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Communication

On the general position number of Mycielskian graphs

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ABSTRACT

The general position problem for graphs was inspired by the no-three-in-line problem from discrete geometry. A set S of vertices of a graph G is a *general position set* if no shortest path in G contains three or more vertices of S. The *general position number* of G is the number of vertices in a largest general position set. In this paper we investigate the general position numbers of the Mycielskian of graphs. We give tight upper and lower bounds on the general position number of the Mycielskian of a graph G and investigate the structure of the graphs meeting these bounds. We determine this number exactly for common classes of graphs, including cubic graphs and a wide range of trees.

1. Introduction

A *graph* G consists of a set of *vertices* $V(G)$ connected by *edges*, which are unordered pairs of vertices. All graphs used here are simple, connected and finite. We write $u \sim v$ if the vertices $u, v \in V(G)$ are adjacent. A *path* $P_{\ell+1}$ of length ℓ in G is a sequence $v_0, v_1, \dots, v_\ell$ of distinct vertices such that v_iv_{i+1} is an edge for $0 \leq i \leq \ell - 1$. If P is a path $v_0, v_1, \dots, v_\ell$, then the *reverse path* $\tilde{P}$ of P is the path $v_\ell, \dots, v_1, v_0$. The *distance* $d_G(u, v)$ between vertices u and v of G is the length of a shortest path from u to v. A shortest path from u to v is also called a u, v-*geodesic*. The *neighbourhood* $N_G(u)$ of $u \in V(G)$ is the set $\{v \in V(G) : v \sim u\}$ of all vertices adjacent to u. More generally, for $t \geq 0$, $N^t(u)$ is the set of all vertices at distance t from u, so that $N_G(u) = N^1(u)$. The *degree* $\deg(u)$ is the number $|N_G(u)|$ of neighbours of u. A *leaf* of G is a vertex with degree one in G and we define the *leaf number* $\ell(G)$ to be the number of leaves in G. A vertex adjacent to a leaf is a *support vertex*. A vertex of degree $n - 1$ in a graph with order n is *universal*.

A cycle C_ℓ of length ℓ is a sequence of vertices $v_0, v_1, \dots, v_{\ell-1}$ such that $v_i \sim v_{i+1}$ for $0 \leq i \leq \ell - 2$ and also $v_0 \sim v_{\ell-1}$. A connected graph with no cycles is a *tree*. The *girth* $g(G)$ of G is the length of a shortest cycle in G (if G is acyclic, then we adopt the convention that $g(G) = \infty$). For $U \subseteq V(G)$ we write the subgraph induced by U as $\langle U \rangle$. An independent set in G is a set of mutually non-adjacent vertices, i.e. a set of vertices that induces an empty subgraph of G, and the cardinality $\alpha(G)$ of a largest independent set in G is the *independence number* of G. By contrast, a *clique* in a graph is a set of mutually adjacent vertices. The complete graph K_n is the graph with order n such that every pair of vertices is adjacent, i.e. the whole vertex set is a clique. A *matching* M in G is an independent set of edges, i.e. if e_1, e_2 are two distinct edges of M, then e_1 and e_2 do not have an endvertex in common. The size of the largest matching in G is the *matching number* of G

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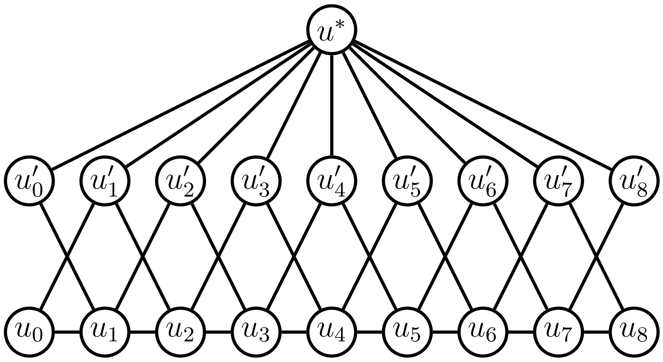

Fig. 1.1. The Mycielskian of a path of length eight.

and is denoted by $\nu(G)$. For any subsets $V_1, V_2 \subseteq V(G)$ we denote the set of edges of G from V_1 to V_2 in G by (V_1, V_2). For any graph-theoretical terminology not defined here we take [6] as our standard.

The *general position problem* originated in Dudeney's no-three-in-line problem and the general position subset selection problem from discrete geometry [11,13,23]. The general position problem was generalised to graph theory independently in [7,20] as follows. A set S of vertices in a graph G is a *general position set*, or is in *general position*, if no shortest path in G contains three or more vertices of G. A largest general position set of G is called a *gp-set* and its cardinality $\mathrm{gp}(G)$ is the *general position number* of G (or *gp-number* for short). The structure of general position sets was characterised in [2]. In particular, it was shown in this article that any general position set induces an *independent union of cliques*, i.e. a disjoint union of one or more cliques $W_1, \ldots, W_t$, $t \geq 1$, with no edges between cliques W_i and W_j if $1 \leq i < j \leq t$. These cliques must also have the *distance-constant* property, meaning that for any cliques W_i and W_j in this collection and any vertices $w_1, w'_1 \in W_i$, $w_2, w'_2 \in W_j$, we have $d_G(w_1, w_2) = d_G(w'_1, w'_2)$.

We make use of two variants of general position sets. The largest number of vertices in a general position set that is also an independent set is called the *independent position number* $\mathrm{ip}(G)$ and was studied in [27]. The article [17] defined a *general d-position set* to be a subset $S \subseteq V(G)$ such that no shortest path of length at most d contains three or more vertices of S. The largest number of vertices in a general d-position set is denoted by $\mathrm{gp}_d(G)$. Other types of position sets have also been investigated, including *monophonic position sets* [28], *mutual visibility sets* [8], *edge general position sets* [21], *Steiner position sets* [16], *mobile position sets* [15], *vertex position sets* [26], *lower general position sets* [9] and others. Some of the most recent papers on this subject include [3,18,19,24,29].

In this paper we discuss the general position numbers of the Myclieskian of a graph. This construction was introduced by Mycielski in [22] in order to produce triangle-free graphs with arbitrarily large chromatic number. Properties of the Mycielskian construction have been investigated extensively, for example dominator colouring number [1], connectivity [5], energy [4], $L(2, 1)$-labelling number [10], circular chromatic number [12], Gromov hyperbolicity [14], wide diameter [25] and Italian domination number [30], amongst many others.

Let G be a graph with vertex set $V = \{u_1, \ldots, u_n\}$ and edge set E. The Mycielskian of G is the graph $\mathcal{M}(G)$ defined as follows. The vertex set of $\mathcal{M}(G)$ is $V \cup V' \cup \{u^*\}$, where $V = \{u_1, \ldots, u_n\}$ is the vertex set of G, $V' = \{u' : u \in V\}$ is a copy of V and u^* is an additional vertex called the *root vertex*. The edge set of $\mathcal{M}(G)$ is $E \cup \{uv' : uv \in E\} \cup \{v'u^* : v' \in V'\}$. In other words, the edge set of $\mathcal{M}(G)$ consists of the edges of G, and for any $u \in V$ we join u' to the root u^* and to each of the neighbours of u in G. The Mycielskian of P_8 is shown in Fig. 1.1 and the Mycielskian of the cycle C_5 in Fig. 5.2.

We note that the Mycielskian is more often denoted by $\mu(G)$, but we avoid this notation here to avoid a clash with the notation used for the mutual visibility number, another position type invariant. We call the vertex u' the $\mathcal{M}$*-twin* of u (and conversely u is the $\mathcal{M}$-twin of u'). For any set $X = \{x_1, \ldots, x_t\}$ of vertices of G we denote the set of $\mathcal{M}$-twins of the vertices in X by $X' = \{x'_1, \ldots, x'_t\}$.

The plan of this paper is as follows. In Section 2 we characterise general position sets of $\mathcal{M}(G)$ that contain the root vertex u^*. In Section 3 we describe general position sets of $\mathcal{M}(G)$ in terms of a 'dual' partition of $V(G)$ into four parts. In Section 4 we give upper and lower bounds for $\mathrm{gp}(\mathcal{M}(G))$ and characterise the graphs that meet our upper bound. We also give exact values of $\mathrm{gp}(\mathcal{M}(G))$ for some common families of graphs. Section 5 presents a bound on the gp-number of the Mycielskian of regular graphs, which we use to classify the gp-numbers of Mycielskians of cubic graphs and all sufficiently large regular graphs. Finally in Section 6 we determine the gp-numbers of the Mycielskians of a wide class of trees.

2. General position sets of $\mathcal{M}(G)$ containing the root u^*

Firstly, we introduce a convention that we use throughout the paper. Given a set S of vertices of a graph, we call a geodesic containing at most two vertices of S *sound* and a shortest path containing at least three vertices of S *unsound*.

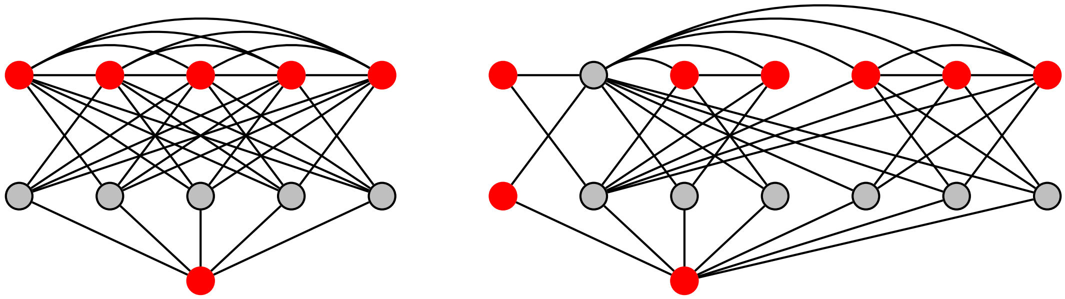

Fig. 2.1. General position sets (in red) used in the proof of Lemma 1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

It is useful to disregard gp-sets of $\mathcal{M}(G)$ that contain the root vertex. In this section we classify the graphs such that $\mathcal{M}(G)$ has a general position set of cardinality at least $n+1$ that does not contain u^*. We also show that for any graph that is not a complete graph there is always a gp-set of $\mathcal{M}(G)$ that does not contain the root u^*.

