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On Some Non-Rigid Unit Distance Patterns

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Abstract
A recent generalization of the Erdős Unit Distance Problem, proposed by Palsson, Senger, and Sheffer, asks for the maximum number of unit distance paths with a given number of vertices in the plane and in 3-space. Studying a variant of this question, we prove sharp bounds on the number of unit distance paths and cycles on the sphere of radius $1/\sqrt{2}$. We also consider a similar problem about 3-regular unit distance graphs in $\mathbb{R}^3$.

Keywords Erdős unit-distance problem · Geometric incidences · Discrete chains

Mathematics Subject Classification 52C10

1 Introduction

The Erdős Unit Distance Problem is one of the most famous unsolved problems in discrete geometry. It asks for $u_2(n)$, the maximum possible number of unit distances among n points in the plane. The best known lower bound, $u_2(n) = \Omega(n^{1+c/\log\log n})$ for some constant c is due to Erdős [5], and the current best upper bound $u_2(n) = O(n^{4/3})$ is due to Spencer et al. [13]. The analogous problem is also interesting, and still far from a resolution, in $\mathbb{R}^3$ and on spheres of most radii. We note that however,

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Dedicated to the memory of Eli Goodman.

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starting from dimension 4, up to the order of magnitude, the question is less interesting, as one can find $\Omega(n^2)$ many unit distances by a well known construction of Lentz [6]. The exact value for large n and even $d \geq 4$ was determined by Brass [2] and Swanepoel [14].

Several variants and generalizations of the unit distance problem have been studied. Recently, Sheffer et al. [11] proposed to find the maximum number of unit distance paths $P_k(n)$ with k vertices in $\mathbb{R}^2$ or $\mathbb{R}^3$. This problem was essentially solved in $\mathbb{R}^2$ in [7] by finding almost sharp bounds for $k = 0, 1 \bmod 3$ and showing that the for $k = 2 \bmod 3$ the problem is essentially equivalent to the Unit Distance problem. Similar questions have been studied for paths and trees determined by dot products by Kilmer et al. [9] and Gunter et al. [8]. Passant [12] obtained results for the corresponding distinct distances problem.

We continue this line of research. First, we study the unit distance path problem on the sphere $\mathbb{S}^2$ of radius $1/\sqrt{2}$. Stereographic projection from the center of the sphere shows that unit distance graphs on a sphere of this radius are very similar to point-line incidence graphs in the plane. The only difference comes from the fact that when we project a line to a great circle on the sphere, we can choose any of its poles to represent the line. Thus, while the maximum number of edges in the two graphs are in within a constant factor, $K_{2,t}$ subgraphs with $t \geq 3$ are not excluded from unit distance graphs on the sphere. In other words, instead of unit distance graphs on the sphere, we could think about point-line incidence graphs in the plane, but allowing every line to be used twice.

For any fixed k, we determine $P_k^S(n)$, the maximum number of unit distance paths with k vertices on the sphere $\mathbb{S}^2$, up to a polylogarithmic factor. We will use notations $\tilde{\Theta}$ and $\tilde{O}$ to hide a poly-logarithmic error term.

Theorem 1.1 *For any fixed $k \geq 1$, the number of paths on k vertices on the sphere is*

$$P_k^S(n) = \begin{cases} \tilde{\Theta}\left(n^{\lfloor 2(k+3)/5 \rfloor}\right), & \text{if } k = 0, 1, 3, 4 \bmod 5, \\ \tilde{\Theta}\left(n^{\lfloor 2(k+3)/5 \rfloor - 2/3}\right), & \text{if } k = 2 \bmod 5. \end{cases}$$

The $k = 2$ case of the theorem above (without the polylogarithmic error), via stereographic projection, is the Szemerédi–Trotter bound for point-line incidences. Note that while the planar quantity, $P_k(n)$ depends on k mod 3, on the sphere the answer depends on k mod 5. The constructions for the lower bounds, described in Sect. 2, will explain this difference. We also remark that the exponent $2k/5$ is very close the non-tight upper bounds from [11] for the planar case, but this appears to be a coincidence.

Next, we study $C_k^S(n)$, the maximum possible number of unit distance cycles with k vertices determined by a set of n points in the sphere $\mathbb{S}^2$. For most cycle lengths we have almost sharp results, however working with short cycles is more difficult. Again, bounding the number of cycles of length $2k$ on the sphere is equivalent to finding a bound on the number of polygons with k vertices in the plane that can be determined by n points and n lines, such that we are allowed to use every line twice.

Theorem 1.2 *We have $C_4^S(n) = \Theta(n^2)$, and for any $k \geq 5$, with the exception of $k = 6, 7, 9$ we have*

$$C_k^S(n) = \begin{cases} \tilde{\Theta}\big(n^{\lfloor 2k/5 \rfloor}\big), & \text{if } k = 0, 1, 3, 4 \bmod 5, \\ \tilde{\Theta}\big(n^{\lfloor 2k/5 \rfloor + 1/3}\big), & \text{if } k = 2 \bmod 5. \end{cases}$$

For $k = 3, 6, 7, 9$ there is a gap between the exponent of the lower and upper bounds. We summarize the (to our knowledge) best bounds for these lengths in Proposition 3.1 in Sect. 3. We note that a related problem about cycles in incidence graphs of points and lines was studied by de Caen and Székely [4]. They conjectured that the maximum number of 6-cycles determined by an incidence graph of n points and m lines is $O(mn)$, which was disproved by Klavík et al. [10].

Next, we turn to a similar question in $\mathbb{R}^3$. We study the maximum number of unit distance subgraphs isomorphic to a given 3-regular graph G.

Theorem 1.3 *Let G be a fixed 3-regular graph on k vertices. The maximum number of unit distance subgraphs isomorphic to G determined by a set of n points in $\mathbb{R}^3$ is $\tilde{O}(n^{k/2})$.*

By slightly modifying the problem, and asking for the maximum number of copies of G with prescribed edge lengths, our upper bound remains valid. In this modified setting, for bipartite graphs we can match this bound by simple constructions.

2 Paths on the Sphere

We begin by recalling the Szemerédi–Trotter bound [15] on the number of point line incidences. For the maximum number of incidences $I(n, m)$ between a set of n points and m lines in the plane we have

$$I(n, m) = \Theta(n^{2/3} m^{2/3} + m + n). \tag{1}$$

Let $u(m, n)$ denote the maximum number of unit distance pairs between a set of n and a set of m points on the sphere. Via stereographic projection, (1) implies

$$u(n, m) = \Theta(n^{2/3} m^{2/3} + m + n). \tag{2}$$

For any $r \geq 1$, we say that a point p on $\mathbb{S}^2$ is *r-rich* with respect to a set of n points $P \subseteq \mathbb{S}^2$, if it is unit distance apart from at least r points of P. A well-known equivalent formulation of (1) gives that the maximum number of r-rich points with respect to P is

$$O\left(\frac{n^3}{r^2} + \frac{n}{r}\right). \tag{3}$$

We now list some simple observations, which will be very helpful in this and in the following section. First, on a sphere of radius $1/\sqrt{2}$, two points p, q are unit distance apart if and only if p lies on the great circle that has q as a pole. Furthermore, if p and q are not antipodal, then there are at most two points unit distance from both of them. However, if p and q are antipodal, then any point lying on their great circle will be unit distance from both. The work of Palsson et al. in [11] and Frankl and Kupavskii in [7] for paths in the plane relies on the fact that in the plane, there are at most two points unit distance from two fixed points. Therefore, we will need to find some way to work around the existence of antipodal pairs in our proofs.

We call a path $(p_1, p_2, \ldots, p_k)$ on the sphere *antipodal-free* if there is no $1 \le i \le k-2$ such that p_i and p_{i+2} are antipodal. From the observations above, [7, Thm. 2] implies the following statement.

