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Mechanical characterisation of novel aperiodic lattice structures

Chikwesiri Imediegwu a,⇑,†, Daniel Clarke b,2, Francesca Carter b,2, Uwe Grimm a,3, Iestyn Jowers b,2, Richard Moat b,2

a The Open University, School of Maths and Stats, Milton Keynes MK2 2FN, UK
b The Open University, School of Engineering and Innovation, Milton Keynes MK2 2FN, UK

HIGHLIGHTS
- Summary of pattern-unique mathematical substitution rules for some aperiodic lattices
- Direction-dependent mechanical properties of aperiodic lattices
- Weight cost of strut-bracing on aperiodic lattices with density fraction variation
- Performance of aperiodic lattice structures against theoretical stiffness limits

GRAPHICAL ABSTRACT

ABSTRACT
This paper compares the mechanical properties of a class of lattice metamaterials with aesthetically-pleasing patterns that are governed by the mathematics of aperiodic order. They are built up of ordered planar rod networks and exhibit higher non-crystallographic rotational symmetries. However, they lack the translational symmetry associated with periodic lattice metamaterials. We present schematics illustrating their development based on pattern-unique mathematical substitution rules and exploit a numerical framework from previous work to demonstrate that they exhibit fascinating near-isotropic properties. The lattice structures are compared to the well-known hexagonal lattice with respect to their elastic anisotropy measure and the proximity of their bulk and shear moduli to Hashin–Shtrikman-Walpole limits. The study lends insight into the cost-constrained benefits of introducing additional connectivities between aperiodically-ordered point sets. The results show that aperiodic lattices have the potential to yield superior mechanical properties to periodic ones subject to the mechanical rigidity of the underlying shapes that constitute the pattern. The inherent ‘near-isotropy’ associated with these aperiodic structures, even with uniform strut thicknesses and at low fractional densities, and the ordered and varied orientation of their lattice struts present them as promising mesoscale architecture for solving complex multi-axially loaded structural optimization problems, providing inspiration for this study.

1. Introduction

Periodic lattice structures are ubiquitous in the design of modern mechanical metamaterials [1]. These are architected materials with properties which differ from the base material they are made from – acquiring their effective bulk material behavior from their smaller scale geometric features [2]. A simple shape can be assembled to fully tessellate a plane (in 2D) or volume (in 3D) to derive
unique metamaterials with intriguing macroscopic elastic behaviors. This regular array of shapes is associated with the concept of periodicity and lattice symmetry and forms the well-established theory uniquely defining crystalline structure [3]. In fact, the structural behavior of many periodic cellular metamaterials is well understood for certain fractional densities [4,5]. By investigating such structures over a single microscale representative unit, homogenization techniques have been used to determine their effective (averaged) mechanical properties [6–8]. These effective properties determine the macroscopic behavior of infinite assemblies of periodic lattices [9] and analytical expressions exist which efficiently describe the mechanical behavior of several periodic metamaterials. Improved multiscale periodic metamaterials have been designed to further push the limits of effective material properties attainable by designing their hierarchical structures over several scales [10,11]. Today, counterintuitive mechanical behavior has been derived for example, by integrating contact non-linearity and deformation mode switching into the microarchitecture of periodic metamaterials used for real-life applications [12,13]. These concepts have been extended to other multiphysics problems and are actively implemented in the aerospace and bio-engineering industry [14–16]. Such practical applications of these lattice metamaterials are based on a periodic array of a recognizable repeating unit of ordered trusses [17]. To explore the effect of irregularity as observed in the natural world’s materials, Liu et al. [18] generated irregular architectural materials using building blocks with arbitrary complexity connected by set adjacency rules that relied on a virtual growth program. This framework yielded metamaterials with combined periodic and non-periodic microarchitecture. However, the underlying network was still constrained to a square grid.

