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The structure of digraphs with excess one

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Abstract

A digraph G is k -geodetic if for any (not necessarily distinct) vertices u, v there is at most one directed walk from u to v with length not exceeding k . The order of a k -geodetic digraph with minimum out-degree d is bounded below by the directed Moore bound $M(d, k) = 1 + d + d^2 + \dots + d^k$. The Moore bound can be met only in the trivial cases $d = 1$ and $k = 1$, so it is of interest to look for k -geodetic digraphs with out-degree d and smallest possible order $M(d, k) + \epsilon$, where ϵ is the excess of the digraph. Miller, Miret and Sillasen recently ruled out the existence of digraphs with excess one for $k = 3, 4$ and $d \geq 2$ and for $k = 2$ and $d \geq 8$. We conjecture that there are no digraphs with excess one for $d, k \geq 2$ and in this paper we investigate the structure of minimal counterexamples to this conjecture. We severely constrain the possible structures of the outlier function and prove the nonexistence of certain digraphs with degree three and excess one, as well closing the open cases $k = 2$ and $d = 3, 4, 5, 6, 7$ left by the analysis of Miller et al. We further show that there are no involutory digraphs with excess one, that is, the outlier function of any such digraph must contain a cycle of length ≥ 3 .

KEYWORDS

Cage, degree/geodecity problem, digraph, excess

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AMS CLASSIFICATION

05C35, 05C20, 90C35

1 | INTRODUCTION

The undirected degree/girth problem asks for the smallest possible order of a graph that has degree d and girth g . If $g = 2k + 1$ is odd, then the order of any such graph is bounded below by the undirected Moore bound $1 + d + d(d - 1) + \dots + d(d - 1)^{k-1}$. A graph that has an order exceeding the Moore bound by ϵ has *excess* ϵ ; a graph that meets the Moore bound is called a *Moore graph*. It was shown in [1, 10, 15] that, with the trivial exception of cycles and complete graphs, for odd $g = 2k + 1$ the Moore bound can be met only for diameter $k = 2$ and degrees $d = 3$ (the Petersen graph), $d = 7$ (the Hoffman–Singleton graph) and possibly $d = 57$. Interestingly, it is even harder for a graph to have excess one than to meet the Moore bound; Bannai and Ito showed in [2] that there are no graphs with excess one and girth $2k + 1 \geq 5$. For a survey of the degree/girth problem, see [11].

A directed graph G is *k-geodetic* if for any pair of (not necessarily distinct) vertices u, v of G there is at most one directed walk from u to v in G with length $\leq k$. The following analogue for directed graphs of the degree/girth problem was raised by Sillassen in [21]:

Problem 1 (Degree/geodeticity problem). What is the smallest possible order of a k -geodetic digraph with minimum out-degree d ?

It was shown in [26] that for any $d, k \geq 1$ there exists a *diregular k-geodetic digraph* with degree d , so this problem is well-posed.

A lower bound on the order of a k -geodetic digraph G with minimum out-degree d can be derived as follows. Fix a vertex u of G ; this will be Level 0 of a directed *Moore tree* of depth k . There are at least d out-neighbours of u ; draw these at Level 1 of the tree with arcs from u at Level 0. In general once we have completed Level t of the tree, where $0 \leq t \leq k - 1$, we draw arcs from each vertex of Level t to each of its out-neighbours at Level $t + 1$. It follows by induction that for $0 \leq t \leq k$ there are at least d^t vertices at Level t of the Moore tree. Moreover by k -geodeticity all of the vertices in this tree must be distinct. It follows that the order of G is bounded below by the *directed Moore bound*

$$M(d, k) = 1 + d + d^2 + \dots + d^k.$$

It was shown by Bridges and Toueg in [6] that the only digraphs that meet the directed Moore bound are directed cycles and complete digraphs. Hence for $d, k \geq 2$ we are interested in k -geodetic digraphs with minimum out-degree d and order $M(d, k) + \epsilon$ for some small *excess* ϵ . A smallest such digraph is a *geodetic cage*.

In this paper we will be interested in digraphs with excess $\epsilon = 1$; we will call such a digraph a $(d, k; +1)$ -digraph, which we will typically denote by the letter G . For every vertex u of such a digraph there is a unique vertex $o(u)$ such that the distance from u to $o(u)$ in G satisfies $d(u, o(u)) \geq k + 1$; we call the associated function $o : V(G) \rightarrow V(G)$ the *outlier function* of G . The following results on the structure of $(d, k; +1)$ -digraphs were proven in [18, 21].

Lemma 1 (Miller et al. [18] and Sillassen [21]). *If G is a $(d, k; +1)$ -digraph, then*

- G is diregular.
- Either G is a $(d, 2; +1)$ -digraph, where d lies in the range $3 \leq d \leq 7$, or a $(d, k; +1)$ -digraph with $d \geq 3$ and $k \geq 5$, or a directed $(k + 2)$ -cycle.
- The outlier function o of G is an automorphism.

Further results on the problem and examples of geodetic cages can be found in [23–26]. As no nontrivial digraphs with excess one have been found, we make the following conjecture.

Conjecture 1. *There are no $(d, k; +1)$ -digraphs with $d, k \geq 2$.*

It is instructive to compare this problem with the state of the art of the directed degree/diameter problem (a survey of which can be found in [19]), which asks for the largest digraphs with given maximum degree d , diameter k and order $M(d, k) - \delta$ for some small defect δ . A digraph with defect $\delta = 1$ is a $(d, k; -1)$ -digraph or an *almost Moore digraph*.

In the Moore tree of a $(d, k; -1)$ -digraph rooted at a vertex u there will be a unique vertex $r(u)$ that is repeated in the tree; this vertex is called the *repeat* of u and the associated *repeat function* $r : V(G) \rightarrow V(G)$ is also a digraph automorphism [4]. All $(d, k; -1)$ -digraphs are diregular [17]. By contrast with the degree/geodeticity problem, for all $d \geq 3$ there is a unique $(d, 2; -1)$ -digraph [12–14]; however there are no $(d, k; -1)$ -digraphs for $d \geq 2$ and diameters $k = 3, 4$ [8, 9]. The existence of $(2, k; -1)$ -digraphs with $k \geq 3$ was ruled out in [16], as were $(3, k; -1)$ -digraphs with $k \geq 3$ in [5].

The plan of this paper is as follows. First in Section 2 we use counting arguments to deduce some strong conditions on digraphs with excess one and a high level of symmetry. In Section 3 we investigate the structure of digraphs with degree three and excess one. Then in Section 4 we use the approach of Sillasen in [22] to analyse the set of vertices fixed by an automorphism of a $(d, k; +1)$ -digraph. As the outlier function o is an automorphism of G these results tell us quite a lot about the structure of o as a permutation. In the final part of this paper, Section 5, we use a spectral approach to exploit the results of Section 4 to prove the nonexistence of certain $(d, k; +1)$ -digraphs.

2 | VERTEX-TRANSITIVE DIGRAPHS WITH EXCESS ONE

All of the known Moore graphs are vertex-transitive. This suggests that it is of interest to look for digraphs with order close to the directed Moore bound that have a high degree of symmetry. In her thesis [20] Sillasen uses this approach on digraphs with defect one; we now emulate this approach for digraphs with excess one.

In [20] as a basis for her counting arguments Sillasen divides the vertices of an almost Moore digraph into two types: Type I and Type II. Adapting this notation, we make the following definition.

Definition 1. A vertex u of a $(d, k; +1)$ -digraph is Type II if $o(u) \rightarrow u$; otherwise u is Type I.

The type of a vertex is preserved by any automorphism of G . This leads us to the following observation.

Observation 1. If a $(d, k; +1)$ -digraph G is vertex-transitive, then either every vertex of G is Type I or every vertex of G is Type II.

If every vertex of G is Type I, then we can obtain a strong divisibility condition on d and k . The reason for this is that all directed $(k + 1)$ -cycles of G are arc-disjoint.

Lemma 2. No arc of G is contained in more than one directed $(k + 1)$ -cycle.

Proof. Suppose that an arc (u, v) is contained in two $(k + 1)$ -cycles. Then there are two distinct k -paths from v to u , which contradicts k -geodesicity. \square

Lemma 3. Any arc (u, v) of G such that $u \neq o(v)$ lies in a unique $(k + 1)$ -cycle.

Proof. Let (u, v) be such an arc. As $u \neq o(v)$ there is a path of length k in G from v to u , so the arc (u, v) is contained in a $(k + 1)$ -cycle, which is unique by Lemma 2. \square

Corollary 1. Suppose that every vertex of G is Type I. Then $(k + 1)$ divides $d(M(d, k) + 1) = 2d + d^2 + d^3 + \dots + d^{k+1}$.

Proof. As all vertices of G are Type I, by Lemma 2 we can partition the arcs of G into disjoint $(k + 1)$ -cycles. Therefore the size $m = d(M(d, k) + 1)$ of G is divisible by $k + 1$. \square

Computer search shows that for k between 2 and 10,000, the following values of d and k satisfy this condition:

$d = 3 : k = 2, 20, 146, 902, 1028, 6320, 7202,$

$d = 4 : k = 3, 7, 87, 171, 472, 2647,$

$d = 5 : k = 4, 84, 114,$

$d = 6 : k = 2, 3, 5, 7, 11, 23, 32, 51, 203, 347, 1095, 3323, 3767, 4903, 9563,$

$d = 7 : k = 6, 76, 118, 2568,$

$d = 8 : k = 3, 7, 9, 15, 87, 463, 1171,$

$d = 9 : k = 2, 8, 68,$

$d = 10 : k = 3, 4, 7, 9, 15, 19, 39, 79, 555, 1069, 2314, 2986, 4659,$

$d = 11 : k = 10,$

$d = 12 : k = 2, 3, 5, 7, 11, 13, 23, 55, 91, 163, 236, 1235, 1356.$

We see that the condition in Corollary 1 is quite strong. On the other hand, if G contains a Type II vertex then these vertices group themselves into cycles in a natural way.