Proposition 2.1 *For any fixed k, the number of antipodal-free k-paths* $(p_1, p_2, \ldots, p_k)$ *determined by a set of n points on the sphere is at most* $P_k(n)$*, the number of k-paths in a set of n points in the plane. That is, the number of antipodal-free k-paths is* $\tilde{O}(n^{\lfloor k/3 \rfloor + 1})$ *for* $k = 0, 1 \bmod 3$*, and* $\tilde{O}(n^{(k+2)/3})$ *for* $k = 2 \bmod 3$.

Proof of Theorem 1.1 We start by proving the upper bounds. The proof is by induction on k. We have to consider several base cases.

- For $k = 1$, we trivially have $P_1^S(n) = n$.
- For $k = 2$, by (2) we have $P_2^S(n) = O(n^{4/3})$.
- For $k = 3$, in a path (p_1, p_2, p_3) either p_1 and p_3 are antipodal, or not. In the first case, after choosing p_1, the antipodal pair p_3 is uniquely determined, giving the bound $O(n^2)$. In the second case, after choosing p_1 and p_3, there are at most two choices for the middle vertex p_2, and we obtain again the bound $O(n^2)$. Overall we still obtain $P_3^S(n) = O(n^2)$.
- For $k = 4$, any path (p_1, p_2, p_3, p_4) is either antipodal-free, or not. If it is not antipodal-free, we may assume without loss of generality that p_2 and p_4 are antipodal. Then after choosing (p_1, p_2, p_3), the last vertex p_4 is uniquely determined. By the $k = 3$ case we have $O(n^2)$ choices for (p_2, p_3, p_4), obtaining the $O(n^2)$ bound. In the antipodal-free case Proposition 2.1 implies the bound $\tilde{O}(n^2)$. Adding together the two cases, we obtain the bound $\tilde{O}(n^2)$.
- For $k = 5, 6, 8$, in any path $(p_1, \ldots, p_k)$ either p_{k-2} and p_k are antipodal, or not. In the first case, after choosing $(p_1, \ldots, p_{k_1})$ the last vertex p_k is uniquely determined, and we are done by the $k-1$ case. In the second case, after choosing $(p_1, \ldots, p_{k-2})$ and p_k, there are at most two options for p_{k-1}, and we are done by the $k-2$ case.
- For $k = 7$, in any path $(p_1, \ldots, p_7)$ at most one of (p_2, p_4) and (p_4, p_6) are antipodal (if both pairs were antipodal, then we would have $p_2 = p_6$, which is forbidden). Without loss of generality, we may assume that p_4 and p_6 are not antipodal. Then after choosing (p_1, p_2, p_3, p_4) and (p_6, p_7) we have at most two options for p_5 and we obtain the bound $\tilde{O}(n^2)\ O(n^{4/3}) = \tilde{O}(n^{10/3})$ bound by the $k = 4$ and $k = 2$ cases.
- For $k = 9$, in any path $(p_1, \ldots, p_9)$ either p_4 and p_6 are antipodal, or not. If they are not antipodal, then after choosing $(p_1, \ldots, p_4)$ and $(p_6, \ldots, p_9)$ we have at most two choices for p_5, and obtain the $\tilde{O}(n^4)$ bound by the $k = 4$ case. If p_4 and p_6 are antipodal, then p_6 and p_8 cannot be antipodal. Then after

choosing (p_1, p_2), (p_5, p_6), and (p_8, p_9), the vertex p_4 is uniquely determined, and we have at most two choices for p_3 and p_7. Thus, we obtain the bound $O(n^{4/3})\ O(n^{4/3})\ O(n^{4/3}) = O(n^4)$ bound by the $k = 2$ case.

For the induction step, we notice that in the bounds we want to prove

$$\text{the difference between the exponent of } P_k^S(n) \text{ and } P_{k-5}^S(n) \text{ is 2 for any } k \geq 6, \text{ and} \tag{4}$$

$$\text{the difference between the exponent of } P_k^S(n) \text{ and } P_{k-8}^S(n) \text{ is at least 3 for any } k \geq 9. \tag{5}$$

In any path $(p_1, \ldots, p_k)$ either one of the pairs (p_4, p_6), (p_{k-3}, p_{k-5}) are antipodal, or none of them are antipodal. We bound the number of each of these type of paths separately.

If p_4 and p_6 are not antipodal, then by the $k = 4$ case there are $\tilde{O}(n^2)$ different ways to choose (p_1, p_2, p_3, p_4). Further, by definition there are $P_{k-5}^S(n)$ ways to choose $(p_6, \ldots, p_k)$. Since p_4 and p_6 are not antipodal, after choosing the first four and the last $k - 5$ vertices, there are at most two different ways to extend it to a path. This gives the bound

$$\tilde{O}(n^2)\ P_{k-5}^S(n) \tag{6}$$

for the number of paths of this type. So, by observation (4) about the exponent of $P_{k-5}^S(n)$ and by induction, in this case we are done. Symmetrically, if the p_{k-3} and p_{k-5} are antipodal, we obtain again the bound (6).

If both p_4, p_6 and p_{k-3}, p_{k-5} are antipodal and $k \geq 10$, then after choosing $(p_5, p_6, \ldots, p_{k-4})$ the vertices p_4 and p_{k-3} are uniquely determined. Further, both (p_1, p_2) and (p_{k-1}, p_k) can be chosen in $O(n^{4/3})$ different ways. Since (p_2, p_4) and (p_{k-3}, p_{k-1}) cannot be antipodal, there are at most two different choices of p_3 through which p_2 and p_4 can be connected, and at most two different choices of p_{k-2} through which p_{k-3} and p_{k-1} can be connected. Together, these imply that the maximum number of paths of this type is

$$O(n^{4/3}) P_{k-8}^S(n) O(n^{4/3}) = O(n^3) P_{k-8}^S(n). \tag{7}$$

From (6) and (7) we obtain that the maximum number of k-paths is bounded by

$$\tilde{O}(n^2) P_{k-5}^S(n) + O(n^3) P_{k-8}^S(n).$$

This, by induction and by observations (4) and (5) about the exponent of $P_{k-5}^S(n)$ and $P_{k-8}^S(n)$ finishes the proof of the upper bound.

We know turn to the lower bound. For the $k = 0, 1, 3, 4 \bmod 5$ cases, we imitate the planar constructions from [11], taking advantage of the antipodal vertices. For an illustration see Fig. 1. Let $m = \lfloor 5n/2k \rfloor$. We take $\lceil 2k/5 \rceil$ great circles $K_0, \ldots, K_{\lfloor 2k/5-1 \rfloor}$ and on each of them we place a set Q_i of $m - 2$ points on each such that:

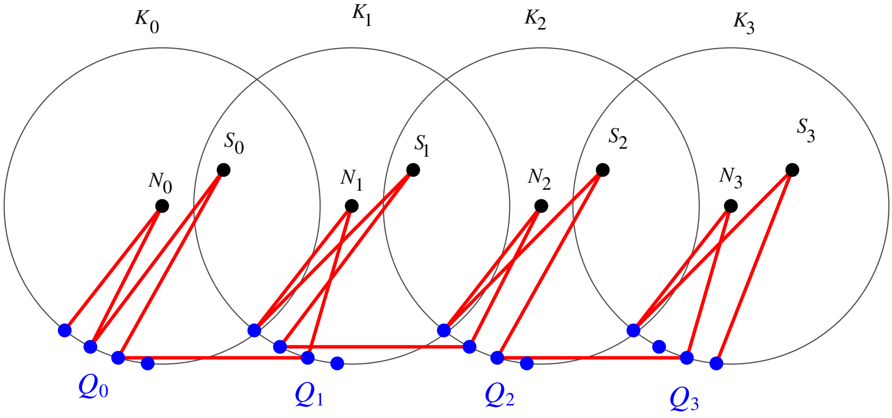

Fig. 1 The circles are the great circles K_i and N_i, S_i are their two poles. A possible path starting with a point from Q_0 and ending in a point in Q_3 is shown in red

- For any $0 \leq i < \lceil 2k/5 \rceil - 1$ and for any $p_i \in Q_i$ there is a point $p_{i+1} \in Q_{i+1}$ at unit distance apart from p_i.
- For any $0 \leq i \leq \lceil 2k/5 \rceil - 1$ the set Q_i does not contain any pole of any circle K_j.

