A class of aperiodic honeycombs with tuneable mechanical properties

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\textbf{A R T I C L E   I N F O}

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metamaterial
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additive manufacturing

\textbf{A B S T R A C T}

Metamaterials are a promising area of research, offering the potential to customise the mechanical properties of designed components to address specific engineering problems. These synthetic materials are engineered structures, the behaviour of which is derived from internal geometry as well as the properties of the base material. It has been shown that designing such structures to give rise to a single desired property is relatively simple, however designing structures that give rise to combinations of desirable properties remains a challenge. This paper is concerned with a class of honeycomb metamaterials that offer the potential to independently and isotropically modify two fundamental mechanical properties, the Poisson’s ratio and the Elastic modulus. The recently discovered ‘hat’ monotile introduced a new aperiodic pattern to investigate as the basis of honeycomb structures, and it has been reported that such structures have zero Poisson’s ratio at a range of relative densities and, consequently, at a range of relative stiffnesses. Unlike most other aperiodic tilings, the ‘hat’ is part of a continuous family of aperiodic tilings, which gives the opportunity to tune combinations of mechanical properties by modifying the geometric properties of the tiling, all while maintaining isotropy. Here we present the full family of tilings and assess their mechanical behaviour both through testing and simulation. Results from computational modelling show that the behaviour of this family of metamaterials is isotropic and they offer a Poisson’s ratio from 0.01 to 0.49 at a range of relative densities, leading to the exciting conclusion that Poisson’s ratio and Elastic modulus can be tuned independently. We envisage that this finding will benefit the design of engineering components, for example by offering the possibility to match mechanical properties of metamaterial components with those of surrounding components or materials to reduce interference stresses.

\section{1. Introduction}

Modifying and designing the mechanical properties of metamaterials is a vibrant field of enquiry with techniques varying from topological geometric design \cite{1,2} and hierarchical structures \cite{3,4} to optimisation algorithms \cite{5,6}. As a measure of recent developments in this field, over 200 topologies of honeycomb are reviewed by Qi et al. \cite{7}. Many of the techniques for generating metamaterials are promising, however methods that focus on optimising a specific mechanical property often result in structures that have compromised performance in other properties \cite{8}. In particular, it is often found that when designing metamaterials to maximise mechanical performance in one direction (e.g. to respond to an expected applied force), the resulting behaviour is optimised to that specific direction, resulting in an highly anisotropic metamaterial. Recently the authors reported an alternative approach, where the behaviour of metamaterials is not designed by optimising the structure, but instead is derived from the underlying geometry of aperiodic tilings \cite{9,10}. It was shown that honeycombs based on aperiodic tilings offer a range of mechanical properties while maintaining isotropy.

Aperiodic tilings were first recognised by Robert Berger in 1966 \cite{11}, while attempting to find an algorithmic process to determine whether a given set of coloured dominoes (or ‘Wang’ tiles) can tile the plane, i.e. cover the plane with no gaps or overlaps. Berger’s proof that such an algorithm cannot exist required the introduction of a set of 20,426 tiles, which were shown to tile the plane aperiodically, without the translational symmetry that is usually apparent in tilings involving finite sets of tiles. Berger conjectured that smaller sets of aperiodic tiles could exist, and the subsequent search captured the interest of the professional and recreational mathematical community. Notably, in 1974 Sir Roger Penrose, the Nobel Laureate in Physics, discovered a set of two patterned aperiodic tiles, the so-called Penrose tiles \cite{12}, giving rise to the long-standing question: does a single tile (a ‘monotile’) exist that can only tile the plane aperiodically?

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2352-9407/© 2024 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).
The ‘hat’ tile shown in Fig. 1, is the first identified example of an aperiodic monotile, discovered after a search of more than 50 years [13]. This elusive monotile (sometimes also referred to as an ‘einstein’, from the German for ‘one stone’) was discovered in 2022, by recreational mathematician David Smith and his collaborators. Unlike the Penrose tiles, the ‘hat’ monotile requires no decoration, but instead forces aperiodic tiling of the plane by geometry alone. In March 2023 the discovery was announced and for a few weeks it was a topic of discussion in a broad range of popular media, inspiring designs of t-shirts, cookie cutters and boardgames.

