On Helly Numbers of Exponential Lattices

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On Helly Numbers of Exponential Lattices

Gergely Ambrus
Department of Geometry, Bolyai Institute, University of Szeged, Szeged, Hungary,
Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Martin Balko
Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University,
Prague, Czech Republic

Nóra Frankl
School of Mathematics and Statistics, The Open University, Milton Keynes, United Kingdom
Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Attila Jung
Institute of Mathematics, ELTE Eötvös Loránd University, Budapest, Hungary

Márton Naszódi
Department of Geometry, ELTE Eötvös Loránd University, Budapest, Hungary,
MTA-ELTE Lendület Combinatorial Geometry Research Group, Budapest, Hungary

Abstract
Given a set $S \subseteq \mathbb{R}^2$, define the Helly number of $S$, denoted by $H(S)$, as the smallest positive integer $N$, if it exists, for which the following statement is true: for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$ such that the intersection of any $N$ or fewer members of $\mathcal{F}$ contains at least one point of $S$, there is a point of $S$ common to all members of $\mathcal{F}$.

We prove that the Helly numbers of exponential lattices $\{\alpha^n : n \in \mathbb{N}_0\}^2$ are finite for every $\alpha > 1$ and we determine their exact values in some instances. In particular, we obtain $H(\{2^n : n \in \mathbb{N}_0\}^2) = 5$, solving a problem posed by Dillon (2021).

For real numbers $\alpha, \beta > 1$, we also fully characterize exponential lattices $L(\alpha, \beta) = \{\alpha^n : n \in \mathbb{N}_0\} \times \{\beta^n : n \in \mathbb{N}_0\}$ with finite Helly numbers by showing that $H(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is rational.

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1 Introduction

*Helly’s theorem* [11] is one of the most classical results in combinatorial geometry. It states that, for each $d \in \mathbb{N}$, if the intersection of any $d + 1$ or fewer members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is nonempty, then the entire family $\mathcal{F}$ has nonempty intersection. There have been numerous variants and generalizations of this famous result; see [1, 13] for example.

One active direction of this research with rich connections to the theory of optimization, in particular to integer programming and LP-type problems [1, 4], is the study of variants of Helly’s theorem with coordinate restrictions, which is captured by the following definition.

Let $d$ be a positive integer. The *Helly number* of a set $S \subseteq \mathbb{R}^d$, denoted by $H(S)$, is the smallest positive integer $N$ if it exists, such that the following statement is true for every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$: if the intersection of any $N$ or fewer members of $\mathcal{F}$ contains at least one point of $S$, then $\bigcap \mathcal{F}$ contains at least one point of $S$. If no such number $N$ exists, then we write $H(S) = \infty$. Helly’s theorem in this language can be restated as $H(\mathbb{R}^d) = d + 1$.

A classical result of this sort is *Doignon’s theorem* [8] where the set $S$ is the integer lattice $\mathbb{Z}^d$. This result, which was also independently discovered by Bell [3] and by Scarf [15], states that $H(\mathbb{Z}^d) \leq 2^d$. This is tight as for $Q = \{0, 1\}^d$ the intersection of any $2^d - 1$ sets in the family $\{\text{conv}(Q \setminus \{x\}) : x \in Q\}$ contains a lattice point, but the intersection of all $2^d$ sets does not.

The theory of Helly numbers of general sets is developing quickly and there are many results of this kind [1, 13]. For example, De Loera, La Haye, Oliveros, and Roldán-Pensado [5] and De Loera, La Haye, Rolnick, and Soberón [6] studied the Helly numbers of differences of lattices and Garber [9] considered Helly numbers of crystals or cut-and-project sets.

The Helly number of a set $S$ is closely related to the maximum size of a set that is empty in $S$. A subset $X \subseteq S$ is *intersect-empty* if $\bigcap_{x \in X} \text{conv}(X \setminus \{x\}) \cap S = \emptyset$. A convex polytope $P$ with vertices in $S$ is *empty in $S$* if $P$ does not contain any points of $S$ other than its vertices. In particular, an empty polytope does not contain points of $S$ in the interior of its edges. For a discrete set $S$, we use $h(S)$ to denote the maximum number of vertices of an empty polytope in $S$. If there are empty polytopes in $S$ with arbitrarily large number of vertices, then we write $h(S) = \infty$.

The following result by Hoffman [12] (which was essentially already proved by Doignon [8]) shows the close connection between intersect-empty sets and empty polygons in $S$ and the $S$-Helly numbers; see also [2].

**Proposition 1** ([12]). If $S \subseteq \mathbb{R}^d$, then $H(S)$ is equal to the maximum cardinality of an intersect-empty set in $S$. If $S$ is discrete, then $H(S) = h(S)$.

Since all the sets $S$ studied in this paper are discrete, we state all of our results using $h(\alpha)$ but, due to Proposition 1, our results apply to $H(\alpha)$ as well.

Very recently, Dillon [7] proved that the Helly number of a set $S$ is infinite if $S$ belongs to a certain collection of *product sets*, which are sets of the form $S = A^d$ with a certain kind of discrete set $A \subseteq \mathbb{R}$. His result shows, for example, that whenever $p$ is a polynomial of degree at least 2 and $d \geq 2$, then $h(\{p(n) : n \in \mathbb{N}_0\}^d) = \infty$. However, there are sets for which Dillon’s method gives no information, for example $\{2^n : n \in \mathbb{N}_0\}^2$. Thus, Dillon [7] posed the following question, which motivated our research.

**Problem 1** (Dillon, [7]). What is $h(\{2^n : n \in \mathbb{N}_0\}^2)$?

In this paper, we study the Helly numbers of exponential lattices $L(\alpha)$ and $L(\alpha, \beta)$ in the plane where $L(\alpha) = \{\alpha^n : n \in \mathbb{N}_0\}^2$ and $L(\alpha, \beta) = \{\alpha^n : n \in \mathbb{N}_0\} \times \{\beta^n : n \in \mathbb{N}_0\}$ for real
numbers $\alpha, \beta > 1$. In particular, we prove that Helly numbers of exponential lattices $L(\alpha)$ are finite and we provide several estimates that give exact values for $\alpha$ sufficiently large, solving Problem 1. We also show that Helly numbers of exponential lattices $L(\alpha, \beta)$ are finite if and only if $\log_\alpha(\beta)$ is rational.

**2 Our results**

For a real number $\alpha > 1$ and the exponential lattice $L(\alpha) = \{\alpha^n : n \in \mathbb{N}_0\}^2$, we abbreviate $h(L(\alpha))$ by $h(\alpha)$.

As our first result, we provide finite bounds on the numbers $h(\alpha)$ for any $\alpha > 1$. The upper bounds are getting smaller as $\alpha$ increases and reach their minimum at $\alpha = 2$.

| Theorem 2. For every real $\alpha > 1$, the maximum number of vertices of an empty polygon in $L(\alpha)$ is finite. More precisely, we have $h(\alpha) \leq 5$ for every $\alpha \geq 2$, $h(\alpha) \leq 7$ for every $\alpha \in [1+\frac{\sqrt{5}}{2}, 2)$, and $h(\alpha) \leq 3 \left\lfloor \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) \right\rfloor + 3$ for every $\alpha \in (1, \frac{1+\sqrt{5}}{2})$. |

We note that if $\alpha = 1 + \frac{1}{x}$ for $x \in (0, \infty)$, then the bound from Theorem 2 becomes $h(1 + \frac{1}{x}) \leq O(x \log_2(x))$. Moreover, we show that the breaking points of $\alpha$ for our upper bounds are determined by certain polynomial equations; see Section 3.