The discovery of quasicrystals in 1982 by Dan Schechtman marked a paradigm shift within the field of crystallography since, prior to his discovery, order and repeatability was associated exclusively to periodicity [19,20]. It took Dan Schechtman two years to convince his colleagues about the existence of quasicrystals, which was eventually recognized by the Nobel Prize in Chemistry in 2011 [21]. A quasicrystalline pattern can continuously fill all available space in an ordered manner (as periodic patterns do), yet lacks the translational symmetry commonly associated with periodic patterns. Where crystallographic restriction limits permissible rotational symmetry to 2-, 3-, 4- and 6-fold symmetries in periodic tessellations, quasicrystalline patterns exhibit much higher ‘periodically forbidden’ rotational symmetries [22,23]. These fascinating tessellations constitute an emerging field formalized in the mathematics of aperiodic order (See examples in Fig. 1). Lattice structures based on these aperiodic arrangements form a class of novel mechanical metamaterials with potentially intriguing elastic behavior. They promise exciting mechanical properties congruous to observations in the elastic, thermal and conductive behavior of quasicrystals [24]. As a consequence, the properties of quasicrystals have led to them being used as thermal insulators and superconductors. These aperiodically-ordered structures are sometimes referred to as quasi-periodic structures in literature [25,26].

An aperiodic tessellation will tile all available space in an ordered fashion, without overlaps or spaces, but will do so without prescribing a repeatable unit tile. The absence of a repeatable unit tile and lack of translational symmetry implies that there exist no general local rules to assemble aperiodic tessellations as do periodic ones. This idea is embedded in the concept of long-range coordination such that local tessellation correlates intrinsically with detached tessellations within the aperiodic system [23]. This intriguing phenomenon of long-range-order also informs how such tessellations are derived.

Recent works around aperiodic metamaterials have focused on their application as phononic [27] and acoustic [28] metamaterials that can be used to design complex wavefront modulations for state-of-the-art multifunctional devices such as acoustic lenses. Chen et al. [29] also used three-dimensional quasicrystal approximants to derive mechanical metamaterials exhibiting nearly isotropic chiral acoustic phonons. More recently, via a design procedure in reciprocal space, Beli et al. [26] showed that near-isotropic wave propagation is observed when quasi-periodic media of high-order symmetries are observed over a broad range of frequencies. For mechanical applications, Somera et al. [30] classified quasi-periodic structures based on their energetic dominance-type (stretch or bend dominance) when subjected to tensile tests. The elementary pattern-shapes were shown to be critical to the mechanical behaviour of quasi-periodic lattice. By compression tests on Penrose P3-type honeycombs, Moat et al. [31] demonstrated that aperiodic honeycombs yield superior deformation modes to those of periodic honeycombs where dimensional stability under plastic deformation is critical. A family of mechanical metamaterials of planar plates layered in transverse quasi-periodic manner were designed by Wang et al. [25] which attained extreme isotropic elastic stiffness. In the context of the vastness of existing knowledge and literature on periodically-ordered metamaterials, these works represent early stages in the exploration of aperiodically-ordered metamaterials and their applications as superior alternatives to periodic ones.

This work extends the knowledge on aperiodic metamaterials by comparing the mechanical behavior of aperiodic lattice structures. First, unique substitution rules governing a developed tessellation algorithm are demonstrated via substitution schematics. The tessellation algorithm yields an aperiodic distribution of vertex points in a plane. Aperiodically-ordered lattice structures are then derived as ordered planar networks connecting the vertices for each pattern. By exploiting the numerical strategy presented by Imediegwu et al. [32] for determining the effective properties of aperiodic structures, we present the direction-dependent properties of these aperiodic lattices in comparison to the well-known hexagonal lattice structure. We demonstrate that even though the feature of high rotational symmetries supports near-isotropic material behavior, the geometry of underlying prototiles constituting the lattice play a significant role in the effective moduli of the lattices. For comparability in tessellation resolution, we ensure that each multi-prototile tessellation of this work is equivalent to the same effective square tessellation with respect to area-averaging of the cells of the tessellation. A brief introduction of the patterns of this work precedes our investigations.