Lemma 1. *Let $G = (V, E)$ be a connected graph with order $n \geq 3$. Then there is a general position set S of $\mathcal{M}(G)$ with $|S| \geq n+1$ that contains u^* if and only if G is a complete graph K_n or the join of K_1 with a disjoint union of cliques, at least one of which is K_1.*

Proof. Suppose that S is a gp-set of $\mathcal{M}(G)$ with $|S| \geq n+1$ that contains the root u^*. Firstly, since $S \cap V(G)$ cannot contain an induced path P_3, the subgraph $\langle S \cap V(G) \rangle$ of G induced by $S \cap V(G)$ must be an independent union of cliques. Also, S can contain at most one vertex of V', since any two vertices of V' are connected by a shortest path through u^*.

Suppose that there is a vertex $u \in V(G)$ such that $u, u' \in S$. By our previous observation, u' is the only vertex of V' in S. Also, no neighbour of u in G lies in $S \cap V(G)$, since if $u \sim v$, then u, v, u' would be an unsound path. We conclude that $|S| \leq n + 2 - \deg_G(u)$. Hence u is a leaf of G and $S \cap V(G) = V(G) \setminus \{v\}$, where v is the support vertex of u. Let W be any clique of $\langle S \cap V(G) \rangle$ apart from the leaf u. As G is connected, there must an edge from some vertex w of W to a vertex outside W; as u is adjacent only to v and $\langle S \cap V(G) \rangle$ is an independent union of cliques, it must be that $w \sim v$. Since $d_{\mathcal{M}(G)}(u, w) = 2$, by the distance-constant property u must be at distance two to every vertex of W. Thus v is a universal vertex and G has the claimed structure. It is easily verified that $\{u^*, u'\} \cup (V(G) \setminus \{v\})$ is in general position. An example is shown on the right of Fig. 2.1 .

Now suppose that there is no vertex u such that $u, u' \in S$. Then we have $|S| = n + 1$ and for any vertex v of G, S contains exactly one of v and v'. If $S \cap V' = \emptyset$, then $V(G) \subset S$ and by our earlier observation G is a clique, in which case $V(G) \cup \{u^*\}$ is in general position. Otherwise S contains a vertex $w' \in V'$ and so $S = \{u^*, w'\} \cup (V(G) \setminus \{w\})$. However, as G is connected, w has a neighbour z in G, and u^*, w', z would be an unsound path, a contradiction. ■

Notice that the general position sets exhibited in the proof of Lemma 1 are not necessarily largest possible. In Corollary 16 we determine the gp-numbers of the Mycielskian of the join of K_1 with a disjoint union of cliques, thereby classifying those graphs for which the Mycielskian has a gp-set containing u^*.

Corollary 2. *For any graph with order $n \geq 3$, the root vertex u^* of $\mathcal{M}(G)$ belongs to every gp-set of $\mathcal{M}(G)$ if and only if G is isomorphic to K_n. For any integer $n \geq 3$, $\text{gp}(\mathcal{M}(K_n)) = n+1$ and the unique gp-set of $\mathcal{M}(K_n)$ is $V \cup \{u^*\}$.*

Proof. If $\text{gp}(\mathcal{M}(G)) = n$, then V' is a gp-set of $\mathcal{M}(G)$ that does not contain u^*, so we can confine our attention to graphs G with $\text{gp}(\mathcal{M}(G)) \geq n+1$. By Lemma 1, if a graph is such that every gp-set of $\mathcal{M}(G)$ contains u^*, then G is either a complete graph or the join of K_1 with an independent union of cliques, at least one of which consists of a single vertex (moreover, the general position sets exhibited in the proof would have to be gp-sets). If G has the latter structure and u is a leaf with universal support vertex, then $V' \cup \{u\}$ is a general position set with the same order. Thus, whether or not the general position sets containing u^* are gp-sets, we see that u^* is not contained in every gp-set of such a graph.

Now suppose that G is a complete graph. It follows from the proof of Lemma 1 that the only general position set of $\mathcal{M}(G)$ containing u^* is $V(G) \cup \{u^*\}$, which has cardinality $n+1$. Suppose that S is a general position set of $\mathcal{M}(G)$ that does not contain u^*. Then there must be a vertex u of K_n such that $u, u' \in S$. Now if v is any vertex of K_n other than u, the path u, v, u' is a geodesic, so $v \notin S$. It follows that $S = V' \cup \{u\}$. However, in this case if $v, w \in V(K_n) \setminus \{u\}$, the path v', u, w' would be unsound, a contradiction. Thus $V(G) \cup \{u^*\}$ is the unique gp-set of $\mathcal{M}(G)$. ■

The gp-set for $\mathcal{M}(K_5)$ is shown on the left of Fig. 2.1. Notice that $\text{gp}(\mathcal{M}(K_n)) = n+1$ is true for $n = 2$, as $\mathcal{M}(K_2) \cong C_5$, but the root vertex u^* is not contained in every gp-set.

It follows from Lemma 1 that the general position sets containing u^* can easily be found in polynomial time and that for any non-complete graph G there is a gp-set of $\mathcal{M}(G)$ that does not contain. Thus in the remainder of this article we can safely ignore gp-sets containing u^*.

3. General position sets of $\mathcal{M}(G)$ in terms of partitions of $V(G)$

In this section we demonstrate a duality between general position sets of $\mathcal{M}(G)$ that do not contain u^* and certain partitions of $V(G)$ into four (possibly empty) parts. This will allow us to calculate $\mathrm{gp}(\mathcal{M}(G))$ working entirely inside G. First we observe the connection between shortest paths in the base graph G and the shortest paths in the Mycielskian $\mathcal{M}(G)$.

Definition 3. The *projection* of a path $P = u_0, u_1, \ldots, u_\ell$ in $V(\mathcal{M}(G)) \setminus \{u^*\}$ is the path with all vertices of P that lie in V' replaced by the corresponding $\mathcal{M}$-twin vertices in V. For example in Fig. 1.1 the projection of the path $u_0, u_1', u_2, u_3, u_4'$ is u_0, u_1, u_2, u_3, u_4. If Q is a path in $V(G)$, then an *expansion* of Q is any path in $\mathcal{M}(G)$ with projection equal to Q.

Observation 4. *The shortest paths in $\mathcal{M}(G)$ between vertices of $V(\mathcal{M}(G)) \setminus \{u^*\}$ have the following form.*

- *If $u, v \in V(G)$ and $d_G(u, v) \leq 3$, then a u, v-path in $\mathcal{M}(G)$ is a shortest path in $\mathcal{M}(G)$ if and only if it is an expansion of a shortest path in G.*
- *If $d_G(u, v) \geq 4$ and P is a shortest u, v-path in $\mathcal{M}(G)$, then either $d_G(u, v) = 4$ and P is an expansion of a u, v-path of length four in G, or else P has the form u, w_1', u^*, w_2', v, where $u \sim w_1$ and $v \sim w_2$ in G.*
- *If $u \neq v$, then the shortest u, v'-paths in $\mathcal{M}(G)$ are either expansions of shortest u, v-paths in G or have the form u, w', u^*, v', where $u \sim w$ in G.*
- *All shortest u, u'-paths in $\mathcal{M}(G)$ have the form u, w, u', where $w \sim u$ in G.*

With any general position set S of $\mathcal{M}(G)$ that does not contain u^* we associate the partition $\pi(S) = (V_1, V_2, V_3, V_4)$ of $V(G)$ where:

- $V_1 = \{u \in V(G) : u, u' \in S\}$,
- $V_2 = \{u \in V(G) : u \notin S, u' \in S\}$,
- $V_3 = \{u \in V(G) : u \in S, u' \notin S\}$, and
- $V_4 = \{u \in V(G) : u, u' \notin S\}$.

Then we have $\mathrm{gp}(\mathcal{M}(G)) = n + n_1 - n_4$, where n is the order of the graph and $n_i = |V_i|$ for $i = 1, 2, 3, 4$. We now identify the essential properties of such a partition that guarantee it has the form $\pi(S)$, where S is a general position set of $\mathcal{M}(G)$. We call such a partition an *$\mathcal{MGP}$-partition* (short for 'Mycielskian general position').

Definition 5. A partition $\pi = (V_1, V_2, V_3, V_4)$ of $V(G)$ into four (possibly empty) sets is an *$\mathcal{MGP}$-partition* if and only if it satisfies the following three conditions:

1. if e is an edge in $\langle V_1 \cup V_3 \rangle$, then e has both endpoints in V_3,
2. if a vertex u in $V_1 \cup V_3$ has a neighbour v in V_2, then $d_G(u, w) = 2$ for all $w \in (V_1 \cup V_2) \setminus \{u, v\}$ and $d_G(u, w) \leq 3$ for all $w \in V_3 \setminus \{u\}$, and
3. if P is a shortest path $u_0, u_1, \ldots, u_\ell$ with length $\ell \leq 4$ in G that passes through three or more vertices of $V_1 \cup V_2 \cup V_3$, then either $u_0, u_\ell \in V_2$ and $\ell \geq 3$, or else P or $\widetilde{P}$ has one of the following forms:
   (a) u_0, u_1, u_2, where $u_0, u_1 \in V_2$,
   (b) u_0, u_1, u_2, u_3, where $u_0, u_1 \in V_2$, $u_2 \notin V_1 \cup V_2 \cup V_3$ and $u_3 \in V_1 \cup V_3$, or
   (c) u_0, u_1, u_2, u_3, u_4, where $u_0 \in V_2$, $u_3 \notin V_1 \cup V_2$ and $u_4 \in V_1 \cup V_3$.

Note the following three important properties of $\mathcal{MGP}$-partitions.