Further, for every i we place two points N_i and S_i in the poles the circle K_i. In this construction, we can find $\Omega(n^{\lfloor 2(k+3)/5 \rfloor})$ many k-paths $(p_1, p_2, \ldots, p_k)$ such that for $i = 5\ell + j$ with $1 \leq j \leq 5$,

- p_i is in Q_ℓ if $j = 1, 3, 5$,
- $p_i = N_\ell$ for $j = 2$,
- $p_i = S_\ell$ for $j = 4$.

Indeed, after choosing p_1 from Q_0 and $p_{5\ell+3}, p_{5\ell+5}$ (for $5\ell + 3, 5\ell + 5 \leq k$) from Q_ℓ arbitrarily for every ℓ, we can extend the resulting set to a k-path.

Finally, we explain the modification to obtain the $n^{1/3}$ improvement for $k = 2 \bmod 5$. We take the construction described previously with the circles K_i, points sets $Q_i \in K_i$ and poles N_i, S_i for $k-2$. Take another point set Q of m points and with $\Omega(m^{4/3}) = \Omega(n^{4/3})$ unit distance pairs (this can be done by the same stereographic projection argument discussed in the introduction). Then modify Q_0 on K_5 such that for any point $q \in Q$ there is a $q_1 \in Q_0$ at unit distance apart from q. Similarly as before, we can find $\Omega\left(n^{\lfloor 2(k+3)/5 \rfloor + 1/3}\right)$ many k-paths $(q_1, q_2, p_1, p_2, \ldots, p_{k-2})$ such that for $i = 5\ell + j$ with $1 \leq j \leq 5$:

- p_i is in Q_ℓ if $j = 1, 3, 5$,
- $p_i = N_\ell$ for $j = 2$,
- $p_i = S_\ell$ for $j = 4$,
- $q_1, q_2 \in Q$.

Indeed, after choosing a unit distance pair (q_1, q_2) from Q, and $p_{5\ell+3}, p_{5\ell+5}$ (for $5\ell + 3, 5\ell + 5 \leq k - 2$) from Q_ℓ arbitrarily for every ℓ, we can extend the resulting set to a k-path. $\square$

3 Cycles on the Sphere of Radius $1/\sqrt{2}$

To obtain the lower bounds, we slightly modify the path construction as follows. We can arrange the points on the last great circle $K_{\lfloor 2k/5-1\rfloor}$ such it has a point at unit distance apart from any point of Q_1 in the $k = 0, 1, 3, 4$ cases, and from any point of Q in the $k = 2$ case. This will allow closing any $(k-1)$-path $(p_1, p_2, \ldots, p_{k-1})$ (in the $k = 0, 1, 3, 4$ cases) or $(q_1, q_2, p_1, p_2, \ldots, p_{k-3})$ (in the $k = 2$ case) to a k-cycle. Notice that in this construction the exponent is one smaller than the corresponding number of paths for each k, thus it indeed matches the claimed bound.

Proof of upper bound in Theorem 1.2 We begin by proving the upper bounds. First we bound the number of those cycles $(p_1, p_2, \ldots, p_k)$ in which there are at most one antipodal pair separated by one other vertex. If $(p_1, \ldots, p_k)$ is antipodal-free, then after choosing $(p_1, \ldots, p_{k-1})$, we have at most two choices for the last vertex p_k. Further, by Proposition 2.1, the number of $(k-1)$-paths $(p_1, \ldots, p_{k-1})$ is bounded by $2P_{k-1}(n)$.

If there is exactly one antipodal pair, say p_1 and p_3, then after choosing the $(k-2)$-path $(p_2, p_3, \ldots, p_{k-1})$, the antipodal pair p_1 of p_3 is uniquely determined. Further, we have at most two choices for the last vertex p_k. Thus, Proposition 2.1, the number of cycles is bounded by $2P_{k-2}(n)$.

Overall, we obtain that the number of such cycles is bounded by $2P_{k-1}(n) + kP_{k-2}(n) \le 2kP_{k-1}(n)$. This, by Proposition 2.1 implies the bound $\tilde{O}(n^{\lfloor k/3\rfloor+1})$ for $k = 0, 1 \bmod 3$, and $\tilde{O}(n^{\lfloor k/3\rfloor+1/3})$ for $k = 2 \bmod 3$. These bounds imply directly the desired bounds. Indeed, for $k \ge 21$ it follows from $k/3 + 1 \le 2\ (k+3)/5 - 1$, and for $k \le 20$ it can be checked (except for $k = 3, 6, 7, 9$) by a brief case analysis.

Thus, we only have to bound the number of cycles under the assumption that there are at least two antipodal pairs. The argument depends on the length of the path up to equivalence mod 5.

$k = 0, 1, 3$ mod 5: In this case we only need the assumption that there is at least one pair of antipodal vertices. If p_1 and p_{k-1} are antipodal, then after choosing a $(k-2)$-path $(p_3, p_4, \ldots, p_k)$, the vertex p_1 is uniquely determined. Further, since p_1 and p_3 cannot be antipodal, there are at most 2 choices of p_2 to extend $(p_1, p_3, p_4, \ldots, p_k)$ to a cycle. Thus the number of cycles in this case is at most twice the number of $(k-2)$-paths, which is $\tilde{O}(n^{\lfloor (2k+1)/5\rfloor}) = \tilde{O}(n^{\lfloor 2k/5\rfloor})$ by Theorem 1.1. As we can argue similarly for any other antipodal pair, overall we obtain the bound $k\tilde{O}(n^{\lfloor 2k/5\rfloor})$ for the number of cycles of this type.

$k = 2$ mod 5: Assume that there are two antipodal pairs (p_1, p_3) and (p_i, p_{i+2}) such that $i \notin \{1, 3\}$. First, we prove the bound in the case when the 5-paths $(p_k, p_1, p_2, p_3, p_4)$ and $(p_{i-1}, p_i, p_{i+1}, p_{i+2}, p_{i+3})$ are disjoint and their complements consist of two non-empty paths $(p_5, \ldots, p_{i-2})$ and $(p_{i+4}, \ldots, p_{k-1})$ of lengths k_1 and k_2 respectively. Since $k_1 + k_2 = 2 \bmod 5$, we may assume without loss of generality that $(k_1, k_2) = (1, 1), (2, 0)$, or $(3, 4) \bmod 5$.

- If $(k_1, k_2) = (1, 1)$ then after choosing $(p_2, p_3, \ldots, p_{i-2}), (p_{i+1}, p_{i+2}, \ldots, p_{k-1})$ arbitrarily, the antipodal pair p_1 of p_3 and p_i of p_{i+2} is uniquely determined. Further, there are at most two choices for p_k and p_{i-1}. Thus, the total number of cycles of this type, using Theorem 1.1, is bounded by

$$P^S_{k_1+3}(n)P^S_{k_2+3}(n) = \tilde{O}\left(n^{\lfloor 2(k_1+6)/5\rfloor+\lfloor 2(k_2+6)/5\rfloor}\right)$$
$$= \tilde{O}\left(n^{2(k_1+4)/5+2(k_2+4)/5}\right) = \tilde{O}\left(n^{(2k-4)/5}\right) = \tilde{O}\left(n^{\lfloor 2k/5\rfloor+1/3}\right).$$

- If $(k_1, k_2) = (2, 0)$ then after choosing $(p_2, p_3, \ldots, p_{i+1})$ and $(p_{i+4}, \ldots, p_{k-1})$ the antipodal pair p_1 of p_3 and p_{i+2} of p_i is uniquely determined. Further, there are at most two choices for p_k and p_{i+3}. Thus, the total number of cycles of this type, using Theorem 1.1, is bounded by

$$P^S_{k_1+6}(n)P^S_{k_2}(n) = \tilde{O}\left(n^{\lfloor 2(k_1+9)/5\rfloor+\lfloor 2(k_2+3)/5\rfloor-2/3}\right) = \tilde{O}\left(n^{(2k+1)/5-2/3}\right)$$
$$= \tilde{O}\left(n^{\lfloor 2k/5\rfloor+1/3}\right).$$