Investigations into honeycombs based on the ‘hat’ tiling show that they also exhibit remarkable mechanical properties, giving rise to isotropic, nominally zero Poisson’s ratio [14]. More than just an isolated case, this zero Poisson’s ratio was found to exist between relative densities of 0.2 and 0.4 (within limitations due to manufacturing technique) meaning zero Poisson’s ratio is possible for a range of effective elastic moduli. In modern engineering, materials with zero Poisson’s ratio are desirable but elusive, mainly found in biological rather than engineering materials [15–17], and this discovery has the potential to lead to innovative solutions, for example in aeronautical and biomedical engineering [18,19].

Perhaps more surprising than the discovery of the ‘hat’, was the subsequent recognition by Smith et al. [13] that it is a member of an infinite family of monotiles which, with the exception of three tiles, all only tile the plane aperiodically. The family of tiles is derived according to the geometry of the ‘hat’, which has fourteen edges of two different lengths, specified by the parameters a and b in Fig. 1. Other members of the family result from changing the relative lengths of a and b, whilst keeping the orientation of the sides constant. In Fig. 1, seven examples of this infinite family of tiles are shown, including the ‘hat’. These tilings are identified using a vector notation (a, b), where a and b denote the changing relative edge lengths, for example in the ‘hat’ the ratio between side lengths is $1 : \sqrt{3}$, so it is identified as $(1, \sqrt{3})$, with $a = 1$ and $b = \sqrt{3}$. For simplicity, one of a and b is always chosen to be 1, as it is the ratio of the two values that is important for the geometry, not their absolute size. Smith et al. [13] identified that when any of $a = 0$, $a = b$, or $b = 0$ hold, then the resulting tiles, $(0, 1)$, $(1,1)$ and $(1,0)$ respectively, can also tile the plane periodically and so fail to be aperiodic monotiles. They also showed that these are the only tiles in the infinite family that tile periodically, and all the other tilings, defined between these extremes by changing the ratio of a to b, are aperiodic.

The discovery of the ‘hat’ family of tilings led Smith et al. [13] to an
patterns [9], including the ‘hat along an irrational direction. Euclidean space down onto the two-dimensional Euclidean plane showing the incommensurability of an underlying lattice structure for mechanism for the underlying aperiodicity of the concerned tilings. By tilings had been discovered before the ‘hat mathematical community. Although parametrised families of aperiodic mechanical properties that have a broad scope of potential applications. can be incorporated in metamaterials with varying geometry of these tilings, defined according to two parameters, eycombs based on the extended ‘hat vestigations into the mechanical properties of an infinite class of hon. lie), they also proved that each of these infinitely many other tiles original proof of aperiodicity that was wholly unexpected by the mathematical community. Although parametrised families of aperiodic tilings had been discovered before the ‘hat family, this was the first time that the existence of the infinite family itself actually explained the mechanism for the underlying aperiodicity of the concerned tilings. By showing the incommensurability of an underlying lattice structure for the (0,1) and (1,0) tilings (a periodic substructure on which the vertices lie), they also proved that each of these infinitely many other tiles (derived from changing the relative lengths of a and b) can only tile the plane aperiodically. The fact that the infinite family exists also guided the study of Baake et al. [20] in showing that there is a higher dimensional super structure that supports all of the tilings in the family at once. They were able to demonstrate that the underlying point-set given by taking the centre of mass of each of the tiles is generated by projecting carefully chosen points from an integer lattice in four-dimensional Euclidean space down onto the two-dimensional Euclidean plane along an irrational direction.