Lemma 6. *If (V_1, V_2, V_3, V_4) is an $\mathcal{MGP}$-partition of G, then V_1 is an independent set of G, $(V_1, V_3) = \emptyset$ and (V_1, V_2) is a matching.*

Proof. That V_1 is independent and $(V_1, V_3) = \emptyset$ follows immediately from Condition 1 in Definition 5. We now show that (V_1, V_2) is a matching. By Condition 2 of Definition 5, a vertex $u \in V_1$ can be at distance one from at most one vertex of V_2, so each vertex of V_1 has at most one neighbour in V_2. Suppose now that there is a vertex $w \in V_2$ with distinct neighbours $u, v \in V_1$. As V_1 is independent, u, w, v would be a shortest path in G of the form forbidden by Condition 3. It follows that the edges in (V_1, V_2) are independent. ■

Given an $\mathcal{MGP}$-partition $\pi = (V_1, V_2, V_3, V_4)$, we associate the subset $\sigma(\pi)$ given by $V_1 \cup V_1' \cup V_2' \cup V_3$.

Lemma 7. *If S is a general position set of $\mathcal{M}(G)$ that does not contain u^*, then $\pi(S)$ is an $\mathcal{MGP}$-partition.*

Proof. We verify the three properties from Definition 5 in turn.

Condition 1: If $u \in V_1$, $v \in V_1 \cup V_3$ and $u \sim v$ in G, then the shortest path u, v, u' in $\mathcal{M}(G)$ would be unsound. Thus any edge of G in $\langle V_1 \cup V_3 \rangle$ has both endpoints in V_3.

Condition 2: Suppose that a vertex $u \in V_1 \cup V_3$ has a neighbour $v \in V_2$. By Condition 1, u has no neighbours in V_1. Furthermore, if u had a second neighbour $v_2 \neq v$ in V_2, then the path v', u, v_2' would be unsound. Thus $d_G(u, w) \geq 2$ for each $w \in (V_1 \cup V_2) \setminus \{u, v\}$. If there is a $w \in (V_1 \cup V_2) \setminus \{u, v\}$ such that $d_G(u, w) \geq 3$, then u, v', u^*, w' would be an unsound geodesic. Therefore $d_G(u, w) = 2$ for all $w \in (V_1 \cup V_2) \setminus \{u, v\}$. Also if $w \in V_3 \setminus \{u\}$ is such that $d_G(u, w) \geq 4$, then u, v', u^*, w_1', w would be an unsound path, where w_1 is any neighbour of w in G.

Condition 3: Suppose that P is a geodesic $u_0, u_1, \ldots, u_\ell$ in G with length $\ell \leq 4$ that has both endpoints and an internal vertex in $V_1 \cup V_2 \cup V_3$. If P has both endpoints in $V_1 \cup V_3$, then P has an unsound expansion that is a geodesic in $\mathcal{M}(G)$, so we can assume that the initial vertex u_0 of P is in V_2. Suppose firstly that $\ell = 2$. Then we must have $u_1 \in V_2$, for if $u_1 \in V_1 \cup V_3$, then u_0', u_1, u_2 is unsound if $u_2 \in V_1 \cup V_3$ and u_0', u_1, u_2' is unsound if $u_2 \in V_2$.

Now suppose that $\ell = 3$ and $u_3 \in V_1 \cup V_3$. As u_0', u_1, u_2, u_3 and u_0', u^*, u_2', u_3 are shortest paths in $\mathcal{M}(G)$, we see that $u_1, u_2 \notin V_1 \cup V_3$ and $u_2 \notin V_2$, so that P must have the form in variant (b) of Condition 3. Lastly, if $\ell = 4$ and $u_4 \in V_1 \cup V_3$, then u_0', u^*, u_3', u_4 is a shortest path in $\mathcal{M}(G)$, so that $u_3 \notin V_1 \cup V_2$ and P has the form of variant (c). ■

We now complete our characterisation of gp-sets of Mycielskians of graphs.

Theorem 8. *Let $\pi = (V_1, V_2, V_3, V_4)$ be an $\mathcal{MGP}$-partition of G. Then $\sigma(\pi) = V_1 \cup V_1' \cup V_2' \cup V_3$ is a general position set of $\mathcal{M}(G)$.*

Proof. We need to verify that all shortest paths between vertices of $S = \sigma(\pi)$ are sound. First let $u, v \in V_1 \cup V_3$. By Condition 3 of Definition 5, no expansion of a shortest u, v-path in G with length at most four contains a third vertex of S. Furthermore, if $d_G(u, v) \geq 4$, then by Observation 4 the only other geodesics to consider are the paths of the form u, w_1', u^*, w_2', v, where $u \sim w_1$ and $v \sim w_2$ in G. By Condition 1 of Definition 5 neither of the vertices w_1 or w_2 is in V_1. Hence we can assume that $w_1 \in V_2$. Then by Condition 2 we would have $d_G(u, v) \leq 3$, so that u, w_1', u^*, w_2', v would not be a shortest path.

Now suppose that $u', v' \in V_1' \cup V_2'$. A shortest u', v'-path P in $\mathcal{M}(G)$ has length two and P is unsound only if it passes through a common neighbour of u and v in $V_1 \cup V_3$. By Condition 2 and Lemma 6 this is impossible.

We can thus assume that P is an unsound geodesic with one endpoint in $V_1 \cup V_3$ and the other in $V_1' \cup V_2'$. If $u \in V_1$, the u, u'-geodesics in $\mathcal{M}(G)$ are the paths u, w, u', where $u \sim w$ in G. By Lemma 6 the vertex w does not lie in S. Hence the endpoints of P are not $\mathcal{M}$-twins of each other. Let us denote the endpoints by u, v', where $u \in V_1 \cup V_3$, $v \in V_1 \cup V_2$ and $u \neq v$. If $\ell(P) = 2$, then by Condition 1 of Definition 5 we must have $u \in V_3$, $v \in V_2$ and the internal vertex of P lies in V_3. However, this form of shortest path is not allowed by Condition 3.

Hence $\ell(P) = 3$. By Observation 4, the u, v'-geodesics in $\mathcal{M}(G)$ are either expansions of shortest paths of length three in G, or else have the form u, w', u^*, v', where $u \sim w$ in G. By Condition 3 the former geodesics are sound and the latter are sound by Condition 2. This shows that all shortest paths in $\mathcal{M}(G)$ are sound and thus S is in general position. ■

Since both maps π and σ are one-to-one, this completes the proof of the claimed duality between general position sets of $\mathcal{M}(G)$ not containing u^* and $\mathcal{MGP}$-partitions of G.

Corollary 9. *Let G be a non-complete graph with order n. Then $\mathrm{gp}(\mathcal{M}(G))$ is the largest value of $n + n_1 - n_4$ over all $\mathcal{MGP}$-partitions (V_1, V_2, V_3, V_4) of G, where $n_i = |V_i|$ for $1 \leq i \leq 4$.*

4. Bounds, extremal cases and exact values

In this section we use Corollary 9 to derive tight upper and lower bounds for $\mathrm{gp}(\mathcal{M}(G))$ and characterise the case of equality with the upper bound. We start with the lower bound. Recall that $\mathrm{ip}(G)$ is the number of vertices in a largest independent set that is also in general position and $\mathrm{gp}_d(G)$ is the number of vertices in a largest subset $S \subseteq V(G)$ such that no shortest path with length at most d passes through three or more vertices in S. To derive bounds for $\mathrm{gp}(\mathcal{M}(G))$ we combine these concepts in the following definition.

Definition 10. An *independent d-position set* is a subset $S \subseteq V(G)$ such that S is an independent set and no shortest path of length at most d passes through three or more vertices of S. We denote that largest number of vertices in an independent d-position set by $\mathrm{ip}_d(G)$.

For any graph G these parameters satisfy the inequalities $\mathrm{ip}(G) \leq \mathrm{gp}(G)$ and $\mathrm{ip}(G) \leq \mathrm{ip}_4(G) \leq \alpha(G)$.

Corollary 11. *For any graph G with order $n \geq 3$,*

$$\mathrm{gp}(\mathcal{M}(G)) \geq \max\{n, 2\mathrm{ip}_4(G)\}.$$

Proof. The set V' is trivially in general position, implying that $\mathrm{gp}(\mathcal{M}(G)) \geq n$ (this is equivalent to using the $\mathcal{MGP}$-partition $V_2 = V(G)$). Let A be a largest independent 4-position set of G. Setting $V_1 = A$ and $V_4 = V(G) \setminus A$ gives an $\mathcal{MGP}$-partition of G, yielding $\mathrm{gp}(\mathcal{M}(G)) \geq 2\mathrm{ip}_4(G)$ by Corollary 9. ■

We call any non-complete graph G with order $n \geq 3$ and $\mathrm{gp}(\mathcal{M}(G)) = \max\{n, 2\mathrm{ip}_4(G)\}$ *meagre*. Otherwise G is *abundant*. We now show that the lower bound in Corollary 11 is tight by exhibiting two families of meagre graphs: graphs with large size, joins of K_1 with disjoint unions of cliques of cardinality at least three, and complete multipartite graphs.

Lemma 12. *If G is an abundant graph with order $n \geq 3$ and $V_4 = \emptyset$, then $V_3 = \emptyset$, $|V_1| = 1$ and the vertex of V_1 is a leaf with support vertex that is a universal vertex. If G has a leaf with a universal support vertex, then* $\mathrm{gp}(\mathcal{M}(G)) \geq n + 1$.

Proof. Suppose that G is abundant and has $\mathcal{MGP}$-partition $(V_1, V_2, V_3, \emptyset)$, where $n_1 \geq 1$. Suppose that $V_3 \neq \emptyset$. As we are dealing with connected graphs, each vertex of V_1 has a path to V_3. Therefore, by Lemma 6, the matching (V_1, V_2) saturates V_1. Then by Condition 2 of Definition 5 we have $d(u, w) \leq 3$ for each $u \in V_1$ and $w \in V_3$. Therefore there is a shortest path from some $u \in V_1$ to a $w \in V_3$ with length two or three with all internal vertices in V_2. However, the existence of such a path contradicts Condition 3 of Definition 5. Thus $V_3 = \emptyset$ and $V(G) = V_1 \cup V_2$. Thus each vertex of V_1 is a leaf.