We also consider the lower bounds on $h(\alpha)$ and provide the following estimate.

| Theorem 3. We have $h(\alpha) \geq 5$ for every $\alpha \geq 2$ and $h(\alpha) \geq 7$ for every $\alpha \in \left[\frac{1+\sqrt{5}}{2}, 2\right)$. |

For every $\alpha \in \left(1, \frac{1+\sqrt{5}}{2}\right)$, we have $h(\alpha) \geq \left\lfloor \sqrt{\frac{1}{\alpha - 1}} \right\rfloor$.

If $\alpha = 1 + \frac{1}{x}$ where $x \in (0, \infty)$, then the lower bound from Theorem 3 becomes $h(1 + \frac{1}{x}) \geq \left\lfloor \sqrt{\frac{x}{x - 1}} \right\rfloor$. So with decreasing $\alpha$, the parameter $h(\alpha)$ indeed grows to infinity.

By combining Theorems 2 and 3, we get the precise value of the Helly numbers of $L(\alpha)$ with $\alpha \geq (1 + \sqrt{5})/2$. In particular, for $\alpha = 2$, we obtain a solution to Problem 1.

| Corollary 4. We have $h(\alpha) = 5$ for every $\alpha \geq 2$ and $h(\alpha) = 7$ for every $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$. |

We prove the following result which shows that even a slight perturbation of $S$ can affect the value $h(S)$ drastically (note that this also follows by adding large empty polygons to $S$ without changing its asymptotic density). The proof is omitted here. We use the Fibonacci numbers $(F_n)_{n \in \mathbb{N}_0}$, which are defined as $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 2$.

| Proposition 5. We have $h(\{F_n : n \in \mathbb{N}_0\}^2) = \infty$. |

We recall that $F_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$ for every $n \in \mathbb{N}_0$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{-1+\sqrt{5}}{2} = 1 - \varphi$ is its conjugate. Since $\psi < 1$, this formula shows that the points of $\{F_n : n \in \mathbb{N}_0\}^2$ are approaching the points of the scaled exponential lattice $\frac{\sqrt{5}}{\varphi} \cdot L(\varphi) = \{\varphi^n : n \in \mathbb{N}_0\}^2$. Thus, Proposition 5 is in sharp contrast with the fact...
that \( h\left(\frac{\phi}{\sqrt{5}} \cdot L(\varphi)\right) = h(\varphi) \leq 7 \), which follows from Theorem 2 and from the fact that affine transformations of any set \( S \subseteq \mathbb{R}^d \) do not change \( h(S) \). We also note Dillon’s method [7] does not imply \( h(F_n : n \in \mathbb{N}_0)^2 = \infty \).

We also consider the more general case of exponential lattices where the rows and the columns might use different bases. For real numbers \( \alpha > 1 \) and \( \beta > 1 \), let \( L(\alpha, \beta) \) be the set \( \{\alpha^n : n \in \mathbb{N}_0\} \times \{\beta^n : n \in \mathbb{N}_0\} \). Note that \( L(\alpha) = L(\alpha, \alpha) \) for every \( \alpha > 1 \).

As our last main result, we fully characterize exponential lattices \( L(\alpha, \beta) \) with finite Helly numbers \( h(L(\alpha, \beta)) \), settling the question of finiteness of Helly numbers of planar exponential lattices completely.

**Theorem 6.** Let \( \alpha > 1 \) and \( \beta > 1 \) be real numbers. Then \( h(L(\alpha, \beta)) \) is finite if and only if \( \log_\alpha(\beta) \) is a rational number.

Moreover, if \( \log_\alpha(\beta) \in \mathbb{Q} \), that is, \( \beta = \alpha^{p/q} \) for some \( p, q \in \mathbb{N} \), then

\[
\frac{1}{pq} \left( 1 - \frac{1}{\alpha^{1/q} - 1} \right) \leq h(L(\alpha, \beta)) \leq pq \cdot h(\alpha^p).
\]

The proof of the 'only if' part of Theorem 6 is based on the theory of continued fractions and Diophantine approximation. The details are discussed in Section 5. The proof of the 'if' part of Theorem 6 is based on Theorem 2 and is omitted here.

**Open problems**

First, it is natural to try to close the gap between the upper bound from Theorem 2 and the lower bound from Theorem 3 and potentially obtain new precise values of \( h(\alpha) \).

Second, we considered only the exponential lattice in the plane, but it would be interesting to obtain some estimates on the Helly numbers of exponential lattices \( \{\alpha^n : n \in \mathbb{N}_0\}^d \) in dimension \( d > 2 \).

We also mention the following conjecture of De Loera, La Haye, Oliveros, and Roldán-Pensado [5], which inspired the research of Dillon [7].

**Conjecture 7** ([5]). If \( \mathcal{P} \) is the set of prime numbers, then \( h(\mathcal{P}^2) = \infty \).

Using computer search, Summers [16] showed that \( h(\mathcal{P}^2) \geq 14 \).

**3 Proof of Theorem 2**

Here, we prove Theorem 2 by showing that the number \( h(\alpha) \) is finite for every \( \alpha > 1 \). This follows from the upper bounds \( h(\alpha) \leq 5 \) for \( \alpha \geq 2 \), \( h(\alpha) \leq 7 \) for every \( \alpha \geq \frac{1 + \sqrt{5}}{2} \), and

\[
h(\alpha) \leq 3 \left[ \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) \right] + 3
\]

for any \( \alpha \in (1, \frac{1 + \sqrt{5}}{2}) \).

We start by introducing some auxiliary definitions and notation. Let \( \alpha > 1 \) be a real number and consider the exponential lattice \( L(\alpha) \). For \( i \in \mathbb{N}_0 \), the \( i \)th column of \( L(\alpha) \) is the set \( \{\alpha^n, \alpha^i : n \in \mathbb{N}_0\} \). Analogously, the \( i \)th row of \( L(\alpha) \) is the set \( \{(\alpha^i, \alpha^n) : n \in \mathbb{N}_0\} \).

For a point \( p \) in the plane, we write \( x(p) \) and \( y(p) \) for the \( x \) and \( y \) coordinates of \( p \), respectively. Let \( P \) be an empty convex polygon in \( L(\alpha) \). Let \( e \) be an edge of \( P \) connecting vertices \( u \) and \( v \) where \( x(u) < x(v) \) or \( y(u) < y(v) \) if \( x(u) = x(v) \). We use \( \pi \) to denote the line determined by \( e \) and oriented from \( u \) to \( v \). The slope of \( e \) is the slope of \( \pi \), that is, \( \frac{y(v)-y(u)}{x(v)-x(u)} \).
We distinguish four types of edges of $P$; see part (a) of Figure 1. First, assume $x(u) \neq x(v)$ and $y(u) \neq y(v)$. We say that $e$ is of type I if the slope of $e$ is negative and $P$ lies to the right of $\tau$. Similarly, $e$ is of type II if the slope of $e$ is positive and $P$ lies to the right of $\tau$. An edge $e$ has type III if the slope of $e$ is negative and $P$ lies to the left of $\tau$. Finally, type IV is for $e$ with positive slope and with $P$ lying to the left of $\tau$. It remains to deal with horizontal and vertical edges of $P$. A horizontal edge $e$ is of type II if $P$ lies below $\tau$ and is of type III otherwise. Similarly, a vertical edge $e$ is of type IV if $P$ lies to the left of $\tau$ and is of type III otherwise.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{(a) The four types of edges of a convex polygon. (b) An illustration of the proof of Lemma 8.}
\end{figure}

Note that each edge of $P$ has exactly one type and that the types partition the edges of $P$ into four convex chains. We first provide an upper bound on the number of edges of those chains of $P$ and then derive the bound on the total number of edges of $P$ by summing the four bounds. We start by estimating the number of edges of $P$ of type I.

\begin{lemma}
The polygon $P$ has at most $\left\lceil \log_\alpha \left( \frac{n}{\alpha - 1} \right) \right\rceil$ edges of type I.
\end{lemma}

\begin{proof}
First, let $r = \left\lfloor \log_\alpha \left( \frac{n}{\alpha - 1} \right) \right\rfloor$ and note that $r \geq 1$ as $\alpha > 1$. Let $e$ be the left-most edge of $P$ of type I and let $u$ and $v$ be vertices of $e$. Since $e$ is of type I, we have $u = (\alpha^k, \alpha^\ell)$ and $v = (\alpha^{k+m}, \alpha^{\ell-n})$ for some positive integers $k$, $\ell$, $m$, and $n$.