Building on investigations into honeycombs based on aperiodic patterns [9], including the ‘hat’ tiling [14,21], this paper reports investigations into the mechanical properties of an infinite class of honeycombs based on the extended ‘hat’ family of tilings. The continuously varying geometry of these tilings, defined according to two parameters, results in honeycombs that exhibit a continuous range of isotropic mechanical properties, which can be incorporated in metamaterials with tuneable mechanical properties that have a broad scope of potential applications.

<table>
<thead>
<tr>
<th>Sample</th>
<th>a length</th>
<th>b length</th>
<th>Orientation (deg)</th>
<th>Density (at 0°)</th>
<th>Density (at 0°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.3</td>
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</tr>
<tr>
<td>(1,4)</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0.3</td>
<td>0.2 to 0.4</td>
</tr>
<tr>
<td>(1,√3)</td>
<td>1</td>
<td>√3</td>
<td>0, 45, 90</td>
<td>0.25, 0.3,</td>
<td>0.2 to 0.4</td>
</tr>
<tr>
<td>(1,1)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.3</td>
<td>0.2 to 0.4</td>
</tr>
<tr>
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<td>0</td>
<td>0.3</td>
<td>0.2 to 0.4</td>
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<tr>
<td>(4,1)</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0.3</td>
<td>0.2 to 0.4</td>
</tr>
<tr>
<td>(1,0)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.3</td>
<td>0.2 to 0.4</td>
</tr>
</tbody>
</table>

Fig. 3. Example toolpaths with circles denoting the start of a path and the arrow the end.

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Building on investigations into honeycombs based on aperiodic patterns [9], including the ‘hat’ tiling [14,21], this paper reports investigations into the mechanical properties of an infinite class of honeycombs based on the extended ‘hat’ family of tilings. The continuously varying geometry of these tilings, defined according to two parameters, results in honeycombs that exhibit a continuous range of isotropic mechanical properties, which can be incorporated in metamaterials with tuneable mechanical properties that have a broad scope of potential applications.

Fig. 2 illustrates how the range of densities and orientation were generated for testing and simulation. Fig. 2a shows how samples were produced to assess the isotropy of the mechanical properties of honeycombs, by extracting 50×50 mm patches of pattern at different orientations; the examples shown are for 0° and 45°. Similarly, Fig. 2b shows how the influence of relative density on the mechanical properties was assessed by taking different sized sub patches and scaling the pattern to 50×50 mm to give an effective change in relative density, resulting in samples with higher or lower relative density, while retaining identical wall thickness.

Physical samples were additively manufactured with dimensions 50×50×50 mm by extruding the two-dimensional patterns into the third dimension. Samples were produced in accordance with ASTM D1621 [23] out of Polylactic acid (PLA) using fused deposition modelling (FDM). To ensure that samples were nominally identical a constant layer height of 0.2 mm was used as well as a wall thickness of 0.5 mm, which resulted in walls consisting of a pair of tool paths, without infill, as shown in Fig. 3.

Honeycombs were compression tested, as illustrated in Fig. 4a at a constant displacement rate of 0.5 mm/min to a maximum strain of 35%. Strain on the sample was measured using digital image correlation (DIC) [24] to determine live strain in two orthogonal directions and to provide spatial quantification of variations in strain. For real time quantitative data analysis, LaVision DaVis 10.2.1.81611 software¹ was used to apply live virtual strain gauges. The vertical gauge was placed between the bottom and top surfaces to measure the normal strain and a box strain gauge was placed between the vertices closest to each side. The positioning of these virtual strain gauges is shown in Fig. 4b. The gauge occupied the central third of the sample vertically so that measurements were not affected by constraints imposed by the top and bottom plates. Poisson’s ratio was calculated using the strain in the horizontal-direction obtained from this gauge and the vertical-direction strain measured using the virtual strain gauge. Full field strain was calculated again with LaVision DaVis 10.2.1.81611. An algorithmic mask was applied to limit calculation to correlating regions of the honeycomb, and displacements were calculated using a 15-pixel window. Two-dimensional maps of strain before the maximum stress were extracted after applying a rigid body motion subtraction while maps of maximum shear stress were extracted after performing a 25-pixel gaussian smoothing step.