Lemma 4. Suppose that G contains a Type II vertex u . Then u has a unique Type II out-neighbour, namely, $o^-(u)$.

Proof. Applying the automorphism o^- to the arc $(o(u), u)$, we see that $(u, o^-(u))$ is also an arc. By inspection, $o^-(u)$ is a Type II vertex. Suppose that u' is an arbitrary Type II out-neighbour of u . As (u, u') is an arc, so is $(o(u), o(u'))$. As u' is Type II, $(o(u'), u')$ is an arc. We therefore have paths $o(u) \rightarrow u \rightarrow u'$ and $o(u) \rightarrow o(u') \rightarrow u'$, so by k -geodesicity we must have $o(u') = u$, that is, $u' = o^-(u)$. \square

It follows immediately that in a vertex-transitive $(d, k; +1)$ -digraph all vertices must be Type I.

Lemma 5. *If G is vertex-transitive, then every vertex of G is Type I.*

Proof. Suppose that G contains a Type II vertex; by vertex-transitivity, every vertex is Type II. But this contradicts Lemma 4. \square

In particular, if G is vertex-transitive, then it must satisfy the divisibility condition in Corollary 1.

Corollary 2. *Let G be a vertex-transitive $(d, k; +1)$ -digraph. Then $(k + 1)$ divides $2d + d^2 + d^3 + \dots + d^{k+1}$.*

In fact, we can significantly extend Corollary 1 using the fact that for any vertex u of a vertex-transitive $(d, k; +1)$ -digraph the vertex $o^-(u)$ cannot be close to u .

Lemma 6. *If G is a vertex-transitive $(d, k; +1)$ -digraph, then for any vertex u of G we have $d(u, o^-(u)) \geq k$.*

Proof. Suppose that $d(u, o^-(u)) = t \leq k - 1$. As G is vertex-transitive, the distance from any vertex v of G to $o^-(v)$ is t . Writing $N^+(u) = \{u_1, u_2, \dots, u_d\}$, let $o^-(u) \in T_{k-2}(u_1)$. As $d(u_2, o^-(u_2)) = t \leq k - 1$, we have $o^-(u_2) \in T(u_2)$. However, as o^- is an automorphism of G , there is an arc $o^-(u) \rightarrow o^-(u_2)$, so that $o^-(u_2)$ also lies in $T(u_1)$. As $o^-(u_2)$ appears twice in the Moore tree, k -geodecity is violated. Hence $t \geq k$. \square

Theorem 1. *For $1 \leq t \leq k - 1$, let Z_{k+t} be the number of distinct directed $(k + t)$ -cycles in a vertex-transitive $(d, k; +1)$ -digraph G . Then for $2 \leq t \leq k - 1$ we have*

$$(M(d, k) + 1)(d^t - d^{t-1}) = Z_{k+t}(k + t). \quad (1)$$

Proof. Let u be any vertex of G and draw the Moore tree of depth k rooted at u . By Lemma 6 all vertices in $T(u)$ have a path of length $\leq k$ to u . For $1 \leq t \leq k - 1$, let us say that a vertex v at Level t of the Moore tree is *short* if $d(v, u) \leq k - 1$ and *long* if $d(v, u) = k$.

All vertices of $N^+(u)$ must be long by k -geodecity. It is easily seen that for $1 \leq t \leq k - 2$ every vertex at Level t has one short out-neighbour and $d - 1$ long out-neighbours at Level $t + 1$. By induction for $2 \leq t \leq k - 1$ there are d^{t-1} short vertices in Level t of the tree. Therefore for $2 \leq t \leq k - 1$ the vertex u is contained in $d^t - d^{t-1}$ closed walks of length $k + t$ and these walks must be cycles by k -geodecity. Equation (1) follows by double-counting pairs (u, Z) , where u is a vertex of G and Z is a directed $(k + t)$ -cycle containing u . \square

Using the strengthened divisibility condition in Theorem 1, computer search shows that the only values of d and k in the range $3 \leq d \leq 12$ and $2 \leq k \leq 10,000$ for which there can exist a vertex-transitive $(d, k; +1)$ -digraph are

$$k = 2 : d = 3, 6, 9, 12,$$

$$k = 3 : d = 6, 10.$$

The results of [18] show that in practice there are no $(d, 3; +1)$ -digraphs for $d \geq 2$ and no $(d, 2; +1)$ -digraphs for $d \geq 8$, so the only remaining values of d and k in this range are

$(d, k) = (3, 2)$ and $(6, 2)$. In Section 7 we will see that such digraphs also do not exist. This scarcity of vertex-transitive $(d, k; +1)$ -digraphs can be taken as evidence in favour of Conjecture 1.

There are some simple number-theoretic conditions on d and k that force G to contain a Type II vertex, so that G cannot be vertex-transitive.

Theorem 2. *If $d \geq 3$ and $k \geq 2$ satisfy any of the following conditions then any $(d, k; +1)$ -digraph contains a Type II vertex:*

- (i) d and k are odd,
- (ii) $d \equiv 1 \pmod{k+1}$ or $d \equiv -1 \pmod{k+1}$,
- (iii) $d^2 | (k+1)$, or
- (iv) there is an odd prime p such that $p | (k+1)$ and $d \equiv 2 \pmod{p}$.

Proof. For part (i), suppose that d and k are odd, but G contains only Type I vertices. By Corollary 1, $(k+1)$ divides $d(M(d, k) + 1)$, so $M(d, k)$ must be odd. However $M(d, k) = 1 + d + d^2 + \dots + d^k$ contains an even number of odd summands and hence is even.

For part (ii), suppose that $d \equiv 1 \pmod{k+1}$. Then

$$d(M(d, k) + 1) = d(2 + d + d^2 + \dots + d^k) \equiv k + 2 \equiv 1 \pmod{k+1},$$

so that $(k+1)$ does not divide $d(M(d, k) + 1)$. Similarly if $d \equiv -1 \pmod{k+1}$, then $d(M(d, k) + 1) \equiv -2 + 1 - 1 + 1 - \dots + (-1)^{k+1} \equiv -2 \pmod{k+1}$ if k is even and $d(M(d, k) + 1) \equiv -1 \pmod{k+1}$ if k is odd.

For part (iii), if d^2 divides $(k+1)$ and $(k+1)$ divides $d(M(d, k) + 1)$, then d divides $M(d, k) + 1$, which implies that $d = 2$. However, we know that there are no $(2, k; +1)$ -digraphs [21].

Finally, for part (iv) suppose that p is an odd prime such that $p | (k+1)$ and $d \equiv 2 \pmod{p}$; then if every vertex is Type I we must have

$$0 \equiv M(d, k) + 1 \equiv M(2, k) + 1 = 2^{k+1} \pmod{p},$$

implying that p is even, a contradiction. □

These results can be extended to larger values of the excess ϵ under the assumption of arc-transitivity.

Lemma 7. *Let $\epsilon < d$ and let G be an arc-transitive $(d, k; +\epsilon)$ -digraph. Then for all $u \in V(G)$ we have $O(u) = O^-(u)$.*

Proof. Write $N^+(u) = \{u_1, u_2, \dots, u_d\}$. Suppose that an element v of $O^-(u)$ lies at distance $t \leq k$ from u . We can assume that $v \in T(u_1)$. By arc-transitivity, there are $d-1$ automorphisms ϕ_i of G , $i = 2, 3, \dots, d$, that map the arc $u \rightarrow u_1$ to the arc $u \rightarrow u_i$. Each image $\phi_i(v)$ of v under these automorphisms must also belong to $O^-(u)$, so each branch contains an element of $O^-(u)$. As $\epsilon < d$, one of these elements must be repeated in the Moore tree rooted at u , violating k -geodesicity. Therefore we must have $O^-(u) \subseteq O(u)$ and the result follows. □

Corollary 3. *If G is an arc-transitive $(d, k; +\epsilon)$ -digraph with excess $\epsilon < d$, then $(k + 1)$ divides $d(M(d, k) + \epsilon)$ and $(k + t)$ divides $(M(d, k) + \epsilon)(d^t - d^{t-1})$ for $2 \leq t \leq k$.*

Proof. Again we denote the number of distinct directed r -cycles in G by Z_r . Using the same reasoning as in the proof of Theorem 1, we see that each vertex of G is contained in d directed $(k + 1)$ -cycles, so that $d(M(d, k) + \epsilon) = Z_{k+1}(k + 1)$, and each vertex is contained in $d^t - d^{t-1}$ directed $(k + t)$ -cycles for $2 \leq t \leq k$, so that $(M(d, k) + \epsilon)(d^t - d^{t-1}) = Z_{k+t}(k + t)$. \square

These results may suggest that the permutation digraphs are smallest possible arc-transitive k -geodetic digraphs for sufficiently large d [26].

3 | DIREGULAR DIGRAPHS WITH EXCESS ONE AND DEGREE THREE

Sillasen has shown that there are no $(2, k; +1)$ -digraphs [21]; therefore a reasonable next step is to ask whether there are any $(3, k; +1)$ -digraphs. It was proven in [5] that there are no $(3, k; -1)$ -digraphs; the strategy of the proof is to show that any two distinct vertices of a $(3, k; -1)$ -digraph can have at most one common out-neighbour (and, conversely, at most one common in-neighbour), classify vertices u according to the distance from u to its repeat $r(u)$ and then count the different types of vertices in two different ways to arrive at a contradiction. In this section we show that the first main result of [5], that any two vertices have at most one common out-neighbour, continues to hold in the setting of digraphs with degree three and excess one.

The fact that a $(d, k; +1)$ -digraph G must be diregular leads to a “duality principle”. This phenomenon has been observed for almost Moore digraphs; in [5] it is shown that taking the converse of a $(d, k; -1)$ -digraph yields another $(d, k; -1)$ -digraph. The proof for digraphs with excess one is trivial and we omit it. The Duality Principle will allow us to interchange out-neighbourhoods and in-neighbourhoods in our results. Recall that σ^- is the inverse function of the outlier function, that is, $\sigma^-(v) = u$ if and only if $v = o(u)$.