We will show that the point $(\alpha^{k+m+r}, 0)$ lies above the line $\tau$. Since there are at most $r - 1$ columns of $L(\alpha)$ between the vertical line containing $v$ and the vertical line containing $(\alpha^{k+m+r}, 0)$ and the point $(\alpha^{k+m+r}, 0)$ is below the lowest row of $L(\alpha)$, it then follows that there are at most $r$ edges of $P$ of type I; see part (b) of Figure 1.

Since the line $\tau$ contains $u$ and $v$, we see that

$$\tau = \{ (x, y) \in \mathbb{R}^2 : (\alpha^{\ell} - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^k)y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n} \}.$$ 

It suffices to check that by substituting the coordinates of the point $(\alpha^{k+m+r}, 0)$ into the equation of the line $\tau$ results in a left side that is at least $\alpha^{k+\ell+m} - \alpha^{k+\ell-n}$. The left side equals $\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r}$ and thus we want

$$\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r} \geq \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.$$ 

By dividing both sides by $\alpha^{k+\ell}$ and by rearranging the terms, we can rewrite this expression as

$$\alpha^{-n}(1 - \alpha^{m+r}) \geq \alpha^m - \alpha^{m+r}.$$ 

Since $m, r > 0$ and $\alpha > 1$, we get $(1 - \alpha^{m+r}) < 0$ and thus the left side is increasing as $n$ increases, so we can assume $n = 1$, leading to

$$\alpha^{-1} - \alpha^{m+r-1} \geq \alpha^m - \alpha^{m+r}.$$
We can again rearrange the inequality as
\[
\alpha^r - \alpha^{r-1} - 1 \geq -\alpha^{-1-m},
\]
where the right side is negative and approaches 0 as \(m\) tends to infinity, so we can replace it by 0, obtaining
\[
\alpha^r - \alpha^{r-1} \geq 1.
\]
This inequality is satisfied by our choice of \(r\).

We now estimate the number of edges of \(P\) that are of type III.

**Lemma 9.** The polygon \(P\) has at most \(2[\log_\alpha (\frac{\alpha+1}{\alpha})] + 1\) edges of type III for \(1 < \alpha < 2\) and at most 2 such edges for \(\alpha \geq 2\).

![Figure 2](image-url) (a) An illustration of the proof of Lemma 9 for \(s = 1 = t\). (b) An illustration of Lemma 10.

**Proof.** Let \(t = [\log_\alpha (\frac{\alpha+1}{\alpha})]\) and \(s = t + 1\) for \(\alpha \in (1, 2)\) and \(t = 1 = s\) for \(\alpha \geq 2\). Suppose for contradiction that there are \(s+t+1\) edges of \(P\) of type III. Let \(v_1, \ldots, v_{s+t+2}\) be the vertices of the convex chain that is formed by edges of \(P\) of type III. We use \(Q\) to denote the convex polygon with vertices \(v_1, \ldots, v_{s+t+2}\). Note that \(Q\) is empty in \(L(\alpha)\) as \(P\) is empty and \(Q \subseteq P\).

Let \(v'\) be the point \((x(v_{s+2}), y(v_{s+2}))\), that is, \(v'\) is the point of \(L(\alpha)\) that lies just above \(v_{s+2}\); see part (a) of Figure 2. We will show that the point \(v'\) lies below the line \(y = -\alpha^{-1-m}\). Since \(v'\) lies in the same column of \(L(\alpha)\) as \(v_{s+2}\), this then implies that \(v'\) lies in the interior of \(Q\), contradicting the fact that \(Q\) is empty in \(L(\alpha)\).

Note that \(x(v') \leq \frac{x(v_{s+2})}{\alpha^1}\) and \(y(v') \leq \frac{y(v_{s+2})}{\alpha^1}\) as all edges \(v_iv_{i+1}\) are of type III and thus the \(x\)- and \(y\)-coordinates decrease by a multiplicative factor at least \(\alpha\) for each step. Since the only vertical edge might be \(v_1v_2\) and the only horizontal edge might be \(v_{s+t+1}v_{s+t+2}\), the \(x\)- or \(y\)-coordinates indeed decrease by the factor \(\alpha\) at each step.

Let \(v_1 = (\alpha^k, \alpha^\ell)\) and \(v_{s+t+2} = (\alpha^{k+m}, \alpha^{\ell-n})\) for some positive integers \(k, \ell, m, n\). Note that \(m, n \geq s + t\). The line determined by \(v_1\) and \(v_{s+t+2}\) is then
\[
\{(x, y) \in \mathbb{R}^2 : (\alpha^\ell - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^k)y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n}\}.
\]
Since \(x(v') \leq \frac{x(v_{s+2})}{\alpha^1}\) and \(y(v') \leq \frac{y(v_{s+2})}{\alpha^1}\), it suffices to check
\[
(\alpha^\ell - \alpha^{\ell-n}) \frac{\alpha^{k+m}}{\alpha^1} + (\alpha^{k+m} - \alpha^k) \frac{\alpha^\ell}{\alpha^s} < \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.
\]
After dividing by $\alpha^{k+\ell+m}$, this can be rewritten as

$$\alpha^{-t} + \alpha^{-s} < 1 - \alpha^{-m-n} + \alpha^{-t-n} + \alpha^{-s-m}.$$  

Since $m, n \geq s + t$, the right hand side is decreasing with increasing $m$ and $n$ and thus we only need to prove

$$\alpha^{-s} + \alpha^{-t} \leq 1.$$  

If $\alpha \geq 2$, then $s = 1 = t$ and this inequality becomes $2/\alpha \leq 1$, which is true. If $\alpha \in (1, 2)$, then $s = t + 1$ and the inequality becomes $1 + 1/\alpha \leq \alpha'$ which holds by our choice of $t$.  

It remains to bound the number of edges of $P$ that are of types II and IV. Observe that if we switch the $x$- and $y$-coordinates of $P$, then edges of type II become edges of type IV and vice versa. Since the exponential lattice $L(\alpha)$ is symmetric with respect to the line $x = y$, we see that it suffices to estimate the number of edges of type II. To do so, we use the following auxiliary result, the proof of which is omitted here.

**Lemma 10.** Let $u$ be a point of $L(\alpha)$ and let $v$ and $v'$ be two points of $L(\alpha)$ that are consecutive in a row $R$ of $L(\alpha)$ that lies above the row containing $u$; see part (b) of Figure 2.