Simulation of the mechanical properties of honeycombs based on aperiodic tilings is complicated by the incompatibility of periodic boundary conditions and the aperiodic structure. A bespoke tool presented by Imiediegwu et al. [25] based on Asymptotic Expansion Homogenisation (AEH) [26] using FEniCS², was employed to perform numerical simulations of the honeycombs. Definition of the topologies was carried out using element-based assignment of material properties where the structures are treated as composites consisting of material and void, the void being assigned as weak isotropic material, in this case E =
100 Pa, also described by Imediegwu et al. in [25]. Instead of constructing CAD models of the honeycombs for modelling, a mesh, consisting of regular triangular elements is created that covers the entire sample area and the structure is created mathematically by assigning properties based on the elements’ proximity to the cell wall. The element nodes are first classified as inside or outside the structure, a node fraction is then assigned to each element based on the fraction of nodes that lie inside the honeycomb structure. Intermediate properties are assigned to elements that do not fully lie within the cell wall based on their node fraction. The necessary number of elements across a strut was determined through mesh sensitivity studies presented in [25], and consequently the initial mesh was generated in order to maintain 7 elements across each strut. The effective properties were approximated by using an adapted method of AEH [25], where an homogenous region is created surrounding the honeycomb region and is given mechanical properties equal to those determined in iteration 0, the effective properties of the domain in its entirety are iteratively re-calculated and assigned to the homogenised region until convergence is obtained. The homogenised properties of this boundary region permit the application of periodic boundary conditions despite the lack of periodicity in the aperiodic tiling itself. In addition to the mech convergence study, a convergence study of the required size of homogenised region is also found in [25]. The honeycombs were simplified to two-dimensional lattices with displacements obtained using Hooke’s law for plane strain conditions. For this study the overall compliance matrix was of most use for comparison of the bulk properties. The compliance matrix from each simulation was transformed for each degree of rotation and the elastic modulus and Poisson’s ratio extracted. By using this method, repeated meshing of the sample is not required for each rotation and the rotational analysis can be automated and carried out with no input from the analyst. This approach has been validated against known mechanical properties of well-understood honeycombs [25], and against results from compression testing [22].

FDM is known to introduce anisotropy into models [27] and printed material may exhibit different mechanical properties depending on toolpath and print parameters such as layer height [28], and we have attempted to minimise this variation as discussed above and shown in Fig. 3. Despite this, FEA models using datasheet values still differ when compared with experimental results [29]. For this reason, when normalising results, the same correction factor has been applied to all experimental data to compensate for deviations from datasheet values, accuracy of manufacturing processes and added stiffness caused by the addition of top and bottom plates.

Fig. 4. a) experimental setup showing compression rig with additively manufactured ‘hat’ tiling based honeycomb and b) an example DIC image with virtual strain gauge placement overlays coloured red for the vertical strain gauge and green for the box gauge.

Fig. 5. a) stress strain curves for the hat family samples orientated at 0 degrees, b) Poisson’s ratio calculated for the elastic region for each of the hat family samples orientated at 0 degrees (strain measured using DIC).
3. Results

Stress-strain curves from mechanical testing of samples with relative density of 0.3 and orientation of 0° are shown in Fig. 5a. With the exception of honeycombs based on the ‘hat’ (1, √3) and also the (1,1) tilings, the stress-strain curves show the typical drop in stress after the initial loading followed by a plateau region as observed in stretch dominated structures [30]. By contrast, the ‘hat’ (1, √3) and the (1,1) honeycombs show a smooth transition into the plateau region typical of bending dominated structures. In samples (0,1) and (1,0) the transition from the plateau to continuously increasing stress is observed at ~15% strain while this transition occurs at ~25% strain for all other tilings. Fig. 5b presents the Poisson’s ratio for each of the stress strain curves calculated using strain measurements made by DIC. The honeycomb based on the ‘hat’ (1, √3) tiling is observed to have Poisson’s ratio just below zero. The honeycomb based on the (1,1) tiling initially has Poisson’s ratio just below zero and increases slightly to around 0.05 during elastic loading. Samples either side of these two tilings show an increase in Poisson’s ratio up to above 0.40 with ~0.42 for the (0,1) honeycomb and ~0.45 for the (1,0) honeycomb. The results presented here do not cover the start of the elastic region because at very low strains the uncertainties in measurements cause large fluctuations in the calculated Poisson’s ratio.