- If $(k_1, k_2) = (3, 4)$ then after choosing $(p_2, p_3, \ldots, p_{i+1})$ and $(p_{i+4}, \ldots, p_{k-1})$ the antipodal pair p_1 of p_3 and p_{i+2} of p_i is uniquely determined. Further, there are at most two choices for p_k and p_{i+3}. Thus, the total number of cycles of this type, using Theorem 1.1, is bounded by

$$P^S_{k_1+6}(n)P^S_{k_2}(n) = \tilde{O}\left(n^{\lfloor 2(k_1+9)/5\rfloor+\lfloor 2(k_2+3)/5\rfloor}\right) = \tilde{O}\left(n^{(2k-4)/5}\right)$$
$$= \tilde{O}\left(n^{\lfloor 2k/5\rfloor+1/3}\right).$$

We also have to consider the more "degenerate" cases, when $(p_k, p_1, p_2, p_3, p_4)$ and $(p_{i-1}, p_i, p_{i+1}, p_{i+2}, p_{i+3})$ are either not disjoint or their complement consists only of one path. That is, up to symmetry we have to consider the cases when $i \in \{2, 4, 5, 6\}$. We obtain the bound $\tilde{O}\left(n^{\lfloor 2(k+3)/5\rfloor+1/3}\right)$

- for $i = 2$ by choosing the $(k-3)$-path $(p_3, p_4, \ldots, p_{k-1})$ first,
- for $i = 4$ by choosing the 4-path (p_2, p_3, p_4, p_5) and $(k-8)$-path $(p_8, \ldots, p_{k-1})$ first,
- for $i = 5$ by choosing the $(k-3)$-path $(p_6, p_7, \ldots, p_k, p_1, p_2)$ first,
- and for $i = 6$ by choosing the 6-path $(p_k, p_1, \ldots, p_5)$ and $(k-10)$-path $(p_8, p_9 \ldots, p_{k-3})$ first.

***k* = 4 mod 5:** Assume that there are two antipodal pairs (p_1, p_3) and (p_i, p_{i+2}) such that $i \notin \{1, 3\}$. We will assume that the 5-paths $(p_k, p_1, p_2, p_3, p_4)$ and $(p_{i-1}, p_i, p_{i+1}, p_{i+2}, p_{i+3})$ are disjoint, and their complement consists of two non-empty paths $(p_5, \ldots, p_{i-2})$ and $(p_{i+4}, \ldots, p_{k-1})$ of lengths k_1 and k_2 respectively. The "degenerate" cases can be done similarly as in the $k = 2$ mod 5 case, thus we omit the details. Since $k_1 + k_2 = 4$ mod 5, we may assume without loss of generality that $(k_1, k_2) = (0, 4)$, $(1, 3)$, or $(2, 2)$ mod 5. The analysis for the $(0, 4)$ and $(1, 3)$ cases are similar to the argument in the $k = 2$ case.

- If $(k_1, k_2) = (1, 3)$ then after choosing $(p_2, p_3, \ldots, p_{i-2}), (p_{i+1}, p_{i+2}, \ldots, p_{k-1})$ arbitrarily, the antipodal pair p_1 of p_3 and p_i of p_{i+2} is uniquely determined. Further, there are at most two choices for p_k and p_{i-1}. Thus, the total number of cycles of this type, using Theorem 1.1, is bounded by

$$\begin{aligned} P^S_{k_1+3}(n) P^S_{k_2+3}(n) &= \tilde{O}\left(n^{\lfloor 2(k_1+6)/5\rfloor + \lfloor 2(k_2+6)/5\rfloor}\right) \\ &= \tilde{O}\left(n^{(2k_1+9)/5+(2k_2+8)/5}\right) = \tilde{O}\left(n^{(2k-3)/5}\right) = \tilde{O}\left(n^{\lfloor 2k/5\rfloor}\right). \end{aligned}$$

- If $(k_1, k_2) = (0, 4)$ then after choosing $(p_2, p_3, \ldots, p_{i+1})$ and $(p_{i+4}, \ldots, p_{k-1})$ the antipodal pair p_1 of p_3 and p_{i+2} of p_i is uniquely determined. Further, there are at most two choices for p_k and p_{i+3}. Thus, the total number of cycles of this type, using Theorem 1.1, is bounded by

$$\begin{aligned} P^S_{k_1+6}(n) P^S_{k_2}(n) &= \tilde{O}\left(n^{\lfloor 2(k_1+9)/5\rfloor + \lfloor 2(k_2+3)/5\rfloor}\right) = n^{(2k_1+15)/5+(2k_2+2)/5} \\ &= \tilde{O}\left(n^{\lfloor 2k/5\rfloor}\right). \end{aligned}$$

- If $(k_1, k_2) = (2, 2)$ an argument similar to the one used for the $(k_1, k_2) = (1, 3)$ and $(0, 4)$ cases is not sufficient, and we need to consider a few other cases depending whether there are other antipodal pairs. (The difficulty of this case comes from the following fact. When we only break up the cycle into two paths, there lengths are in a way that the product of the number of paths of the corresponding length is too large). Assume first that there are no other antipodal pairs. After choosing $(p_2, p_3, \ldots, p_{i+1})$ and $(p_{i+4}, \ldots, p_{k-1})$, the antipodal pair p_1 of p_3 and p_{i+2} of p_i is uniquely determined. Further, there are at most 2 choices for p_k and p_{i+1}. We bound the number of paths $(p_2, p_3, \ldots, p_{i+1})$ and $(p_{i+4}, \ldots, p_{k-1})$ by Proposition 2.1, and obtain the bound

$$4P_{k_1+2} P_{k_2} = \tilde{O}\left(n^{\lfloor 2k/5\rfloor + 1/3}\right).$$

Indeed, for $k \geq 19$ it follows from

$$4P_{k_1+2} P_{k_2} = \tilde{O}\left(n^{(k_1+6)/3+1} n^{k_2/3+1}\right) = \tilde{O}\left(n^{(k-3)/3+2}\right) \leq \tilde{O}\left(n^{\lfloor 2k/5\rfloor}\right),$$

and for $k \leq 15$ it follows by using the exact bounds from Proposition 2.1. Thus, we may assume that without loss of generality there is a third antipodal pair (p_j, p_{j+1}) with $2 \leq j \leq i-1$. Again, we assume that the 5-paths $(p_k, p_1, p_2, p_3, p_4)$, $(p_{j-1}, p_j, p_{j+1}, p_{j+2}, p_{j+3})$, and $(p_{i-1}, p_i, p_{i+1}, p_{i+2}, p_{i+3})$ are pairwise disjoint, and their complement consists of non-empty paths $(p_5, \ldots, p_{j-2})$, $(p_{j+4}, \ldots, p_{i-2})$, and $(p_{i+4}, \ldots, p_{k-1})$, of lengths ℓ_1, ℓ_2, and k_2 respectively. Without loss of generality we may assume that $(\ell_1, \ell_2, k_2) = (0, 2, 2)$, $(1, 1, 2)$, or $(3, 4, 2)$. In any of these cases, using the antipodal pairs (p_1, p_3) and (p_j, p_{j+1}), we can proceed as in the $(k_1, k_2) = (0, 4)$ or $(1, 3)$ cases. Finally, we note that the degenerate cases, when two antipodal pairs are too close to each other, can be done similarly as in the $k = 2 \bmod 5$ case. □

The next proposition summarizes the bounds for short cycles.