Lemma 8 (Duality principle). *Let G be a $(d, k; +1)$ -digraph with outlier function o . Taking the converse of G yields another $(d, k; +1)$ -digraph G^- with outlier function o^- .*

We begin with a lemma that holds generally for $(d, k; +1)$ -digraphs; it describes the situation of vertices with identical out-neighbourhoods.

Lemma 9. *Let u and v be vertices of a $(d, k; +1)$ -digraph such that $N^+(u) = N^+(v)$, where $u \neq v$. Then $v = o(u)$ and $u = o(v)$, that is, the outlier function o transposes u and v . The same result holds if $N^-(u) = N^-(v)$.*

Proof. Suppose that $N^+(u) = N^+(v)$, but $u \neq v$. Draw the Moore tree of depth k rooted at u ; by k -geodeticity, u appears only at Level 0, the root position, of this tree. As the Moore tree rooted at v differs from the Moore tree rooted at u only at Level 0, and $u \neq v$, it follows that v cannot reach u by a path of length $\leq k$, so $o(v) = u$ and, by symmetry, $o(u) = v$. The result for in-neighbourhoods follows by the Duality Principle. \square

For the remainder of this section let G be a diregular $(3, k; +1)$ -digraph with outlier function o . Our first goal is to show that no pair of distinct vertices can have identical out-neighbourhoods; to achieve this we need a lemma for pairs of vertices with exactly two common out-neighbours.

Lemma 10. *Let u, v be distinct vertices of G with exactly two common out-neighbours, that is, $|N^+(u) \cap N^+(v)| = 2$. If we write $N^+(u) = \{u_1, u_2, u_3\}$ and $N^+(v) = \{v_1, v_2, v_3\}$, where $u_1 = v_1, u_2 = v_2$ and $u_3 \neq v_3$, then $o(u) = v_3$ and $o(v) = u_3$.*

Proof. Let u and v be as described. This configuration is shown in Figure 1. By k -geodeticity $u_3 \notin T(u_1) \cup T(u_2)$. Hence there are three possible positions for the vertex u_3 in the Moore tree rooted at v : (i) $u_3 = v$, (ii) $u_3 \in T(v_3) - \{v_3\}$ or (iii) $u_3 = o(v)$. If $u_3 = v$, then we have paths $u \rightarrow u_2$ and $u \rightarrow u_3 \rightarrow u_2$, which is impossible for $k \geq 2$.

Suppose that $u_3 \in T(v_3) - \{v_3\}$. Put $\ell = d(v_3, u_3)$, so that $1 \leq \ell \leq k - 1$. Let w be a vertex in $N^{k-1-\ell}(u_3)$; then $w \in N^{k-1}(v_3)$. The vertex w has three out-neighbours w_1, w_2 and w_3 . By k -geodeticity none of these out-neighbours can lie in $T(v_3)$. At most two of the out-neighbours can lie in $\{v, o(v)\}$, so it follows that w has an out-neighbour, say w_1 , that lies in $T(v_1) \cup T(v_2) = T(u_1) \cup T(u_2)$; without loss of generality $w_1 \in T(u_1)$. Hence there is a path of length $\leq k$ from u to w_1 via u_1 . There is also a path from u to w_1 with length $\leq k$ formed from the arc $u \rightarrow u_3$, followed by the path from u_3 to w with length $k - 1 - \ell$ and the arc $w \rightarrow w_1$. This violates k -geodeticity. It follows that option (iii) must hold, that is, $u_3 = o(v)$. Similarly $v_3 = o(u)$. □

Corollary 4. *No pair u, v of distinct vertices of G have identical out-neighbourhoods.*

Proof. Suppose that $u \neq v$ but $N^+(u) = N^+(v) = \{u_1, u_2, u_3\}$. The setup is shown in Figure 2. By Lemma 9 we know that $v = o(u)$ and $u = o(v)$. For $i = 1, 2, 3$ denote the in-neighbour of u_i that does not lie in $\{u, v\}$ by u_i^* . We cannot have $u_1^* = u_2^* = u_3^*$, for otherwise by Lemma 9 we would have $o(u) = u_1^* = v$.

By the Duality Principle taking the converse G^- of G yields a diregular $(3, k; +1)$ -digraph with outlier function $o' = o^-$. In G^- we have $N^+(u_i) = \{u_i^*, u, v\}$ for $i = 1, 2, 3$.

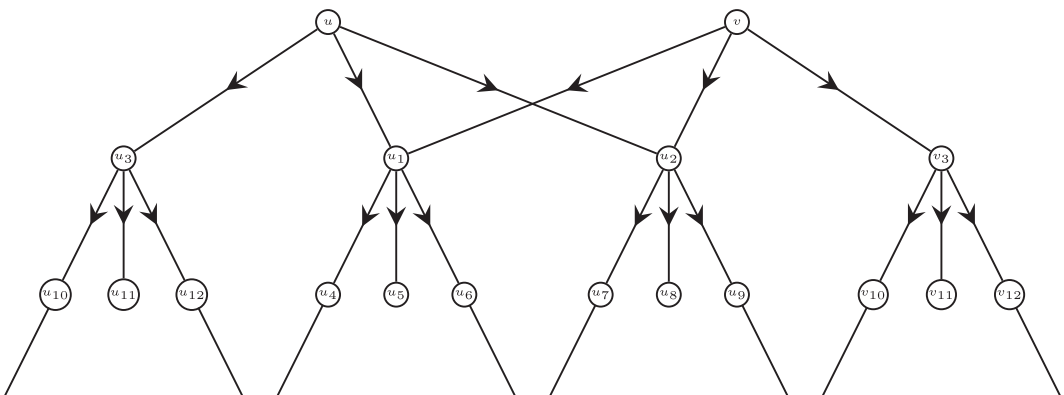


FIGURE 1 Vertices with two common out-neighbours.

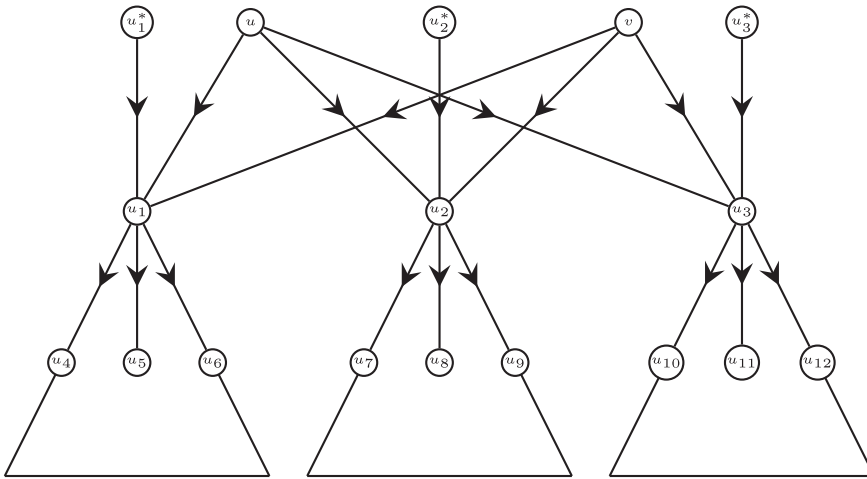


FIGURE 2 Configuration for Corollary 4.

Suppose that $u_i^* \neq u_j^*$. Then in G^- the pair of vertices u_i, u_j has exactly two common out-neighbours, so that by Lemma 10 we obtain $o^-(u_i) = u_j^*$ and $o^-(u_j) = u_i^*$. If u_1^*, u_2^*, u_3^* are all distinct, we would then obtain $o^-(u_1) = u_2^* = u_3^*$, a contradiction.

We can thus assume that $u_1^* = u_2^* \neq u_3^*$. Applying Lemma 10 to the pairs u_1, u_3 and u_2, u_3 in turn, we deduce that $o^-(u_1) = o^-(u_2) = u_3^*$, again a contradiction, as o is a permutation. □

Having ruled out identical out-neighbourhoods, we can complete the proof of our desired result.

Theorem 3. Any two distinct vertices of a $(3, k; +1)$ -digraph G have at most one common out-neighbour and at most one common in-neighbour.

Proof. Suppose that u, v are distinct vertices with more than one out-neighbour in common. By Corollary 4, u and v must have exactly two common out-neighbours. Write $N^+(u) = \{u_1, u_2, u_3\}$ and $N^+(v) = \{v_1, v_2, v_3\}$, where $u_1 = v_1, u_2 = v_2$, but $u_3 \neq v_3$ (see Figure 1). By Lemma 10 we know that $o(u) = v_3$ and $o(v) = u_3$.

Let $w \in T(u_3) \cap T(v_3)$, with $d(u_3, w) = s$ and $d(v_3, w) = t$. Suppose that $s > t$. Consider the set $N^{k-s}(w)$. By construction, $N^{k-s}(w) \subseteq N^k(u_3)$, so $N^{k-s}(w) \cap T(u_3) = \emptyset$. We have $k + t - s \leq k - 1$, so $N^{k-s}(w) \subseteq T(v_3)$. Hence by k -geodesicity $N^{k-s}(w) \cap (T(u_1) \cup T(u_2)) = \emptyset$. As no vertex of $N^{k-s}(w)$ can lie in any of the branches of the Moore tree rooted at u , we must have $N^{k-s}(w) \subseteq \{u, o(u)\}$. Thus the size of the set $N^{k-s}(w)$ satisfies $|N^{k-s}(w)| = 3^{k-s} \leq 2$, which is impossible for $s \leq k - 1$. Therefore $d(u_3, w) = d(v_3, w)$ for every $w \in T(u_3) \cap T(v_3)$.

