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Scarcity and stability in a very simple general equilibrium model

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Abstract
This paper examines the relation between ‘scarcity’ prices and stability in a competitive equilibrium model by analysing the structure of interdependence between the good for which there is a parameter shift, and all other goods in the model. It is shown that if a knowledge of stability is supplemented only with the sign of the own-price partial derivative of the excess demand function of the good in question, the direction of the comparative static outcome is known.
The results of general competitive analysis have demonstrated that two apparently plausible economic intuitions are without foundation. First, the intuition that prices in a general competitive model reflect 'scarcity' has been shown to hold only in the presence of a strong assumption such as that of gross substitutes.\(^1\) In the absence of such an assumption, a good's price will not necessarily move in the expected direction following an exogenous shift in preferences, endowments or technology. In this case, for example, the presence of 'perverse' prices cannot be ruled out because the price of a good may fall relative to other goods even though preferences have shifted in its favour. Second, Paul Samuelson's intuition that the assumption of stability would entail determinate comparative static outcomes (otherwise known as the Correspondence Principle),\(^2\) has also proved unfounded.\(^3\)

This paper will present a very simple comparison of equilibria in a competitive model subject to a binary change by focussing on the structure of interdependence between a single good subject to an exogenous change and all other goods taken together. This will allow a reformulation of both the comparative static issue of the presence of 'scarcity' prices and the dynamic process of price adjustment. In this model, the Correspondence Principle is shown to hold on the basis only of prior knowledge of the sign of the own-price partial derivative of the excess demand function for the single good subject to the parameter shift. Part II presents the comparative static analysis, Part III presents the dynamic analysis, and Part IV draws out some conclusions.

**PART II**

In an \(n\) good plus numeraire model, the \(n\) excess demand functions, \(Z_i\), link together the \(n\) endogenous variables, \(p_i\), with the \(m\) exogenous variables, \(a_s\).

\[ Z_i = f_i(p, a) \quad \text{where } i = 1, 2, ..., n. \]

\(p\) and \(a\) are vectors.

In this very simple model, it is assumed that only one element of \(a\), say \(a_s\), is non-zero. The effect on the price structure of a change in the exogenous variable can be determined from the following:

\[
\frac{dZ_i}{dp} = \sum_{j=1}^{n} \left[ (\frac{\partial Z_i}{\partial p_j}) dp_j^* \right] + (\frac{\partial Z_i}{\partial a}) da.
\]
As $Z_i = 0$ at equilibrium, then $dZ_i = 0$. Hence

$$
\sum_{j=1}^{n} [(\partial Z_i / \partial p_j) dp_j^*] = -(\partial Z_i / \partial a) da.
$$

In matrix notation

$$Zdp = - \alpha \quad (1)$$

where

$Z$ is an $n \times n$ matrix of the partial derivatives of the excess demand functions with respect to price, i.e., $z_{ij} = \partial Z_i / \partial p_j$.

dp is an $n \times 1$ column vector of the changes in the equilibrium price, i.e. $dp_i = dp_j^*$.

and

$\alpha$ is an $n \times 1$ column vector of the exogenous changes in the excess demand functions, i.e. $\alpha_i = (\partial Z_i / \partial a) da$.

In comparing equilibria, the object is to find the effect on prices of a change in the exogenous variable. Provided that $Z$ is non-singular, (1) may be rewritten as:

$$dp = - Z^{-1} \alpha.$$

In the binary change case, $x_j$ is the only good (excluding numeraire) to be affected by the exogenous change, i.e., $\partial Z_i / \partial a = 0$ for all $i = 1, 2, ..., j-1, j+1, ..., n$. This means that all elements of $\alpha$ are equal to zero except the element $\alpha_j = (\partial Z_j / \partial a) da$.

In this binary change case, the change in the equilibrium price of any good, $x_i$, is given by

$$dp_i^* = - \frac{\bar{z}_{ij} (\partial Z_j / \partial a) da}{|Z|} \quad (2)$$

where $\bar{z}_{ij}$ is the transpose of the cofactor of $z_{ij}$ in $Z$, and where $z_{ij} = \partial z_i / \partial p_j$. 


From (2) it follows that

\[ dp^*_{j} = - \frac{\bar{z}_{jj} (\partial Z_j/\partial a)}{|Z|} \]  

(3)

where \( \bar{z}_{jj} \) is the transpose of the cofactor of \( z_{jj} \) in \( Z \), and where \( z_{jj} = \partial z_j/\partial p_j \).