Then, all points of $L(\alpha)$ that lie above $R$ in the interior of the wedge $W$ spanned by the lines $\overline{uv}$ and $\overline{uv'}$ lie on at most $\left\lceil \log_\alpha \left( \frac{\alpha}{\alpha-1} \right) \right\rceil$ lines containing the origin.

Now, we can apply Lemma 10 to obtain an upper bound on the number of edges of $P$ of type II.

**Lemma 11.** The polygon $P$ has at most $\left\lceil \log_\alpha \left( \frac{\alpha}{\alpha-1} \right) \right\rceil + 1$ edges of type II.

**Proof.** Again, let $r = \left\lceil \log_\alpha \left( \frac{\alpha}{\alpha-1} \right) \right\rceil$. Let $u$ be the leftmost vertex of the convex chain $C$ determined by the edges of $P$ of type II. Similarly, let $v$ be the second leftmost vertex of $C$.

Note that since the edge $uv$ is of type II, the vertex $v$ lies in a row $R$ of $L(\alpha)$ above the row containing $u$. Let $v'$ be the point $(\alpha \cdot x(v), y(v))$, that is, point of $L(\alpha)$ that is to the right of $v$ on $R$.

Then, by Lemma 10, all points of $L(\alpha)$ that lie above $R$ and in the interior of the wedge $W$ spanned by the lines $\overline{uv}$ and $\overline{uv'}$ lie on at most $r$ lines containing the origin.

Since $P$ is empty in $L(\alpha)$, all vertices of $C$ besides $u$ and $v$ and possibly $v'$ lie in $W$ above $R$. Since all edges of $C$ are of type II, every line determined by the origin and by a point of $L(\alpha)$ from the interior of $W$ contains at most one vertex of $C$.

Note that if $v'$ is a vertex of $C$, then the only vertices of $C$ are $u, v, v'$. Thus, in total $C$ has at most $r + 2$ vertices and therefore at most $r + 1$ edges.  

We recall that, by symmetry, the same bound applies for edges of type IV and thus we get the following result.

**Corollary 12.** The polygon $P$ has at most $\left\lceil \log_\alpha \left( \frac{\alpha}{\alpha-1} \right) \right\rceil + 1$ edges of type IV.
as \( \log_x \left( \frac{x}{r} \right) \geq \log_x \left( \frac{1 + 1}{x} \right) \) for every \( x > 1 \). In particular, this gives \( h(2) \leq 8 \) and \( h \left( \frac{1 + \sqrt{5}}{2} \right) \leq 13 \). To obtain better bounds that are tight for \( \alpha \geq 1 + \sqrt{5} \), we observe that not all types can appear simultaneously. To show this, we will use one last auxiliary result.

Let \( p \) and \( q \) be (not necessarily different) points lying on the same row \( R \) of \( R(\alpha) \), each contained in an edge of \( P \). Let \( L \) and \( L' \) be two lines containing \( p \) and \( q \), respectively. If the slopes of \( L \) and \( L' \) are negative, then we call the part of the plane between \( L \) and \( L' \) below \( R \) a slice of negative slope; see part (a) of Figure 3. Analogously, a slice of positive slope is the part of the plane between \( L \) and \( L' \) above \( R \) if \( L \) and \( L' \) have positive slope.

![Figure 3](a) An example of a slice of negative slope. The slice is denoted by dark gray stripes. (b) An illustration of the proof of Lemma 13 for negative slopes.

Lemma 13. If the empty polygon \( P \) is contained in a slice of negative slope, then there is no non-vertical edge of \( P \) of type IV. Similarly, if \( P \) is contained in a slice of positive slope, then there is no edge of type I.

Proof. By symmetry, it suffices to prove the statement for slices of negative slope. Suppose for contradiction that there is a non-vertical edge \( uv \) of type IV in a slice of negative slope determined by lines \( L \) and \( L' \) and points \( p \) and \( q \) as in the definition of a slice. Without loss of generality, we assume \( x(u) < x(v) \).

Consider the point \( w = (x(u), y(v)) \) of \( L(\alpha) \). Since \( uv \) is non-vertical, we have \( w \notin \{u, v\} \).

We claim that \( w \) is in the interior of \( P \), contradicting the assumption that \( P \) is empty in \( L(\alpha) \). Since \( uv \) is of type IV, the point \( u \) lies below the row containing \( w \). However, since \( p \) is contained in an edge of \( P \) and \( P \) is in the slice, the boundary of \( P \) intersects this row to the left of \( w \). Analogously, \( v \) is to the right of the column containing \( w \) and thus the boundary of \( P \) intersects this column above \( w \). Then, however, \( w \) lies in the interior of \( P \).

Finally, we can now finish the proof of Theorem 2.

Proof of Theorem 2. First, we observe that if all vertices of \( P \) lie on two columns of \( L(\alpha) \), then \( P \) can have at most four vertices. So we assume that this is not the case. Let \( u \) be the leftmost vertex of \( P \) with the highest y-coordinate among all leftmost vertices of \( P \). Let \( e_1 \) and \( e_2 \) be the edges of \( P \) incident to \( u \). We denote the other edge of \( P \) incident to \( e_1 \) as \( e \). We also use \( t_I, t_{II}, t_{III}, \) and \( t_{IV} \) to denote the number of edges of \( P \) of type I, II, III, and IV, respectively.

First, assume that \( e_1 \) is vertical. If \( e_2 \) is horizontal, then, since \( u \) is the top vertex of \( e_1 \) and \( P \) is not contained in two columns of \( L(\alpha) \), the point \( (\alpha \cdot x(u), y(u) / \alpha) \) of \( L(\alpha) \) lies in the interior of \( P \), which is impossible as \( P \) is empty in \( L(\alpha) \).

If \( e_1 \) is vertical and the slope of \( e_2 \) is negative, then there is no edge of type II. Thus, the edge \( e \) intersects the row \( R \) of \( L(\alpha) \) containing the other vertex of \( e_1 \) and \( \overline{e} \) has negative
slope. Then, the part of $P$ below $R$ is contained in the slice of negative slope determined by
$ar{e_2}$ and $\tau$; see part (a) of Figure 4. By Lemma 13, there is no non-vertical edge of type IV in $P$. By Lemmas 8 and 9, the total number of edges of $P$ is thus at most
\[
 t_I + t_{III} + 1 \leq \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) + 2 \log_\alpha \left( \frac{\alpha + 1}{\alpha} \right) + 2
\]
for $\alpha \in (1,2)$ and is by one smaller for $\alpha \geq 2$.

If $e_1$ is vertical and the slope of $e_2$ is positive, then, since $P$ is empty, there is no edge of
type III besides $e_1$ as otherwise the point $(\alpha \cdot x(u), y(u))$ of $L(\alpha)$ is in the interior of $P$. The
dge $e$ intersects the row $R$ of $L(\alpha)$ containing $u$ and $\tau$ has positive slope. Thus, the part
of $P$ above $R$ is contained in the slice of positive slope determined by $\bar{e_2}$ and $\tau$; see part (b)
of Figure 4. By Lemma 13, there is no edge of type I in $P$. By Lemma 11 and Corollary 12,
the total number of edges of $P$ is then at most
\[
 t_{II} + t_{IV} \leq 2 \left( \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) \right) + 3.
\]
In the rest of the proof, we can now assume that none of the edges $e_1$ and $e_2$ is vertical.
We can label them so that the slope of $e_1$ is larger than the slope of $e_2$.