Consider $N^+(u_3)$ and $N^+(v_3)$. By k -geodesicity $N^+(u_3) \cap (T(u_1) \cup T(u_2)) = \emptyset$. Also $v_3 \notin N^+(u_3)$, as $v_3 = o(u)$, and $o(v) = u_3 \notin N^+(u_3)$. Thus $N^+(u_3) \subset \{v\} \cup N^+(v_3)$ and similarly $N^+(v_3) \subset \{u\} \cup N^+(u_3)$. By Corollary 4 we cannot have $N^+(u_3) = N^+(v_3)$, so we can assume that $u_{10} = v_{10}, u_{11} = v_{11}, u_{12} = v$ and $v_{12} = u$. If $k \geq 3$ then u will have distinct $\leq k$ -paths to u_1 (and u_2), namely, $u \rightarrow u_1$ and $u \rightarrow u_3 \rightarrow v \rightarrow u_1$, so $k = 2$. The resulting configuration is displayed in Figure 3.

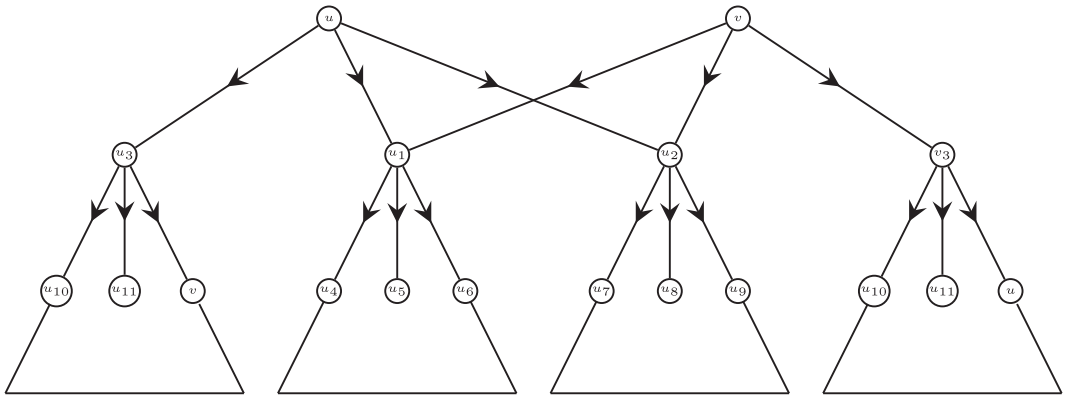


FIGURE 3 Configuration for Theorem 3.

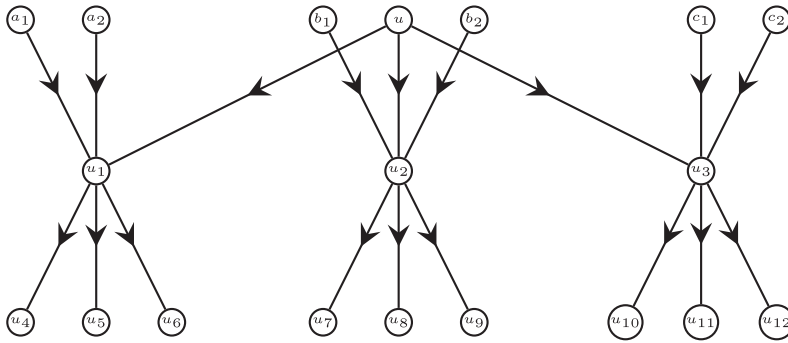


FIGURE 4 The Moore tree for Theorem 4.

Observe that now u_3 and v_3 have two out-neighbours in common, namely, u_{10} and u_{11} , so by Lemma 10 we have $o(u_3) = u$ and $o(v_3) = v$. Applying the outlier automorphism to the arcs incident with u , we deduce that $o(u) = v_3$ has arcs to $o(u_3) = u$ and $o(u_1)$ and $o(u_2)$, so $\{o(u_1), o(u_2)\} = \{u_{10}, u_{11}\}$. By 2-geodcity u_{10} can have arcs only to $N^+(u_1)$ and $N^+(u_2)$. As u_{10} has three out-going arcs and cannot have the same out-neighbourhood as u_1 or u_2 by Corollary 4, it follows that u_{10} must have two common out-neighbours with either u_1 or u_2 ; without loss of generality $N^+(u_{10}) = \{u_4, u_5, u_7\}$. Applying Lemma 10 to the pair u_1, u_{10} we see that $o(u_1) = u_7$. As we have already determined that $o(u_1) \in \{u_{10}, u_{11}\}$, this is a contradiction. The last part of the theorem follows by the Duality Principle. \square

Theorem 3 allows us to prove our first nonexistence result for degree three, namely, that there are no $(3, 2; +1)$ -digraphs.

Theorem 4. *There are no $(3, 2; +1)$ -digraphs.*

Proof. Suppose that G is a diregular $(3, 2; +1)$ -digraph. Fix an arbitrary vertex u of G with $N^+(u) = \{u_1, u_2, u_3\}$ and draw the Moore tree rooted at u as shown in Figure 4. We set $N^-(u_1) = \{u, a_1, a_2\}$, $N^-(u_2) = \{u, b_1, b_2\}$ and $N^-(u_3) = \{u, c_1, c_2\}$.

At least one of the vertices c_1, c_2 is not equal to $o(u)$, say $c_1 \neq o(u)$. By 2-geodecty we can assume that $c_1 = u_4$. By Theorem 3 c_1 has no arcs to $T(u) - \{u_3\}$, at most one arc to $N^+(u_2)$ and by 2-geodecty has no arcs to $N^+(u_1) \cup N^+(u_3)$. It follows that c_1 must have exactly one arc to $N^+(u_2)$ as well as an arc to $o(u)$. If $c_2 = o(u)$ this would yield two paths of length ≤ 2 from c_1 to u_3 , so $c_2 \neq o(u)$. By the same reasoning c_2 has an arc to $o(u)$; however, we now have two distinct vertices with at least two common out-neighbours, contradicting Theorem 3. \square

4 | AUTOMORPHISMS OF DIGRAPHS WITH EXCESS ONE

In [22] Sillasen uses counting arguments to deduce information on the form of the subdigraph of an almost Moore digraph that is induced by the set of vertices fixed by an automorphism. Using the same approach we can deduce a strong result on the action of automorphisms of a digraph with excess one. This will later help us to analyse the structure of the outlier function of a $(d, k; +1)$ -digraph.

Let G be a $(d, k; +1)$ -digraph and $\phi \in \text{Aut}(G)$ a nonidentity automorphism of G . Denote by $\text{Fix}(\phi)$ the set of vertices of G that are fixed by ϕ and let $\text{FIX}(\phi)$ be the subdigraph induced by $\text{Fix}(\phi)$.

First we show that the fix-set $\text{Fix}(\phi)$ is closed under the action of the outlier automorphism.

Lemma 11. *If $u \in \text{Fix}(\phi)$, then $o^j(u) \in \text{Fix}(\phi)$ for all $j \in \mathbb{N}$.*

Proof. Let $u \in \text{Fix}(\phi)$. If $d(u, v) \leq k$, then $d(\phi(u), \phi(v)) = d(u, \phi(v)) \leq k$, so the only vertex of $V(G)$ that lies at distance $\geq k + 1$ from u is $\phi(o(u))$ and so $\phi(o(u)) = o(u)$ and $o(u) \in \text{Fix}(\phi)$. Iteration of o implies the result. \square

As the outlier automorphism is fixed-point-free, it follows from Lemma 11 that any fix-set $\text{Fix}(\phi)$ cannot consist of a single vertex and, if ϕ fixes just two vertices u, u' of G , then these two vertices are outliers of each other, that is, $o(u) = u'$ and $o(u') = u$.

Corollary 5. *If $|\text{Fix}(\phi)| \leq 2$, then either $\text{Fix}(\phi) = \emptyset$ and $\text{FIX}(\phi)$ is the null digraph, or $|\text{Fix}(\phi)| = 2$ and $\text{FIX}(\phi) \cong 2K_1$.*

We will now assume that $\text{Fix}(\phi)$ contains at least three vertices.

Lemma 12. *If $u, v \in \text{Fix}(\phi)$ and P is a path of length $\leq k$ from u to v , then all vertices of P are contained in $\text{Fix}(\phi)$.*

Proof. Let u, v and P be as described. Suppose that there is a vertex $u' \in V(P)$ that is not fixed by ϕ . Then P and $\phi(P)$ are distinct $\leq k$ -paths from u to v , contradicting k -geodecty. \square

Lemma 13. *The digraph $\text{FIX}(\phi)$ is diregular.*

Proof. For any vertex $u \in \text{Fix}(\phi)$ we will denote the out-degree and in-degree of u in the subdigraph $\text{FIX}(\phi)$ by $d_\phi^+(u)$ and $d_\phi^-(u)$, respectively. We will show that for

any two (not necessarily distinct) vertices $u, v \in \text{Fix}(\phi)$ we have $d_{\phi}^+(u) = d_{\phi}^-(v)$; this implies the desired result. For this pair u, v we will write $N^+(u) = \{u_1, u_2, \dots, u_d\}$ and $N^-(v) = \{v_1, \dots, v_d\}$.

Assume that $v \notin o(N^+(u))$. Then for $1 \leq i \leq d$ there is a unique $\leq k$ -path P_i from u_i to v . Suppose that $u \not\rightarrow v$. By k -geodesicity, none of the paths P_i pass through the same in-neighbour of v , so without loss of generality there is a $\leq (k-1)$ -path from u_i to v_i for $1 \leq i \leq d$. By Lemma 12, it follows that $u_i \in \text{Fix}(\phi)$ if and only if $v_i \in \text{Fix}(\phi)$, so that $d_{\phi}^+(u) = d_{\phi}^-(v)$. If $u \rightarrow v$, then repeating this reasoning for the out-neighbours of u other than v shows that we still have $d_{\phi}^+(u) = d_{\phi}^-(v)$.