Fig. 6 presents rotational plots for the normalised elastic modulus, $E/E_s$, and Poisson’s ratio, $v$, calculated from simulations. Fig. 6a shows the normalised elastic modulus for different tiling patterns and relative densities. The normalised elastic modulus increases with increasing relative density for all tiling patterns, as shown in Fig. 6a. Fig. 6b presents the Poisson’s ratio for each tiling pattern, with the ‘hat’ (1, √3) tiling showing the lowest Poisson’s ratio of 0.006 and the (0,1) and (1,0) honeycombs showing the highest, fluctuating between 0.432 and 0.491. All other honeycombs fit somewhere between 0.006 and 0.491. A key finding here is that for any tiling, the Poisson’s ratio of the resulting honeycomb is essentially independent of relative density, within the range of densities simulated (i.e. between 0.2 and 0.4).

Fig. 7a and b presents three-axis plots of the complete set of simulations showing normalised elastic modulus, $E/E_s$, and Poisson’s ratio, $v$, for different tiling patterns and relative densities. Fig. 7a shows the influence of tiling pattern on the normalised elastic modulus, $E/E_s$, with lines denoting simulated data and filled circular markers representing results from physical testing. Results are shown for (a) normalised elastic modulus with changing edge length ratio, (b) normalised elastic modulus for honeycombs based on the ‘hat’ (1, √3) tiling with changing density, (c) Poisson’s ratio with changing edge length ratio and (d) Poisson’s ratio for honeycombs based on the ‘hat’ (1, √3) tiling with changing density.

<table>
<thead>
<tr>
<th>Tiling (a,b)</th>
<th>$I$</th>
<th>$\tau_{max} - \tau_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>0.042</td>
<td>0.026</td>
</tr>
<tr>
<td>(1,4)</td>
<td>0.026</td>
<td>0.034</td>
</tr>
<tr>
<td>(1, √3)</td>
<td>0.780</td>
<td>0.031</td>
</tr>
<tr>
<td>(1,1)</td>
<td>0.176</td>
<td>0.031</td>
</tr>
<tr>
<td>(√3,1)</td>
<td>0.076</td>
<td>0.039</td>
</tr>
<tr>
<td>(4,1)</td>
<td>0.043</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Fig. 7b shows simulated average Poisson’s ratio for each honeycomb. The honeycomb based on the ‘hat’ (1, √3) tiling has the lowest Poisson’s ratio of 0.006 and the (0,1) and (1,0) honeycombs show the highest, fluctuating between 0.432 and 0.491. All other honeycombs fit somewhere between 0.006 and 0.491. A key finding here is that for any tiling, the Poisson’s ratio of the resulting honeycomb is essentially independent of relative density, within the range of densities simulated (i.e. between 0.2 and 0.4).