First, assume that the slope of $e_1$ is positive and the slope of $e_2$ is negative. Then, since
the vertices of $P$ do not lie on two columns of $L(\alpha)$, the point $(\alpha \cdot x(u), y(u))$ is contained in
the interior of $P$, which is impossible as $P$ is empty in $L(\alpha)$.

If the slopes of $e_1$ and $e_2$ are both non-positive, then there is no edge of type II besides
the possibly horizontal edge $e_1$ as $u$ is the leftmost vertex of $P$. By Lemma 13, there is also
no non-vertical edge of type IV as $P$ is contained in the slice of negative slopes determined
by $\bar{e_2}$ and $\bar{e_1}$ or by $\bar{\tau}$ and $\bar{e_2}$ if $e_1$ is horizontal; see part (c) of Figure 4. Thus, by Lemmas 8
and 9, the number of edges of $P$ is at most
\[
 t_I + t_{III} + 1 \leq \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) + 2 \log_\alpha \left( \frac{\alpha + 1}{\alpha} \right) + 3
\]
for $\alpha \in (1,2)$ and is by one smaller for $\alpha \geq 2$.

If the slopes of $e_1$ and $e_2$ are both non-negative, then there is no edge of type III besides
the possibly horizontal edge $e_2$ (note that a vertical edge of type III would have $u$ as its
bottom vertex, which is impossible by the choice of $u$). Then, $P$ is contained in the slice of
positive slope determined by $\bar{e_2}$ and $\bar{e_1}$ or, if $e_2$ is horizontal, by $\bar{e_1}$ and $\bar{\tau}$; see part (d) of
Figure 4. Lemma 13 then implies that there is also no edge of type I. We thus have at most
\[
 t_{II} + t_{IV} \leq 2 \left( \log_\alpha \left( \frac{\alpha}{\alpha - 1} \right) \right) + 3
\]
edges of $P$ by Lemma 11 and Corollary 12.

Altogether, the upper bound on the number of edges of $P$ is

$$\max \left\{ \left\lfloor \log_{\alpha} \left( \frac{\alpha}{\alpha - 1} \right) \right\rfloor + 2 \left\lfloor \log_{\alpha} \left( \frac{\alpha + 1}{\alpha} \right) \right\rfloor + 3, 2 \left\lfloor \log_{\alpha} \left( \frac{\alpha}{\alpha - 1} \right) \right\rfloor + 3 \right\}$$

for $\alpha \in (1, 2)$ and the first term is smaller by 1 for $\alpha \geq 2$. This becomes 5 for $\alpha \geq 2$, $h(\alpha) \leq 7$ for $\alpha \geq \left[ \frac{1 + \sqrt{5}}{2} \right]$, and at most $3 \left\lfloor \log_{\alpha} \left( \frac{\alpha}{\alpha - 1} \right) \right\rfloor + 3$ otherwise, since $\left\lfloor \log_{\alpha} \left( \frac{\alpha + 1}{\alpha} \right) \right\rfloor \leq \left\lfloor \log_{\alpha} \left( \frac{\alpha}{\alpha - 1} \right) \right\rfloor$ for every $\alpha \in (1, \frac{1 + \sqrt{5}}{2})$.

4 Proof of Theorem 3

We prove the lower bounds on $h(\alpha)$ through the following three propositions.

- Proposition 14. For every $\alpha \geq 2$, we have $h(\alpha) \geq 5$.

Proof. It is easy to check that $\text{conv}\{(1, \alpha^2), (\alpha, \alpha), (\alpha^2, 1), (\alpha, \alpha), (\alpha, \alpha^2)\}$ is an empty polygon in $L(\alpha)$ with 5 vertices for any $\alpha$.

- Proposition 15. For every $\alpha \in \left[ \frac{1 + \sqrt{5}}{2}, 2 \right)$, we have $h(\alpha) \geq 7$.

Figure 5 An illustration of the proof of Proposition 15.

Proof. Let $k = k(\alpha)$ be a sufficiently large integer, and let

$$Q(\alpha) = \{(1, \alpha^k), (\alpha^{k-2}, \alpha^{k-1}), (\alpha^{k-1}, \alpha^{k-2}), (\alpha^{k-2}, 1), (\alpha^k, \alpha), (\alpha^{k-1}, \alpha), (\alpha, \alpha^k)\};$$

see Figure 5. We will show that $\text{conv}(Q(\alpha))$ is an empty polygon in $L(\alpha)$ with 7 vertices.

First, we show that $Q(\alpha) \setminus \{(\alpha^{k-1}, \alpha^{k-1})\}$ is in convex position. For this, by symmetry, it is enough to check that the vector $(\alpha^{k-1}, \alpha^{k-2}) - (\alpha^{k-1}, 1)$ is to the left of $(1, \alpha^k) - (\alpha^k, 1)$. This is the case exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-2} - 1 < 0$. By rearranging we get $\alpha^{k-2}(\alpha + 1 - \alpha^2) < 1$, which holds for any $k$, since $\alpha + 1 - \alpha^2 \leq 0$ as $\alpha \geq (1 + \sqrt{5})/2$.

Now, to show that the set $Q(\alpha)$ is in convex position, it is sufficient to check that $(\alpha^{k-1}, \alpha^{k-1}) - (\alpha^k, \alpha)$ is to the left of $(1, \alpha^k) - (\alpha^k, \alpha)$. This holds exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-2} - \alpha \geq 0$. By rearranging we get $2\alpha^{k-2}(2 - \alpha) \geq 1$. Since $1 < \alpha < 2$, this holds if $k$ is sufficiently large.

Thus, $\text{conv}(Q(\alpha))$ has 7 vertices. To show that $\text{conv}(Q(\alpha))$ is empty in $L(\alpha)$, we remark that points of the exponential lattice $L(\alpha)$ with at least one coordinate smaller than $\alpha^{k-1}$ are below the line through $(\alpha^{k-1}, \alpha^{k-2})$ and $(\alpha^{k-2}, \alpha^{k-1})$. Further, points with at least one coordinate larger than $\alpha^{k-1}$ are either above the line through $(1, \alpha^k)$ and $(\alpha, \alpha^k)$ or to the right of the line through $(\alpha^k, 1)$ and $(\alpha^k, \alpha)$.


Proposition 16. For every \( \alpha > 1 \), we have \( h(\alpha) \geq \sqrt[2\alpha-1]{\frac{1}{\alpha}} \).

Proof. For a positive integer \( k \), let \( P(k) = \{ (\alpha^i, \alpha^{k-i}) : 1 \leq i \leq k \} \). Since \( P(k) \) is contained in the hyperbola \( h = \{ (x, y) \in \mathbb{R}^2 : x, y > 0, xy = \alpha^k \} \), the points of \( P(k) \) are in convex position, and \( \text{conv}(P(k)) \) has \( k \) vertices. We will show that if \( k \leq \sqrt[2\alpha-1]{\frac{1}{\alpha}} \), then \( \text{conv}(P(k)) \) is empty.

For points \((x, y)\) of \( L(\alpha) \) above \( h \), we have \( xy \geq \alpha^{k+1} \). Further, points \((x, y)\) of \( L(\alpha) \) with \( xy \geq \alpha^{k+2} \) are separated from \( h \) by the hyperbola \( h' = \{ (x, y) \in \mathbb{R}^2 : x, y > 0, xy = \alpha^{k+1} \} \).