In the next section, unique tessellation parameters for each aperiodic lattice of this investigation are presented, supported by substitution schematics. Section 3 describes the strategy employed to enforce pattern resolution equivalence. Section 4 and 5 presents results and deductions relating to directional effective Young’s moduli and Poisson’s ratios for a range of fractional densities. The performance of the lattices is compared to Hashin–Shtrikman-Walpole limits. The hexagonal lattice structure is included in the investigations for reference. In Section 6, an elastic anisotropy measure indicates comparative proximity to isotropic mechanical behaviour. Finally, Section 7 summarizes the paper and lends insight into prospective future research directions.

2. Pattern generation

Approaches for the generation and analyses of aperiodic tessellation are restricted by the absence of a repeating unit cell and lack
of translational symmetry. As a consequence, conventional methods for creating and analyzing periodic lattice structures are inadequate for this work. There exist no general rules for assembling aperiodic lattice structures. Two main approaches for generating aperiodic tessellation have been presented by Baake et al. [22,23] – the first approach is a ‘cut and project’ strategy that creates planar tilings by cutting a high-dimension periodic lattice with a two-dimensional plane and projecting the lattice points unto the cutting plane [33,34]. A second method, used in this work for its simplicity, is a substitution strategy that defines specific matching rules for ‘substituting’ prototiles of a pattern with scaled-down prototiles following pattern-unique scaling ratios. To assist visualization, pattern-unique substitution schematics are presented. These substitution schematics of Figs. 2–8 show the prototile types in rows (A, B, C, …) and prototile generations in columns (I, II, III) based on the pattern-unique deflation rules. The starting prototiles, denoted ‘prime prototiles’ are in column I. The next generation-II shows the positions and orientations of deflated prototiles according to unique substitution rules. Generation-III excludes orientation labeling but demonstrates the next-level deflation for each prototile in generation-II following the same substitution rules. Observe that in generations-II and -III, the preceding generation outline is shown in gray lines to facilitate comprehension. By following these substitution rules over several deflation generations, the aperiodic pattern is generated. The deflation scaling ratio across each pattern is unique and clearly defined below. The rest of this section presents briefly, the aperiodic patterns modelled in this work. Detailed descriptions of these patterns (and many more) can be found as presented by Baake et al. [22].

2.1. The Penrose tilings

The Penrose aperiodic tilings are arguably the most popular substitution tilings in aperiodic tiling literature [35]. Though there
exists multiple versions, the Penrose Kite and Dart (Penrose-P2) and the Penrose Rhomb (Penrose-P3) shown in Figs. 1a and 1b respectively are suited to the investigations of this work, ie. only tilings with a maximum of two shapes within the aperiodic pattern have been included in investigations.

As a consequence, a third variant of the Penrose tilings, the Pentagon boat star (Penrose-P1), has not been included in this study. The deflations over two generations are shown for the P2 and P3 aperiodic tilings in Figs. 2 and 3 respectively. The P2 comprises ‘kite and dart’ prototiles while the P3 comprises ‘fat and thin’ (72° and 36°) rhombi. Other Penrose tilings investigated in this work include the Robinson’s Triangle (RT) and the Tübingen Triangle (TT) tilings shown in Figs. 1c and 1d respectively. Both comprise isosceles triangles referred to as the Golden Triangle and the Golden Gnomon, so called because the ratio of the lengths of their equal sides to the base are the golden ratio, \(\phi = \frac{1}{2}(1 + \sqrt{5})\) and inverse of the golden ratio, \(\frac{1}{\phi}\) respectively. Deflation generations for the RT and TT are shown in Figs. 4 and 5 respectively.

Across all the Penrose tilings, the deflation scaling ratio is also the golden ratio. The ratio of the larger prototiles to the smaller (eg. the area of the kite to the dart for the P2 or the area of the fat rhombus to the thin rhombus for the P3) is also the golden ratio. This is also the case for the ratio of the number of larger prototiles to the number of smaller prototiles, for an infinitely extended observation of the pattern. All Penrose tilings exhibit non-crystallographic 5-fold rotational symmetry.

