | DUAL RELATIONSHIPS

I start "out of order" with Theorem 1, part 2, of Peyhardi (2023), that is, (2). This is because it gives perhaps the simplest dual of the three relationships. Parts 1 and 3 of Theorem 1 of Peyhardi (2023), that is, (1) and (3), respectively, give only slightly more complex duals which are of a similar nature to each other, and are treated thereafter.

3.1 | Dual of (2)

To obtain the dual of result (2), let $K|N = n \sim \mathcal{C}_n(\theta, \gamma)$ and $N \sim \mathcal{APS}(\theta + \gamma, g(\alpha))$. Then observe that the conditional pmf of $N|K = k$ is proportional to

$$\frac{a_\gamma(n-k)}{a_{\theta+\gamma}(n)} \times a_{\theta+\gamma}(n)\{g(\alpha)\}^n \propto a_\gamma(n-k)\{g(\alpha)\}^{n-k}, \qquad k = 0, \ldots, n.$$

But this is the distribution of $k + N_\gamma$ where $N_\gamma \sim \mathcal{APS}(\gamma, g(\alpha))$ which shows that the dual of (2) is none other than

$$N_\theta + N_\gamma \overset{d}{=} N_{\theta+\gamma}, \tag{4}$$

where N_θ and N_γ are independent; this is the convolution closure property of the additive modified power series distribution given in Proposition 1 of Peyhardi (2023). In this sense, the convolution closure and convolution thinning closure results (4) and (2) are two sides of the same coin, arising from marginal and conditional distributions of the bivariate power series distribution with pmf.

$$P(K = k, N = n) = \frac{a_\theta(k)a_\gamma(n-k)\{g(\alpha)\}^n}{h_{\theta+\gamma}(\alpha)}, \qquad k = 0, \ldots, n. \tag{5}$$

Note also that (2) says that $N_\theta | N_{\theta+\gamma} = n \sim \mathcal{C}_n(\theta, \gamma)$.

Taking on the specific example alluded to in Section 1, the binomially thinned Poisson special case in Corollary 1, part 2, of Peyhardi (2023) and in Rao (1965), namely,

$$\mathcal{B}_n(\pi) \underset{n}{\wedge} \mathcal{P}(\lambda) = \mathcal{P}(\pi\lambda),$$

has as its dual the convolution property of the Poisson distribution:

$$P_{\pi\lambda} + P_{(1-\pi)\lambda} \overset{d}{=} P_\lambda,$$

where $P_\lambda \sim \mathcal{P}(\lambda)$ and $P_{\pi\lambda}$ and $P_{(1-\pi)\lambda}$ are independent.

Similarly, the beta binomially thinned negative binomial special case of Corollary 2, part 2, of Peyhardi (2023), which reads

$$\beta\mathcal{B}_n(\theta, \gamma) \underset{n}{\wedge} \mathcal{NB}(\theta + \gamma, p) = \mathcal{NB}(\theta, p), \tag{6}$$

has as its dual the convolution property of the negative binomial distribution:

$$N_{\theta,p} + N_{\gamma,p} \overset{d}{=} N_{\theta+\gamma,p},$$

where $N_{\theta,p} \sim \mathcal{NB}(\theta, p)$ and $N_{\theta,p}$ and $N_{\gamma,p}$ are independent. Note also that (6) encompasses the fact that $N_{\theta,p}|N_{\theta+\gamma,p} = n \sim \beta\mathcal{B}_n(\theta, \gamma)$.

3.2 | Dual of (1)

Theorem 1, part 1, of Peyhardi (2023), that is, (1), concerns convolution distributions only. To obtain the dual of this result, observe that the conditional pmf of $N|K = k$ is proportional to

$$\frac{a_\gamma(n-k)}{a_{\theta+\gamma}(n)} \times a_{\theta+\gamma}(n)a_\lambda(m-n) = a_\gamma(n-k)a_\lambda(m-n), \qquad n = k, \dots, m.$$

Now, this is the distribution of $k + N_k$ where $N_k \sim \mathcal{C}_{m-k}(\gamma, \lambda)$. Using the obvious abuse of notation of $k + \mathcal{C}_{m-k}(\gamma, \lambda)$ for the distribution of $k + N_k$, this shows that the dual of (1) is

$$\{k + \mathcal{C}_{m-k}(\gamma, \lambda)\} \underset{k}{\wedge} \mathcal{C}_m(\theta, \gamma + \lambda) = \mathcal{C}_m(\theta + \gamma, \lambda). \tag{7}$$

Of course, this is only a “convolution-type" result because the distribution of N_k depends on k unlike the independence of the random variables in (4).

