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Kirstie Asciak, Marston D.E. Conder, Soňa Pavlíková, Jozef Širáň

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Orientable and non-orientable regular maps with given exponent group

Kirstie Asciak
Open University, UK

Marston D.E. Conder
University of Auckland, Anckland, New Zealand

Soňa Pavlíková
A. Dubček University in Trenčín, Trenčín, Slovakia

Jozef Širáň
Open University, UK, and Slovak University of Technology, Bratislava, Slovakia

Abstract

With the help of the parallel product (also known as the join) of maps given by subgroups of triangle groups, and some facts about automorphisms of products of simple groups, we extend a 2016 theorem of Conder and Širáň on exponent groups of orientable maps, by proving that for every $d \geq 3$ and every group U of units mod d containing -1, there exist infinitely many non-orientable regular maps of valency d with exponent group equal to U.

Keywords: Map; regular map; exponent of a map; exponent group.

1 Introduction

Embeddings of connected graphs on surfaces (more commonly known as maps) with the highest level of symmetry have attracted considerable interest as objects linking graph theory, low-dimensional topology, group theory and the theory of complex functions; see [22] for a relatively recent survey. The 'level of symmetry' of a structure is given by the type of action of its automorphism group on some given building blocks for the structure.

For maps, these building blocks can be taken as the flags (which except in degenerate cases can be viewed as the incident vertex-edge-face triples), and it is well known that the automorphism group of a map acts fixed-point-freely on the set of its flags. If this action is transitive, and hence regular, then the map is called *regular*. Also when the carrier surface of a map is orientable, one may focus on the group of all its orientation-preserving automorphisms, which acts fixed-point-freely on arcs of the underlying graph, and if this action is transitive, then map is called *orientably-regular*.

Accordingly, regular and orientably-regular maps are the 'most symmetric' maps, when measured by the action of their automorphism groups on flags and/or arcs. Some of these maps, however, may be more symmetric than others, in a sense that we will clarify. For example, if a regular map is self-dual or self-Petrie-dual, then this extra level of symmetry is not induced by internal means (namely via automorphisms), but by isomorphisms among the maps coming from external operations such as duality or Petrie duality. For recent advances in the study of self-dual and self-Petrie-dual regular maps, we refer to [13, 16] and to the survey [22].

Another form of external symmetry of regular maps can arise from taking rotational powers, also known as 'hole operators' (thanks to Wilson [25], although the notion has been attributed to Coxeter). Suppose that one re-embeds the underlying graph of an orientably-regular map of valency d into a (possibly different) surface in such a way that one replaces the cyclic rotation of arcs emanating from each vertex by its jth power, for some fixed $j \in \{1, 2, \dots, d-1\}$ that is relatively prime to d. The new map will still be orientably-regular, but it might not be isomorphic to the original map. If it is, then the power j is called an *exponent* of the original map, a term coined in [19]. Exponents of such a map of valency d form a subgroup of the multiplicative group U_d units mod d, called the *exponent group*. The analogous concept can be defined for non-orientable regular maps, and we do so in Section 2.

As we will focus on regular and orientably-regular maps with given exponent group, we now take time to give just a brief survey of known facts about extreme cases. Extending earlier discoveries made in [2] and [23], it was shown in [9] by permutation techniques and in [15] by holomorphic differentials that for every hyperbolic pair (d, ℓ) there exists an orientably-regular map of type (d, ℓ) not admitting the exponent -1; such maps are known as *chiral*. A further examination [3] of the chiral maps from [9] arising from quotient digraphs of permutation groups revealed that they actually have *trivial* exponent group. At the other extreme, as observed in [24], residual finiteness of triangle groups implies the existence of infinitely many orientably-regular maps of given valency $d \geq 3$ admitting the 'full' exponent group U_d. The same was proved in [1] for regular maps of arbitrary even valency on orientable surfaces, by lifting techniques. Eventually, a common generalisation was obtained in [11], namely that for any $d \geq 3$ and every subgroup U of the group U_d of units mod d, there exists an orientably-regular map of valency d with exponent group equal to U. This was proved by constructing suitable normal subgroups of the free product of a pair of cyclic groups of order 2 and d.