### 2.2. The Ammann-Beenker Tiling

The Ammann-Beenker (AB) tiling exhibits 8-fold rotational symmetry comprising of square and 45°-rhombic prototiles, shown in Fig. 1e. The deflations over two generations are shown for the AB in Fig. 6. The deflation scaling ratio across the generations for the AB is given as, \(\phi_A = 1 + \sqrt{2}\). The ratio of number of squares to the number of rhombi is also given by \(\phi_A\), for an infinitely extended observation of the pattern.

---

![Fig. 5. Tübingen Triangle deflation.](image)

![Fig. 6. Ammann-Beenker deflation.](image)

![Fig. 7. Square-Triangle deflation.](image)

![Fig. 8. Pinwheel deflation.](image)
2.3. The Square-Triangle tiling

The Square-Triangle (ST) tiling exhibits remarkable 12-fold rotational symmetry comprising square and equilateral triangle prototiles shown in Fig. 1f. The deflations over two generations are shown in Fig. 7 with a deflation scaling ratio of \( \varphi_s = 2 + \sqrt{3} \). Within an infinite observation of the ST, the area covered by all square prototiles equals the area covered by the triangles. The ratio of the number of triangle prototiles to the number of square prototiles is also given by \( \varphi_s \).

2.4. The Pinwheel tiling

The Pinwheel (PIN) tiling is one of a few substitution tilings which exhibit \( \infty \)-fold rotational symmetry, comprising of a right-angled triangle prototile of side length ratio \( \sqrt{5}\cdot2:1 \) and its reflection shown in Fig. 1g. The triangle has irrational edge lengths and incommensurate angles with deflations over two generations shown in Fig. 8. The pattern has deflation scaling ratio, \( \varphi_p = \sqrt{5} \). An infinite observation of the pinwheel tiling shows that the orientations of triangles in the tiling are distributed uniformly in all directions [22].

For comparison purposes, the periodic hexagonal pattern with crystallographic 6-fold rotational symmetry has been included in this work. There exist no higher symmetries associated with periodic tilings. Periodic tilings are easy to generate since they possess translational symmetry and there exists a repeatable unit prototile. The next section describes considerations to account for pattern resolution equivalence.

3. Pattern resolution equivalence

A requirement for the correct implementation of homogenization of architectured lattice materials is the existence of sufficient scale separation between the structure patch and the microarchitectures that build it up. Previous studies shown by Cheng et al. [36] have indicated that a square pattern requires about 6 layers of a square architecture in each dimension for full scale computations to agree to within 8% of homogenized elastic properties. In this work, we ensure that the coarsest pattern is equivalent to a square lattice pattern of at least 10 layers of square architecture in each dimension. For cellular lattice structures of this work of ‘single-prototile area’ type such as the pinwheel and the hexagonal lattices (Figs. 1g and 1h), we ensure pattern resolution equivalence for both by imposing that their average prototile areas (at deflation termination) are equal to that of an equivalent square lattice resolution. Mathematically, we stipulate that

\[
a_0 = A_s
\]

as shown in Fig. 9, using the Pinwheel tiling as the exemplar so that \( a_0 \) is the area of the pinwheel prototile and \( A_s \) is the area of the equivalent square prototile. For ‘double-prototile area’ type patterns, such as the Penrose-P2 consisting of kites and darts, we ensure that the area average across all cells in an infinite array of the pattern is equivalent to that of the square lattice as above. For the P2 example in Fig. 9, mathematically,

\[
\Phi a_0 + \Phi a_k = A_s
\]

where \( \Phi = \varphi \) is the deflation scaling ratio for the exemplar P2. The values \( \varphi, \varphi_A, \varphi_k \) and \( \varphi_s \), being irrational is a defining characteristic of all aperiodic patterns. Aperiodic patterns of this work have been limited to those of ‘single or double-prototile area’ types and no more.