Now suppose that $v \in o(N^+(u))$; say $v = o(u_1)$. If $u \not\rightarrow v$, then for $2 \leq i \leq d$ we can assume that there is a $\leq (k-1)$ -path from u_i to v_i and as before $u_i \in \text{Fix}(\phi)$ if and only if $v_i \in \text{Fix}(\phi)$ for $2 \leq i \leq d$. There is an arc $u \rightarrow u_1$, so as o is an automorphism there exists an arc $o(u) \rightarrow o(u_1) = v$, giving $o(u) \in N^-(v)$. There are $\leq k$ -paths from u to each v_i for $2 \leq i \leq d$, so we must have $o(u) = v_1$. As $o(u) \in \text{Fix}(\phi)$ by Lemma 11 we have $v_1 \in \text{Fix}(\phi)$. Also by Lemma 11 we have $o^-(v) = u_1 \in \text{Fix}(\phi)$, so again we see that $d_{\phi}^+(u) = d_{\phi}^-(v)$. Again the case $u \rightarrow v$ is similar.

It follows that $\text{FIX}(\phi)$ is diregular. \square

Lemma 14. *The digraph $\text{FIX}(\phi)$ is an isometric subdigraph of G and has diameter $k + 1$.*

Proof. As $\text{Fix}(\phi)$ is a subdigraph of G we certainly have $d_{\text{FIX}(\phi)}(u, v) \geq d_G(u, v)$ for all $u, v \in \text{Fix}(\phi)$. Let $u, v \in \text{Fix}(\phi)$ be arbitrary. If $v \in T_k(u) \cap \text{Fix}(\phi)$, then by Lemma 12 all the vertices of the path of length $\leq k$ from u to v in G also belong to $\text{Fix}(\phi)$, so that $d_G(u, v) = d_{\text{FIX}(\phi)}(u, v)$.

By Lemma 13, $\text{FIX}(\phi)$ is diregular with degree ≥ 1 (as we are assuming that $|\text{Fix}(\phi)| \geq 3$). Hence if $v = o(u)$ in G , then $o(u)$ has an in-neighbour v' in $\text{Fix}(\phi)$, so that by the preceding argument $d_G(u, v') = d_{\text{FIX}(\phi)}(u, v')$ and thus $d_{\text{FIX}(\phi)}(u, o(u)) = k + 1$. Therefore $\text{FIX}(\phi)$ is an isometric subdigraph of G and, since $o(u) \in \text{Fix}(\phi)$ for any $u \in \text{Fix}(\phi)$, the diameter of $\text{FIX}(\phi)$ is exactly $k + 1$. \square

Corollary 6. *The digraph $\text{FIX}(\phi)$ is a $(d', k; +1)$ -digraph for some d' in the range $1 \leq d' \leq d - 1$.*

Proof. As a subdigraph of G , $\text{FIX}(\phi)$ is k -geodesic. By Lemma 13, $\text{FIX}(\phi)$ is diregular with degree d' . We are assuming that $\text{Fix}(\phi)$ contains at least three vertices, so by Lemma 12, $\text{FIX}(\phi)$ contains a path and $d' \geq 1$. We are also assuming that ϕ is not the identity automorphism, so ϕ does not fix all vertices of G and $d' \leq d - 1$. By diregularity and Lemma 14, it follows that $\text{FIX}(\phi)$ has order $M(d', k) + 1$, so $\text{FIX}(\phi)$ is a $(d', k; +1)$ -digraph. \square

As there are no diregular $(2, k; +1)$ -digraphs [22], we have the following result.

Corollary 7. *If G is a $(d, k; +1)$ -digraph and ϕ is a nonidentity automorphism of G , then $\text{FIX}(\phi)$ is either the null digraph, a pair of isolated vertices, a directed $(k + 2)$ -cycle or a $(d', k; +1)$ -digraph, where $3 \leq d' \leq d - 1$.*

5 | STRUCTURE OF THE OUTLIER FUNCTION

We will now make use of some of the results from the preceding sections to deduce useful information on the permutation structure of the outlier function of a digraph with excess one. Let G be a $(d, k; +1)$ -digraph. By Lemma 1, G is diregular and the outlier function of G is an automorphism. Therefore every vertex u of G has an associated order $\omega(u)$, which is the smallest integer such that $o^\omega(u) = u$. Cholily et al. have used the vertex orders of the repeat function to successfully analyse the structure of digraphs with defect one in such papers as [3] and [7]. We can immediately apply the method of [7] to make a connection between the vertex orders for the outlier function and the existence of short paths in G .

Lemma 15. *Let u_0, u_1, \dots, u_r be a path of length r in G , where $r \leq k$, and put $t = \text{lcm}(\omega(u_0), \omega(u_r))$. Then $\omega(u_i)$ divides t for $1 \leq i \leq r - 1$.*

Proof. Suppose that for some $1 \leq i \leq r - 1$ the order of u_i does not divide t . Then $o^t(u_i) \neq u_i$, so we obtain two $\leq k$ -paths $u_0, u_1, \dots, u_i, \dots, u_r$ and $o^t(u_0), o^t(u_1), \dots, o^t(u_i), \dots, o^t(u_r) = u_0, o^t(u_1), \dots, o^t(u_i), \dots, u_r$ from u_0 to u_r , a contradiction. \square

Corollary 8. *If p is the minimum vertex order of G and W is a walk of length $\leq k$ between two vertices u, v with order p , then every vertex on W has order p .*

Proof. Suppose that there is a vertex w on W such that $\omega(w) > p$. Then W and $o^p(W)$ are two distinct walks of length $\leq k$ between u and v , contradicting k -geodecity. \square

We now make two definitions that will help us to analyse the structure of the permutation o .

Definition 2. The *index* $\omega(G)$ of a $(d, k; +1)$ -digraph G is the value of the smallest vertex order in G , that is, $\omega(G) = \min\{\omega(u) : u \in V(G)\}$.

Definition 3. A $(d, k; +1)$ -digraph is *outlier-regular* if its outlier function o is a regular permutation. If each vertex of G has order ω , then G is ω -*outlier-regular*.

As o is an automorphism it follows that any power o^r of o is also an automorphism of G . In Section 4 we classified the possible fixed sets of any nonidentity automorphism of G . We therefore record the following implication of Corollary 7.

Corollary 9. *For any integer $r \geq 2$, the set of vertices of G with order dividing r induces one of the following:*

- the entire digraph G ,
- the empty digraph,
- a pair of vertices that form a transposition in o ,
- a directed $(k + 2)$ -cycle, or
- a $(d', k; +1)$ -digraph, where $3 \leq d' \leq d - 1$.

Conjecture 1 claims that there are no nontrivial $(d, k; +1)$ -digraphs; one approach to proving this conjecture is to study the properties of a minimal counterexample. Let $k \geq 2$ and

$k \neq 3, 4$. Suppose that there exists a $(d, k; +1)$ -digraph with $d \geq 2$ and let d' be the smallest possible value of $d \geq 3$ such that there exists a $(d, k; +1)$ -digraph; then we will refer to a $(d', k; +1)$ -digraph as a *minimal* $(d, k; +1)$ -digraph. For a fixed k , Corollary 9 strongly restricts the structure of the outlier automorphism of a minimal $(d, k; +1)$ -digraph.

Corollary 10. *A minimal $(d, k; +1)$ -digraph G satisfies one of the following:*

- G is outlier-regular,
- the outlier function o of G contains a unique transposition, or
- the vertices of G with order $\omega(G)$ form a directed $(k + 2)$ -cycle.

In particular this holds for any $(3, k; +1)$ -digraph.

Proof. By Corollary 9 the automorphism $o^{\omega(G)}$ fixes either (i) every vertex of G , in which case every vertex of G has order $\omega(G)$ and G is outlier-regular, (ii) two vertices that are outliers of each other, so that $\omega(G) = 2$ and o contains a unique transposition, or (iii) a $(k + 2)$ -cycle. \square

If a minimal $(d, k; +1)$ -digraph is not outlier-regular, then Corollary 10 allows us to deduce the subdigraph induced by the set of vertices with smallest order.

Lemma 16. *If a minimal $(d, k; +1)$ -digraph G is not outlier-regular, then either its outlier function o contains a unique transposition, or else $\omega(G) = k + 2$ and the vertices with order $k + 2$ induce a directed $(k + 2)$ -cycle.*

Proof. Suppose that G is a nonoutlier-regular minimal $(d, k; +1)$ -digraph with outlier function that does not contain a unique transposition. Then by Corollary 10 the vertices with order equal to the index $\omega(G)$ of G induce a directed $(k + 2)$ -cycle C . For any vertex u in the cycle its outlier $o(u)$ also has order $\omega(G)$, so the outlier of u must be the vertex preceding u on the cycle C ; it follows that $\omega(G) = k + 2$. \square

We will find the following classification of this behaviour convenient.

Definition 4. A minimal $(d, k; +1)$ -digraph G such that the vertices of G with order $\omega(G)$ form a directed $(k + 2)$ -cycle is *Type A*, whereas if the outlier function o of G contains a unique transposition, then G is *Type B*.

According to this definition, every minimal $(d, k; +1)$ -digraph is either Type A, Type B or outlier-regular. Let us now return to the problem of digraphs with degree three and excess one; if any such digraph exists it is minimal.

Lemma 17. *Let G be a $(3, k; +1)$ -digraph of Type A. Then $k + 2$ divides $\frac{M(3, k) - k - 1}{2}$.*

Proof. By Lemma 16 we have $\omega(G) = k + 2$ and the vertices with order $k + 2$ induce a $(k + 2)$ -cycle C . Pick a vertex u on C and write $N^+(u) = \{u_1, u_2, u_3\}$, where u_1 also lies on C . The automorphism $o^{(k+2)}$ fixes u and u_1 , but not u_2 and u_3 , so $o^{(k+2)}$ transposes u_2 and

u_3 . Thus $o^{2(k+2)}$ fixes every vertex in $T_1(u)$ and by Corollary 9 every vertex of G has order either $k + 2$ or $2(k + 2)$. If there are r cycles in o with length $2(k + 2)$, then we obtain

$$M(3, k) + 1 = k + 2 + 2r(k + 2)$$

and $k + 2$ divides $\frac{M(3, k) - k - 1}{2}$. □

Lemma 18. *Let G be a $(3, k; +1)$ -digraph of Type B. Then $k \not\equiv 3, 5 \pmod{6}$. If $k \equiv 0, 2 \pmod{6}$, then G contains two vertices of order two, with all other vertices of G having order six.*

Proof. Assume that G is a nonoutlier-regular $(3, k; +1)$ -digraph with outlier function o containing a unique transposition. Let u and $o(u)$ be the vertices of G with order two, where $N^+(u) = \{u_1, u_2, u_3\}$. The automorphism o^2 fixes u , but fixes no vertex in $\{u_1, u_2, u_3\}$. We can thus assume that o^2 permutes u_1, u_2, u_3 in a 3-cycle $(u_1 u_2 u_3)$.