Thus, it is sufficient to check that \( h' \) is above the line \( \ell \) connecting \((1, \alpha^k)\) with \((\alpha^k, 1)\). The closest point of \( h' \) to \( \ell \) is \((\alpha^{(k+1)/2}, \alpha^{(k+1)/2})\), thus it is sufficient to check that this point is above \( \ell \). This holds if \( 2\alpha^{(k+1)/2} - \alpha^k - 1 \geq 0 \) and we show that this inequality is satisfied for \( k \leq \sqrt[2\alpha-1]{\frac{1}{\alpha}} \).

Let \( \alpha = 1 + s^2 \) with some \( s \in (0, 1) \). In this notation, \( k \leq 1/s \) and we need to prove that
\[
2(1 + s^2)^{(k+1)/2} \geq (1 + s^2)^{k+1}.
\]
Since \((1 + s^2)^{(k+1)/2} \geq 1 + s^2 + \frac{k+1}{2}\) by the Bernoulli inequality, and \((1 + s^2)^k \leq e^{s^2 k} \), it is sufficient to prove the stronger inequality \( 2(1 + s^2 + \frac{k+1}{2}) \geq e^{s^2 k} + 1 \).

The worst case, when \( k = 1/s \), is equivalent to \( 1 + s + s^2 \geq e^s \), which holds for \( s \in (0, 1) \) as can be seen by the Taylor expansion of \( e^s \).

5 Proof of 'only if part' of Theorem 6

Let \( \alpha, \beta > 1 \) be two real numbers. We prove that if \( \log_\alpha(\beta) \) is irrational, then \( h(L(\alpha, \beta)) \) is not finite.

To do so, we will find a subset of \( L(\alpha, \beta) \) forming empty convex polygon in \( L(\alpha, \beta) \) with arbitrarily many vertices. To do so, we use a theory of continued fractions, so we first introduce some definitions and notation.

5.1 Continued fractions

Here, we recall mostly basic facts about so-called continued fractions, which we use in the proof. Most of the results that we state can be found, for example, in the book by Khinchin [14].

For a positive real number \( r \), the (simple) continued fraction of \( r \) is an expression of the form
\[
r = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},
\]
where \( a_0 \in \mathbb{N}_0 \) and \( a_1, a_2, \ldots \) are positive integers. The simple continued fraction of \( r \) can be written in a compact notation as
\[
[a_0; a_1, a_2, a_3, \ldots].
\]

For every \( n \in \mathbb{N}_0 \), if we denote \( \frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n] \) and set \( p_{-1} = 1 \), \( p_0 = a_0 \), \( q_{-1} = 0 \), \( q_0 = 1 \), then the numbers \( p_n \) and \( q_n \) satisfy the recurrence
\[
p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}
\]
for each \( n \in \mathbb{N} \). Observe that if \( r \) is irrational, then its continued fraction has infinitely many coefficients. Also, it follows from (1) that \( \frac{p_n}{q_n} < r \) for \( n \) even and \( \frac{p_n}{q_n} > r \) for \( n \) odd.
For example, if \( r = \log_2(3) \), we get the continued fraction \([1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]\)
and the sequence \( \left( \frac{p_n}{q_n} \right)_{n \in \mathbb{N}_0} = \left( \frac{1}{1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{5}, \frac{8}{13}, \frac{18}{31}, \frac{44}{79}, \ldots \right) \). For \( r = \frac{1+\sqrt{5}}{2} \), we have
\[ [1; 1, 1, \ldots] \text{ and } \left( \frac{p_n}{q_n} \right)_{n \in \mathbb{N}_0} = \left( \frac{1}{1}, \frac{1}{1}, \frac{2}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \frac{55}{89}, \ldots \right). \]

We will call the fractions \( \frac{p_n}{q_n} \) the convergents of \( r \). A semi-convergent of \( r \) is a number \( \frac{\pm p + \alpha}{\pm q + \beta} \), where \( \alpha \) and \( \beta \) are numbers that are exactly those numbers \( p \) and \( q \) in the partial denominators of \( r \). The names are motivated by the use of convergents and semi-convergents as rational approximations of an irrational number \( r \).

A rational number \( \frac{p}{q} \) is a best approximation of an irrational number \( r \), if any fraction \( \frac{p'}{q'} \) with \( q' < q \) satisfies
\[ |q' \left( r - \frac{p'}{q'} \right) | > | q \left( r - \frac{p}{q} \right) | . \]
A rational number \( \frac{p}{q} \) is a best lower approximation of \( r \) if
\[ q' \left( r - \frac{p'}{q'} \right) > q \left( r - \frac{p}{q} \right) \geq 0 \]
for all rational numbers \( \frac{p'}{q'} \) with \( \frac{p'}{q'} \leq r \), \( \frac{p}{q} \neq \frac{p'}{q'} \), and \( 0 < q' \leq q \). Similarly, \( \frac{p}{q} \) is a best upper approximation of \( r \) if
\[ q' \left( r - \frac{p'}{q'} \right) < q \left( r - \frac{p}{q} \right) \leq 0 \]
for all rational numbers \( \frac{p'}{q'} \) with \( \frac{p'}{q'} \geq r \), \( \frac{p}{q} \neq \frac{p'}{q'} \), and \( 0 < q' \leq q \).

It is a well known fact that convergents are best approximations of \( r \) [14]. The following lemma about best lower and upper best approximations is a recent result of Hančel and Turek [10].

**Lemma 17** [10]. Let \( r \) be a real number with \( r = [a_0; a_1, a_2, \ldots] \) and let \( \frac{p_n}{q_n} \) be the \( n \)-th convergent of \( r \) for each \( n \in \mathbb{N}_0 \). Then, the following three statements hold.

1. The set of best lower approximations of \( r \) consists of semi-convergents \( \frac{p_n+1+ip_n}{q_n+1+iq_n} \) of \( r \) with \( n \) odd and \( 0 \leq i < a_{n+1} \).
2. The set of best upper approximations of \( r \) consists of semi-convergents \( \frac{p_n+1+ip_n}{q_n+1+iq_n} \) of \( r \) with \( n \) even and \( 0 \leq i < a_{n+1} \), except for the pair \((n, i) = (0, 0)\).

Finally, a real number \( r \) is restricted if there is a positive integer \( M \) such that all the partial denominators \( a_i \) from the continued fraction of \( r \) are at most \( M \). The restricted numbers are exactly those numbers \( r \) that are badly approximable by rationals [14], that is, there is a constant \( c > 0 \) such that for every \( \frac{p}{q} \in \mathbb{Q} \) we have \( |r - \frac{p}{q}| > \frac{c}{q^2} \).

We divide the rest of the proof of Theorem 6 into two cases, depending on whether \( \log_\alpha(\beta) \) is restricted or not.

### 5.2 Unrestricted case

First, we assume that \( \log_\alpha(\beta) \) is not restricted. Let \( [a_0; a_1, a_2, a_3, \ldots] \) be the continued fraction of \( \log_\alpha(\beta) \) with \( \frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n] \) for every \( n \in \mathbb{N}_0 \). Then, for every positive integer \( m \), there is a positive integer \( n(m) \) such that \( a_{n(m)+1} \geq m \). We use this assumption to construct, for every positive integer \( m \), a convex polygon with at least \( m \) vertices from \( L(\alpha, \beta) \) that is empty in \( L(\alpha, \beta) \).
For a given \( m \), consider the integer \( n(m) \) and let \( W \) be the set of points

\[
w_i = \left( \alpha^{p_{n(m)-1} + p_{n(m)}}, \beta^{q_{n(m)-1} + q_{n(m)}} \right)
\]

where \( i \in \{0, 1, \ldots, n(m)+1\} \). That is, we consider points where the exponents form semi-convergents \( \frac{p_{n(m)-1} + p_{n(m)}}{q_{n(m)-1} + q_{n(m)}} \) to \( \log_\alpha(\beta) \). We abbreviate \( p_{n,i} = p_{n(m)-1} + ip_{n(m)} \) and \( q_{n,i} = q_{n(m)-1} + iq_{n(m)} \). Observe that \( |W| \geq m \). We will show that \( W \) is the vertex set of an empty convex polygon in \( L(\alpha, \beta) \). To do so, we assume without loss of generality that \( n(m) \) is even so that \( \frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} > 1 \). The other case when \( n(m) \) is odd is analogous.