By Condition 2 of Definition 5 any vertex of V_1 is at distance two from every vertex of V_2, with the exception of its support vertex. Therefore any pair of vertices from V_1 are at distance three from each other, with shortest path passing through V_2. This contradicts Condition 3 of Definition 5 unless $n_1 = 1$. By Condition 2 of Definition 5 it follows that the support vertex of the vertex of V_1 is universal.

Conversely, if $u \in V(G)$ is a leaf with a universal support vertex, then $(\{u\}, V(G) \setminus \{u\}, \emptyset, \emptyset)$ is an $\mathcal{MGP}$-partition and $n + n_1 - n_4 = n + 1$. ■

Theorem 13. *For $n \geq 3$, the largest size of a graph with order n that satisfies $\mathrm{gp}(\mathcal{M}(G)) > n$ is $\binom{n}{2} + 1$. Any abundant graphs with size $\binom{n}{2} + 1$ are either a clique K_{n-1} with a leaf attached, the gem graph $K_1 \vee P_4$, or $3K_1 \vee K_2$. The latter graph is not abundant, for the former two are.*

Proof. Let G be an abundant graph with order n and size at least $\binom{n}{2} + 1$. Let (V_1, V_2, V_3, V_4) be an $\mathcal{MGP}$-partition corresponding to a gp-set of $\mathcal{M}(G)$. By Lemma 6 each vertex u of V_1 has degree at most $1 + n_4$ if u has an edge to V_2 and degree at most n_4 otherwise. By Lemma 12, if $n_1 \geq 2$, then $n_1 > n_4 > 0$.

Suppose that $n_1 \geq 3$ and let $u_1, u_2, u_3 \in V_1$. Let $t = |(\{u_1, u_2, u_3\}, V_2)|$. Then, taking care not to count missing edges between u_1, u_2 and u_3 twice, we see that there are at least

$$3(n - n_4 - 1) - t - 3 = (n - 2) + (2n - 3n_4 - t - 4)$$

missing edges in G. As $n_2 \geq t$ and $n_1 > n_4$, we have

$$2n - 3n_4 - t - 4 = 2n_1 + 2n_2 + 2n_3 - n_4 - t - 4 \geq n_1 + n_2 + 2n_3 - 3 \geq 0.$$

Moreover, for equality to hold, we would need $n_1 = 3, n_2 = n_3 = 0, n_4 = 2$. As all other edges must be present, G is isomorphic to $3K_1 \vee K_2$ and V_1 is an independent 4-position set, so that we would have $\mathrm{gp}(\mathcal{M}(G)) = 2\mathrm{ip}_4(G)$ and G would not be abundant.

Now suppose that $n_1 = 2$, where $V_1 = \{u_1, u_2\}$. Hence $n_4 = 1$ and $n = n_2 + n_3 + 3$. Set $t = |(V_1, V_2)|$. There are at least $2n_2 + 2n_3 + 1 - t$ missing edges with endpoints in V_1. We have

$$2n_2 + 2n_3 + 1 - t = (n - 2) + (n_2 + n_3 - t) \geq n - 2,$$

since $n_2 \geq t$. To have equality, we must have $n_2 = t$ and $n_3 = 0$. As all other edges must be present, if $t = 0, 1$, then G is a clique with a leaf attached. If $t = 2$, then G is isomorphic to the gem graph, which is readily verified to be abundant, since its independence number is two. A gp-set of the Mycielskian of the gem graph is shown in Fig. 4.1.

Finally if $n_1 = 1$, then we must have $n_4 = 0$ and Lemma 12 tells us that the vertex of V_1 is a leaf. The edges missing to the leaf account for all $n - 2$ missing edges, so G is a clique with a leaf attached as claimed. ■

Theorem 14. *Let $K_{r_1,r_2,\ldots,r_k}$ be a complete k-partite graph, where $2 \leq k < n = r_1 + \cdots + r_k$ and $r_1 \geq r_2 \geq \cdots \geq r_k$. Then*

$$\mathrm{gp}(\mathcal{M}(K_{r_1,r_2,\ldots,r_k})) = \max\{n, 2r_1\}.$$

Proof. Since any partite set of $G = K_{r_1,r_2,\ldots,r_k}$ is an independent position set, by Corollary 11 we have $\mathrm{gp}\mathcal{M}(G) = \max\{n, 2r_1\}$. Suppose for a contradiction that (V_1, V_2, V_3, V_4) is an $\mathcal{MGP}$-partition of G with $n_i = |V_i|$ for $i = 1, 2, 3, 4$ corresponding to a gp-set of $\mathcal{M}(G)$ containing more than $\max\{n, 2r_1\}$ vertices.

As V_1 is an independent set, it must be contained entirely within some partite set X of G. By Lemma 6 there are no edges from V_1 to V_3, so we have $V_1 \cup V_3 \subseteq X$. If V_2 lies entirely in X, then $\mathrm{gp}(\mathcal{M}(K_{r_1,r_2,\ldots,r_k})) = 2n_1 + n_2 + n_3 \leq 2r_1$, so assume that there is a vertex $y \in V_2 \setminus X$. Then the vertex y is adjacent to every vertex of X. As (V_1, V_2) is a matching by Lemma 6, we must have $n_1 = 1$ and hence $n_4 = 0$. It now follows from Lemma 12 that the vertex of V_1 is a leaf, so there is just one vertex of G outside X and hence G is a star, i.e. a tree of order n with a universal vertex. The associated general position set of $\mathcal{M}(G)$ contains $n + 1$ vertices. However, for $n \geq 3$ this would be smaller than $2r_1 = 2(n - 1)$, a contradiction. It follows that the graph is meagre. ■

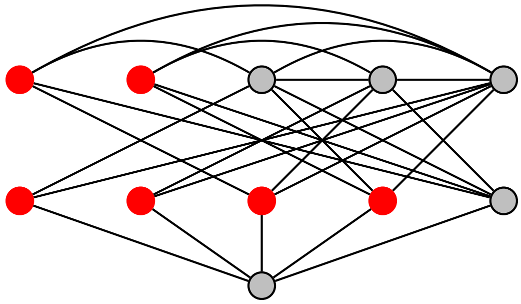

Fig. 4.1. A gp-set (in red) of the Mycielskian of the gem graph. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Theorem 15. *Let G be the join of K_1 with a disjoint union of $t \geq 2$ cliques $W = \dot{\bigcup}_{i=1}^{t} W_i$. Suppose that W contains t_1 cliques of order at most two and t_2 cliques of order at least three. Then* $\mathrm{gp}(\mathcal{M}(G)) = n + t_1$.

Proof. Let (V_1, V_2, V_3, V_4) be an $\mathcal{MGP}$-partition corresponding to a gp-set S of $\mathcal{M}(G)$ not containing the root u^*. It follows from Lemma 1 that $|S| \geq n + 1$, so we can assume that $n_1 > n_4$. Let u be the universal vertex of G. If $u \in V_3$, then by Definition 5 V_1 is empty, contradicting $|S| > n$. If $u \in V_1$, then by Condition 1 of Definition 5 there are no further vertices in $V_1 \cup V_3$. By Condition 3, there cannot be vertices of V_2 in two different cliques of W, so V_4 is non-empty and we would have $|S| \leq n$, a contradiction. If $u \in V_2$, then, assuming that W_1 contains a vertex v of V_1, Definition 5 shows that G contains no further vertices of V_1, so $V_4 = \emptyset$ and Lemma 12 shows that $|W_1| = 1$ and $|S| = n + 1$. Hence we can assume that $u \in V_4$.

Any clique W_i contains at most one vertex of V_1, so can contribute at most one to the difference $n_1 - n_4$. By Definition 5, if any of the t_2 cliques with order at least three contains a vertex of V_1, then it must also contain a vertex of V_4, and so cannot contribute to $n_1 - n_4$. Hence $|S| \leq n + t_1$. Conversely, setting $V_4 = \{u\}$, choosing one vertex from each of the t_1 cliques of order at most two to be in V_1 and setting all other vertices of W to belong to V_2 gives an $\mathcal{MGP}$-partition. Comparing the cardinality of the general position sets obtained, including those containing the root u^* from Lemma 1, we conclude that $\mathrm{gp}(\mathcal{M}(G)) = n + t_1$. ■

Therefore we obtain a family of meagre graphs by letting all of the cliques in W have order at least three.

Corollary 16. *A graph G has a gp-set containing the root vertex u^* if and only if G is a clique or G is the join of K_1 with a disjoint union of $t \geq 2$ cliques $W_1, \ldots, W_t$, where one of the cliques has order one and all of the other cliques have order at least three.*

Later in the paper we will see other examples of meagre graphs: paths (Lemma 31), all cycles apart from C_3 and C_5 (Theorem 23), all cubic graph with two exceptions (Theorem 24) and in general all sufficiently large regular graphs (Theorem 25). It also follows from Corollary 18 that for $n \geq 2$ the star graph S_n is meagre (in fact for this graph the bounds in Corollaries 11 and 18 coincide). We now derive a tight upper bound on $\mathrm{gp}(\mathcal{M}(G))$ in terms of the independence number $\alpha(G)$.