These are standard results in comparative statics. In order to analyse the complex structure of interdependence when comparing equilibria, it is proposed here to decompose \( dp^*_{j} \) into two parts:

(i) that part which corresponds to the partial equilibrium outcome and which is unaffected by the interdependence between \( x_j \) and all other goods, and

(ii) the remaining part which results from the interdependence between \( x_j \) and all other goods.

The first component is here called the direct effect \( (D_j) \) and the second component is called the indirect effect \( (I_j) \), that is:

\[ dp^*_{j} = D_j + I_j \]  

(4)

The direct effect is given by the following:

\[ D_j = - (\partial Z_j/\partial a) da \]  

(5)

In this case, as \( (\partial Z_j/\partial a) da \) is positive by supposition, the sign of \( D_j \) depends on the sign of \( \partial Z_j/\partial p_j \). Figure 1 sketches three possible outcomes where \( D_j \) is positive, i.e., where the own-price excess demand function, \( \partial Z_j/\partial p_j \), is negative.

In Figure 1a, the indirect effect is positive and reinforces the direct effect; here the normal price outcome occurs where a shift in preferences in favour of a good results in an increase in its price. In Figure 1b, the indirect effect is negative but smaller than the direct effect; here the normal price outcome occurs but the final increase in \( p^* \) is smaller than the direct effect. In Figure 1c, the indirect effect is both negative and larger than the direct effect; in
this case the 'perverse' price effect occurs and $p^*_j$ falls even though preferences have shifted in favour of $x_j$.

Note:
$D_o$ is initial excess demand
$D^*$ is final excess demand
$D^D$ is the shift in excess demand that reflects only the direct effect
This underlines the importance of the indirect effect in potentially overturning the results derived from a partial equilibrium setting, and shows how even a downward-sloping own-price excess demand function can result in 'perverse' prices in a general equilibrium setting. A 'perverse' indirect effect may, however, result in orthodox 'scarcity' prices if the own-price excess demand function is upward sloping. This is shown as case (c) in Figure 2.

In Figure 2a the indirect effect works in the same direction as the direct effect, whereas in Figure 2b it works in the opposite direction but is too small to outweigh the direct effect.

The indirect effect, \( I_j \), derives from the complex structure of interdependence between \( x_j \) and all other goods. It may be represented in general terms as some proportion, \( \lambda_j \), of the total change in price,

\[
i.e. \ I_j = \lambda_j \cdot dp^*_j. \tag{6}\]

Combining equations (4), (5) and (6) gives:

\[
dp^*_j = - \frac{(\partial Z_j / \partial a) \ da}{\partial Z_j / \partial p_j} + \lambda_j \cdot dp^*_j. \tag{7}\]

Rearranging terms gives:

\[
dp^*_j = - \frac{(\partial Z_j / \partial a) \ da}{\partial Z_j / \partial p_j} \cdot \frac{1}{1 - \lambda_j}. \tag{8}\]

i.e. \( dp^*_j = D_j \cdot 1/ (1 - \lambda_j). \)

Thus, for any given \( D_j \), the change in the equilibrium price of \( x_j \) depends on the sign and size of \( \lambda_j \), which may be regarded as a coefficient of interdependence between \( x_j \) and all other goods.

Combining (3) and (7) gives the following expression for \( \lambda_j \):

\[
\lambda_j = 1 - \frac{|Z|}{z_{jj} \cdot z_{jj}}. \tag{9}\]
Note:
$D_0$ is initial excess demand
$D^*$ is final excess demand
$D^D$ is the shift in excess demand that reflects only the direct effect
After some manipulation for the case of a three good model (excluding numeraire) where \( j = 2 \), equation (9) becomes:

\[
\lambda_2 = \frac{z_{12} \cdot z_{21}}{z_{11} \cdot z_{22}} + \frac{z_{13} \cdot z_{31}}{z_{11} \cdot z_{33}} + \frac{z_{23} \cdot z_{32}}{z_{22} \cdot z_{33}} - \frac{z_{12} \cdot z_{23} \cdot z_{31}}{z_{11} \cdot z_{22} \cdot z_{33}} - \frac{z_{13} \cdot z_{32} \cdot z_{21}}{z_{11} \cdot z_{33} \cdot z_{22}}
\]

(10)

This expression for \( \lambda \) can be understood intuitively as a measure of diagonal dominance; if the \( Z \) matrix is characterised by diagonal dominance, then \( \lambda \) will be less than one.

Returning to equation (8), it is clear that \( d\pi^*_j \) will depend on the various combinatorial outcomes of \( D_j \) and \( \lambda_j \). In the case considered here where \( (\partial Z_j / \partial a) \, da > 0 \) by supposition, the outcome depends on \( \partial Z_j / \partial p_j \) and \( \lambda_j \).