Proposition 3.1 *We have*

$$C_3(n) = \Omega(n) \quad and \quad C_3(n) = O(n^{4/3}),$$
$$C_6(n) = \Omega(n^2 \log\log n) \quad and \quad C_6(n) = \tilde{O}(n^{20/9}),$$
$$C_7(n) = \Omega(n^{7/3}) \quad and \quad C_7(n) = \tilde{O}(n^{8/3}),$$
$$C_9(n) = \Omega(n^3) \quad and \quad C_9(n) = \tilde{O}(n^{10/3}).$$

It would be interesting to find sharp bounds for short cycles too. The lower bound $C_6(n) = \Omega(n^2 \log\log n)$, via stereographic projection, is an immediate corollary of a construction by Klávik et al. [10] giving $\Omega(n^2 \log n \log n)$ 6-cycles in an incidence graph of n points and n lines. This disproved a conjecture of de Caen and Székely [4] that the maximum number of 6-cycles is $O(n^2)$ in the point-line incidence graph. It is not hard to see that the number of those 6-cycles that contain an antipodal pair in the unit distance graph on the sphere is $O(n^2)$. Thus, up to the order of magnitude, the problem of bounding the number of 6-cycles in point line incidence graphs and in unit distance graphs on the sphere, are equivalent.

There is a similar explanation why the C_7 and C_9 cases are more difficult than the longer cycles: It is not hard to prove upper bounds matching the lower bounds for the number of those cycles, in which there are is least one antipodal pair. Thus, again, for C_7 and C_9 the most difficult types of cycles to bound are those in which there are no antipodal pairs. For longer cycles, there is no similar issue. This is because no antipodal pair means we can use the bound from Proposition 2.1 for long sub-paths, and even with a wasteful estimate we will obtain sufficiently good bounds.

Proof of Proposition 3.1 We start with the lower bounds. For $k = 3$ finding linear lower bounds is trivial, and for $k = 6$ it follows from [10] combined with stereographic projection. For $k = 7, 9$ we can use the same constructions as for the $k \geq 10$ case. We now turn to the lower bound.

***k* = 3:** The upper bound follows from the Szemerédi–Trotter bound, since every edge can be extended in at most two different ways to a triangle. For the other cases we use a nested dyadic decomposition argument, similar to the one used by [7]. The proof for the $k = 6, 7, 9$ cases are all similar, but the $k = 6$ cases is somewhat harder. Thus we only spell out the proof for the $k = 6$ case and omit the details for $k = 7, 9$. Recall that by (3) that for any $0 \leq \alpha \leq 1$ the maximum number of n^α-rich points is $O(\max\{n^{3-2\alpha}, n^{1-\alpha}\})$. We call a point *usual* if it is n^α-rich for some $0 \leq \alpha \leq 1/2$ and *very rich* if it is n^α-rich for some $1/2 \leq \alpha \leq 1$.

***k* = 6:** Bounding the number of those 6-cycles in which there are antipodal pairs, can be done similarly as for the $k \geq 10$ cases. Thus, we may assume that there are no antipodal pairs. First, we bound the number of those 6-cycles in which there are no four consecutive usual points. In any such cycle, we can find two disjoint edges, separated by another vertex in each direction, that both have a very rich endpoint. Assume that they are n^{α_1} and n^{α_2}-rich respectively. Since there are no antipodal pairs in the cycle, after choosing the two disjoint edges, there are at most four different ways to extend it to a 6-cycle. Thus, by using dyadic decomposition and (3), we obtain the bound

$$\sum_{(\alpha_1,\alpha_2\in\Lambda)} O(n^{1-\alpha_1}n^{\alpha_1}n^{1-\alpha_2}n^{\alpha_2}) = \tilde{O}(n)^2,$$

where $\Lambda = \{(i,j) : i,j \in \{\lfloor(\log_2 n)/2\rfloor, \lfloor(\log_2 n)/2\rfloor+1,\dots,\lfloor\log_2 n\rfloor\}\}$.

Next, we bound the number of those 6-cycles $(p_1,p_2,p_3,p_4,p_5,p_6)$, in which there are four consecutive usual points, say (p_2,p_3,p_4,p_5). For some $0 \le \alpha_2, \alpha_3,\alpha_4,\alpha_5 \le 1/2$ let $Q_2(\alpha_2)$ be the set of those points that are at least n^{α_2}-rich and at most $2n^{\alpha_2}$-rich, and let $Q_5(\alpha_5)$ be the set of those points that are at least n^{α_5}-rich and at most $2n^{\alpha_5}$-rich. Further, let $Q_3(\alpha_2,\alpha_3)$ be the set of those points that are at least n^{α_3}-rich and at most $2n^{\alpha_3}$ with respect to $Q_2(\alpha_3)$, and let $Q_4(\alpha_4)$ be the set of those points that are at least n^{α_4}-rich and at most $2n^{\alpha_4}$-rich with respect to $Q_5(\alpha_5)$. Finally, let $Q_1(\alpha_2,\alpha_3)$ be the union of the second neighborhoods of points in $Q_3(\alpha_3)$, and $Q_6(\alpha_4,\alpha_5)$ be the union of the second neighborhoods of the points in $Q_4(\alpha_4)$.

Using dyadic decomposition, it is sufficient to show that for any fixed $0 \le \alpha_2, \alpha_3,\alpha_4,\alpha_5 \le 1/2$ the number of those 6-cycles $(p_1,\dots,p_6)$ be such that $p_i \in Q_i(\alpha_i)$ for $i \in \{2,3,4,5\}$ is $O(n^{20/9})$. Note that for any such cycle p_1 must be in $Q_1(\alpha_2,\alpha_3)$ and p_6 must be in $Q_6(\alpha_4,\alpha_5)$. In the rest of the proof we will use the notation $Q_i = Q_i(\alpha_i)$ for $i \in \{2,3,4,5\}$, and $Q_6 = Q_6(\alpha_4,\alpha_5)$, $Q_1 = Q_1(\alpha_2,\alpha_3)$.

Let $0 \le x_1,\dots,x_6 \le 1$ be such that $|Q_i| = n^{x_i}$. First, we bound the number of 6-cycles in the case when at least two Q_i is of size $O(n^{1/2})$. If there are two cyclically adjacent indices, say 1 and 2, such that Q_1 and Q_2 are of size $O(n^{1/2})$, then we obtain the bound

$$4u(n^{x_1},n^{x_6})\; u(n^{x_3},n^{x_4}) = O(n^{2/3}n^{4/3}) = O(n^2),$$

by picking (p_1,p_6), (p_3,p_4) and extending (p_2,p_3,p_4,p_6) to a 6-cycle in at most four different ways. If there are two such non-adjacent i and j such that Q_i and Q_j are of size $O(n^{1/2})$, then we will find two disjoint pairs of indices, separated by one index in each direction, say $(1,6)$ and $(3,4)$, such that $u(n^{x_1},n^{x_6}) = O(n)$ and $u(n^{x_3},n^{x_4}) = O(n)$, and obtain the bound $O(n^2)$ again.

Next, we bound the number of 6-cycles in the case when at least one Q_i, say Q_1 is of size $O(n^{2/9})$. In this case, by picking (p_3,p_4,p_5) and p_1 first, we can extend it to a 6-cycle in at most four different ways, and obtain the bound $O(n^2)n^{2/9} = O(n^{20/9})$.

From now on, we assume that there are at most one Q_i with $|Q_i| = O(n^{1/2})$, and every Q_i is of size $\Omega(n^{2/9})$. We count the 6-cycles by picking (p_3,p_4) first. Then p_1 must be in the second neighborhood of p_3, and p_6 must be in the second neighborhood of p_4. Further, once (p_1,p_3,p_4,p_6) is picked, there are at most four different ways to finish the cycles. With this, we obtain the bound

$$u(n^{x_3},n^{x_4})\, u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}). \tag{8}$$

We also have

$$n^{\alpha_2} \le \frac{u(n^{x_2},n)}{n^{x_2}}, \quad n^{\alpha_3} \le \frac{u(n^{x_2},n^{x_3})}{n^{x_3}}, \quad n^{\alpha_4} \le \frac{u(n^{x_4},n^{x_5})}{n^{x_4}}, \quad n^{\alpha_5} \le \frac{u(n^{x_5},n)}{n^{x_5}}. \tag{9}$$

By (2) we have $u(m,n) = O(m^{2/3}n^{2/3} + n + m) = O(\max\{m^{2/3}n^{2/3}, n, m\})$. We will distinguish a few cases based on which term the maximum is taken in (8) and in the inequalities in (9).