Theorem 17. *For any non-complete graph G with order n and minimum degree $\delta \geq 1$ we have*

$$\mathrm{gp}(\mathcal{M}(G)) \leq n + \max\{0, \mathrm{ip}_4(G) - \delta + 1\}. \quad (1)$$

Proof. Let (V_1, V_2, V_3, V_4) be the $\mathcal{MGP}$-partition associated with a gp-set S of $\mathcal{M}(G)$. If V_1 is empty, then $\mathrm{gp}(\mathcal{M}(G)) \leq n$, so assume that $V_1 \neq \emptyset$. It follows from the conditions in Definition 5 and Lemma 6 that V_1 is an independent 4-position set of G. Thus $n_1 \leq \mathrm{ip}_4(G)$. Also by Lemma 6, any vertex in V_1 has at most one neighbour in V_2 and hence has at least $\delta - 1$ neighbours in V_4. Thus $n_4 \geq \delta - 1$ and so $\mathrm{gp}(\mathcal{M}(G)) = n + n_1 - n_4 \leq n + \mathrm{ip}_4(G) - \delta + 1$. To meet this bound V_1 must be a largest independent 4-position set of G, which implies that $n_1 \geq 2$, and each vertex of V_1 must have a neighbour in V_2, so by Condition 2 of Definition 5 any pair of vertices in V_1 has a common neighbour in V_4, implying that $\delta \geq 2$. Thus if $\delta = 1$ we can improve the bound to $\mathrm{gp}(\mathcal{M}(G)) \leq n + \mathrm{ip}_4(G) - 1$. ■

Theorem 17 also gives an upper bound in terms of the more familiar independence number.

Corollary 18. *For any non-complete graph G,* $\mathrm{gp}(\mathcal{M}(G)) \leq n + \alpha - 1$.

We now characterise the graphs that meet these upper bounds.

Theorem 19. *An abundant graph G satisfies $\mathrm{gp}(\mathcal{M}(G)) = n + \mathrm{ip}_4(G) - \delta + 1$ if and only if $V(G)$ can be partitioned into three sets V_1, V_2 and V_4 such that*

- $|V_1| = |V_2| > |V_4| = \delta - 1$,
- *V_1 is an independent set,*
- *(V_1, V_2) is a matching,*
- *the vertices of V_2 have degree at least δ, and*
- *any vertex of V_2 that is not universal in $\langle V_2 \rangle$ has an edge to V_4.*

Proof. Suppose that G meets the upper bound in Theorem 17. Let (V_1, V_2, V_3, V_4) be an $\mathcal{MGP}$-partition associated with a gp-set of G. The proof of Theorem 17 shows that V_1 is a largest independent 4-position set, the matching (V_1, V_2) saturates V_1, each vertex of V_1 has degree δ and $n_4 = \delta - 1$. Write $V_1 = \{u_1, \dots, u_r\}$ and $V_2 = \{v_1, \dots, v_s\}$, where $s \geq r$ and $u_i \sim v_i$ for $1 \leq i \leq s$. Hence for $1 \leq i \leq r$ we have $N_G(u_i) = \{v_i\} \cup V_4$.

We have $\mathrm{ip}_4(G) \geq 2$ for any non-complete graph with order $n \geq 3$, so $n_1 \geq 2$. By Condition 2 of Definition 5, any pair of vertices in V_1 has a common neighbour in V_4, implying that $\delta \geq 2$. Thus V_4 is non-empty.

Consider the vertices of V_3. As V_1 is a maximum independent 4-position set and $(V_1, V_3) = \emptyset$, it follows for that any vertex w of V_3 there is a shortest path P of length four containing w and two vertices of V_1. Observe that any pair of vertices from V_1 are at distance two from each other, since they have a common neighbour in V_4. Also by Condition 2 of Definition 5 each vertex of V_1 is at distance at most three from any vertex of V_3. It follows that there is no such path P and $V_3 = \emptyset$. Similarly, if $s > r$ and v is a vertex of $\{v_{r+1}, \dots, v_s\}$, then there would have to be a shortest path in G of length four that contains v and two vertices of V_1. This is impossible, since the vertices of V_1 are at distance two apart and by Condition 2 of Definition 5 each vertex of V_1 is at distance two from v. Therefore $r = s$. ■

Corollary 20. *A non-complete graph G with order $n \geq 3$ has $\mathrm{gp}(\mathcal{M}(G)) = n + \alpha(G) - 1$ if and only if G is either a clique K_{n-1} with a leaf attached, or has the following structure:*

- $V(G) = V_{1,1} \cup V_{1,2} \cup V_2 \cup \{x\}$,
- *$V_{1,1} \cup V_{1,2}$ is an independent set,*
- *x is adjacent to every vertex of $V_{1,1} \cup V_{1,2}$, if $V_{1,1} \neq \emptyset$, then x is a universal vertex,*
- *$(V_{1,2}, V_2)$ is a perfect matching, and*
- *any vertex of V_2 that is not universal in $\langle V_2 \rangle$ is adjacent to x.*

Proof. It follows from the proof of Theorem 19 that if any vertex of V_1 has degree at least three, then

$$\mathrm{gp}(\mathcal{M}(G)) \leq n + \mathrm{ip}_4(G) - 3 + 1 \leq n + \alpha(G) - 2.$$

Hence we can assume that each vertex of V_1 has degree one or two. To have $\mathrm{gp}(\mathcal{M}(G)) = n + \alpha - 1$, either (a) V_1 is a maximum independent set and $n_4 = 1$, or (b) V_1 is an independent set of order $\alpha - 1$ and $V_4 = \emptyset$.

In case (b) we have $\alpha = 2$ and Lemma 12 tells us that there are vertices $u, v \in V(G)$ such that u is a leaf and v is a universal vertex, and $V_1 = \{u\}$, $V_2 = V(G) \setminus \{u\}$ and $V_3 = \emptyset$. Also, if there is any pair of vertices $w, z \in V_2 \setminus \{v\}$ that are non-adjacent, then the independence number of G would be at least three. Thus G is a clique K_{n-1} with a leaf attached, which meets the upper bound $n + \alpha - 1$ by Theorem 15.

Now consider option (b). As V_1 is a maximum independent set, every vertex of $V_2 \cup V_3 \cup V_4$ has an edge to V_1. By Definition 5 it follows that $V_3 = \emptyset$ and the matching (V_1, V_2) saturates V_2. Let x be the vertex of V_4. If $V_2 = \emptyset$, then each vertex of V_1 is a leaf attached to x and G is a star. The $n - 1$ leaves of the star form an independent 4-position set, so by Corollary 11 the star meets the upper bound $n + \alpha - 1$. Hence assume that $V_2 \neq \emptyset$.

We split V_1 into three parts $V_{1,1}$, $V_{1,2}$ and $V_{1,3}$, where

- $V_{1,1}$ are the vertices of V_1 adjacent only to x,
- $V_{1,2}$ are the vertices of V_1 with degree two, and
- $V_{1,3}$ are the vertices of V_1 adjacent only to a vertex V_2.

As the matching (V_1, V_2) saturates V_2, there is a perfect matching between $V_{1,2} \cup V_{1,3}$ and V_2. If $u \in V_{1,3}$, then there is no path of length two from u to any vertex of $V_1 \setminus \{u\}$, so that we would have $V_1 = \{u\}$ by Condition 2 of Definition 5. As $(V_{1,2} \cup V_{1,3}, V_2)$ is a perfect matching, V_2 would also have just one vertex and G must be the path P_3. Upon setting $V_2 = \emptyset$ and $|V_{1,1}| = 2$ in the statement of the corollary, P_3 has the claimed form. Hence we can now assume that $V_{1,3} = \emptyset$.

Suppose that there is a pair of vertices v_1, v_2 of V_2 such that $v_1 \not\sim v_2$, and let u_1, u_2 be the neighbours of v_1, v_2 in V_1 respectively. Then by Condition 2 of Definition 5 we have $d_G(u_1, v_2) = d_G(u_2, v_1) = 2$, so that both v_1 and v_2 are adjacent to x. It follows that any vertex $v \in V_2$ that is not a universal vertex in V_2 is adjacent to x. Finally, suppose that $V_{1,1} \neq \emptyset$. Then if any $v \in V_2$ is not adjacent to x, the shortest paths from x to $V_{1,1}$ would have length three and pass through $V_{1,2}$, which is not allowed by Condition 3 of Definition 5. Hence if $V_{1,1} \neq \emptyset$, then x is a universal vertex.

Conversely, if all of the above conditions are satisfied, then the partition is an $\mathcal{MGP}$-partition. ■

Notice that if $V_2 = \emptyset$ in the second family described in Corollary 20, then we obtain a star. Some examples of the constructions from Corollary 20 are shown in Fig. 4.2.

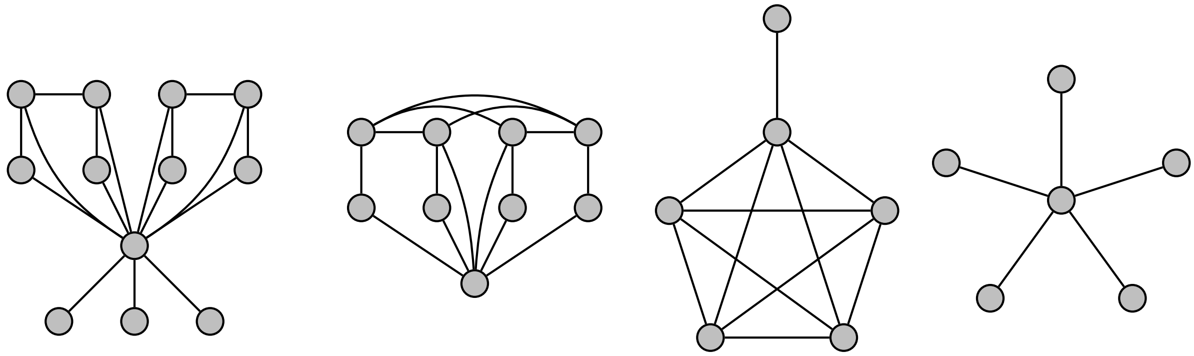

Fig. 4.2. Some graphs with $\mathrm{gp}(\mathcal{M}(G)) = n + \alpha - 1$.

5. Regular graphs

It was indicated in Section 4 that the regular graphs provide a rich source of meagre graphs. In this section we show that there is a strong upper bound on $\mathrm{gp}(\mathcal{M}(G))$ for regular graphs G and that for any d there are finitely many abundant d-regular graphs.