By Theorem 3 u and $o(u)$ have at most one common out-neighbour. Suppose that $N^+(u) \cap N^+(o(u)) \neq \emptyset$; we can assume that u_1 is the common out-neighbour of u and $o(u)$. However, applying the automorphism o^2 to G shows that $o^2(u_1)$ is a common out-neighbour of u and $o(u)$ and, since G is Type B we have $o^2(u_1) \neq u_1$, thereby violating Theorem 3. It follows that o contains the 6-cycle $(u_1, o(u_1), u_2, o(u_2), u_3, o(u_3))$.

Therefore $\{u\} \cup N^+(u) \subseteq \text{Fix}(o^6)$ and hence o^6 fixes every vertex of G , so that the order of every vertex apart from u and $o(u)$ is either 3 or 6.

Suppose that there is a vertex with order 3. Then o^3 fixes a $(k + 2)$ -cycle. If there are r cycles in o of length 6, then

$$M(3, k) + 1 = 2 + (k + 2) + 6r.$$

Hence $6 | (M(3, k) - k - 3)$. This implies that $k \equiv 1 \pmod{3}$.

On the other hand, suppose that all vertices of G have order six, with the exception of the two vertices with order two. Then $6 | (M(3, k) - 1)$, which implies that k is even.

Thus if $k \equiv 3 \pmod{6}$ or $k \equiv 5 \pmod{6}$, then no such digraph can exist. □

Corollary 11. *If $k \geq 2$ is such that*

- $k \equiv 3$ or $5 \pmod{6}$,
- $k + 2$ does not divide $\frac{M(3, k) - k - 1}{2}$, and
- $M(3, k) + 1$ is prime,

then there is no $(3, k; +1)$ -digraph.

Proof. Assume that G is a $(3, k; +1)$ -digraph such that k satisfies each of these conditions. By Lemmas 17 and 18, G is neither Type A nor Type B and hence must be outlier-regular. As the order of G is prime, it follows that its outlier function o consists of a single cycle of length $M(3, k) + 1$; thus G is vertex-transitive. However, any vertex-transitive digraph with prime order is a circulant digraph, which is not k -geodetic for $k \geq 2$. It follows that there is no $(3, k; +1)$ -digraph for such k . □

The first k for which Corollary 11 applies are $k = 3, 15$ and 63 . This provides an independent proof of the nonexistence of $(3, 3; +1)$ -digraphs [19], as well as ruling out the existence of $(3, k; +1)$ -digraphs for some larger k .

Corollary 12. *There are no $(3, 3; +1)$, $(3, 15; +1)$ - or $(3, 63; +1)$ -digraphs.*

6 | SPECTRAL RESULTS

Now that we have more information about the permutation structure of o , we can apply some more powerful spectral results developed in [18]. If A is the adjacency matrix of a $(d, k; +1)$ -digraph G with order n , J is the $n \times n$ all-one matrix and P is the permutation matrix associated with the permutation o , then counting the paths of length $\leq k$ we have that

$$I + A + A^2 + \cdots + A^k = J - P. \quad (2)$$

We will now exploit the connection in Equation (2) between the permutation structure of the outlier function o of a $(d, k; +1)$ -digraph G and the spectrum of G . We will use the following concise description of the permutation structure of the outlier function o from [18].

Definition 5. For any $(d, k; +1)$ -digraph G and $1 \leq j \leq M(d, k) + 1$, the number of cycles of length j in the permutation o will be denoted by m_j . The $(M(d, k) + 1)$ -tuple $(m_1, m_2, \dots, m_{M(d, k)+1})$ is the *permutation vector* of G . For $1 \leq j \leq M(d, k) + 1$ we define

- $m'(j)$ is the number of odd cycles in the permutation o with length divisible by j ,
- $m''(j)$ is the number of even cycles in the permutation o with length divisible by j , and
- $m(j) = m'(j) + m''(j)$ is the total number of cycles in o with length divisible by j .

Note that as the outlier function is fixed-point-free we always have $m_1 = 0$. We will also need the following family of polynomials derived from the cyclotomic polynomials.

Definition 6. For $n, k \geq 1$ the polynomial $F_{n,k}(x)$ is defined by

$$F_{n,k}(x) = \Phi_n(1 + x + x^2 + \cdots + x^k),$$

where $\Phi_n(x)$ is the n th cyclotomic polynomial.

In [18] Miller et al. derive the following relation between the characteristic polynomial of $J - P$ and the permutation vector of G .

Lemma 19. (Miller et al. [18]). *The characteristic polynomial of $J - P$ is*

$$(x - M(d, k))(x + 1)^{-1} \prod_{j \geq 2, j \text{ even}} (x^j - 1)^{m_j} \prod_{j \geq 3, j \text{ odd}} (x^j + 1)^{m_j}.$$

Theorem 5. *There are no 2-outlier-regular $(d, k; +1)$ -digraphs.*

Proof. Assume that G is a 2-outlier-regular $(d, k; +1)$ -digraph with order $n = M(d, k) + 1$, that is, the outlier function o of G contains only transpositions. Thus $m_2 = \frac{n}{2}$ and $m_i = 0$ for $i \neq 2$. Therefore by Lemma 19 the characteristic polynomial of $J - P$ is

$$(x - M(d, k))(x + 1)^{-1}(x^2 - 1)^{n/2} = (x - M(d, k))(x - 1)^{\frac{n}{2}}(x + 1)^{\frac{n}{2}-1}.$$

It follows from Equation (2) that the spectrum of G consists of

- one eigenvalue d ,
- $\frac{n}{2}$ eigenvalues $\lambda_i, 1 \leq i \leq \frac{n}{2}$, such that $1 + \lambda_i + \lambda_i^2 + \dots + \lambda_i^k = 1$ for $1 \leq i \leq \frac{n}{2}$, and
- $\frac{n}{2} - 1$ eigenvalues μ_i such that for $1 \leq i \leq \frac{n}{2} - 1$ we have $1 + \mu_i + \mu_i^2 + \dots + \mu_i^k = -1$.

For any integer $r \geq 0$ we define

$$\Lambda_r = \sum_{i=1}^{\frac{n}{2}} \lambda_i^r$$

and

$$M_r = \sum_{i=1}^{\frac{n}{2}-1} \mu_i^r.$$

As for all vertices u of G we have $o^-(u) = o(u)$, the reasoning of Theorem 1 shows that each vertex of G is contained in d directed $(k + 1)$ -cycles. By k -geodesicity, any closed $(k + 1)$ -walk must be a cycle, so it follows by counting walks of length $\leq k + 1$ that

$$\text{Tr}(A^r) = d^r + \Lambda_r + M_r = 0, \quad 1 \leq r \leq k, \tag{3}$$

and

$$\text{Tr}(A^{k+1}) = d^{k+1} + \Lambda_{k+1} + M_{k+1} = dn. \tag{4}$$

Each eigenvalue λ_i satisfies $1 + \lambda_i + \lambda_i^2 + \dots + \lambda_i^k = 1$; summing this geometric series and rearranging we obtain $\lambda_i(\lambda_i^k - 1) = 0$, so each λ_i is either zero or a k th root of unity. Hence for $1 \leq r \leq k$ we have $\Lambda_{k+r} = \Lambda_r$. Similarly each eigenvalue μ_i satisfies $\mu_i^{k+1} = 2 - \mu_i$, so that for $1 \leq r \leq k$ we have $\mu_i^{k+r} = 2\mu_i^{r-1} - \mu_i^r$. Thus for $1 \leq r \leq k$ the numbers M_{k+r} satisfy $M_{k+r} = -M_r + 2M_{r-1}$.

In particular $M_{k+1} = -M_1 + 2M_0 = -M_1 + n - 2$ and $\Lambda_{k+1} = \Lambda_1$. Therefore by Equation (4) we have

$$\Lambda_1 - M_1 = dn - d^{k+1} - n + 2 = d.$$

Subtracting this from Equation (3) with $r = 1$ yields

$$M_1 = -d.$$

For any prime p the polynomial $p + x + x^2 + \cdots + x^k$ is irreducible over \mathbb{Q} [14]. As each μ_i is a solution of $2 + x + x^2 + \cdots + x^k = 0$, it follows that the roots of $2 + x + x^2 + \cdots + x^k$ must appear with equal multiplicity among the μ_i . Therefore k must divide $\frac{n}{2} - 1$. The sum of the roots of $2 + x + x^2 + \cdots + x^k$ is -1 ; therefore it follows that

$$-d = M_1 = -\frac{1}{k} \left[\frac{n}{2} - 1 \right].$$

Rearranging, we obtain

$$n = 2 + d + d^2 + d^3 + \cdots + d^k = 2kd + 2,$$

or, simplifying,

$$2k = 1 + d + d^2 + \cdots + d^{k-1},$$

which is impossible for $d, k \geq 2$. \square

In particular, it follows from Lemma 7 and Theorem 5 that no $(d, k; +1)$ -digraph with $d, k \geq 2$ can be arc-transitive.

7 | TWO-GEODETIC DIGRAPHS WITH EXCESS ONE

In [18] the authors use spectral techniques to show that there are no 2-geodetic digraphs with excess one and degree $d \geq 8$. Sillasen's first paper on the subject [21] proves that there are no $(2, 2; +1)$ -digraphs. Theorem 4 of the present work further showed that there are no $(3, 2; +1)$ -digraphs. This leaves open the existence of $(d, 2; +1)$ -digraphs for $d = 4, 5, 6$ and 7 . We will see that no $(d, 2; +1)$ -digraphs exist for these values of d . We first rule out the existence of outlier-regular $(d, 2; +1)$ -digraphs, then use an inductive approach to deal with the remaining cases.

