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The work of Jesse Douglas on Minimal Surfaces

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This paper is dedicated to the memory of our student, Adam Merton (1920-1999) without whose interest in minimal surfaces and the work of Jesse Douglas we would never have embarked on this study.

1   Introduction

The demonstration of the existence of a least-area surface spanning a given contour has a long history. Through the soap-film experiments of Plateau, this existence problem became known as the Plateau Problem. By the end of the 19th century, the list of contours for which the Plateau problem could be solved was intriguingly close to that considered by Plateau, namely polygonal contours, lines or rays extending to infinity and circles. Free boundaries on planes were also considered. But even these contours were required to have some special symmetry, for example a pair of circles had to be in parallel planes. These solutions were obtained by finding the pair of holomorphic functions (which, for the experts, we shall refer to as the conformal factor f and the Gauss map g) required in the so-called Weierstrass-Enneper representation of a minimal surface. In 1928, Garnier completed a programme started independently by Riemann and Weierstrass in the 1860s and then continued by Darboux towards the end of the 19th Century for finding f and g for an arbitrary polygonal contour. In his 92-page paper [8] (which is very difficult to read and which, apparently has never been fully checked) Garnier also employed a limiting process to extend his solution of the Plateau problem to contours which consisted of a finite number of unknotted arcs with bounded curvature. Garnier's work was soon eclipsed by the works of Radó and Jesse Douglas about which there is considerable amount of inaccurate information in the literature and on which we hope to shed some light in this article. In particular, we challenge the popular belief that Douglas arrived at his functional for solving the Plateau Problem by direct consideration of Dirichlet's integral and its relation to the area functional. Douglas was awarded one of the first Fields Medals for his work on the Plateau problem. There are many amusing aspects of the Fields Medal ceremony at which Douglas was awarded his prize. We simply mention that Wiener collected the medal on behalf of Douglas, even though Douglas did attend the International Congress. And in the address, Carathéodory described a method for finding a minimal surface that is due to Radó (different from the one sketched below) but gave the impression that it was due to Douglas! Full historical and mathematical details will appear in [9].

2   Minimal Surfaces as Conformal Harmonic Maps

Recall that $S \subset \mathbb{R}^n$ is called a regular (immersed) surface if there exists a surface (a manifold of real dimension 2) Σ, possibly with boundary, and a map $\mathbf{r} \colon \Sigma \to \mathbb{R}^n$ of class C^1 up to the boundary such that $\mathbf{r}(\Sigma) = S$ and the differential of $\mathbf{r}$ has maximal rank (and so, the induced

map on tangent spaces is an injection).[1] The map $\mathbf{r}$ is called a parameterisation of S. The first fundamental form, I, of $\mathbf{r}$ with respect to local coordinates (u, v) on Σ is the symmetric matrix defined by:

$$I := \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{where} \quad E := \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u}, \ F := \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}, \ G := \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}. \tag{2.1}$$

The condition that $\mathbf{r}$ is of maximal rank is equivalent to the positive definiteness of I. The area of S is defined by the integral

$$\text{Area}(S) := \iint_\Sigma \sqrt{g} \, du \, dv \quad \text{where} \quad g := EG - F^2 = \det I. \tag{2.2}$$

A *critical point* of this area functional is called a *minimal* surface. It has zero mean curvature, i. e. the shape operator B (presently defined for a surface S in $\mathbb{R}^3$) has zero trace. The Gauss map $G \colon S \to S^2$, where S^2 is the round unit sphere in $\mathbb{R}^3$, is simply defined by $G(p) := \mathbf{n}(p)$, where $\mathbf{n}$ is the positively oriented unit normal of S. The tangent space of S^2 at $G(p)$ and the tangent space $T_p S$ of S at p may be identified because both are orthogonal to $\mathbf{n}(p)$. Therefore, the differential of G at p may be viewed as an endomorphism $B(p)$ of $T_p(S)$. The second fundamental form II and the shape operator B are related via the first fundamental form I by $II(X, Y) = -I(BX, Y) \ \forall \, X, Y \in T_p S$.

It is important (for the purposes of §4.3) to realise that a minimal surface need not actually minimise area in the same way that a geodesic need not minimise length. For instance, consider a catenoid and a cylinder with a common axis of rotation and which intesect in a pair of circles. If the radius of the cylinder is not too large, the portion of the catenoid within the cylinder will have less area than the portion of the cylinder bounded by the circles. But if the radius of the cylinder is sufficiently large, the opposite will be true, and the area minimiser is then another catenoid which is 'fatter' than the one initially considered.