Theorem 21. *If G is a d-regular graph with order $n \geq 3$, then*

$$\mathrm{gp}(\mathcal{M}(G)) \leq n + \left\lfloor \frac{d-1}{2} + \frac{1}{d} \right\rfloor.$$

Proof. Let G be a d-regular graph, S be a gp-set of $\mathcal{M}(G)$ that does not include u^* and (V_1, V_2, V_3, V_4) be the corresponding $\mathcal{MGP}$-partition. Suppose that $(V_1, V_2) = \emptyset$. Then all neighbours of vertices in V_1 lie in V_4. This implies that $n_4 d \geq n_1 d$, yielding $n_4 \geq n_1$, so that $\mathrm{gp}(\mathcal{M}(G)) = n$.

Hence we may assume that $|(V_1, V_2)| = r > 0$. As (V_1, V_2) is a matching, there are $n_1 d - r$ edges in (V_1, V_4), implying that $n_4 \geq n_1 - \frac{r}{d}$. Thus

$$\mathrm{gp}(\mathcal{M}(G)) = n + n_1 - n_4 \leq n + \frac{r}{d}.$$

By Condition 2 of Definition 5, if $u \in V_1$ is adjacent to a vertex $v \in V_2$, then all vertices of $(V_1 \cup V_2) \setminus \{u, v\}$ lie in $N^2(u)$. There are at least $2r$ vertices in $V_1 \cup V_2$ and at most $d^2 - d$ vertices in $N^2(u)$ and so $2r \leq d^2 - d + 2$. It follows that

$$\mathrm{gp}(\mathcal{M}(G)) \leq n + \frac{d-1}{2} + \frac{1}{d}.$$

In particular, for $d \geq 3$ we have $\mathrm{gp}(\mathcal{M}(G)) \leq n + \frac{d-1}{2}$. ■

Theorem 21 suggests the following construction of abundant regular graphs.

Theorem 22. *For all $d \geq 2$, there is an abundant d-regular graph $G(d)$ with order $n = 3d - 1$ and $\mathrm{gp}(\mathcal{M}(G(d))) = n + 1$.*

Proof. We form the graph $G(d)$ as follows. Take three sets V_1, V_2, V_4 of vertices, where $|V_1| = |V_2| = d$ and $|V_4| = d - 1$. Form the complete bipartite graph $K_{d,d-1}$ with partite sets V_1 and V_4. Make V_2 into a clique K_d by adding an edge between every pair of vertices in V_2. Finally join V_1 and V_2 by a matching of size d. The graph $G(5)$ is shown in Fig. 5.1. As suggested by our labelling of the sets, $(V_1, V_2, \emptyset, V_4)$ is an $\mathcal{MGP}$-partition of $G(d)$. As $\mathrm{ip}_4(G(d)) = \alpha(G(d)) = d$, these graphs are abundant. ■

We now classify the abundant 2- and 3-regular graphs. It turns out that the graphs $G(2)$ and $G(3)$ are the only abundant regular graphs with degree three or less.

Theorem 23. *For $n \geq 3$, the Mycielskian of the cycle C_n has general position number*

$$\mathrm{gp}(\mathcal{M}(C_n)) = \begin{cases} n + 1, & \text{if } n = 3 \text{ or } 5 \\ n, & \text{otherwise.} \end{cases}$$

Proof. The result for $n = 3$ follows from Corollary 2. For $n \geq 4$ let S be a gp-set of $\mathcal{M}(C_n)$ that does not include u^* and consider the corresponding $\mathcal{MGP}$-partition (V_1, V_2, V_3, V_4) of $V(C_n)$. Suppose that $\mathrm{gp}(\mathcal{M}(C_n)) \geq n + 1$. By Theorem 21 we

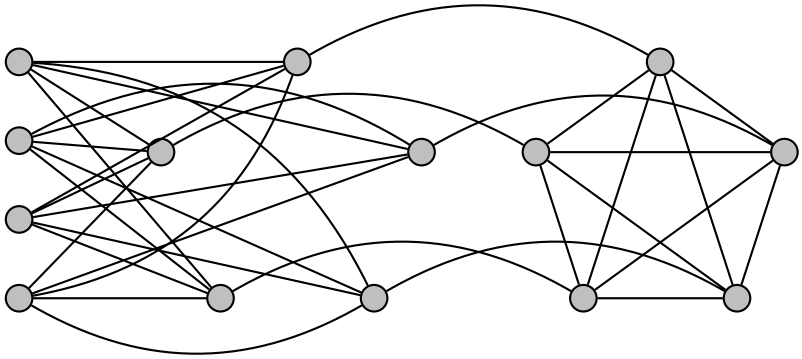

Fig. 5.1. A 5-regular graph G with $\mathrm{gp}(\mathcal{M}(G)) = n + 1$.

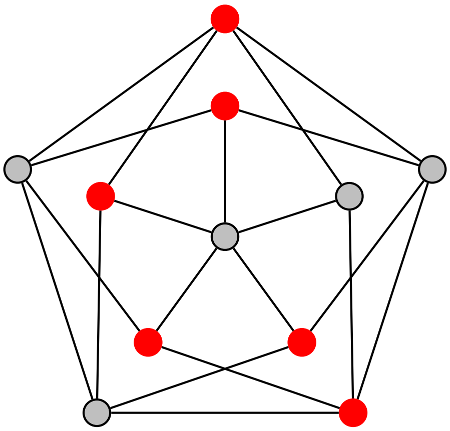

Fig. 5.2. A gp-set (in red) for $\mathcal{M}(C_5)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

have $\mathrm{gp}(\mathcal{M}(C_n)) \leq n + 1$, so we must have exactly $\mathrm{gp}(\mathcal{M}(C_n)) = n + 1$. As we have equality in the bound of Theorem 21, it follows that the matching (V_1, V_2) has size $r = 2$ and $n_4 = n_1 - 1$.

Let $u_1, u_2 \in V_1$ and $v_1, v_2 \in V_2$, where $u_1 \sim v_1$, $u_2 \sim v_2$ are the edges of (V_1, V_2). It follows from Theorem 21 that $\{u_2, v_2\} = N^2(u_1)$. Since $u_2 \sim v_2$, the cycle has length five and all cycles with length $n = 4$ or $n \geq 6$ are meagre. However, as can be seen in Fig. 5.2, this argument does yield a gp-set of cardinality six in the Grötzsch graph $\mathcal{M}(C_5)$. Notice that C_5 is isomorphic to $G(2)$ from Theorem 22. ■

Theorem 24. *The graph $G(3)$ is the unique non-complete abundant cubic graph.*

Proof. Let G be a non-complete cubic graph with order n and $\mathrm{gp}(\mathcal{M}(G)) \geq n + 1$. The proof of Theorem 21 shows that $\mathrm{gp}(\mathcal{M}(G)) = n + 1$ and the matching (V_1, V_2) has size $r = 3$ or 4.

Suppose firstly that $r = 4$ and let $u_i \sim v_i$ for $i = 1, 2, 3, 4$ be the edges of (V_1, V_2). By Condition 2 of Definition 5 we have $\{u_2, u_3, u_4, v_2, v_3, v_4\} \subseteq N^2(u_1)$. As there are at most six vertices in $N^2(u_1)$, it follows that $\{u_2, u_3, u_4, v_2, v_3, v_4\} = N^2(u_1)$, $n_1 = n_2 = 4$ and $n_4 = 3$. Again by Condition 2 of Definition 5 any vertex w of V_3 lies in $N^3(u)$. However, a shortest u_1, w-path would pass through $V_1 \cup V_2$, which is forbidden by Condition 3 of Definition 5. Thus $V_3 = \emptyset$. However, this implies that G has order $n = n_1 + n_2 + n_4 = 11$, whereas any cubic graph has even order, a contradiction.

Hence we can assume that $r = 3$. Write $V_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$, where $u_i \sim v_i$ are the edges of the matching (V_1, V_2) for $i = 1, 2, 3$. As above, $(V_1 \cup V_2) \setminus \{u_1, v_1\} \subseteq N^2(u_1)$, so have $n_1 + n_2 \leq 8$. Since $n_1, n_2 \geq r = 3$, n_1 lies in the range $3 \leq n_1 \leq 5$. If $n_1 + n_2 = 8$, then $(V_1 \cup V_2) \setminus \{u_1, v_1\} = N^2(u_1)$ and as before V_3 would be empty.

Suppose that $n_1 = 5$. Then as $3 \leq n_2 \leq 8 - n_1 = 3$, we have $n_2 = 3$, $n_3 = 0$ and $n_4 = 4$, giving $n = 12$. For $i = 1, 2, 3$, we have $V_1 \cup V_2 = \{u_i, v_i\} \cup N^2(u_i)$. As each vertex of V_2 has just one edge to V_1, both vertices of $N_G(v_i) \cap N^2(u_i)$ must lie in V_2. Hence V_2 induces a triangle in G. Thus there is no path of length at most three from v_1 to u_4 via $\{v_2, v_3\}$, so the path v_1, u_1, w, u_4, where w is a neighbour of u_1 in V_4, is a shortest path of the form forbidden by Condition 3 of Definition 5, a contradiction.

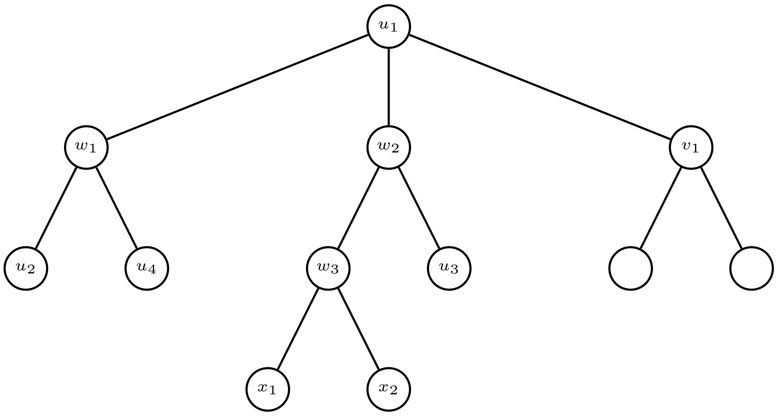

Fig. 5.3. Configuration for $n_1 = 4, n_2 = 3, n_3 = 2, n_4 = 3$.