In the non-orientable case, regular maps with the smallest exponent group $\{\pm 1\}$ were constructed in [3] for every hyperbolic type (d, ℓ) with at least one even entry, and for infinitely many types with both entries odd. This was achieved by considering maps with automorphism group isomorphic to a linear fractional group. At the other end of the spectrum, the existence of non-orientable regular maps with full exponent group U_d for valency d of the form $2^{2n} - 1$ follows from [16], by a very general analysis of joins of maps with automorphism group isomorphic to $\mathrm{SL}(2, 2^n)$, and similarly for some other odd valencies by using Suzuki groups.

Given the current state of knowledge, it is natural to ask if the findings of [11] extend to non-orientable regular maps. We answer this question in the affirmative, by proving that for every integer $d \geq 3$ and every subgroup U of the group U_d of units modulo d such that $-1 \in U$, there exists a non-orientable regular map of valency d with exponent group U.

Inspired by the construction of 'super-symmetric' maps in [16, Section 9], our method combines parallel products of maps [26] in their later treatment as joins [16, 14], with knowledge of the structure of the automorphism groups of direct products of finite simple groups [5]. Using this approach, we also give an alternative proof of the existence of at least one orientably-regular map of any given valency $d \geq 3$ and with a given exponent group $U \leq U_d$ [11].

Unlike what has been achieved for trivial exponent group for orientably-regular maps and on exponent group $\{\pm 1\}$ for regular maps on non-orientable surfaces, which apply to all hyperbolic types (d, ℓ), however, our approach to construction of such maps for a given subgroup of the group of units modulo d does not offer control over the face length ℓ of the maps. In fact, it turns out that is not possible to prescribe such a subgroup for a given pair (d, ℓ) in general, and we comment on this issue in the final section.

The rest of this paper is organised as follows. In the next section we summarise aspects of the algebraic theory of (orientably-) regular maps that are relevant for our purposes. In Section 3 we introduce the operation of parallel product (or, equivalently, join) of maps, and with its help we give our new proof of the main theorem of [11] on the existence of orientably-regular maps with prescribed exponent group (given as a subgroup U of U_d). Then we extend this in Section 4 to prove our main theorem, on the existence of non-orientable regular maps with any given exponent group $U < U_d$ containing the unit -1, and we conclude the paper with some further observations in Section 5.

2 Algebraic preliminaries

We begin by reviewing a few basic facts from the theory of regular maps, collected from the influential papers [17] and [6] in the orientable case and the general case, respectively; some of these may also be found in the survey [22].

We recall that a *map* M is a cellular embedding of a connected graph on a closed surface. The *flags* of M are the oriented triangles in the barycentric subdivision of M (each made up of a vertex v of M, the mid-point of an edge e incident with v, and the centre of a face f incident with e), and except in rare degenerate cases, these can be viewed as the incident vertex-edge-faces triples (v, e, f) of M.

When the carrier surface is orientable, M is said to be *orientably-regular* if the group $\mathrm{Aut}^+(M)$ of all its orientation-preserving automorphisms is transitive, and hence regular, on the set of all arcs of M, while in the more general case of an arbitrary carrier surface (orientable or not), M is said to be *fully regular*, or simply *regular*, if its full automorphism group $\mathrm{Aut}(M)$ is regular on the set of all flags of M. Here we note that for a regular map on an orientable surface, $\text{Aut}^+(M)$ is a subgroup of index 2 in $\text{Aut}(M)$.

Under both forms of regularity, every vertex of M has the same valency say, d, and every face of M is bounded by a closed walk in the underlying graph of M having the same length, say ℓ, and then we say that the map M has *type* (d, ℓ), and that the type of M is *spherical*, *Euclidean* or *hyperbolic* when $1/2 + 1/d + 1/\ell$ is greater than 1, equal to 1 or less than 1, respectively.

If M is orientably-regular and has type (d, ℓ), then one of the standard presentations of the group $\text{Aut}^+(M)$ can be derived as follows. Choose any vertex v together with an arc e incident to v in M. As M is orientably-regular, the group $\text{Aut}^+(M)$ contains an involution x acting locally as a 180-degree rotation of M about the centre of e, and an element y of order d acting locally as a d-fold rotation of M about v, consistent with the orientation of the carrier surface of the map, and such that xy acts locally as an ℓ-fold rotation of M about the centre of a face incident with e.