We shall now assume S to be the image of an immersion $\mathbf{r} \colon D \to \mathbb{R}^n$ of the open unit disc $D \subset \mathbb{R}^2 = \mathbb{C}$. It is a fundamental theorem in surface theory (due to Gauss in the analytic case) that S can be realised as the image of a conformal map of D — so we may take the immersion $\mathbf{r}$ to be conformal in u and v. This means that $E = G$ and $F = 0$. The coordinates (u, v) are then called isothermal parameters. The condition that S be minimal, i. e. that its mean curvature vanishes, is equivalent to requiring each component x_i, $1 \leqslant i \leqslant n$, of $\mathbf{r}$ be a harmonic function of the isothermal coordinates; the map $\mathbf{r}$ is then called a harmonic map (called a harmonic vector in the days of Douglas). If the condition of regularity of $\mathbf{r}$ is dropped, i. e., $\mathbf{r} \colon D \to \mathbb{R}^n$ is harmonic and satisfies $E = G$ and $F = 0$ but E (and therefore G) is allowed to vanish, then $\mathbf{r}$ parameterises, in modern terminology, a *generalised, or branched, minimal disc*. The (isolated) points at which E vanishes are called branch points.

After the work of Riemann and Weierstrass, the Plateau problem acquired the following formulation (with $n = 3$):

Plateau Problem *Given a contour $\Gamma \subset \mathbb{R}^n$, span Γ by a (possibly) branched minimal disc, i. e., find $\mathbf{r} \colon \overline{D} \to \mathbb{R}^n$ which is continuous on the closed unit disc $\overline{D}$, harmonic and conformal (away from branch points) on the interior D and whose restriction to ∂D parameterises Γ.*

As noted above, only modest progress was made on this problem in the several decades from the 1860s until the end of the 1920s. Then two mathematicians working independently and with quite different methods were able to make decisive progress and produce general methods for solving it. They were Tibor Radó in Europe and Jesse Douglas in the United States.

3 Radó's solution of the Plateau problem

Radó based his solution of the Plateau problem in [17] on the uniformisation theorem of Koebe.

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[1] Not much is lost for the present purposes if one restricts attention to the simple case when Σ is a domain in $\mathbb{R}^2$.

Given a polygonal contour $\Gamma \subset \mathbb{R}^3$, define

$$\lambda := \inf\{\operatorname{Area}(\Pi) \mid \Pi \text{ is a polyhedral surface spanning } \Gamma\}.$$

Then, for each $\sigma > 0$, there exists a polyhedron Π_σ spanning Γ whose area is less than $\lambda + \sigma$. By the uniformisation theorem of Koebe (see Koebe's [12] and [13]), Π_σ admits an isothermic parameterisation $\bar{\mathbf{r}}_\sigma \colon \overline{D} \to \mathbb{R}^3$. Let $\mathbf{r}_\sigma$ be the harmonic extension of $\bar{\mathbf{r}}_\sigma$ restricted to the unit circle C, the boundary of $\overline{D}$. By a lemma on harmonic surfaces, Radó asserted the existence of a polyhedron Π_σ^* whose area differs from that of $\mathbf{r}_\sigma(\overline{D})$ by no more than σ. Let $\sqrt{EG - F^2}$ and $\sqrt{\bar{E}\bar{G} - \bar{F}^2}$ denote the area elements of $\mathbf{r}_\sigma$ and $\bar{\mathbf{r}}_\sigma$ respectively. Since the Dirichlet energy $\frac{1}{2} \iint_{\overline{D}} \bar{E} + \bar{G}$ of $\bar{\mathbf{r}}_\sigma$ is greater than or equal to the Dirichlet energy $\frac{1}{2} \iint_{\overline{D}} E + G$ of $\mathbf{r}_\sigma$ we have the following chain of inequalities:

$$\begin{aligned} \lambda + \sigma > \iint_{\overline{D}} \sqrt{\bar{E}\bar{G} - \bar{F}^2} &= \frac{1}{2} \iint_{\overline{D}} \bar{E} + \bar{G} \\ &\geqslant \frac{1}{2} \iint_{\overline{D}} E + G \geqslant \iint_{\overline{D}} \sqrt{EG - F^2} > \operatorname{Area}(\Pi_\sigma^*) - \sigma \geqslant \lambda - \sigma. \end{aligned}$$