First, we show that \( W \) is in convex position. In fact, we prove that all triples \((w_i, w_{i+1}, w_{i+2})\) with \( i_1 < i_2 < i_3 \) are oriented counterclockwise. It suffices to show this for every triple \((w_i, w_{i+1}, w_{i+2})\). To do so, we need to prove the inequality

\[
\frac{y(w_{i+2}) - y(w_{i+1})}{x(w_{i+2}) - x(w_{i+1})} = \frac{\beta^{q_{n,i+2}} - \beta^{q_{n,i+1}}}{\alpha^{p_{n,i+2}} - \alpha^{p_{n,i+1}}} > \frac{\beta^{q_{n,i}} - \beta^{q_{n,i-1}}}{\alpha^{p_{n,i}} - \alpha^{p_{n,i-1}}} = \frac{y(w_{i+1}) - y(w_i)}{x(w_{i+1}) - x(w_i)}.
\]

After dividing by \( \frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} - \frac{\beta^{q_{n(m)-1}}}{\alpha^{p_{n(m)-1}}} \), this can be written as

\[
\frac{\beta^{q_{n,i+2}} - \beta^{q_{n,i+1}}}{\alpha^{p_{n,i+2}} - \alpha^{p_{n,i+1}}} - \frac{\beta^{q_{n,i}} - \beta^{q_{n,i-1}}}{\alpha^{p_{n,i}} - \alpha^{p_{n,i-1}}} = \frac{\beta^{q_{n,i}} - \beta^{q_{n,i-1}}}{\alpha^{p_{n,i}} - \alpha^{p_{n,i-1}}} \cdot \frac{\beta^{p_{n,i+1}} - \beta^{p_{n,i}}}{\alpha^{q_{n,i+1}} - \alpha^{q_{n,i}}}.
\]

If divide both sides by \( \frac{\beta^{p_{n,i+1}}}{\alpha^{q_{n,i+1}}} - \frac{\beta^{p_{n,i}}}{\alpha^{q_{n,i}}} \), then the above inequality becomes

\[
\frac{\beta^{q_{n,i}}}{\alpha^{p_{n,i}}} > 1.
\]

This is true as \( n(m) \) is even.

It remains to prove that the polygon \( Q \) with the vertex set \( W \) is empty in \( L(\alpha, \beta) \). Suppose for contradiction that there is a point \((\alpha^p, \beta^q)\) of \( L(\alpha, \beta) \) lying in the interior of \( Q \). Let \( i \) be the minimum positive integer from \( \{1, \ldots, n(m)+1\} \) such that \( q < q_{n,i} \). Such an \( i \) exists as \((\alpha^p, \beta^q)\) is in the interior of \( Q \). We then have \( q_{n,i-1} < q < q_{n,i} \). Since \((\alpha^p, \beta^q)\) is in the interior of \( Q \) and \( W \) lies below the line \( x = y \), we have \( \frac{p}{q} > \log_\alpha(\beta) \). So it is enough to prove that \((\alpha^p, \beta^q)\) lies above the line \( w_{i-1}w_i \).

We have \( p_{n,i-1} - \log_\alpha(\beta)q_{n,i} < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1} \) as \( \frac{p_{n,i}}{q_{n,i}} \) is a best upper approximation of \( \log_\alpha(\beta) \) and \( q_{n,i} - 1 < q_{n,i} \). This implies \( \frac{\beta^{p_{n,i-1}}}{\alpha^{q_{n,i-1}}} < \frac{\beta^{p_{n,i}}}{\alpha^{q_{n,i}}} \), or equivalently that \( w_i \) lies above the line determined by \( w_{i-1} \) and the origin.

Now if \((\alpha^p, \beta^q)\) lies above the line \( w_{i-1}w_i \), then it also lies above the line determined by \( w_{i-1} \) and the origin. Thus, \( \frac{\beta^{p_{n,i-1}}}{\alpha^{q_{n,i-1}}} < \frac{\beta^{p_{n,i}}}{\alpha^{q_{n,i}}} \), implying

\[
p - \log_\alpha(\beta)q < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1}.
\]

which means that \( \frac{p}{q} \) is a better upper approximation of \( \log_\alpha(\beta) \) than \( \frac{p_{n,i-1}}{q_{n,i-1}} \). Thus, there exists a best upper approximation \( \frac{p'}{q'} \) of \( \log_\alpha(\beta) \) with \( q_{n,i-1} < q' < q_{n,i} \). This contradicts part (c) of Lemma 17 as \( \frac{p}{q} \) is not a semi-convergent of \( \log_\alpha(\beta) \).

### 5.3 Restricted case

Now, assume that the number \( \log_\alpha(\beta) \) is restricted. Let \([a_0; a_1, a_2, a_3, \ldots] \) be the continued fraction of \( \log_\alpha(\beta) \) with \( \frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n] \) for every \( n \in \mathbb{N}_0 \). Let \( M = \{M(\alpha, \beta)\} \) be a number satisfying

\[
a_0 \leq M
\]
for every \( n \in \mathbb{N}_0 \) and let \( c = c(\alpha, \beta) > 0 \) be a constant such that
\[
\left| \log_\alpha(\beta) - \frac{p}{q} \right| > \frac{c}{q^2}
\] (3)
holds for every \( \frac{p}{q} \in \mathbb{Q} \). Recall that \( \frac{\alpha^n}{\beta^n} < 1 \) for even \( n \) and \( \frac{\alpha^n}{\beta^n} > 1 \) for odd \( n \). Note also that the sequence \( \left( \frac{\alpha^n}{\beta^n} \right)_{n \in \mathbb{N}_0} \) converges to 1 as \( \left( \frac{\alpha^n}{\beta^n} \right)_{n \in \mathbb{N}_0} \) converges to \( \log_\alpha(\beta) \). Moreover, the terms of \( \left( \frac{\alpha^n}{\beta^n} \right)_{n \in \mathbb{N}_0} \) with odd indices form a decreasing subsequence and the terms with even indices determine an increasing subsequence.

Let \( n_0 = n_0(\alpha, \beta) \) be a sufficiently large positive integer and let \( V \) be the set of points \( v_n = (\alpha^n, \beta^n) \) for every odd \( n \geq n_0 \). Note that \( V \) is a subset of \( L(\alpha, \beta) \).