4. Results and Discussions

The numerical simulations of this work involve an element-based material assignment which facilitates the modelling of arbitrary lattice structures as composites comprising base isotropic material and void (very weak isotropic material). This strategy for determining linear elastic properties is based on a validated adaptation to asymptotic expansion homogenization where a homogenized phase bounds the lattice pattern to permit the implementation of periodic boundary conditions. Homogenized properties are then derived for the extended domain by iterative fashion. For details on the strategy, the reader is referred to earlier work by Imediegwu et al. [32]. The following sections show results of this investigation with subsequent deductions.

4.1. Directional Young’s moduli and Poisson’s ratios against Density fraction

Rotational analyses of the properties of aperiodic lattices presented here are shown in Fig. 10. The second row of the figure shows the directional Young’s moduli for the different lattice structures at density fraction, \( \rho / \rho_s = 0.5 \). It can be observed that all the aperiodic lattices demonstrate near-isotropic properties with varying levels of isotropy and stiffness. The P3 and AB lattices have the two lowest Young’s moduli with \( E / E_s < 0.1 \). Both lattices include rhombus-type prototiles within their structure. We account the low Young’s moduli for these two lattices to the presence of these rhombi when they are not braced [37]. Observe that the RT is structurally equivalent to introducing bracing struts to the rhombi-based structure of the P3. The higher Young’s modulus of the RT shown in Fig. 10 in comparison to the P3 supports the fact that, at this density fraction regime, introducing braces to the P3 lattice stiffens the lattice, making the rhombi more rigid under load for the same pattern resolution and density fraction. This means that the benefit (with regards to stiffness) of introducing bracing struts outweighs the cost of the accompanying increase in density fraction. Similarly, the Young’s modulus of the TT, shown in Fig. 10 is higher than that of the P2 for the same pattern resolution and density fraction, with the TT similar to the P2 but with additional bracing struts. The TT and ST exhibit the highest Young’s moduli shown in Fig. 10. Of the aperiodic lattices, the PIN is least isotropic at \( \rho / \rho_s = 0.50 \). Quantitative measures of the level of anisotropy across the lattices is shown in the Universal Elastic Anisotropy Index (\( A_s^{AE} \)) plot of Fig. 16 and discussed in Section 6.

The final row of Fig. 10 shows the directional Poisson’s ratio for the different lattice structures at density fraction, \( \rho / \rho_s = 0.5 \). In general, all the aperiodic structures of this investigation have positive Poisson’s ratios and the lattice structures with the lowest
Young’s moduli exhibit the highest effective Poisson’s ratios, i.e., under loading, the least stiff lattice structures strain the most in the orthonormal direction to the direction of loading and vice versa. For any density fraction (See Fig. 15), the P3 and AB exhibit lower Poisson’s effect than the base material. Young’s moduli and Poisson’s ratios as shown in Figs. 11 and 12 respectively and show that comparative mechanical performance varies in the direction perpendicular to the specific direction of loading. The Poisson effect on these lattice structures is more significant than that exhibited by the base material from which they are made, \( v_i = 0.3 \). Interestingly, the periodic HEX also exhibits higher effective Poisson’s ratio than that of the base material. All other aperiodic lattices of this work exhibit lower Poisson effect than that of the base material, at the density fraction, \( q_s = 0.75 \). However, at \( q_s = 0.25 \), Figs. 11a and 12a show a reduced stiffness properties of the periodic HEX, with Young’s modulus only superior to those of the P3 and AB. This is a significant result concerning the mechanical performance of thin-walled honeycombs. Instead, aperiodic lattice structures such as the ST and TT exhibit far higher moduli than the HEX in this low density fraction regime with all the other aperiodic lattices showing higher Young’s moduli than the HEX at this density fraction. It is important to note that there exists more significance to the inclusion of bracing struts at lower density fractions (thinner struts) as shown by comparing Young’s moduli for the P2 and P3 against the TT and RT respectively. At this lower density fraction, the Poisson’s ratios of the aperiodic lattice structures (except the P3 and AB) remain below that of the base material, \( v_i \). All structures are less isotropic with decreasing \( \rho/\rho_i \), the PIN being by far the most anisotropic.