Therefore, by taking σ sufficiently small, one can find, for each $\varepsilon > 0$, a harmonic vector $\mathbf{r}_\varepsilon \colon D \to \mathbb{R}^3$ which

(i) extends continuously to the closed unit disc $\overline{D}$ so that its restriction to ∂D parameterises Γ and

(ii) is approximately conformal in the sense that

$$\iint_{\overline{D}} |F| < \varepsilon \qquad \text{and} \qquad \iint_{\overline{D}} (E^{1/2} - G^{1/2})^2 < \varepsilon. \tag{3.1}$$

Let (ε_n) be a sequence of positive numbers decreasing to 0 and denote $\mathbf{r}_{\varepsilon_n}$ more simply by $\mathbf{r}_n$. Radó showed that a subsequence of $(\mathbf{r}_n)$ converges, up to reparameterisation by Möbius transformations of D, to a generalised minimal surface spanning Γ. This limiting argument is one of Radó's major achievements. He had previously established the following variant of it in [16].

Approximation Theorem *Let Γ_n be a sequence of simple closed curves of uniformly bounded length, for each of which the Plateau Problem is solvable. If Γ_n converges (in the sense of Fréchet) to a simple closed curve Γ then the Plateau problem for Γ is solvable.*

The proof of this Approximation Theorem was not only *much* simpler than Garnier's limiting argument but, together with [17], it also provided a solution to Plateau's problem for any rectifiable contour.

Radó's use of polyhedral surfaces is highly reminiscent of Lebesgue's definition of area of a surface as the infimum, over all sequences, of $\liminf$ of the areas of a sequence of polyhedra tending to the surface. (The analogous definition for curves coincides with the usual limsup definition. If the surface admits a Lipschitz parameterisaton, or even a parameterisation which has square integrable first derivatives, then Lebesgue's definition agrees with the one given by the usual double integral.) It is surprising that Radó did not connect his method of proof with Lebesgue's definition of area until he gave a colloquium on his result at Harvard. We refer the reader to [9] for details of this story. We simply note here that, once Radó made this connection, he wrote [18] in which he showed that his solution of the Plateau problem has least area among discs spanning Γ.

Radó's strategy marked a total departure from the Riemann-Weierstrass-Darboux programme and it was strikingly original at the time. We now turn to the solution of Jesse Douglas, who also abandoned the Riemann-Weierstrass-Darboux programme but in a direction different from that of Radó.

4 Douglas's solution of the Plateau problem

Douglas's first full account of his solution of the Plateau problem appears in his 59-page paper 'Solution of the Problem of Plateau', published in the *Transactions of the American Mathematical Society* in January 1931. However, he had made several announcements previously at meetings of the American Mathematical Society, starting in late 1926. In these announcements, which have essentially been forgotten, Douglas made clear that his plan was to find a parameterisation $\mathbf{g}^*$ of the given contour Γ so that the harmonic extension $\mathbf{r}^*$ of $\mathbf{g}^*$ is conformal. In the famous 1931 paper, Douglas achieved this by minimising his so-called A-functional:

$$A(\mathbf{g}) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{\sum_{i=1}^n [g_i(\theta) - g_i(\varphi)]^2}{4\sin^2 \frac{\theta-\varphi}{2}} \, d\theta d\varphi. \tag{4.1}$$

He gives no clue as to how he thought of this functional but, in Part III of his paper, he does show that $A(\mathbf{g})$ is equal to the Dirichlet energy $\frac{1}{2}\iint_{\overline{D}}(E+G)$ of the harmonic extension $\mathbf{r_g}$ of $\mathbf{g}$. It has always been assumed that this was Douglas's starting point but his announcements in the *Bulletin* indicate otherwise. Moreover, Douglas repeatedly refuted claims made by Radó and Courant that Douglas's method essentially amounted to implementing Dirichlet's principle. It must be remembered that, in 1931, Dirichlet's principle was not yet firmly established and Douglas was keen, indeed all too keen, to emphasize that his A-functional, being a 1-dimensional integral which did not involve any derivatives, did not suffer from all the difficulties that plagued Dirichlet's integral at the time. Specifically, as Douglas remarked, the Dirichlet integral could not be shown to attain its lower bound, whereas his A-functional, being lower semi-continuous on a sequentially compact space, necessarily attained its minimum. Details of the exchanges between Radó and Douglas and Courant and Douglas can be found in [9].