As a first special case, the binomial version of Corollary 1, part 1, of Peyhardi (2023) and of Rao (1965), namely,

$$\mathcal{B}_n(\pi) \underset{n}{\wedge} \mathcal{B}_m(p) = \mathcal{B}_m(p\pi),$$

has as its dual

$$\left\{k + \mathcal{B}_{m-k}\left(\frac{p(1-\pi)}{1-p\pi}\right)\right\} \underset{k}{\wedge} \mathcal{B}_m(p\pi) = \mathcal{B}_m(p).$$

Notice that, while N represents the number of “1”s in the first m realizations of a Bernoulli process with probability of success p, $k + N_k^{\mathcal{B}}$ where $N_k^{\mathcal{B}} \sim \mathcal{B}_{m-k}(p(1-\pi)/\ (1-p\pi))$ represents the number of “1”s in the first m realizations of a sequence comprising k “1”s followed by a Bernoulli process with success probability $p(1-\pi)/(1-p\pi) < p$, $k = 0, \dots, m$.

Similarly, as a second special case, the beta-binomial version of Corollary 2, part 1, of Peyhardi (2023), namely,

$$\beta\mathcal{B}_n(\theta, \gamma) \underset{n}{\wedge} \beta\mathcal{B}_m(\theta + \gamma, \lambda) = \beta\mathcal{B}_m(\theta, \gamma + \lambda),$$

has as its dual

$$\{k + \beta\mathcal{B}_{m-k}(\gamma, \lambda)\} \underset{k}{\wedge} \beta\mathcal{B}_m(\theta, \gamma + \lambda) = \beta\mathcal{B}_m(\theta + \gamma, \lambda).$$

3.3 | Dual of (3)

Theorem 1, part 3, of Peyhardi (2023), that is (3), concerns inverse convolution distributions, and leads to further convolution-like relationships. Its dual arises from the conditional pmf of $N|K = k$ being proportional to

$$\frac{a_\gamma(n-k)}{a_{\theta+\gamma}(n)} \times \frac{r}{n+r}\,\frac{a_{\theta+\gamma}(n)}{a_{\theta+\gamma+\lambda}(n+r)} = \frac{r}{(n-k)+(r+k)}\,\frac{a_\gamma(n-k)}{a_{\theta+\gamma+\lambda}((n-k)+(r+k))},$$

$k = 0, \dots, n$. This is the distribution of $k + N_k$ where $N_k \sim \mathcal{IC}(r+k; \gamma, \theta+\lambda)$, so the dual of (3) is that

$$\{k + \mathcal{IC}(r+k; \gamma, \theta+\lambda)\} \underset{k}{\wedge} \mathcal{IC}(r; \theta, \lambda) = \mathcal{IC}(r; \theta+\gamma, \lambda). \tag{8}$$

Negative binomial distributions feature in the first special case of this, Corollary 1, part 3, of Peyhardi (2023) and Rao (1965), namely,

$$\mathcal{B}_n(\pi) \underset{n}{\wedge} \mathcal{NB}(r; p) = \mathcal{NB}\left(r; \frac{\pi p}{\pi p + 1 - p}\right).$$

Its dual:

$$\{k + \mathcal{NB}(r+k; p(1-\pi))\} \underset{k}{\wedge} \mathcal{NB}\left(r; \frac{\pi p}{\pi p + 1 - p}\right) = \mathcal{NB}(r; p).$$

Similar results concern the beta-negative binomial distribution as in Corollary 2, part 3, of Peyhardi (2023), namely,

$$\beta\mathcal{B}_n(\theta, \gamma) \underset{n}{\wedge} \beta\mathcal{NB}(r; \theta+\gamma, \lambda) = \beta\mathcal{NB}(r; \theta, \lambda),$$

has as its dual

$$\{k + \beta\mathcal{NB}(r+k; \gamma, \theta+\lambda)\} \underset{k}{\wedge} \beta\mathcal{NB}(r; \theta, \lambda) = \beta\mathcal{NB}(r; \theta+\gamma, \lambda).$$