On the other hand at higher density fractions, say \( \rho/\rho_i = 0.75 \), the HEX lattice exhibits remarkable Young’s modulus, closely followed by the ST. We highlight here that the ST’s mechanical performance (with respect to comparable stiffness) is sustained for all density fractions. Also, there exists no improvements to Young’s moduli for the P3 due to the inclusion of bracing struts in this density fraction regime. In fact, the Young’s modulus of the TT is marginally worse than that of the P2 at this higher density fraction, suggesting that the material-cost of including bracing struts exceeds the improvements in mechanical performance. The PIN exhibits the lowest Young’s modulus at this density fraction. Above \( \rho/\rho_i = 0.75 \), the lattice structures begin to lose their pattern-dependent mechanical properties as their properties converge to those of base material. This is emphasized in Fig. 12c which shows the Poisson’s ratios of all the lattice structures approaching that of the isotropic base material, \( v_i = 0.3 \).

### 4.2. Performance against Hashin–Shtrikman Walpole (HS-W) bounds

The HS-W bounds show the theoretical limits for the effective elastic moduli of multiphase materials where the materials may be considered to be composed of a mechanical mixture of isotropic and homogeneous elastic phases. In its most general form, the HS-W lower and upper bounds of effective bulk and shear moduli for d-dimensional isotropic materials with \( n \) phases are given by [38,39] as:

\[
K_{\text{min}} \leq K \leq K_{\text{max}} \leq \left[ \sum_{i=1}^{n} \rho_i (K_i^{\text{min}} + K_i) \right]^{-1} - K_{\text{min}}
\]

and

\[
G_{\text{min}} \leq G \leq G_{\text{max}} \leq \left[ \sum_{i=1}^{n} \rho_i (G_i^{\text{min}} + G_i) \right]^{-1} - G_{\text{min}}
\]

where \( K_i, G_i \) and \( \rho_i \) are the shear modulus, bulk modulus and density fraction of phase \( i \). \( K_{\text{min}}, K_{\text{max}} = 2d^{-1}(d-1)G_{\text{min}}, K_{\text{max}} = 2d^{-1}(d-1)G_{\text{min}}, G_{\text{max}} = \min(h_i), G_{\text{max}} = \max(h_i) \), and

\[
h_i = \frac{d^2 K_i + 2(d + 1)(d - 2)G_i}{2d(K_i + 2G_i)}
\]

The patterns of this work consists of two phases (\( n = 2 \)): base isotropic material and void (modelled as a very weak isotropic mate-
and Young's modulus apply so that $G_i = \frac{d}{2}$. The patterns of this investigation are restricted to two dimensions so that $d = 2$ and the isotropic relations for bulk modulus, shear modulus and Young's modulus apply so that $C_i = \frac{d}{2(1-\nu)}$ and $K_i = \frac{d}{3(1-\nu)}$.

Consider the plots of Fig. 13 which show the variation of effective bulk and shear moduli of the lattice structures of this work, for a range of density fractions. The plots also show the HS-W theoretical upper bounds achievable. We seek to compare the proximity of these mechanical properties to these bounds for each lattice structure of our study. Fig. 13a shows the variation of effective bulk modulus with density fraction. In general, the HEX shows the closest proximity to the theoretical limits for $\rho/\rho_b > 0.25$. Within this density fraction range, the PIN performs worst, with the least effective bulk modulus for any given density fraction. For $\rho/\rho_b < 0.25$, the ST demonstrates superior effective bulk modulus to the HEX. The P2 and P3 perform just as poorly as the PIN. All other aperiodic lattices have effective bulk moduli between those of the PIN and HEX. Within the range $0.3 < \rho/\rho_b < 0.6$ shown more clearly in the X-X region of Fig. 13d, the RT has higher effective bulk modulus than the P3, but only up to $\rho/\rho_b \approx 0.4$, beyond which the cost of including bracing struts to the P3 outweighs the improvements in effective bulk modulus. The inclusion of bracing struts of the TT remains beneficial in comparison to the P2. Interestingly, the AB exhibits the third closest proximity to the bulk moduli limits until just below $\rho/\rho_b = 0.4$ were the TT outperforms it.