Rotational plots of elastic modulus and Poisson’s ratio are presented in Fig. 7. Table 2 shows the isotropy factor and spread in Poisson’s ratio for each tiling determined from modelling at a density of 0.3.
circles showing discrete points corresponding to physical test data points. All simulations show close to isotropic behaviour with similar trends at all orientations to those shown for a single orientation in Fig. 6. There is good agreement between results from measurements and simulations. Poisson’s ratio results are shown in Fig. 7c; these also show slight deviations from isotropic behaviour but, as with the elastic modulus, these deviations are not sufficient to cause discrepancies from the trends observed in Fig. 6, with the lowest Poisson’s ratio from the honeycomb based on the ‘hat’ (1,√3) tiling. Figs. 7b and d show the normalised elastic modulus and the Poisson’s ratio for honeycombs based on the ‘hat’ (1,√3) tiling for different densities. Again, the behaviour observed in Fig. 6 is apparent, with different densities giving...
rise to variations in normalised elastic modulus, but for the Poisson’s ratio nominally identical behaviour for all densities.

All data presented in Fig. 7 shows a low level of anisotropy. However, there are deviations from the perfectly circular plots needed for ideal isotropy. The extent of similar deviations observed in periodic honeycombs have been quantified in [22] using an isotropy factor defined as

\[ I = \frac{\nu_{\text{max}} - \nu_{\text{min}}}{\nu_{\text{max}} + \nu_{\text{min}}} \]

where \( \nu_{\text{max}} \) and \( \nu_{\text{min}} \) are the maximum and minimum effective Poisson’s ratio for a specific honeycomb across all in-plane orientations, respectively. The isotropy factors for honeycombs based on seven tilings are found in Table 2 where values closer to 1 represent higher levels of isotropy. Because the numerator of isotropy factor is highly dependent on the maximum effective Poisson’s ratio, honeycombs with low Poisson’s ratio show higher isotropy factor than for honeycombs with high Poisson’s ratio with a similar spread. Therefore, Table 2 also shows the spread in Poisson’s ratio. The spread in Poisson’s ratio is between 0.013 for honeycombs based on the (1,4) tiling and 0.039 for honeycombs based on the (\( \sqrt{3},1 \)) tiling while the isotropy factor has a much larger range between 0.0176 for the (1,1) honeycombs and 0.780 for the (1,\( \sqrt{3} \)) honeycombs. When considering isotropy in terms of the potential impact it may have on mechanical performance of honeycombs loaded in different orientations, the spread in Poisson’s ratio appears to be a better representation. Both representations of isotropy are included in Table 2 for completeness.

To further explore these results, two-dimensional maps of transverse displacement (colour maps) and total displacement (arrows) are shown in Fig. 8. These show the deformation of the honeycombs, under uniaxial compressive loading to global strains of 1% and 3%. The honeycombs based on the (0,1) and (1,0) tilings show the greatest displacement with both honeycombs showing greater than 0.15 mm of displacement at each side of the sample. The honeycomb based on the ‘hat’ (1,\( \sqrt{3} \)) tiling is observed to have almost negligible lateral displacement, as evidenced by the colour map and the absence of any deviation from vertical in the displacement arrows. In contrast the (1,0) and (0,1) honeycombs show a strong curve in the arrows from the loading direction at the top and bottom towards the transverse direction at the midpoint of each side of the sample. The honeycomb based on the (1,1) tiling shows very little overall transverse deformation, but in contrast to the ‘hat’ honeycomb, there is some evidence of bands of highly opposing transverse displacement. In the 3% strain data, this can be seen as the red and blue regions within the central region of the sample, but these average to a value close to zero. Compare this to the central region of the honeycomb based on the ‘hat’ tiling which has few regions exceeding 0.02 mm of displacement. The honeycomb based on the (1,4) tiling, which is geometrically between the ‘hat’ and the (0,1) tiling is observed to have behaviour somewhere between the two. Similarly, the honeycombs based on the (\( \sqrt{3},1 \)) and (4,1) tilings show a gradual transition from the banding in the (1,1) tiling to the regions of opposing transverse displacement on each side of the sample observed in the honeycomb based on the (1,0) tiling.

Fig. 9 shows the yield strength calculated from the testing data for honeycombs at a relative density of 0.3. The highest yield strength of 1.4 MPa was obtained from the honeycomb based on the (1,0) tiling with second highest at the other extreme in the (0,1) honeycomb. A ‘V’ shaped trend is observed with respect to the tiling parameters with a decreasing yield strength moving away from the extremes with the lowest of ~0.75 MPa for the central, (1,1) honeycomb.