1. \( \partial Z_j / \partial p_j < 0 \)

The 'normal' case of a negative own-price excess demand function is illustrated in Figure 3 which shows how \( d\pi^*_j \) depends on the value of \( \lambda_j \).

If \( \lambda_j = 0 \) then the outcome is the same as the partial equilibrium result. If \( 0 < \lambda_j < 1 \), then the price outcome is the same as that shown in figure 1a. If \( \lambda_j < 0 \), then the outcome corresponds with figure 1b. The price outcome is 'perverse' iff \( \lambda_j > 1 \) as shown in figure 1c, where the equilibrium price of \( x_j \) falls as a result of an exogenous shift in its favour; in this case there is a high degree of complementarity and an absence of diagonal dominance. (At \( \lambda_j = 1 \), the \( Z \) matrix is singular).
2. $\partial Z_j / \partial p_j > 0$
This corresponds to a positive diagonal element in the $Z$ matrix owing to a positively sloped own-price demand function, a negatively sloped own-price supply function, or both. In a partial equilibrium framework where $\lambda_j = 0$, an increase in demand for $x_j$ is compatible only with a lower equilibrium price, whilst the equilibrium quantity traded may be higher or lower. This partial equilibrium case has long been recognised but, in a general equilibrium context, such an excess demand function will not result in 'perverse' prices iff $\lambda_j > 1$. This case is illustrated in Figure 4.
Gross substitutes
It is known that the assumption of gross substitutes is sufficient to prevent the 'perverse' price effect, so this raises the question of the relation between the GS condition and the conditions listed above. The GS assumption excludes any complementarity and therefore excludes all outcomes falling outside $0 < \lambda < 1$ as well as all positive diagonal elements in the $Z$ matrix. In addition, some outcomes falling within $0 < \lambda < 1$ may exhibit some weak complementarity and would therefore be excluded too. (That is, although $0 < \lambda < 1$ and negative diagonals are jointly necessary for GS, they are not sufficient.) Further, the GS assumption constrains all excess demand functions whereas the two conditions presented here are specified in terms of the $j$th commodity only and so are less restrictive in their implications for the remaining commodities.

Thus, in this simple general equilibrium model, prices reflect scarcity under conditions that are much less restrictive than the GS assumption. For any good, $x_j$, price changes reflect scarcity following a parameter change iff either:

(i) $\frac{\partial Z_j}{\partial p_j} < 0$ and $\lambda_j < 1$, or
(ii) $\frac{\partial Z_j}{\partial p_j} > 0$ and $\lambda_j > 1$.

This simple model may also shed some light on the discussion as to whether prices in a general competitive model do reflect 'scarcity'. In this context, 'scarcity' refers to the exogenous conditions of tastes, endowments and technology; prices are thought not to reflect 'scarcity' if the change in the equilibrium price does not correspond with the exogenous change. This accords with an exogenous notion of 'scarcity' as determined by the environment of the model. On the other hand, there is an alternative concept of scarcity as 'endogenous scarcity' which is given by the interaction between the exogenous and endogenous variables of the model. The exogenous change may have been, say, a shift in tastes in favour of coffee, but if, at the end of the interdependent process of adjustment, the demand for coffee falls below its original level, the resulting fall in price is fully in line with this reduction in 'endogenous scarcity'. In other words, it is 'endogenous scarcity' that underlies the excess demand function, and competitive equilibrium prices can never be out of line with this scarcity. The extent to which this scarcity corresponds with 'exogenous scarcity' depends of course on income effects and the relations of interdependence; in other words, it depends on $\frac{\partial Z_j}{\partial p_j}$ and $\lambda_j$. 
PART III
The above analysis concentrated on the comparative static results. To consider the issue of stability, let the model be divided into two subsets where the first subset is composed only of \( x_j \), and the other subset contains all other goods. In other words, the first subset contains \( x_j \), and the second subset contains \( x_i \) where \( i = 1, 2, ..., j-1, j+1, ..., n \). The auctioneer fixes prices in each subset in turn; this means that prices are fixed to eliminate excess demand in the one subset given the price(s) existing in the other subset, and so on with the auctioneer turning from one subset to the other and back again to the first subset until all prices are fixed at the equilibrium level.