Case 1: Both in (8) and (9) the maximum is taken on the first term everywhere. Then we obtain the bound

$$\begin{aligned} &u(n^{x_3}, n^{x_4})\, u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) \\ &\quad = O\big(n^{2(x_3+x_4)/3} n^{2(4/3+(x_2+x_5)/3-(x_3+x_4)/3)/3}\big) = O(n^{20/9}). \end{aligned}$$

Case 2: $u(n^{x_3}, n^{x_4}) = O(\max\{n^{x_3}, n^{x_4}\})$. Without loss of generality we may assume that $u(n^{x_3}, n^{x_4}) = O(n^{x_4})$. Note that this implies $n^{x_3} = O(n^{1/2})$. Then we may assume that $u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) = O(n^{2(\alpha_2+\alpha_3+\alpha_4+\alpha_5)/3})$, otherwise we would obtain the bound $O(n^2)$. Similarly, we may assume that the maximum in the bound for $u(n^{x_2}, n), u(n^{x_4, x_5}), u(n^{x_5}, n)$ the maximum is taken on the first term, otherwise we would obtain two parts Q_i of size $O(n^{1/2})$.

- If $u(n^{x_2}, n^{x_3}) = O(n^{x_3})$, then $n^{x_2} = O(n^{1/2})$, giving a Q_i of size $O(n^{1/2})$.
- If $u(n^{x_2}, n^{x_3}) = O(n^{x_2})$, then we obtain the bound

$$\begin{aligned} u(n^{x_3}, n^{x_4})\, u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) &= O\big(n^{x_4} n^{2(4/3+2x_2/3-x_3-x_4/3+x_5/3)/3}\big) \\ &= O(n^{20/9}), \end{aligned}$$

  using the assumption that $n^{x_3} = \Omega(n^{2/9})$.
- Finally, if $u(n^{x_2}, n^{x_3}) = u(n^{2(x_2+x_3)/3})$, then we obtain

$$\begin{aligned} u(n^{x_3}, n^{x_4})\, u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) &= O\big(n^{x_4} n^{2(4/3+(x_2+x_5)/3-(x_3+x_4)/3)/3}\big) \\ &= O(n^{20/9}). \end{aligned}$$

Case 3: $u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) = O(\max\{n^{\alpha_2+\alpha_3}, n^{\alpha_4+\alpha_5}\})$. Without loss of generality we may assume that $u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) = O(\max\{n^{\alpha_2+\alpha_3}\})$. This implies $n^{\alpha_4+\alpha_5} = O(n^{1/2})$. Similarly as in Case 2, we may assume that in the bound for $u(n^{x_2}, n), u(n^{x_2}, n^{x_3}), u(n^{x_3}, n^{x_4})$ the maximum is taken in the first term. Then regardless on which term the maximum is taken in $u(n^{x_5}, n)$, we obtain the bound

$$u(n^{x_3}, n^{x_4})\, u(4n^{\alpha_2+\alpha_3}, 4n^{\alpha_4+\alpha_5}) = O\big(n^{2(x_3+x_4)/3} n^{2/3+x_2/3-x_3/3}\big) = O(n^{20/9}).$$

***k* = 7, 9:** The proof is by using dyadic decomposition in a similar way as in the $k = 6$ case. The reason why there is 3 in the denominator of the exponent instead of 9 is that we do not consider $Q_1(\alpha_2, \alpha_3)$ type of sets, only $Q_i(\alpha_i)$. □

4 3-Regular Graphs in $\mathbb{R}^3$

The main goal of this section is to prove Theorem 1.3. As it does not affect the answer up to the order of magnitude, we switch to the multipartite version of the problem. For a fixed 3-regular graph G on k vertices, and for k sets $P_1, \ldots, P_k \subseteq \mathbb{R}^3$ we denote by $F(P_1, \ldots, P_k)$ the maximum number of k-tuples $(p_1, \ldots, p_k)$ such that the unit distance graph determined by them is isometric to G, and $p_i \in P_i$ for every $i \in [k]$. Further, we use the notation

$$f(n_1, \ldots, n_k) = \max |F(P_1, \ldots, P_k)|,$$

and $f(n) = f(n, \ldots, n)$.

To prove Theorem 1.3, we follow a divide and conquer strategy of Agarwal and Sharir [1]. The strategy uses *cuttings*, a partitioning technique, which was a precursor to the more recent polynomial partitioning method. We say that a sphere S *crosses* a subset $\tau \subseteq \mathbb{R}^n$ if $S \cap \tau \neq \emptyset$, but $\tau \not\subset S$. To follow usual terminology, we will call the subsets in the partition *cells*. (Note that usually a *cell* in this context, means a more specific subset described by a bounded number of polynomials. However, since we only use the cutting results as a black-box in a very specific case, for simplicity, we do not define cells here more properly.) The following lemma was proved in [1].

Lemma 4.1 (Cutting Lemma) *Given a set of points P and ℓ sets $Q_0, \ldots, Q_\ell$ of spheres in $\mathbb{R}^d$, for any $1 \leq r \leq n$ we can partition the $\mathbb{R}^d$ into cells such that the following three conditions hold.*

- *The number of cells is $\tilde{O}(r^d)$.*
- *The number of points in each cell is $O(|P|/r^d)$.*
- *For every $i \in [\ell]$ each cell is crossed by $O(|Q_i|/r)$ many spheres from Q_i.*

Further, if P is contained in a $(d-1)$-sphere $\mathbb{S}^{d-1}$, then we can partition $\mathbb{S}^{d-1}$ into cells such that the following three conditions holds.

- *The number of cells is $\tilde{O}(r^{d-1})$.*
- *The number of points in each cell is $O(|P|/r^{d-1})$.*
- *For every $i \in [\ell]$ each cell is crossed by $O(|Q_i|/r)$ many spheres from Q_i.*

To illustrate the method, we first prove the following simpler proposition, which gives the best upper bound we have been able to prove on the number of 4-cycles in $\mathbb{R}^3$.

Proposition 4.2 *The maximum number of unit 4-cycles determined by a set of n points in $\mathbb{R}^3$ is $\tilde{O}(n^{12/5})$.*

It would be interesting to find better estimates. The best lower bounds we found is $\Omega(n^2)$, given by the same construction used on the sphere.

Problem 4.3 *Find the maximum possible number of 4-cycles determined by a set of n points in $\mathbb{R}^3$.*

Proof of Proposition 4.2 We again switch to the multipartite version of the problem, and for sets P_1, P_2, P_3, P_4 we denote by $C_4(P_1, P_2, P_3, P_4)$ the maximum number of unit 4-cycles (p_1, p_2, p_3, p_4) with $p_i \in P_i$. Further, let $c_4(n_1, n_2, n_3, n_4) = \max |C_4(P_1, P_2, P_3, P_4)|$ where the maximum is taken over all P_i with $|P_i| \leq n_i$, and let $c_4(n) = c_4(n, n, n, n)$. For a point p we also denote by $S(p)$ the unit sphere centered at p. We will bound $C_4(P_1, P_2, P_3, P_4)$ with $|P_i| \le n$ for all $i \in [4]$.