It was observed that localised plastic deformation started to occur in all samples beyond 3% strain. Fig. 10 shows images of the deformed samples at 5% and 10% strain, with the maximum shear strain superimposed as a colour map. The scale of the colour map is semi-transparent below 5% strain, highlighting regions of high magnitude of shear strain. At a global strain of 5%, localised shear strain is present in multiple regions in all samples with the exception of the honeycomb based on the (1,0) tiling, which exhibits a thin, diagonal band of shear strain. At 10% global strain the distribution of shear strain has separated into two groups. Honeycombs based on the (1,0), (0,1), and (1,4) tilings show distinct, thin diagonal bands of localised shear strain with very little present elsewhere, while honeycombs based on the (1,3), (1,1), (\( \sqrt{3},1 \)) and (4,1) tilings show varying levels of spatially dispersed shear strain, with significantly less localisation. There are also fewer occurrences of diagonal bands and more horizontal bands for tilings closer to the (1,1) tiling.

4. Discussion

The ‘hat’ family of aperiodic tilings has been used as the basis of additively manufactured honeycombs with a variety of aperiodic geometries. As a direct consequence of the continuous nature of the ‘hat’ family, it is possible to alter the geometry of the tiling in infinitesimally small increments and therefore alter the mechanical properties continuously. Results presented in Fig. 6 show that by changing the tiling parameters, a Poisson’s ratio between 0.006 and 0.491 can be obtained. These values of Poisson’s ratio also show levels of isotropy comparable to honeycombs reported in [22] (see Fig. 7 and Table 2). Also, Fig. 5b and Fig. 8 confirm that the Poisson’s ratio is maintained throughout the elastic region. In addition to the tiling parameters, varying the relative densities between 0.2 to 0.4 results in an approximately constant Poisson’s ratio for each set of tiling parameters (see Fig. 6b) while maintaining isotropy (see Fig. 7 and Table 2). Despite the constant Poisson’s ratio the normalised elastic modulus varies significantly between the range of 0.003 and 0.056, as a consequence of the change in density as shown in Fig. 6a. More importantly, all simulated Poisson’s ratio values are available at a normalised elastic modulus of below 0.025. This finding is significant because it means that, within the densities simulated, the Poisson’s ratio and the normalised elastic modulus of these honeycombs are independently tunable within the intervals [0.006, 0.491] and [0.003, 0.025] respectively.

The two-dimensional maps of lateral displacement in Fig. 8 confirm qualitatively that the same distribution of strain is observed at both 1% and 3% strain, but with an increase in magnitude. For honeycombs with positive Poisson’s ratio, such as those based on the (1,0) and (0,1) tilings a lobe of lateral translation is present on either side of the central region, and this is a typical deformation, for example in keeping with the hexagonal honeycombs presented by Clarke et al. [14]. The low Poisson’s ratio honeycombs, i.e. those based on the (1,1) and (1,\( \sqrt{3} \)) tilings, do not show this deformation at either 1% or 3% global strain; instead a horizontal banded structure of alternating displacement of various
magnitudes is observed, as illustrated in Fig. 11. This finding can explain the presence of low Poisson’s ratio in these honeycombs. During compressive loading, the space between the cell walls reduces and the lateral bulging observed in many structures is the physical response to the reduction of available space, with material being pushed to the sides. In the honeycombs based on the (1,1) and (1,√3) tilings however, some material is able to move one way, while adjacent material moves the other way. This appears to be possible due to the flexibility in the internodal struts, which in these cases are made of a three-strut spring-like assembly. These ‘springs’ are much less stiff than a continuous strut and can accommodate large amounts of deformation by crumpling, hinging and to a lesser extent, by straightening.