Fig. 13b shows the variation of the shear moduli of lattice structures with density fraction. The TT and ST exhibit closest proximity to the theoretical limits where $\rho/\rho_b < 0.6$. For $\rho/\rho_b > 0.6$, the HEX presents the highest shear modulus for given density fraction. We highlight here, that the shear modulus of the HEX relative to the other lattice structures drops off rapidly for $\rho/\rho_b < 0.5$, so that it shows better shear modulus properties than only the P3 and AB.

These two lattices exhibit the least shear moduli for $0 < \rho/\rho_b < 0.7$. For $\rho/\rho_b > 0.7$, the PIN exhibits the least shear modulus. Throughout the range of density fractions, the inclusion of bracing struts significantly improves effective shear modulus as can be observed in the Y-Y region of Fig. 13c.

Finally, Figs. 13c and 13d show the trajectories of the properties of the lattice structures through the plane of attainable effective shear and bulk moduli, as the density fractions are increased. For all the lattice structures, changes in effective shear and bulk moduli are less sensitive to variations in density fraction at lower density fractions. At higher density fractions, the properties are much more sensitive to variations in density fraction. More specifically, at lower density fractions shown in the Z-Z region of Fig. 13f, the P3 and AB exhibit the least shear moduli for a given bulk modulus value followed by the HEX. Every other aperiodic lattice of this work exhibits more than proportional shear modulus for given bulk modulus in this density fraction regime. The trajectories of each the properties eventually converge to those of the isotropic base material as the density fractions of the patterns tend to unity as shown in Fig. 13c.

5. Young's modulus and Density fraction against Poisson's ratio

Consider the plot of Figs. 14 showing the trajectory of each lattice structure on the Young's modulus - Poisson's Ratio plane, as the density fraction is incremented. The plot shows the Young's modulus in the x-direction. Apart from the P3 and AB, the aperiodic lattices of this study exhibit Poisson effect lower than that of the base material, regardless of the density fraction. Of these aperiodic structures, the ST maintains the greatest stability in Poisson effect with Poisson's ratio values close to $\nu_i$ in the low density fraction range ($\rho/\rho_b < 0.4$). This is consistent with its effective bulk and
shear moduli performances shown in Fig. 13. Interestingly, the P3, AB and the periodic HEX show significant instability in this low density fraction range. The other aperiodic structures exhibit Poisson effect below $v_s$. Fig. 15 shows the variation of the Poisson’s ratios of the lattices with density fraction. This plot emphasizes the critical density fraction where the P3, AB and HEX exhibit the same Poisson effect as the base material. Below this critical density fraction, these three lattice structures exhibit higher Poisson effect than $v_s$. In general, the results suggest that aperiodic lattice structures have the potential to yield interesting mechanical properties (here, stability in Poisson effect with changing density fraction). However, the underlying shapes of the aperiodic lattice system play a significant role in the mechanical behaviour observed as demonstrated by the aperiodic P3 and AB.

Fig. 13. Variation of effective bulk and shear moduli against density fraction ((a) and (b)), Trajectory through the effective bulk and shear modulii plane with increasing density fraction for all lattices (c). Figures (d), (e) and (f) show zoomed in regions X-X, Y-Y and Z-Z of (a), (b) and (c) respectively to facilitate discussions.

Fig. 14. $E/E_s$ vs $v$.

Fig. 15. $\rho/\rho_s$ vs $v$. 