4.1 Conformality of a harmonic surface via an integral equation for the parameterisation of the boundary

In the Abstract (1927, 32, 143-4)[2] Douglas claimed that, if $t \mapsto \mathbf{g}(t) \colon \mathbb{R} \cup \{\infty\} \to \Gamma \subset \mathbb{R}^3$ is a parameterisation of Γ and if $\varphi \colon \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ is a homeomorphism which solves the following integral equation,

$$\int_\Gamma \frac{K(t,\tau)}{\varphi(t) - \varphi(\tau)} d\tau = 0, \qquad \text{where } K(t,\tau) = \mathbf{g}'(t) \cdot \mathbf{g}'(\tau) \tag{4.2}$$

then the harmonic extension of $\mathbf{g} \circ \varphi^{-1}$ to the upper half plane defined by means of Poisson's integral is the required conformal harmonic representation of a minimal surface spanning Γ. (Douglas was actually less precise than this in the abstract but it is clear that this is what he meant.) If Γ is parameterised by a map from the unit circle C (and a surface S spanning Γ is parameterised by a map from the disc) then the integral equation (4.2) becomes

$$\int_0^{2\pi} K(t,\tau) \cot(\tfrac{1}{2}\varphi(t) - \tfrac{1}{2}\varphi(\tau)) \, d\tau = 0. \tag{4.3}$$

Douglas never published a proof of his claim. A derivation of (4.3) can be found in [9].

Integral equations were a highly active topic of research at the time. Therefore, the formulation of the conformality requirement on the parameterisation of the minimal surface in terms of an integral equation would have been perceived as valuable, even though the integral equation itself is intractable except for special contours. More importantly, it was a *new, non-algebraic* formulation of the conformality condition. As already stated, seeking the necessary special parameterisation of the contour as the solution of an integral equation was a significant departure from the then current strategy, adopted by Garnier, of seeking Weierstrass-Enneper data from the contour.

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[2]Abstracts will be referred to by the year of publication and number in the Bulletin of the American Mathematical Society, and listed separately at the end of this paper.

4.2 The integral equation as the Euler-Lagrange equation of a functional

Douglas was stuck for a while on how to solve (4.3) for a general contour. His important break-through, which he announced in his [6] published in the July-August 1928 issue of the *Bulletin of the American Mathematical Society*, came when he realised that (4.3) is the Euler-Lagrange equation of the first version of his later-to-be-famous A-functional:

$$A(\varphi) := -\int_0^{2\pi} \int_0^{2\pi} K(t,\tau) \log \sin \tfrac{1}{2}|\varphi(t) - \varphi(\tau)| \, dt \, d\tau \,. \tag{4.4}$$

Furthermore, he stated that he could use Fréchet's compactness theory of curves to assert the existence of a minimizer φ^* of the A-functional, at least in the case that $K(t,\tau)$ is positive for all values of t and τ. This positivity requirement on K rendered this variational problem inapplicable to the Plateau problem but he overcame this problem when he discovered that he could employ the functional (4.1) instead of (4.4).

Douglas gave many talks on his solution of the Plateau problem, especially in 1929 during his European tour. A 4-page document written in April 1929 seems to be his notes for one of these talks. It is contained in a black notebook, now in City College New York, headed Lebesgue 4ième conference, 13 Dec 1928. The relevant pages are between material dated 11 April 1929 and 29 April 1929 and the first page is reproduced on the front cover. The significance of seeking the minimal surface in any number of dimensions and of enriching the problem of Plateau by requiring that the minimal surface be a conformal image of a domain is explained in the next subsection. On the second page, he wrote: "This additional requirement is a <u>natural</u> complement of the usual enunciation which demands only M, the minimal surface, in the sense that the problem is really simpler with it, than without it" [emphasis in original]. The document ends with the integral equation (4.3). Presumably, he would have gone on to state that (4.3) is the Euler-Lagrange equation of (4.4) but there is no evidence that he had discovered his famous A-functional (4.1) at this stage. His audience, especially in Göttingen, was not always convinced that he had sorted out all the details. These criticisms were not put to rest until the discovery of (4.1) and the publication of [1]. [9] contains a fuller discussion of the CCNY notebook, the development of Douglas's ideas and the reception he got at various seminars.

4.3 The Riemann Mapping Theorem and the Least Area Property

An unexpected bonus of Douglas's method is a proof of the Riemann-Carathéodory-Osgood Theorem which follows simply by taking $n = 2$. (A little work using the argument principle is required to establish univalency of the map.) Douglas was rightly proud that his solution not only did not require any theorems from conformal mapping but that some such theorems could, in fact, be proved using his method.