We first show that \( V \) is in convex position. In fact, we prove a stronger claim by showing that the orientation of every triple \( (v_{n_1}, v_{n_2}, v_{n_3}) \) with \( n_1 < n_2 < n_3 \) is counterclockwise. It suffices to show this for every triple \( (v_{n-4}, v_{n-2}, v_n) \). To do so, we prove that the slopes of the lines determined by consecutive points of \( V \) are increasing, that is,
\[
\frac{y(v_n) - y(v_{n-2})}{x(v_n) - x(v_{n-2})} = \frac{\beta^n - \beta^{n-2}}{\alpha^n - \alpha^{n-2}} > \frac{\beta^{n-2} - \beta^{n-4}}{\alpha^{n-2} - \alpha^{n-4}} = \frac{y(v_{n-2}) - y(v_{n-4})}{x(v_{n-2}) - x(v_{n-4})}
\]
for every \( n \geq n_0 \). By dividing both sides of the inequality with \( \frac{\beta^n}{\alpha^n - \alpha^{n-2}} \), we rewrite this expression as
\[
\frac{\beta^n - \beta^{n-2}}{\alpha^n - \alpha^{n-2}} > \frac{1}{1 - \frac{\beta^{n-2} - \beta^{n-4}}{\alpha^{n-2} - \alpha^{n-4}}}.
\]
Using (1), this is the same as
\[
\frac{\beta^n - \beta^{n-2}}{\alpha^n - \alpha^{n-2}} > \frac{1}{1 - \frac{\beta^{n-2} - \beta^{n-4}}{\alpha^{n-2} - \alpha^{n-4}}}.
\]
The above inequality can be rewritten as
\[
\frac{\beta^n - \beta^{n-2}}{\alpha^n - \alpha^{n-2}} > (\alpha^{n-2} - \beta^{n-2} - \beta^{n-3}),
\]
where \( \beta^n - 1 > \alpha^{n-2} - \beta^{n-2} - \beta^{n-3} > 1 > \alpha^{n-3} - \beta^{n-3} > 0 \) as \( n-1 \) and \( n-3 \) are even. Therefore, if the above inequality holds for \( a_n = 1 = a_{n-2} \), then it holds for any \( a_n \) and \( a_{n-1} \) as both numbers are always at least 1. Thus, it suffices to show
\[
(\beta^n - 1)(1 - \alpha^{-n-2} - \beta^{-n-3}) > (\alpha^{n-2} - 1)(1 - \beta^{-n-3}).
\]
We prove this using the following simple auxiliary lemma.

Lemma 18. Consider the function \( f: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) given by \( f(x, y) = (x-1)(1-1/y) \). Let \( x, y, x', y' > 1 \) be real numbers such that \( 1 - \frac{1}{y} - \frac{x}{y} > 0 \). Then, \( f(x', y) > f(x, y') \).

Proof. We have
\[
f(x', y) - f(x, y') = (x' - 1) \left( 1 - \frac{1}{y} \right) - (x - 1) \left( 1 - \frac{1}{y'} \right)
\]
\[
= x' - \frac{x' - 1}{y} - x + \frac{x - 1}{y'} > x' - \frac{x - 1}{y'} = x - x' \left( 1 - \frac{1}{y} - \frac{x}{y'} \right) > 0,
\]
where the last inequality follows from \( 1 - \frac{1}{y} - \frac{x}{y'} > 0 \).

\( \square \)
Now, by choosing \( x = \alpha^{p_{n-1}}, \ x' = \beta^{p_{n-1}}, \ y = \alpha^{p_{n-3}}, \) and \( y' = \beta^{p_{n-3}}, \) the inequality (4) becomes \( f(x', y') > f(x, y'). \) In order to prove it, we just need to verify the assumptions of Lemma 18. We clearly have \( x, x', y, y' > 1. \) It now suffices to show \( 1 - \frac{1}{y} - \frac{\alpha}{\beta} > 0. \) By (3), we obtain that \( g_{n-1} \logn(\beta) - p_{n-1} \geq c/q_{n-1}, \) thus

\[
\frac{x}{x'} = \frac{\alpha^{p_{n-1}}}{\beta q_{n-1}} \leq \alpha^{-c/q_{n-1}}.
\]

Now, to bound \( q_{n-1} \) in terms of \( p_{n-3}, \) equation (1) gives

\[
q_{n-1} = a_{n-3} - q_{n-3} \leq (M + 1)q_{n-2} = (M + 1)(a_{n-2}q_{n-3} + q_{n-4}) \leq (M + 1)^2 q_{n-3} \leq 2 \logn(\alpha)(M + 1)^2 p_{n-3},
\]

where we used (2) and \( q_{n-4} \leq q_{n-3} \leq q_{n-2}. \) \( q_{n-3} \leq 2 \logn(\alpha)p_{n-3} \) for \( n \) large enough. It follows that \( q_{n-1} \leq M' p_{n-3} \) for a suitable constant \( M' = M'(\alpha, \beta) > 0. \) Thus,

\[
1 - \frac{1}{y} - \frac{x}{x'} \geq 1 - \alpha^{-p_{n-3}} - \alpha^{-c/q_{n-1}} \geq 1 - \alpha^{-p_{n-3}} - \alpha^{-c/(M' p_{n-3})},
\]

which is at least

\[
\frac{c \logn}{2M' p_{n-3}} - \frac{1}{\alpha^{p_{n-3}}}
\]

as \( 1 - \frac{c \logn}{(2M' p_{n-3})} \geq e^{-2c \logn(\alpha)/(2M' p_{n-3})} = \alpha^{-c/(M' p_{n-3})} \) if \( 0 < c \logn(\alpha)/(2M' p_{n-3}) < 1/2. \)

The last expression is positive if \( n \geq n_0 \) and \( n_0 \) is sufficiently so that \( p_{n-3} \) is large enough.

It remains to show that the convex polygon \( P' \) with the vertex set \( V \) is empty in \( L(\alpha, \beta). \)

We proceed analogously as in the unrestricted case. Suppose for contradiction that there is a point \((\alpha^p, \beta^q)\) of \( L(\alpha, \beta) \) lying in the interior of \( P. \) Then, let \( v_n = (\alpha^{p_n}, \beta^{q_n}) \) be the lowest vertex of \( P \) that has \((\alpha^p, \beta^q)\) below. Such a vertex \( v_n \) exists, as \( V \) contains points with arbitrarily large \( y \)-coordinate. By the choice of \( v_n, \) we obtain \( q_{n-2} < q < q_n. \) Since \((\alpha^p, \beta^q)\) is in the interior of \( P \) and \( V \) lies below the line \( x = y, \) we have \( \frac{q}{q} > logn(\alpha) > \frac{p_{n-1}}{q_{n-1}}. \)

Moreover, since all triples from \( V \) are oriented counterclockwise, the point \((\alpha^p, \beta^q)\) lies above the line \( \overline{v_{n-2}v_n}. \)

Let

\[
w_i = (\alpha^{p_{n-2}+ip_{n-1}}, \beta^{q_{n-2}+iq_{n-1}})
\]

where \( i \in \{0, 1, \ldots, a_n\} \) similarly as in the proof of the unrestricted case. There, it was shown that all the triples \( w_{i-1}, w_i, w_{i+1} \) are oriented counterclockwise, thus all the points \( w_i \) with \( i \in \{1, \ldots, a_n - 1\} \) lie below the line \( \overline{v_{n-2}v_n}. \) Thus, if \((\alpha^p, \beta^q)\) lies above the segment connecting \( v_{n-2} \) and \( v_n, \) then there is an \( i \) such that \((\alpha^p, \beta^q)\) lies above the segment connecting \( w_{i-1} \) and \( w_i. \) As in the last two paragraphs of the proof of the unrestricted case, the position of \((\alpha^p, \beta^q)\) implies the inequality \( p - \logn(\beta)q + p_{n-1} - \logn(\beta)q_{n-1}, \) and the contradiction follows from part (c) of Lemma 17, as there can be no best upper approximation of \( \logn(\beta) \) which is not a semi-convergent of \( \logn(\beta). \)

References

On Helly Numbers of Exponential Lattices