6. Elastic anisotropy measure

To investigate the level of anisotropy associated with the different lattice structures, we leverage the elastic anisotropic measure presented by Li et al. [40] where given the elasticity matrix of a 2D lattice system, a universal elastic anisotropy index is given as:

\[
A^{SU} = \left( \frac{1}{2}(C_{11} + C_{22} + 2C_{12})(S_{11} + S_{22} + 2S_{12}) - 1 \right)^2 + \frac{2}{\pi}(C_{11} + C_{22} - 2C_{12} + 4C_{66}) \\
\times (S_{11} + S_{22} - 2S_{12} + S_{66}) - 1 \right)^{\frac{1}{2}}
\]

where the \( C_{\alpha} \) terms are the components of the stiffness matrix and the \( S_{\alpha} \) terms are the components of the compliance matrix for the lattice system. A perfectly isotropic lattice system would have \( A^{SU} = 0 \), with increasing \( A^{SU} \) values representing increasing anisotropy.

Fig. 16 shows the universal elastic anisotropy measure for the aperiodic lattices of this study at \( \rho/\rho_s = 0.25, 0.50 \) and 0.75. The figure confirms that the PIN exhibits the least isotropic performance for all density fractions despite having infinite rotational symmetry. All the studied lattice structures converge to a homogeneous isotropic material of the base material as density fraction tends to unity. Interestingly, the RT exhibits best isotropic performance across the density fraction range.

7. Conclusion

7.1. Summary

The linear elastic properties of seven aperiodically-ordered lattice structures were determined and compared with that of the hexagonal lattice structure. The investigation demonstrated unique substitution rules for the generation of each aperiodic tessellation of the study. Pattern resolution equivalence was ensured by imposing that their average prototile areas were equal to that of an equivalent square lattice resolution. The direction-dependent properties were compared against each other to investigate their level of anisotropy. Also, these properties were compared against Hashin-Shtrikman-Walpole theoretical bounds. Of the aperiodic lattice structures studied, the ST demonstrated the best mechanical properties for all densities. The well known hexagonal lattice structure outperformed the aperiodic structures with respect to bulk modulus but exhibited worse shear modulus for most of the aperiodic lattice structures in the low density fraction range \( (\rho/\rho_s < 0.4) \). Only the P3 and AB exhibited less shear moduli in this density fraction regime - both of which feature mechanically unstable rhombi prototiles. The results suggest that there exists no direct mapping of the number of rotational symmetries to isotropy measure, especially for finite patches of the aperiodic patterns. This is further emphasized by the PIN’s comparatively low modulus and poor isotropic performance, despite having infinite rotational symmetry. Elementary shapes of each pattern play a key role in the overall mechanical behaviour of the pattern as suggested by Somera et al. [30]. Finally, for constant density fraction and pattern resolution, including bracing struts to patterns of known mechanical properties does not guarantee improved mechanical performance. The weight-cost of introducing bracing struts can overcome any potential improvements in mechanical properties so that bracing is detrimental to the strength/weight ratio. Knowledge of the lattice-specific critical density fraction would be essential to understanding the effect of including bracing struts to the lattice.

7.2. Future research directions

Aperiodic lattice structures have the potential to outperform the traditional periodic cellular lattice structures. Though investigations here were limited to aperiodic lattices with uniform strut thickness, the isotropic nature of these aperiodic lattices suggest their suitability for energy-absorption and uncertainty mechanics problems. The high variation in strut orientations in each pattern suggests that anisotropic mechanical behaviour can be more readily programmed in the microarchitecture of the structures by spatially varying geometry parameters such as strut thickness. A spatially-varying deflation substitution scheme can also be introduced to facilitate functional grading as can be found in many natural metamaterials. Nature teaches us that introducing structural hierarchy has the potential to instill mechanical robustness and multifunctionality [41]. Such capability will find applications within the biomedical and aerospace industry such as in the optimal design of patient-specific bone implants or aerospace structures subject to complex stress states or loading uncertainty. Importantly, the concepts for developing aperiodic patterns herein generalize to three-dimensions. These ideas motivate our future research directions.

8. Data Availability

The raw/processed data required to reproduce these findings cannot be shared at this time as the data also forms part of an ongoing study.

Data availability

Data will be made available on request.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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