Fig. 10. Images of progression of the accommodation of shear strain for each of the tested samples.
This opposing behaviour is evident in the bands of lateral deformation in Fig. 8, and is also evident beyond the elastic limit as shown in Fig. 10 where bands of horizontal deformation are observed in the honeycombs based on the (1,1) and (1,√3) tilings. Fig. 11 shows this in detail with an example of the (1,√3) honeycomb showing crumpling of a cell and the (1,1) honeycomb showing straightening. Honeycombs based on other tilings in the family do not have such flexibility in the ‘spring’ because of the ratio of side lengths.

As a further consequence of the spring-like struts, the honeycombs based on tilings close to the (1,1) tiling have significantly lower yield strength compared to those based on (1,0) and (0,1) tilings. In future work it will be interesting to see how much of the crumpling can be recovered during cyclic loading and how dependant this is on material of manufacture.

Unlike many low-Poisson’s ratio metamaterials, such as honeycombs based on re-entrant or double-V patterns [22], the honeycombs based on the (1,1) and (1,√3) tilings have the ‘spring’ like struts in many directions and orientations. For example, the re-entrant pattern also has a spring-like strut assemblies, but only on two sides of the unit cell, thus resulting in low Poisson’s ratio only in very limited orientations. Initial investigations into isotropic low Poisson’s ratio behaviour of honeycombs based on the ‘hat’ tiling, first presented by Clarke et al. [14], did not identify a single mechanism to explain this behaviour. The ‘hat’ tiling is based on an underlying hexagonal grid, and it was hypothesised that this could be a contributing factor to the isotropic nature of the behaviour of the honeycomb. However, this same explanation does not hold true for honeycombs based on other members of the ‘hat’ family, which are not all based on an underlying hexagonal grid. Of the infinite family, only the (0,1), (1,√3), (√3,1), and (1,0) tilings have this underlying hexagonal grid, but near isotropic behaviour is observed in all tilings from the family. Therefore, it is highly probable that it is the aperiodicity of the geometry of the patterns that governs the isotropic behaviour rather than the underlying hexagonal grid, as suggested by Imediegwu et al. [9].

5. Summary

The findings reported in this manuscript form an experimental and computer simulation study into the mechanical properties of additively manufactured honeycombs with geometries derived from the ‘hat’ family of aperiodic tiling. In recent work [22] it was reported that honeycombs based on the ‘hat’ tiling resulted in an isotropic zero Poisson’s ratio. Along with the ‘hat’ tiling, Smith et al. [13] also reported a continuous family of infinite tilings that were also aperiodic and geometrically related to the ‘hat’ but with different ratios of side lengths. By altering the length of the two side components, an infinitely variable set of tilings can be constructed. It is as a direct consequence of this that honeycombs with geometry based on this family of tilings has yielded a continuously variable set of mechanical properties. Specifically, Poisson’s ratio can be varied from between 0.49 and 0.01 by altering the relative lengths of the sides. Furthermore, between densities of 0.2 and 0.4, the Poisson’s ratio for honeycomb with any specific set of side lengths remains constant. Although mathematically there is an infinite number of options between the (0,1) and (1,0) tilings, in reality the possible number that could be created into honeycombs is finite and limited by the resolution of the manufacturing technique used.

A metamaterial with tuneable mechanical properties has broad scope for application, however the relatively low normalised modulus lends the tilings to biomedical applications where low stiffness and patient-to-patient tailoring is desirable. For example, surgical meshes [31], orthopaedic implants [32] and spinal disk replacements [33] all suffer from potential failure as a result of overly stiff materials, and their performance could potentially be improved if their design included metamaterials based on honeycombs derived from tilings taken from the infinite ‘hat’ family of monolites.

CRediT authorship contribution statement

Richard J. Moat: Writing – original draft, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. Daniel John Clarke: Writing – review & editing, Visualization, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. Francesca Carter: Writing – review & editing, Conceptualization. Dan Rust: Writing – review & editing. Iestyn Jowers: Writing – review & editing, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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