Accordingly we shall now assume that any $(d, 2; +1)$ -digraph is outlier-regular. Then the index $\omega(G)$ of the digraph G must be a nonunit divisor of the order $2 + d + d^2$ of the $(d, 2; +1)$ -digraph. These divisors are displayed in Table 1.

Theorem 5 shows that there are no 2-outlier-regular digraphs, so we have already dealt with the divisors in red. Furthermore, if the index $\omega(G)$ of the digraph is equal to the order $2 + d + d^2$ of the digraph G , then G is vertex-transitive and by Corollary 2 the size $2d + d^2 + d^3$ of G must be divisible by 3; this precludes the existence of outlier-regular digraphs with the divisors written in

TABLE 1 Nontrivial divisors of the orders of the $(d, 2; +1)$ -graphs.

d	Order	Divisors >1
4	22	2,11,22
5	32	2,4,8,16,32
6	44	2,4,11,22,44
7	58	2,29,58

blue in Table 1. For the remaining possible structures of the outlier function we will need the exact factorisation of the characteristic polynomial of a $(d, 2; +1)$ -digraph from [18].

Lemma 20. (Miller et al. [18]). *The characteristic polynomial of a $(d, 2; +1)$ -digraph factorises in $\mathbb{Q}[x]$ as*

$$(x - d)x^{a_1}(x + 1)^{a_2}(x^2 + x + 2)^{\frac{m(2)+m'(1)-1}{2}}(x^2 + 1)^{m(4)} \times \prod_{j \geq 3, j \text{ odd}} F_{j,2}(x)^{\frac{m'(j)}{2}} F_{2j,2}(x)^{\frac{m'(j)}{2}} \prod_{j \geq 6, j \text{ even}} F_{j,2}(x)^{\frac{m'(j)}{2}},$$

where a_1 and a_2 are nonnegative integers that satisfy the simultaneous equations

$$a_1 + a_2 = m''(1) \quad \text{and} \quad d^2 - d + 1 = a_1 - a_2 + 2m(4). \tag{5}$$

Lemma 21. *An outlier-regular $(d, 2; +1)$ -digraph cannot have odd index $\omega(G)$.*

Proof. Suppose that $\omega(G)$ is odd. Then $m(4) = m''(1) = 0$ and $a_1 = a_2 = 0$. Equation (5) then gives $d^2 - d + 1 = 0$, which has no real solutions. □

Lemma 21 disposes of all of the green entries in Table 1.

Lemma 22. *There are no 22- or 44-outlier-regular $(6, 2; +1)$ -digraphs.*

Proof. If a $(6, 2; +1)$ -digraph G is 22-outlier-regular, then $m(4) = 0$ and $m''(1) = 2$, so the simultaneous equations in Equation (5) give $a_1 + a_2 = 2$ and $a_1 - a_2 = 31$, which has no solution in nonnegative integers.

Similarly, if G is a 44-outlier-regular $(6, 2; +1)$ -digraph, then $m''(1) = 1$ and $m(4) = 1$, so that Equation (5) yields $a_1 + a_2 = 1$ and $a_1 - a_2 = 29$, which again does not have nonnegative solutions. □

Lemma 22 disposes of the pink divisors in Table 1.

Lemma 23. *There are no outlier-regular $(d, 2; +1)$ -digraphs.*

Proof. First let $d = 5$ and $\omega \in \{4, 8, 16\}$. Then $m''(1) = m(4) = \frac{32}{\omega}$ and $d^2 - d + 1 = 21$. Equation (5) yields $a_1 + a_2 = \frac{32}{\omega}$ and $a_1 - a_2 = 21 - \frac{64}{\omega}$. Solving for a_1 shows that

$$a_1 = \frac{1}{2} \left[21 - \frac{32}{\omega} \right],$$

which is not an integer for $\omega \in \{4, 8, 16\}$. This gets rid of all of the orange entries in Table 1.

The only remaining option for an ω -outlier-regular $(d, 2; +1)$ -digraph G is that $d = 6$ and $\omega = 4$. Then $m(4) = m''(1) = 44/4 = 11$ and $d^2 - d + 1 = 31$. Equation (5) becomes $a_1 + a_2 = 11$ and $a_1 - a_2 = 9$, which has solution $a_1 = 10$ and $a_2 = 1$. It follows from Lemma 20 that the spectrum of G is

$$\left\{ 6^{(1)}, 0^{(10)}, -1^{(1)}, \left[\frac{-1 + \sqrt{7}i}{2} \right]^{(5)}, \left[\frac{-1 - \sqrt{7}i}{2} \right]^{(5)}, i^{(11)}, -i^{(11)} \right\},$$

where multiplicities are indicated in round brackets. Summing the third powers of the eigenvalues, it follows that

$$\text{Tr}(A^3) = 240.$$

A vertex is contained in six directed triangles if it is Type I and five directed triangles if it is Type II. If there are α Type I vertices and β Type II vertices in G , it follows that $6\alpha + 5\beta = 240$ and $\alpha + \beta = 44$. Solving these equations, we have $\alpha = 20$ and $\beta = 24$. Let A be the subdigraph of G induced by the Type I vertices and B the subdigraph induced by the Type II vertices. By Lemma 4 it follows that B consists of a collection of six directed 4-cycles. Thus $|E(A, B)| = |E(B, A)| = 120$. Thus each vertex in A has out-neighbourhood entirely contained in B . Each vertex in B has just one out-neighbour in B and so each arc from a vertex u of A allows it to reach just 12 vertices of B by paths of length ≤ 2 , which is impossible. \square

It follows by Lemma 16 that any minimal $(d, 2; +1)$ -digraph must be either Type A (with a directed 4-cycle C of vertices with order 4 and all other vertices with order greater than 4) or Type B. We will take an inductive approach. Theorem 4 shows that there is no $(3, 2; +1)$ -digraph, so we can take any $(4, 2; +1)$ -digraph to be minimal, which allows us to show that $(4, 2; +1)$ -digraphs do not exist, so that any $(5, 2; +1)$ -digraph is minimal and so on. We make the following two observations from Lemma 20 and Corollary 7, respectively.

Lemma 24. *If G is a minimal $(d, 2; +1)$ -digraph, then $m''(1)$ is odd.*

Proof. By Lemma 20 we have $a_1 + a_2 = m''(1)$ and $a_1 - a_2 = d^2 - d + 1 - 2m(4)$, so $m''(1)$ has the same parity as $d^2 - d + 1$, which is odd. \square

Lemma 25. *There are exactly two nonzero entries in the permutation vector of a minimal $(d, 2; +1)$ -digraph and both cycle lengths of o are even.*

Proof. By Lemma 23, a minimal $(d, 2; +1)$ -digraph is either Type A, in which case the smallest nonzero entry of the permutation vector is $m_4 = 1$, or Type B, in which case the smallest entry is $m_2 = 1$.

By Corollary 7, for each $r \geq 1$ the automorphism o^r has a fix-set of size 0, 2 or 4, or else fixes every vertex of G . Therefore if for some $r \geq 2$ the automorphism o^r fixes 3 or ≥ 5 vertices of G , then o^r is the identity automorphism and every vertex of G has order dividing r . Suppose that the permutation vector contains a nonzero entry $m_j = 0$, where $j \geq 3$ is odd. Then o^j fixes either 3 or ≥ 5 vertices of G , but not the vertices with even order, which is impossible. Likewise, if $i < j$ are both even, i is greater than $\omega(G)$ and m_i and m_j are both nonzero, then o^i fixes at least six vertices with order $\omega(G)$ or i , but not the vertices with order j , again a contradiction. \square

Theorem 6. *There are no $(4, 2; +1)$ -digraphs.*

Proof. Suppose that G is a $(4, 2; +1)$ -digraph. Assume first that G is Type A. Let u be a vertex on the 4-cycle C of vertices with order 4, with $N^+(u) = \{u_1, u_2, u_3, u_4\}$, where u_1 also lies on C . The automorphism o^4 fixes u and u_1 , but has no fixed points in $\{u_2, u_3, u_4\}$, so o^2 permutes u_2, u_3 and u_4 in a 3-cycle, say $(u_2 u_3 u_4)$. Thus o^{12} fixes every vertex of $T(u)$; by Corollary 7, o^{12} fixes every vertex of G and so every vertex of G has order 4, 6 or 12. G has order 22, so we either have $m_4 = 1, m_6 = 3$ or $m_4 = 1, m_6 = 1, m_{12} = 1$. Both are impossible by Lemmas 24 and 25.

Thus we can assume G to be Type B. Let u be one of the two vertices of G with order 2. The automorphism o^2 fixes u , but permutes its out-neighbours u_1, u_2, u_3 and u_4 without fixed points. Hence without loss of generality o^2 permutes these vertices either as $(u_1 u_2)(u_3 u_4)$ or $(u_1 u_2 u_3 u_4)$; in either case o^8 fixes every vertex of $T(u)$ and hence all of G , so every vertex has order 2, 4 or 8. Hence $22 = 2 + 4m_4 + 8m_8$ or $5 = m_4 + 2m_8$. There are three solutions of this equation: (i) $m_4 = 1, m_8 = 2$, (ii) $m_4 = 3, m_8 = 1$ and (iii) $m_4 = 5, m_8 = 0$. By Lemma 25 only option (iii) can hold. Thus the two nonzero entries of the permutation vector of G are $m_2 = 1, m_4 = 5$ and $m''(1) = 6$, which is even, contradicting Lemma 24. \square

Having proved that there are no $(4, 2; +1)$ -digraphs, we know that any $(5, 2; +1)$ -digraph is minimal.