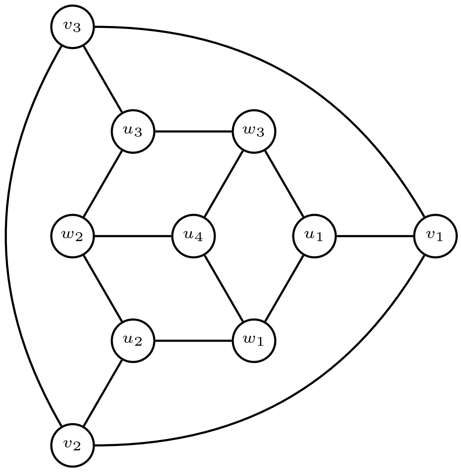

Fig. 5.4. Cubic graph for $n_1 = 4, n_2 = 3, n_3 = 0, n_4 = 3$.

Now suppose that $n_1 = 4$, so that $n_4 = 3$ and $n_2 \in \{3, 4\}$. If $n_2 = 4$, then $n_3 = 0$, giving $n = 11$, whereas G must have even order. Thus $n_2 = 3$. By Condition 2 of Definition 5 all vertices of n_3 lie at distance at most three from u_i for $i = 1, 2, 3$. There is just one vertex x_i of $N^2(u_i)$ that does not belong to $V_1 \cup V_2$ for $i = 1, 2, 3$ and x_i is the only vertex of $N^2(u_i)$ that can have neighbours in V_3 by Condition 3 of Definition 5. Thus $n_3 \leq 3$ and, by the parity of n, either $n_3 = 0$ or $n_3 = 2$. Write $N_G(x_i) \cap N^3(u_i) = \{y_1, y_2\}$. If $x_i \in V_3$, then by Condition 3 of Definition 5 neither y_1 nor y_2 is in V_3, contradicting the parity requirement. Thus $x_i \in V_4$ and for each of u_1, u_2, u_3 two vertices of V_4 lie in $N_G(u_i)$ and one vertex of V_4 lies in $N^2(u_i)$. Write $V_4 = \{x_1, w_1, w_2\}$, where $N_G(u_1) = \{w_1, w_2, v_1\}$.

If $n_3 = 2$, then, by Condition 3 of Definition 5, $\{x_1\} \cup (V_1 \setminus \{u_1\}) \cap N_G(v_1) = \emptyset$. Assume without loss of generality that $x_1 \sim w_2$. Then at least one vertex of $\{u_2, u_3\}$, say u_2, lies in $N_G(w_1)$, and we obtain the configuration in Fig. 5.3 (although note that the positions of u_3 and u_4 might be swapped). But now u_2 has at most one neighbour in V_4, contradicting our previous deduction. Therefore $n_3 = 0$.

Thus suppose that $n_1 = 4$, $n_2 = n_4 = 3$ and $n_3 = 0$. Write $V_4 = \{w_1, w_2, w_3\}$. Then each of u_1, u_2, u_3 has one edge to V_2 and all other edges incident with V_1 are to V_4. In particular, $N_G(u_4) \subset V_4$. Thus $|(V_1, V_4)| = 9$ and every vertex of V_4 has neighbourhood entirely contained in V_1. Hence each vertex of V_2 must have one neighbour in V_1 and two neighbours in V_2, so V_2 induces a triangle. This implies that G is isomorphic to the graph in Fig. 5.4. However, this graph contains a shortest path v_1, u_1, w_3, u_4, contradicting Condition 3 in Definition 5.

Finally, let $n_1 = 3, n_4 = 2$. Each of u_1, u_2, u_3 has one edge to V_2 and two edges to V_4, making six edges from V_1 to V_4. Hence both vertices of V_4 have neighbourhood contained in V_1. For each of u_1, u_2, u_3 to reach each vertex of V_2 in distance at most two, we see that $n_2 = 3$ and V_2 induces a triangle in G, so that $V_3 = \emptyset$. Hence G is isomorphic to $G(3)$, completing the proof. ■

Theorem 25. *Any d-regular graph with order* $\geq d^3 - 2d^2 + 2d + 2$ *is meagre.*

Proof. Let G be an abundant d-regular graph. We can assume that $r = |(V_1, V_2)| \geq d > 0$. Let $u_1 \sim v_1$ be an edge of the matching (V_1, V_2), where $u_1 \in V_1$, $v_1 \in V_2$. Consider a breadth-first search tree T of depth three rooted at u_1. By Condition 2 in Definition 5, all vertices of $(V_1 \cup V_2) \setminus \{u_1, v_1\}$ are contained in $N^2(u)$ and V_3 is contained in $N^2(u) \cup N^3(u)$. There are at most $d^2 - d$ vertices at distance two from u_1 and at most $d(d-1)^2$ vertices in $N^3(u_1)$.

Moreover there are at least $2d - 2$ vertices of $V_1 \cup V_2$ in $N^2(u)$ and no such vertex can have descendants at distance three from u_1 in V_3 by Condition 3 in Definition 5. Therefore we see that

$$|V_1 \cup V_2 \cup V_3| \leq 2 + d^2 - d + [d^2 - d - (2d-2)](d-1) = d^3 - 3d^2 + 4d.$$

Since we are assuming that G is abundant, we can bound n_4 as follows:

$$n_4 \leq n_1 - 1 \leq 1 + (d^2 - d) - (d-1) - 1 = d^2 - 2d + 1.$$

In total then the order of G is bounded above by $d^3 - 2d^2 + 2d + 1$. ■

This shows that for any value of d there are only a finite number of abundant d-regular graphs. This raises the question whether we can strengthen this result by making the same conclusion for graphs with given maximum degree. The answer is no, as evidenced by the following construction. Let the maximum degree be $\Delta \geq 4$. Let P be a path $u_0, u_1, \ldots, u_r$, where $r \geq 3$, and take the Cartesian product $P \,\square\, K_2$ to form a ladder graph. If we identify the vertices of K_2 with $\{0, 1\}$, then we can write the vertices of the ladder as (u_i, j) for $0 \leq i \leq r$ and $j = 0, 1$. Finally add $\Delta - 3$ leaves to each vertex $(u_i, 0)$ for $1 \leq i \leq r$ and $\Delta - 4$ leaves to $(u_0, 0)$ and $(u_r, 0)$. By taking the set of leaves to be V_1, $V_2 = V(P) \times \{1\}$ and $V_4 = V(P) \times \{0\}$ we obtain an $\mathcal{MGP}$-partition that shows that the graph is abundant. By increasing r we see that the number of vertices in this family is unbounded for fixed Δ.

For degrees two and three, we have shown that our construction $G(d)$ is the unique abundant graph. Also, for both of these graphs the general position number of the Mycielskian is just one more than the order of the graph. This suggests the following problem.

Problem 26. For given $d \geq 4$, what is the largest value of $\mathrm{gp}(\mathcal{M}(G)) - n$ over all d-regular graphs? Is $G(d)$ the unique abundant graph?

6. The Mycielskian of trees and graphs with large girth

We now investigate the general position number of the Mycielskian of trees and, as a by-product, give an upper bound on $\mathrm{gp}(\mathcal{M}(G))$ for graphs G without short cycles. Let T be any tree with order $n \geq 3$ and L be the set of all $\ell(T)$ leaves of T. Recall that T is *abundant* if $\mathrm{gp}(\mathcal{M}(T)) > \max\{n, 2\mathrm{ip}_4(T)\}$ and otherwise is *meagre*. We present an exact expression for $\mathrm{gp}(\mathcal{M}(T))$ for a wide class of trees.

A vertex of T is a *support vertex* if it is adjacent to a leaf of T. We denote the set of support vertices of T by $\Sigma = \{y_1, y_2, \ldots, y_s\}$ and the number of support vertices of T by $s = |\Sigma|$, whilst $Z = \{z_1, z_2, \ldots, z_t\}$ is the set of vertices in $V(T) \setminus (L \cup \Sigma)$, so that $n = \ell + s + t$. We associate with each support vertex y_i the set L_i of leaves to which it is adjacent in T. Let $D_T = \min\{d_T(l, l') : l \in L_i, l' \in L_j, i \neq j\}$, where d_T represents the distance in T. If $s = 1$, then T is a star and by Corollary 20 we have $\mathrm{gp}(\mathcal{M}(T)) = 2\ell(T) = 2\mathrm{ip}_4(T)$, so that T is meagre. Hence we can assume that $s \geq 2$ and D_T is defined.

We begin with a lower bound on $\mathrm{gp}(\mathcal{M}(T))$ for trees with support vertices at distance at least three from each other and then proceed to prove that this bound is sharp.

Lemma 27. *If T has ℓ leaves and s support vertices and also $D_T \geq 5$, then the general position number of the Mycielskian $\mathcal{M}(T)$ satisfies*

$$\mathrm{gp}(\mathcal{M}(T)) \geq n + \ell - s.$$

Proof. Recall that L is the set of leaves of T, Σ the set of support vertices and Z is $V(T) \setminus (L \cup \Sigma)$. $(L, Z, \emptyset, \Sigma)$ is an $\mathcal{MGP}$-partition, so it follows from Corollary 9 that $S = L \cup L' \cup Z'$ is in general position in $\mathcal{M}(T)$. This implies that $\mathrm{gp}(\mathcal{M}(T)) \geq 2\ell + t = n + \ell - s$. ■

In the following, let T be an abundant tree and S be a gp-set of $\mathcal{M}(T)$, which we can assume by Lemma 1 does not contain u^*. We now present an upper bound on $\mathrm{gp}(\mathcal{M}(G))$ for graphs with girth at least six. Recall by our convention that the girth of a tree is ∞, so the following result applies in particular to trees.