For some parameter r we partition $\mathbb{R}^3$ into $\tilde{O}(r^3)$ cells as in Lemma 4.1 with P_1 as the set of points, and $\{S(p_2) : p_2 \in P_2\}$ and $\{S(p_4) : p_4 \in P_4\}$ as the sets of spheres. For a cell τ let $P_1^\tau = P_1 \cap \tau$, further, for $i = 2, 4$ let $Q_i^\tau \subseteq \{S(p_i) : p_i \in P_i\}$ be the set of those spheres that cross τ, and $R_i^\tau \subseteq \{S(p_i) : p_i \in P_i\}$ be the set of those spheres that contain τ. Summing over all cells τ we obtain

$$C_4(P_1, P_2, P_3, P_4) \le \sum_\tau \big(C_4(P_1^\tau, Q_2^\tau, P_3, Q_4^\tau) + C_4(P_1^\tau, R_2^\tau, P_3, P_4) + C_4(P_1^\tau, P_2, P_3, R_4^\tau)\big). \qquad \square$$

Proposition 4.4 *$C_4(P_1^\tau, R_2^\tau, P_3, P_4) = \tilde{O}(n^2)$ and $C_4(P_1^\tau, P_2, P_3, R_4^\tau) = \tilde{O}(n^2)$.*

Proof We will only prove $C_4(P_1^\tau, R_2^\tau, P_3, P_4) = \tilde{O}(n^2)$, which is sufficient by symmetry. This means we have to bound the number of 4-cycles under the condition that all points in P_1^τ are contained in the intersection of unit spheres centered around the points of R_2^τ. Since in $\mathbb{R}^3$ for any three points there are at most two other points unit distance apart from each, we either have $|P_1^\tau| \le 2$ or $|R_2^\tau| \le 2$. Thus, $C_4(P_1^\tau, R_2^\tau, P_3, P_4)$ is bounded by four times the maximum number of 3-paths in $\mathbb{R}^3$, which is $\tilde{O}(n^2)$ by [7]. $\square$

By the properties of the cutting, $C_4(P_1^\tau, Q_2^\tau, P_3, Q_4^\tau) \le c_4(n/r^3, n/r, n, n/r)$. Thus, we obtain

$$C_4(P_1, P_2, P_3, P_4) \le \sum_\tau \left(c_4\left(\frac{n}{r^3}, \frac{n}{r}, n, \frac{n}{r}\right) + \tilde{O}(n^2)\right).$$

Repeating a similar analysis three more times with cyclic shifts (in the next round P_2 plays the role of P_1, P_3 plays the role of P_2, P_4 play the role of P_3, and P_1 plays the role of P_4, and so on), and using that the number of cells in the cutting is $\tilde{O}(r^3)$, we obtain the recurrence

$$c_4(n) \le \tilde{O}(r^{12})\, c_4\left(\frac{n}{r^5}\right).$$

With an appropriate choice of r this recursion yields $c_4(n) \le \tilde{O}(n^{12/5})$. $\square$

Notice that in the proof of Proposition 4.2 in each round when the Cutting Lemma is applied, for every cell we have to consider two different situations and split into two subproblems. For those spheres that cross the cell, we directly plug in the bound on the number of crossings from the lemma, and obtain the main term of the recursion. Those spheres that contain the cell are accounted for in Proposition 4.4. Proving Proposition 4.4 was simple, and we could deal with it by 'hand'. However, when we work with a large 3-regular graph, knowing information only about one local containment situation does not make the problem sufficiently simpler.

Therefore, we will follow further ideas of Agarwal and Sharir that they developed for bounding the number of k-simplices in higher dimension. We will sketch these ideas for completeness with incorporating the sufficient changes in the method, adjusting it to our problem. Notice that for $d = 0, 1$ Lemma 4.1 is trivial. Yet, we will utilize these trivial cases, as they will give a convenient uniform way to handle the containment situations mentioned in the previous paragraph.

Proof of Theorem 1.3 We will use the Cutting Lemma in k rounds to derive a recurrence for $f(n)$, following [1, Sect. 5], with making some suitable changes. We will assume that the vertex set of G is $[k]$.

Let $P_1, \dots, P_k \subseteq \mathbb{R}^3$ be point sets with $|P_i| \le n_i$ for every $i \in [k]$. We denote by $S(p)$ the unit sphere centered at p. Without loss of generality, we may assume that 2, 3, 4 are the neighbors of 1.

In the first round, we use the Cutting Lemma with some parameter r, with P_1 as the set of points, and $Q_i = \{S(p) : p \in P_i\}$ as the sets of spheres for $i = 2, 3, 4$. For a cell τ let $P_1^\tau = P_1 \cap \tau$. Further, for $i = 2, 3, 4$ let $Q_i^\tau \subseteq Q_i$ be the set of those spheres that cross τ, and R_i^τ be the set of those spheres that contain τ. Summing over all cells τ we obtain

$$F(P_1, \dots, P_k) = \sum_\tau \big(F(P_1^\tau, Q_2^\tau, Q_3^\tau, Q_4^\tau, P_5, \dots, P_k) + F(P_1^\tau, R_2^\tau, P_3, \dots, P_k) + F(P_1^\tau, P_2, R_3^\tau, P_4, \dots, P_k) + F(P_1^\tau, P_2, P_3, R_4^\tau, P_5 \dots, P_k) \big). \tag{10}$$

We will bound the first term in each summand by plugging in the information from the Cutting Lemma. In the proof of Proposition 4.2 we bounded the terms similar to the other three terms in Proposition 4.4, by observing that they correspond to bounding cycles in a geometrically constrained situation. While it is still true here that for example either $|P_1^\tau| \le 2$ or $|R_2^\tau| \le 2$, for large k it does not constrain the geometry sufficiently to bound $F(P_1, R_2^\tau, P_3, \dots, P_k)$ easily.

Thus, we will introduce new sub-problems, where we will keep track of containments that occur between certain parts by a weighted auxiliary graph H. Notice that we can say more than just $|P_1^\tau| \le 2$ or $|R_2^\tau| \le 2$. It is also true that if $|P_1^\tau| \ge 3$, then P_1 is contained in a 2-sphere, and a similar observation holds for R_2.

For a subgraph H of G we say that a k-tuple of points $(P_1, \dots, P_k)$ is of *type* H, if for every edge (i, j) of H the distance between every point of P_i and P_j is the one. We denote by

$$F^H(n_1, \dots, n_k) = \max F(P_1, \dots, P_k),$$

where the maximum is taken over all k-tuples $(P_1, \dots, P_k)$ of type H with $|P_i| \le n_i$. With this notation, (10) implies

$$F(n_1,\ldots,n_k)=\tilde{O}(r^3)\left(F\left(\frac{n_1}{r^3},\frac{n_2}{r},\frac{n_3}{r},\frac{n_4}{r},n_5,\ldots,n_k\right)\right.$$
$$\left.+F^{H_2}(n_1,\ldots,n_k)+F^{H_3}(n_1,\ldots,n_k)+F^{H_4}(n_1,\ldots,n_k)\right),$$

where H_i is the graph with a single edge $(1,i)$.

Refining the notion further, for a vector $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_k)\in\{0,1,2,3\}^k$ we say that a k-tuple $(P_1,\ldots,P_k)$ is of type $(H,\boldsymbol{\lambda})$,

- if $(P_1,\ldots,P_k)$ is of type H, and
- if $\lambda_i\le 2$, then P_i is contained in a λ_i-sphere but is not contained in a (λ_i-1)-sphere, further
- if $\lambda_i=3$, then P_i is not contained in a sphere.

We define $F^{H,\boldsymbol{\lambda}}(n_1,\ldots,n_k)$ analogously to $F^H(n_1,\ldots,n_k)$. Further, we say that a pair $(H,\boldsymbol{\lambda})$ is *realizable*, if there exists a k-tuple $(P_1,\ldots,P_k)$ of type $(H,\boldsymbol{\lambda})$.