Theorem 7. *There is no $(5, 2; +1)$ -digraph.*

Proof. Assume that G is a $(5, 2; +1)$ -digraph. G has order 32. By Theorem 6, G is minimal and hence by Lemma 23 is either Type A or Type B. Suppose that G is Type A. As in Theorem 6, fix a vertex u on the cycle C of vertices with order 4 and set $N^+(u) = \{u_1, u_2, u_3, u_4, u_5\}$, where $u_1 \in V(C)$. The automorphism o^4 must permute u_2, u_3, u_4 and u_5 amongst themselves without fixed points, so o^4 acts on these vertices either as $(u_2 u_3 u_4 u_5)$ or $(u_2 u_3)(u_4 u_5)$; in either case o^{16} fixes all vertices of G and every vertex order is 4, 8 or 16. We have $32 = 4 + 8m_8 + 16m_{16}$ or $7 = 2m_8 + 4m_{16}$. However, the right-hand side is even and the left odd.

Now suppose that G is Type B and let u be a vertex of G belonging to the unique transposition of o . o^2 permutes the vertices of $N^+(u) = \{u_1, u_2, u_3, u_4, u_5\}$ without fixed points; without loss of generality, o^2 acts on these vertices either as (i) $(u_1 u_2 u_3 u_4 u_5)$ or (ii) $(u_1 u_2 u_3)(u_4 u_5)$.

In Case (i) o^{10} fixes every vertex of G and every vertex order is 2, 5 or 10, where $30 = 5m_5 + 10m_{10}$. By Lemma 25, $m_5 = 0$, so the nonzero entries of the permutation vector are $m_2 = 1, m_{10} = 3$, yielding $m''(1) = 4$, contradicting Lemma 24.

In Case (ii) o^{12} is the identity and every vertex has order 2, 3, 4, 6 or 12. Lemma 25 shows that $m_3 = 0$. We have $30 = 4m_4 + 6m_6 + 12m_{12}$ and Lemma 25 shows that just one of m_4, m_6 and m_{12} is nonzero. By Lemma 25, as 12 and 4 do not divide 30, we have $m_4 = m_{12} = 0$. If $m_6 > 0$, then $m_2 = 1$ and $m_6 = 5$, giving an even value of $m''(1)$, which is impossible. \square

Theorem 8. *There is no $(6, 2; +1)$ -digraph.*

Proof. Suppose that there exists a $(6, 2; +1)$ -digraph G with order 44. Suppose that G is Type A; as before, let $u \rightarrow u_1$ be an arc of the 4-cycle of vertices with order 4. We can

assume that the automorphism o^4 permutes the other out-neighbours $\{u_2, u_3, u_4, u_5, u_6\}$ either as (i) $(u_2 u_3 u_4 u_5 u_6)$ or (ii) $(u_2 u_3 u_4)(u_5 u_6)$. In Case (i) every vertex of G has order 4, 5, 10 or 20; by Lemma 25 $m_5 = 0$. If $m_{20} > 0$, then by Lemma 25 $m_4 = 1$, $m_{10} = 0$, $m_{20} = 2$. By Lemma 20, this yields $a_1 + a_2 = 3$ and $a_1 - a_2 = 25$, which would imply that a_2 is negative. Therefore $m_{10} = 4$, $m_{20} = 0$, giving $a_1 + a_2 = 5$, $a_1 - a_2 = 29$, which again is impossible.

In Case (ii) every vertex order is 4, 6, 8, 12 or 24. We have $40 = 6m_6 + 8m_8 + 12m_{12} + 24m_{24}$. As none of 6, 12 or 24 are divisors of 40, Lemma 25 shows that $m_8 = 5$, $m_6 = m_{12} = m_{24} = 0$, so that $m''(1) = 6$ is even, violating Lemma 24.

Now suppose that G is Type B. Let u be a vertex in the unique transposition of o . We can assume that o^2 permutes the vertices of $N^+(u)$ in one of four ways: (i) $(u_1 u_2 u_3 u_4)(u_5 u_6)$, (ii) $(u_1 u_2 u_3 u_4 u_5 u_6)$, (iii) $(u_1 u_2 u_3)(u_4 u_5 u_6)$ or (iv) $(u_1 u_2)(u_3 u_4)(u_5 u_6)$. For Case (i), every vertex has order 2, 4 or 8, but neither 4 nor 8 divides 42, contradicting Lemma 25.

In Cases (ii), (iii) and (iv) each vertex order is 2, 3, 4, 6 or 12, with $42 = 3m_3 + 4m_4 + 6m_6 + 12m_{12}$. By Lemma 25 we know that $m_3 = 0$ and just one of m_4 , m_6 and m_{12} is nonzero. As 4 and 12 do not divide 42, we must have $m_4 = m_{12} = 0$ and $m_6 = 7$, so that $m''(1) = 8$ is even, a contradiction. \square

Theorem 9. *There are no $(7, 2; +1)$ -digraphs.*

Proof. Assume that G is a $(7, 2; +1)$ -digraph. G has order 58. Suppose that G is Type A and $u \rightarrow u_1$ is an arc of the 4-cycle of vertices with order 4. Then we can assume that o^4 permutes $\{u_2, u_3, u_4, u_5, u_6, u_7\} = N^+(u) - \{u_1\}$ in one of the following ways: (i) $(u_2 u_3 u_4 u_5)(u_6 u_7)$, (ii) $(u_2 u_3)(u_4 u_5)(u_6 u_7)$, (iii) $(u_2 u_3 u_4 u_5 u_6 u_7)$ or (iv) $(u_2 u_3 u_4)(u_5 u_6 u_7)$.

In Cases (i) and (ii) every vertex order is 4, 8 or 16 and $54 = 8m_8 + 16m_{16}$. However neither 8 nor 16 divides 54, violating Lemma 25. We can thus assume that either Case (iii) or (iv) holds and every vertex order is 4, 6, 8, 12 or 24, with $54 = 6m_6 + 8m_8 + 12m_{12} + 24m_{24}$. Lemma 25 shows that $m_8 = m_{12} = m_{24} = 0$ and $m_6 = 9$. Then $m''(1) = 10$ is even, contradicting Lemma 24.

Therefore assume that G is Type B and let u be a vertex with order 2. o^2 permutes the elements of $N^+(u) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ in one of the following ways: (i) $(u_1 u_2 u_3 u_4 u_5 u_6 u_7)$, (ii) $(u_1 u_2 u_3 u_4 u_5)(u_6 u_7)$, (iii) $(u_1 u_2 u_3 u_4)(u_5 u_6 u_7)$ or (iv) $(u_1 u_2 u_3)(u_4 u_5)(u_6 u_7)$.

In Case (i) all vertex orders are 2, 7 or 14, so by Lemma 25, $m_2 = 1$, $m_7 = 0$, $m_{14} = 4$. Then $m''(1) = 5$ and $m(4) = 0$, so that $a_1 + a_2 = 5$ and $a_1 - a_2 = 43$, which has no suitable solutions.

In Case (ii) all vertex orders are 2, 4, 5, 10 or 20, $m_5 = 0$ and $56 = 4m_4 + 10m_{10} + 20m_{20}$. The numbers 10 and 20 do not divide 56, so $m_2 = 1$, $m_4 = 14$. In Cases (iii) and (iv) all vertex orders are 2, 3, 4, 6, 8, 12 or 24 and $56 = 3m_3 + 4m_4 + 6m_6 + 8m_8 + 12m_{12} + 24m_{24}$. Lemma 25 shows that the only valid solutions are $m_2 = 1$ and $m_8 = 7$ and again $m_2 = 1$, $m_4 = 14$. In the former case $m''(1)$ is even, so we have shown that we can assume that $m_2 = 1$ and $m_4 = 14$. Thus $m''(1) = 15$ and $m(4) = 14$, giving $a_1 + a_2 = 15$, $a_1 - a_2 = 43 - 28 = 15$, so that $a_1 = 15$ and $a_2 = 0$. It follows from Lemma 20 that the spectrum of G is, counted by multiplicity,

$$\left\{ 7, 0^{(15)}, \left[\frac{-1 + \sqrt{7}i}{2} \right]^{(7)}, \left[\frac{-1 - \sqrt{7}i}{2} \right]^{(7)}, i^{(14)}, -i^{(14)} \right\}.$$

Adding up the third powers of these eigenvalues, we see that the trace of A^3 is $\text{Tr}(A^3) = 378$. A vertex lies in seven directed triangles if it is Type I and six if it is Type II; therefore if there are α Type I vertices in G and β Type II vertices, then counting paths we see that

$$7\alpha + 6\beta = 378, \alpha + \beta = 58.$$

Solving these equations gives $\alpha = 30, \beta = 28$. As a vertex has the same type as its outlier, it follows that the subdigraph B induced by the Type II vertices consists of seven directed 4-cycles and there is a set A of 28 vertices v such that $d(v, o^-(v)) = 2$, the other two Type I vertices being u and $o(u)$.

B has size 28 and $(B, V(G) - B) = (V(G) - B, B) = 168$. This means that the subdigraph induced by $A' = A \cup \{u, o(u)\}$ has size 42. Fix a vertex $v \in A'$. The outlier $o(v)$ of v also lies in A' , so v can reach every vertex of B by ≤ 2 -paths. The vertex v has seven out-going arcs. Suppose that the out-degree of v in the subdigraph induced by A' is ≤ 2 . Each arc from v to B allows v to reach two vertices of B ; therefore the largest possible number of vertices of B that v could reach by ≤ 2 -paths would be achieved if v has two out-neighbours in A' , each of which has out-neighbourhood contained in B ; however, this would still only allow v to reach 24 of the 28 vertices of B . It follows that the minimum out-degree in A' is ≥ 3 , which implies that the size of the subdigraph induced by A' is ≥ 90 , a contradiction. \square

This completes the remaining cases in the classification of $(d, 2; +1)$ -digraphs from [18]. Combined with the results of [18], we see that to find a digraph with excess one, we must look at digraphs that have degree at least three and are at least 5-geodetic.

Theorem 10. *If $d, k \geq 2$ and $\epsilon(d, k) = 1$, then $d \geq 3$ and $k \geq 5$.*

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