In \( t = 0 \), the auctioneer observes the set containing \( x_j \) whose demand has increased following some exogenous change. In this conjugate pairs case, no markets in the other subset containing \( x_i \) are affected by the exogenous change. The auctioneer therefore changes \( p_j \) to eliminate excess demand in the market for \( x_j \). That is, \( p_j \) is changed by \( D_j \) where

\[
D_j = \frac{(\partial Z_j / \partial a) da}{\partial Z_j / \partial p_j}.
\]

In \( t = 1 \), the auctioneer turns to the second set which is now out of equilibrium as a result of the change in \( p_j \). The auctioneer therefore changes \( p_i \) until all excess demands \( Z_i = 0 \), where \( i = 1, 2, ..., j-1, j+1, ..., n \).

In \( t = 2 \), the auctioneer returns to the first subset where \( Z_j \) is no longer zero as a result of the price changes in the second set made in \( t = 1 \). The auctioneer then changes \( p_j \) to eliminate the excess demand for \( x_j \), given the prices existing in the second subset.

In \( t = 3 \), the second set contains non-zero excess demands as a result of the change in \( p_j \) in the previous period. Therefore \( p_i \) are adjusted to eliminate excess demand given the existing \( p_j \) established in the previous period.

This process of fixing prices alternately in each subset continues until equilibrium is restored in both subsets.

Consider the price changes in the \( x_j \) market only:
In $t = 0$, 

$$dp_j = D_j = -\frac{(\partial Z_j/\partial a) da}{\partial Z_j/\partial p}$$

Thereafter, the indirect effect only will be operative, as outlined above in section II, where the coefficient of interdependence $\lambda_j$ measures the feedback effect on the price of $x_j$ for a unit change in $p_j$;

i.e. $dp_j(t=0) = D_j$
$$dp_j(t=2) = \lambda_j D_j$$
$$dp_j(t=4) = \lambda_j^2 D_j$$
and so on.

Thus the change in the equilibrium price is given by:

$$dp^*_j = D_j (1 + l_j + l_j^2 + ...).$$

The stability of this process depends on whether the series $(1 + \lambda_j + \lambda_j^2 + ...)$ converges; that is, this process is stable iff $|\lambda_j| < 1$.

**Summarising the various outcomes with respect to stability:**

**a) Stable adjustment processes**

1) $\partial Z_j/\partial p_j < 0$ and $-1 < \lambda_j < 1$. The adjustment process is stable, even though complementarity is not ruled out. Here the price change reflects ‘exogenous scarcity’.

2) $\partial Z_j/\partial p_j > 0$ and $-1 < \lambda_j < 1$. The process is stable but the price change is ‘perverse’.

**b) Unstable adjustment processes**

1) price change reflects ‘exogenous scarcity’:

   (a) $\partial Z_j/\partial p_j < 0$ and $\lambda_j < -1$. The adjustment process is unstable with an explosive price spiral oscillating between positive and negative changes.

   (b) $\partial Z_j/\partial p_j > 0$ and $\lambda_j > 1$. This is the case with a positively sloping excess demand function for $x_j$ but where a high degree of complementarity in the system can compensate for this and produce an orthodox comparative static price result where price reflects exogenous scarcity.

2) price change is ‘perverse’:

   (i) $\partial Z_j/\partial p_j < 0$ and $\lambda_j > 1$. The price change is perverse and the process is unstable.

   (ii) $\partial Z_j/\partial p_j > 0$ and $\lambda_j < -1$. The price change is perverse and the process is unstable with prices oscillating between positive and negative changes.
These results summarise the relation between stability and 'scarcity' prices. Information that a system is stable, is by itself insufficient to provide comparative static results, but if this information is supplemented only with knowledge of the sign of the own-price partial derivative of the excess demand function for the good subject to the parameter shift, then the comparative static outcome is known. Thus, in this simple model, Samuelson's intuition that stability conditions and comparative static outcomes may be connected is shown to be not entirely without foundation.

PART IV
This paper has re-examined Paul Samuelson's intuition that stability conditions and comparative static outcomes may be intimately linked. The paper has approached this question by decomposing any given price change into two parts, that which corresponds to the partial equilibrium outcome, and that which results from the structure of interdependence between the good for which there has been a parameter shift, and all other goods in the model. This allowed a simple reformulation of the comparative static outcome in terms of the good's own-price excess demand function and $\lambda$, a coefficient of interdependence between that single good and all other goods in the model. When reformulated in this way, it was shown that prices reflect 'exogenous scarcity' under necessary and sufficient conditions that are much less restrictive than the gross substitutes assumption. Further, it was shown that the information that a system is stable, needs only to be supplemented with a knowledge of the own-price partial derivative of the excess demand function for the good subject to the parameter shift, in order to deduce the direction of the comparative static result.

* 
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1 Arrow and Hahn (1971) chapter 10.
2 Samuelson (1947) chapter 9.

References

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