4 | CONSEQUENCES AND ANALOGUES

4.1 | INAR(1) processes

Any bivariate distribution gives rise to an integer-valued autoregressive process of order 1, INAR(1), with a specified marginal distribution, by taking the random variable N_t, say, following said distribution, and to construct, first, K_t following the distribution of $K|N_t = n$ and then N_{t+1} following the distribution of $N|K_t = k$. As all the relationships considered in this note correspond to specific bivariate distributions (e.g. (5)), all correspond to specific INAR(1) processes. The simplest, linear, ones correspond to N in (2) and (4). In this case, if $N_t \sim \mathcal{APS}(\theta+\gamma, g(\alpha))$, then $K_t|N_t = n \sim \mathcal{C}_n(\theta, \gamma)$, $N_{t+1}|K_t = k \sim k + \mathcal{APS}(\gamma, g(\alpha))$ so that $N_{t+1} \sim \mathcal{APS}(\theta+\gamma, g(\alpha))$ also. Setting $\theta = \pi\lambda, \gamma = (1-\pi)\lambda$, $0 < \pi < 1, \lambda > 0$, and concatenating the above steps, this results in the INAR(1) process $N_{t+1} = K_t + \epsilon_t$ having $\mathcal{APS}(\lambda, g(\alpha))$ marginal distribution where $K_t \sim \mathcal{C}_{N_t}(\pi\lambda, (1-\pi)\lambda)$ independently of $\epsilon_t \sim \mathcal{APS}((1-\pi)\lambda, g(\alpha))$. This process, produced via convolution thinning of a power series distribution, is essentially that of section 4.3.2 of Peyhardi (2023), who lists five special cases of this, four of which are familiar INAR(1) models from the 1980s and 1990s. One of these is the Poisson INAR(1) model (McKenzie, 1985) in which $K_t \sim \mathcal{B}_{N_t}(\pi)$ and $\epsilon_t \sim \mathcal{P}((1-\pi)\lambda)$ independently so that, marginally, $N_{t+1} = K_t + \epsilon_t \sim \mathcal{P}(\lambda)$.

An alternative INAR(1) process with power series marginals is available by focussing on the marginal distribution of K rather than N. In this case, if $K_t \sim \mathcal{APS}(\lambda, g(\alpha))$, then

$N_t|K_t = k \sim k + \epsilon_t$ where $\epsilon_t \sim \mathcal{APS}(\gamma, g(\alpha))$ and so $K_{t+1} \sim \mathcal{C}_{K_t+\epsilon_t}(\lambda, \gamma)$ also has the $\mathcal{APS}(\lambda, g(\alpha))$ distribution marginally. Notice that the order of convolution and convolution thinning has been reversed here relative to the process derived in the previous paragraph. For example, the Poisson INAR(1) process has $K_{t+1} \sim \mathcal{B}_{K_t+\epsilon_t}(1/(2-\pi))$ and ϵ_t has the same distribution as before. I am not aware of this process or those that follow in the remainder of this subsection having been previously mentioned in the literature.

Similarly, (1) and (7) can be combined in each of two ways to provide two INAR(1) processes with convolution distribution marginals. Briefly, for $\mathcal{C}_m(\mu, \nu)$ marginals, these are: (i) $N_{t+1} = K_t + \mathcal{C}_{m-K_t}((1-\pi)\mu, \nu)$ where $K_t \sim \mathcal{C}_{N_t}(\pi\mu, (1-\pi)\mu)$; and (ii) $K_{t+1} \sim \mathcal{C}_{K_t+\epsilon_t}(\mu, \pi\nu)$ where $\epsilon_t \sim \mathcal{C}_{m-K_t}(\pi\nu, (1-\pi)\nu)$. In particular, INAR(1) processes with binomial, $\mathcal{B}_m(p)$, marginals are: (a) $N_{t+1} = K_t + \mathcal{B}_{m-K_t}((1-\pi)p/(1-p\pi))$ where $K_t \sim \mathcal{B}_{N_t}(\pi)$; and (b) $K_{t+1} \sim \mathcal{B}_{K_t+\epsilon_t}(p/(p+\pi(1-p)))$ where $\epsilon_t \sim \mathcal{B}_{m-K_t}(\pi)$.

And again, (3) and (8) can be combined in each of two ways to provide two INAR(1) processes with inverse convolution distribution marginals. The resulting $\mathcal{NB}(r;p)$-distributed INAR(1) processes are: (I) $N_{t+1} = K_t + \mathcal{NB}(r + K_t; p(1-\pi))$ where $K_t \sim \mathcal{B}_{N_t}(\pi)$; and (II) $K_{t+1} \sim \mathcal{B}_{K_t+\epsilon_t}(\pi)$ where $\epsilon_t \sim \mathcal{NB}(r + K_t; p(1-\pi)/\{\pi + p(1-\pi)\})$.

4.2 | Continuous analogues

Corollary 4 of Peyhardi (2023) lists three mixture relationships in the continuous case involving beta and gamma random variables which are analogous to those in the discrete case in his Corollary 1. Each is well known, the first being due to Rao (1949), the third to Jambunathan (1954). The duals of all three of these relationships have been explored in Jones (2022).

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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How to cite this article: Jones, M. C. (2024). Duals of convolution thinned relationships. *Statistica Neerlandica*, 1–6. https://doi.org/10.1111/stan.12337