However, Douglas did have to use Koebe's theorem in order to establish that his solution had least area among discs spanning Γ. He had hoped to fix this blemish but he never succeeded. That had to wait for contributions from Morrey [15] and, more recently, from Hildebrandt and von der Mosel [10]. Again, details of this matter and how it contributed to the Douglas-Radó controversy are fully discussed in [9].

4.4 Solutions of the Plateau Problem of infinite area

In the final section of [1], Douglas indicated that there were contours for which every spanning surface has infinite area. Nevertheless, he could prove the existence of a minimal surface spanning such a contour Γ as a limit of minimal surfaces spanning polygonal contours which converge to Γ. Douglas was very cross that Radó regarded the Plateau problem as meaningless for contours which could only bound surfaces of infinite area. He compared the situation to that in Dirichlet's problem, for which Hadamard had earlier constructed continuous boundary values for which the boundary value problem is solvable, even though the Dirichlet functional is identically $+\infty$. The reader is referred to [9] for a fuller account of this issue.

4.5 Higher Connectivity and Higher Genus

Even before working out all the details for the disc case, Douglas was considering the Plateau problem for surfaces of higher connectivity and higher genus. For instance, at the February 1927 meeting of the American Mathematical Society, [5], he wrote down two integral equations that have to be satisfied to solve the Plateau problem for the case of two contours in $\mathbb{R}^n$, *n arbitrary.* There is a third equation that has to be solved; it determines the conformal type of the annulus. This is another major contribution to the theory of minimal surfaces. As Douglas pointed out in [2], this form of the Plateau problem had only been raised in very special cases before (Riemann's investigation of two parallel circles, and two polygons in parallel planes) so his was the first general account. It is also amusing to note that in [2], Douglas anticipated a result reproved by Frank Morgan 50 years later in [14]!

As early as 26 October 1929, Douglas announced that his methods could be extended to surfaces of arbitrary genus, orientable or not, with arbitrary many boundary curves, in a space of any dimension.[3] He may well have had a programme at this early stage but it is doubtful that he had complete proofs. Even when he did publish details in [3], the arguments are so cumbersome as to be unconvincing. One should remember that Teichmüller theory was still being worked out at that time and that the description of a Riemann surface as a branched cover of the sphere is not ideally suited for the calculation of the dependence of the A-functional on the conformal moduli of the surface. Courant's treatment in [7] was more transparent but still awkward. The proper context in which to study minimal surfaces of higher connectivity and higher genus had to wait until the works of Sacks-Uhlenbeck [19], Schoen-Yau [20], Jost [11] and Tomi-Tromba [21]. Sacks-Uhlenbeck and Schoen-Yau introduced important methods which enabled them to establish the existence of closed minimal surfaces (without boundary) in a closed Riemannian manifold whose universal cover is not contractible or whose fundamental group contains a surface subgroup. Jost extended these methods to the boundary value problem for minimal surfaces in Riemannian manifolds. His paper is the first published complete solution of the Plateau-Douglas problem. The approach of Tomi-Tromba makes use of a differential geometric treatment of Teichmüller space.

5 The Fields medal

If priority is assigned on the basis of published papers alone, then Radó was the first to put into print a comprehensible solution of the Plateau problem in anything like generality. Douglas's announcements give the impression that he was occasionally cavalier about what he could achieve. Most of the time he delivered on his claims, but it may not always be appropriate to use the timing of his claims to determine priority issues.

In [9] we conclude that Radó and Douglas share equal credit for solving the Plateau problem for disc-like surfaces spanning a single contour which bounds at least one disc-like surface of finite area. Radó deserves full credit for solving the least area problem. Douglas, on the other hand, was the first to solve the Plateau problem in complete generality, that is, for an arbitrary contour, including ones that bound only surfaces of infinite area. He was also the only one to consider more general types of surface than the disc, to which Radó's attentions were exclusively confined. Radó's method for solving the Plateau problem shifts almost all the difficulty onto problems in conformal mapping whose solution for higher topological types was certainly not available at the time. By contrast, Douglas's method even helped solve some of these problems in conformal mapping. Thus, Douglas's contributions to the Plateau problem are more major, broader and deeper than those of Radó. Douglas's ideas, as developed later by Courant (who brought the Dirichlet integral back to the forefront), have remained important in the theory of minimal surfaces up to the present. The reader is referred to [9] for a fuller account.

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