For any subgraph H of G, and any vector $\boldsymbol{\lambda}$ by applying the Cutting, Lemma with the ith part playing the role of the points, and with some parameter r_i, we obtain

$$F^{H,\boldsymbol{\lambda}}(n_1,\ldots,n_k)=\tilde{O}(r^3)\big(F^{H,\boldsymbol{\lambda}}(m_1,\ldots,m_k)+F^{H_1,\boldsymbol{\lambda}_1}(n_1,\ldots,n_k)$$
$$+F^{H_2,\boldsymbol{\lambda}_2}(n_1,\ldots,n_k)+F^{H_3,\boldsymbol{\lambda}_3}(n_1,\ldots,n_k)\big),$$

where

- $m_i=n^i/r^{\lambda_i}$,
- $m_j=n_j/r$ if $(i,j)\in G\setminus H$,
- $m_j=n_j$ otherwise,
- each H_i is H extended by an edge connecting i with one of its neighbours in G, and
- $\boldsymbol{\lambda}_i$ is $\boldsymbol{\lambda}$ modified suitably along the new edge.

By applying the Cutting Lemma similarly k times such that in the ith round we have the points of the ith part as the set of points, the spheres centred in the neighbours (in G) of i as spheres, and $r_i=r^{x_i}$ as parameter, for any subgraph H we obtain

$$F^{H,\boldsymbol{\lambda}}(n_1,\ldots,n_k)=\tilde{O}\left(\prod_i r^{x_i}\right)\left(F^{H,\boldsymbol{\lambda}}(m_1,\ldots,m_1)+\sum_{H'}F^{H'}(n_1,\ldots,n_k)\right),$$

where the sum is taken over all subgraphs H' of G with strictly more edges than H, and where

$$m_i=\frac{n_i}{r^{x_i\lambda_i+\sum_{(i,j)\in G\setminus H}x_j}}.$$

Let $\xi(H, \boldsymbol{\lambda})$ be the solution of the following linear optimization problem:

$$\begin{aligned} &\min \sum x_i \lambda_i && \text{subject to:} && \\ &x_i \geq 0 && \text{for } 1 \leq i \leq k && (11) \\ &\lambda_i x_i + \sum_{(i,j)\in G\setminus H} x_j \geq 1 && \text{for } 1 \leq i \leq k. && (12) \end{aligned}$$

With this, we obtain

$$F^{H,\boldsymbol{\lambda}}(n) = \tilde{O}(r^{\xi(H,\boldsymbol{\lambda})}) \left(F^{H,\boldsymbol{\lambda}}\left(\frac{n}{r}\right) + \sum_{H'} F^{H'}(n) \right). \tag{13}$$

Let $\xi = \max_{H,\boldsymbol{\lambda}} \xi(H, \boldsymbol{\lambda})$, where the maximum is taken over all realizable $(H, \boldsymbol{\lambda})$. By induction on the number of edges in $G\setminus H$, and with an appropriate choice of r, using (13) one can show that we have $F(n) = \tilde{O}(n^{\xi})$. Note that the starting case of the induction is $G = H$, for which one can show directly that $F^G(n) = O(n^{k/2})$. Indeed, if $(P_1, \ldots, P_k)$ is of type G, then $|P_i| \leq 2$ for at least $k/2$ indices i. Thus, it is sufficient to show that the solution of the linear optimization problem for any realizable pair $(H, \boldsymbol{\lambda})$ is at most $k/2$.

We make some geometric observations about realizable pairs $(H, \boldsymbol{\lambda})$:

(i) If $\lambda_i = 3$, then i is an isolated vertex of H.
(ii) If $\lambda_i = 2$, then i has exactly one neighbour j in H, for which we must have $\lambda_j = 0$. (As P_j must be in the center of a 2-sphere).
(iii) If $\lambda_i = 1$, then for any neighbour j of i in H we must have $\lambda_j = 0$. (As there are at most two points at a given distance from all points of a circle.)
(iv) If i is an isolated vertex in H, then we may assume that for every neighbour j of i in G we have $\lambda_j \geq 1$.
(v) We may assume that if $\lambda_i = 0$, then $\deg_H(i) = 3$. (Indeed, if $\lambda_i = 0$, and $(P_1, \ldots, P_n)$ is of type $(H, \boldsymbol{\lambda})$, then $|P_i| \leq 2$. Since we only want to bound $F^{H,\boldsymbol{\lambda}}(n)$ up to the order of magnitude, we may assume that $|P_i| = 1$. Then if $(i, j) \in G$, we may discard those $p_j \in P_j$ that are not at distance one from the single point of P_i, as they cannot be part of any copy of G.)

Let us define $\mathbf{x} = (x_1, \ldots, x_k)$ as

$$x_i = \begin{cases} \dfrac{1}{2\lambda_i} & \text{if } \lambda_i \neq 0 \text{ and } \deg_H(i) = 0, \\[2ex] \dfrac{1 - (3 - \deg_H(i))/6}{\lambda_i} & \text{if } \lambda_i \neq 0 \text{ and } \deg_H(i) \neq 0, \\[2ex] 0 & \text{if } \lambda_i = 0. \end{cases}$$

We will show that this $\mathbf{x}$ satisfies the constraints in (12) in the linear optimisation problem, and gives optimal-function value $k/2$ in (11). This will finish the proof.

To check (12) we point out that each vertex i gets a contribution from those vertices j that are neighbours if i in G but not in H. Notice that $x_j \geq 1/6$ if $\lambda_j \neq 0$. This, together with (v) implies that

$$\lambda_i x_i + \sum_{(i,j)\in G\setminus H} x_j \geq 1 - \frac{3-\deg_H(i)}{6} + \frac{3-\deg_H(i)}{6} = 1.$$

To finish, we show that $\Sigma_{i=1}^{k}\lambda_i x_i \leq k/2$. Let

$$\begin{aligned} k_0 &= |\{i : \deg_H(i) = 0\}|, & k_1 &= |\{i : \deg_H(i) = 1,\ \lambda_i = 2\}|,\\ k_2 &= |\{i : \deg_H(i) = 3,\ \lambda_i = 1\}|, & k_3 &= |\{i : \deg_H(i) = 1,\ \lambda_i = 1\}|,\\ k_4 &= |\{i : \deg_H(i) = 2,\ \lambda_i = 1\}|, & k_5 &= |\{i : \lambda_i = 0\}|. \end{aligned}$$

Then (ii), (iii), and (v) together with a double counting implies that

$$3k_5 = k_1 + 3k_2 + k_3 + 2k_4,$$

which gives that

$$k = k_0 + k_1 + k_2 + k_3 + k_4 + k_5 = k_0 + \frac{4k_1}{3} + 2k_2 + \frac{4k_3}{3} + \frac{5k_4}{3}.$$

From this, and the definition of **x** it follows that

$$\sum_{i=1}^{k}\lambda_i x_i = \frac{k_0}{2} + \frac{k_1}{3} + k_2 + \frac{2k_3}{3} + \frac{5k_4}{6} \leq \frac{k}{2}.$$

□

We close this section by describing constructions for bipartite G that match the upper bound in the slightly modified setting, when we count the number of copies of G with prescribed edge lengths. We choose $k/2$ points on a line ℓ, Then, we fix a circle C in a plane orthogonal to ℓ, and centred in a point of ℓ, and place $n-k/2$ points on it. Now it is easy to check that by picking the $k/2$ points from ℓ, and any $k/2$ points from C, we obtain a copy of G.

5 Concluding Remarks and Further Problems

While we could find sharp bounds for unit distance k-cycles for most k, on $\mathbb{R}^2$ and in $\mathbb{R}^3$ the problem seems more difficult. Proving good bounds for short cycles would be particularly interesting. In the plane, for $k=3$ an easy upper bound is $u_2(n)$, and the best lower bound is $ne^{\Omega(\log n/\log\log n)}$. (See discussion in [3, Chap. 6].) For $k=4$ in the plane, we can construct $\Omega(u_2(n))$ many 4-cycles by using two translated copies of a set with optimally many unit distances, and the best upper bound we could prove is $\tilde{O}(n^{5/3})$.

For 3-regular graphs in $\mathbb{R}^3$, our bounds are sharp for bipartite graphs for the modified setting. It would be interesting to find sharp bounds in the general case, or at least for some small non-bipartite 3-regular graphs.

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