Theorem 28. *Let G be a graph with order $n \geq 4$, girth $\mathrm{g}(G) \geq 6$ and matching number $\nu(G)$. Then*

$$\mathrm{gp}(\mathcal{M}(G)) \leq 2n - 2\nu(G).$$

Also, if G satisfies $\mathrm{gp}(\mathcal{M}(G)) > n$ and $\pi = (V_1, V_2, V_3, V_4)$ is an $\mathcal{MGP}$-partition corresponding to a gp-set S of $\mathcal{M}(G)$ that does not contain u^*, then $(V_1, V_2) = (V_2, V_3) = \emptyset$.

Proof. Let G, S and π be as described. Suppose that the matching (V_1, V_2) contains an edge $u \sim v$, where $u \in V_1$ and $v \in V_2$. By Condition 2 of Definition 5, all vertices of $(V_1 \cup V_2) \setminus \{u, v\}$ are contained in $N^2(u)$.

Suppose that $n_1 \geq 2$ and let $w \in V_1 \setminus \{u\}$. By Conditions 1 and 2 of Definition 5, $w \notin N_G(u) \cup N_G(v)$, so G contains a path w, x, u, v, where $x \in N_G(u) \setminus \{v\}$. By Condition 3 of Definition 5, this cannot be a shortest path, so there must be a path of length two from w to v, implying the existence of a cycle of length at most five, a contradiction.

Thus in this case $n_1 = 1$. Thus if $\text{gp}(\mathcal{M}(G)) > n$, then $n_4 = 0$. As u has no neighbours in $V_3 \cup V_4$ and just one neighbour in V_2, u is a leaf with support vertex v. Thus any of the paths from u to V_3 guaranteed by Condition 2 must pass through v_2, violating Condition 3. Thus $V_3 = \emptyset$. As G is triangle-free, it follows that G is a star. However, by Corollary 18, the star S_n has $\text{gp}(\mathcal{M}(S_n)) = 2n - 2$, whereas in the partition described above all but one vertex of G lies in V_2. Since S is assumed to be a gp-set, this implies that $\text{gp}(\mathcal{M}(G)) = n + 1 = 2n - 2$, yielding $n = 3$. We conclude that $(V_1, V_2) = \emptyset$.

Suppose now that there is an edge $v \sim w$ in G, where $v \in V_2$ and $w \in V_3$. By Condition 2 of Definition 5, all vertices of V_1 lie in $N^2(w)$. By Condition 2, $N_G(v) \cap V_1 = \emptyset$. Thus, for any $u \in V_1$, there is a path u, x, w, v for some $x \in V(G)$, and by Condition 3 this cannot be a shortest path. Again this implies the existence of a cycle with length at most five if $V_1 \neq \emptyset$.

If $\text{gp}(\mathcal{M}(G)) = n$, then the claimed bound certainly holds, so assume that $\text{gp}(\mathcal{M}(G)) > n$. Let M be a maximum matching of G. If $x \sim y$ is any edge in M, then it follows from $(V_1, V_2) = (V_1, V_3) = \emptyset$ that one of x, y is in V_4 and the vertices $\{x, y, x', y'\}$ of $\mathcal{M}(G)$ can contribute at most two to $\text{gp}(\mathcal{M}(G))$. Summing over all edges of M, we see that the vertices saturated by M and their $\mathcal{M}$-twins contribute at most $2\nu(G)$ to $\text{gp}(\mathcal{M}(G))$. Assuming that every other vertex lies in V_1, we obtain the upper bound $\text{gp}(\mathcal{M}(G)) \leq 2\nu(G) + 2(n - 2\nu(G)) = 2n - 2\nu(G)$. ■

The star S_n shows that the bound of Theorem 28 is tight.

Corollary 29. *If G is a graph with order $n \geq 4$ and girth $g(G) \geq 6$ that has a perfect matching, then $\text{gp}(\mathcal{M}(G)) = n$.*

Corollary 30. *If T is a tree with order $n \geq 3$ and matching number $\nu(T)$, then $\text{gp}(\mathcal{M}(T)) \leq 2n - 2\nu(T)$. If $\text{gp}(\mathcal{M}(T)) > n$, then (V_1, V_2) and (V_2, V_3) are empty.*

Lemma 31. *Let T be any tree with order $n \geq 3$ and $\ell = s$, i.e. each support vertex of T is adjacent to exactly one leaf. Then $\text{gp}(\mathcal{M}(T)) = n$.*

Proof. Let T be as described and assume for a contradiction that $\text{gp}(\mathcal{M}(T)) > n$. By Lemma 6 and Corollary 30, every vertex of V_1 has neighbourhood lying in V_4. Consider the subgraph $\langle V_1 \cup V_4 \rangle$ of T. As $n_1 > n_4$, there is no matching in this subgraph that saturates V_1 and we can choose a smallest subset $\widetilde{V_1} \subseteq V_1$ such that $\widetilde{V_4} = N_T(\widetilde{V_1})$ satisfies $|\widetilde{V_4}| < |\widetilde{V_1}|$. In particular, the subtree $\langle \widetilde{V_1} \cup \widetilde{V_4} \rangle$ is connected.

If any $v \in \widetilde{V_4}$ has just one neighbour u in $\widetilde{V_1}$, then we could delete u and v from $\widetilde{V_1}$ and $\widetilde{V_4}$ to obtain a smaller pair with $|\widetilde{V_1} \setminus \{u\}| < |\widetilde{V_4} \setminus \{v\}|$, contradicting minimality of $\widetilde{V_1}$. Thus let $u_1, u_2 \in \widetilde{V_1}$, $w_1 \in \widetilde{V_4}$, where $u_1, u_2 \in N_T(w_1)$. By our assumption $\ell = s$, not both of u_1, u_2 are leaves, so there is a $w_2 \neq w_1$ with $u_2 \sim w_2$. Now by the preceding argument w_2 must have a neighbour $u_3 \neq u_2$ in $\widetilde{V_1}$. To avoid cycles, we also have $u_3 \neq u_1$. Therefore we have constructed a path u_1, w_1, u_2, w_2, u_3, where $u_1, u_2, u_3 \in V_1$. This path is a geodesic, and so violates Condition 3 of Definition 5. This contradiction establishes the result. ■

Lemma 31 shows in particular that paths P_n with $n \geq 4$ are meagre (and P_3 is also easily seen to be meagre).

Corollary 32. *If T is any tree with order $n \geq 3$, leaf number ℓ and s support vertices, then $\text{gp}(\mathcal{M}(T)) \leq n + \ell - s$.*

Proof. We prove the result by induction on $\ell - s$. If $\ell = s$, then Lemma 31 shows that the result is true. Suppose that T satisfies $\text{gp}(\mathcal{M}(T)) > n$ and construct the sets $\widetilde{V_1}$ and $\widetilde{V_4}$ as in the proof of Lemma 31. As in the previous lemma, we can assume that there is a vertex $w_1 \in \widetilde{V_4}$ that is adjacent to at least two leaves u_1, u_2 in $\widetilde{V_1}$. Set $\widetilde{T} = T \setminus \{u_1\}$. By induction, $\text{gp}(\mathcal{M}(\widetilde{T})) \leq (n-1) + (\ell - 1) + s$. Adding back the leaf u_1, the vertices u_1, u_1' can contribute at most two to $\text{gp}(\mathcal{M}(T))$, so $\text{gp}(\mathcal{M}(T)) \leq (n - 2 + \ell - s) + 2 = n + \ell - s$. ■

Combining the upper bound from Corollary 32 with the lower bound from Lemma 27 we obtain the exact value of $\text{gp}(\mathcal{M}(T))$ when $D_T \geq 5$.

Corollary 33. *If T is any tree with order $n \geq 3$, $D_T \geq 5$, leaf number ℓ and s support vertices, then $\text{gp}(\mathcal{M}(T)) = n + \ell - s$.*

We show finally that $\text{gp}(\mathcal{M}(T)) = n + \ell - s$ also holds for some trees with $D_T \leq 4$. A *caterpillar* is a tree formed by attaching leaves to the vertices of a path (called the *central path*).

Corollary 34. *If T is a tree in which every vertex is either a leaf or a support vertex, then $\text{gp}(\mathcal{M}(T)) = 2\ell(T)$ and T is meagre.*

Proof. If every vertex of T is either a leaf, or a support vertex, then $n = \ell + s$ and the upper bound in Corollary 32 gives $\text{gp}(\mathcal{M}(T)) \leq (\ell + s) + \ell - s = 2\ell$. It is shown in [7] that the set of leaves L is a general position set of T and hence an independent 4-position set. Therefore the lower bound 2ℓ from Corollary 11 matches the upper bound $n + \ell - s$ and T is meagre. ■

Corollary 34 shows in particular that all caterpillars with a leaf adjacent to each vertex of the central path is meagre. Equality in the bound $\mathrm{gp}(\mathcal{M}(T)) \leq n + \ell - s$ does not always hold, as can be easily seen by attaching several leaves to the endpoints of P_3.

7. Conclusion

We conclude with two promising directions for further research. The first obvious question is the complexity of finding the gp-number of Mycielskians. The proof of Lemma 1 shows that any general position sets containing u^* and having cardinality at least $n+1$ can be found easily in polynomial time. Therefore, given the duality between general position sets of $\mathcal{M}(G)$ and $\mathcal{MGP}$-partitions of $V(G)$, the complexity of finding $\mathrm{gp}(\mathcal{M}(G))$ is the same as the complexity of finding the largest value of $n_1 - n_4$ over all $\mathcal{MGP}$-partitions (V_1, V_2, V_3, V_4) of G.

Problem 35. What is the complexity of finding the general position number of the Mycielskian $\mathcal{M}(G)$ of an arbitrary graph G?

Secondly, it was mentioned in Section 1 that the general position number is one amongst a variety of 'position-type' parameters. We suggest that it would be of interest to find the values of some of these other position numbers on Mycielskians of graphs, in particular the following three numbers.

Problem 36. What are the mutual visibility, lower general position and monophonic position numbers of Mycielskians of graphs?

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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