By connectedness of M, it follows easily that the group $\text{Aut}^+(M)$ is generated by the pair $\{x, y\}$, and admits a presentation of the form

$$\text{Aut}^+(M) = \langle\, x, y \mid x^2 = y^d = (xy)^\ell = \cdots = 1 \,\rangle. \quad (1)$$

Furthermore, using the regular action of $\text{Aut}^+(M)$ on the arcs of M, the map M itself can be identified with this presentation of $\text{Aut}^+(M)$, with the arcs, edges, vertices and faces of M corresponding to right cosets of the subgroups $\langle 1\rangle$, $\langle x\rangle$, $\langle y\rangle$ and $\langle xy\rangle$, and with incidence between pairs of these elements given by non-empty intersection of the corresponding cosets. Also $\text{Aut}^+(M)$ acts simply by right multiplication on cosets. In such a situation, we can write $M = (G; x, y)$ to indicate a regular map M associated with a group G having a presentation in the form (1).

Continuing our explanation, we note that the group $G = \text{Aut}^+(M)$ is a quotient of the $(2, d, \infty)$-*triangle group* $\Delta^+(2, d, \infty) = \langle\, X, Y \mid X^2 = Y^d = 1 \,\rangle$, which is isomorphic to the free product $C_2 * C_d$ of a pair of cyclic groups of orders 2 and d. In particular, $G \cong \Delta^+(2, d, \infty)/N^+(M)$ where $N^+(M)$ is the kernel of the epimorphism from $\Delta^+(2, d, \infty)$ to G taking (X, Y) to (x, y). We will call $N^+(M)$ the *map subgroup.*

It follows that the orientably-regular map $M = (G; x, y)$ is completely determined by the map subgroup $N^+(M)$, which is a normal subgroup of finite index in $\Delta(2, d, \infty)$ containing $(xy)^\ell$. Also two orientably-regular maps $M_1 = (G_1; x_1, y_1)$ and $M_2 = (G_2; x_2, y_2)$ with valency d are isomorphic if and only if there exists a group isomorphism from G_1 to G_2 taking (x_1, y_1) to (x_2, y_2), or equivalently, if and only if $N^+(M_1) = N^+(M_2)$ in $\Delta(2, d, \infty)$.

Fully regular maps allow for a similar kind of algebraic description. If M is such a map of type (d, ℓ), then a standard presentation of its full automorphism group $\text{Aut}(M)$ can be obtained by choosing fixing a flag Φ of M, and letting a, b and c be involutory automorphisms of M that act locally as reflections in the three sides of that flag, in such a way that the composites ab, bc and ca act locally as an ℓ-fold rotation about the face-centre of Φ, a d-fold rotation about the vertex of Φ, and a two-fold rotation about the edgemidpoint of Φ, respectively. Again by connectedness of M, these three automorphisms generate $\mathrm{Aut}(M)$, which admits a presentation of the form

$$\mathrm{Aut}(M) = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^\ell = (bc)^d = (ca)^2 = \cdots = 1\,\rangle. \tag{2}$$

Also by letting $x = ac$ and $y = cb$, we find that the group $\mathrm{Aut}(M)$ has an alternative presentation of the form

$$\mathrm{Aut}(M) = \langle\, x, y, c \mid x^2 = y^d = (xy)^\ell = c^2 = (xc)^2 = (cy)^2 = \cdots = 1\rangle \tag{3}$$

Aside from degenerate cases which will not be considered here, this fully regular map M is orientable if and only if the subgroup $\langle x, y\rangle$ has index 2 in $\mathrm{Aut}(M)$, in which case $\langle x, y\rangle = \mathrm{Aut}^+(M)$, with presentation of the form (1). In contrast, when M is non-orientable, $\langle a, b, c\rangle = \langle ac, cb\rangle$, or equivalently, $\langle x, y, c\rangle = \langle x, y\rangle$, so that $c \in \langle x, y\rangle$.

Just as in the orientably-regular case, the regular action of $\mathrm{Aut}(M)$ on flags of a regular map M of type (d, ℓ) allows us to identify the map M with the presentation of $\mathrm{Aut}(M)$ in the form (2), by regarding flags, arcs, edges, vertices and faces of M as right cosets of the subgroups $\langle 1\rangle$, $\langle c\rangle$, $\langle ac\rangle = \langle x\rangle$, $\langle cb\rangle = \langle y\rangle$ and $\langle ab\rangle = \langle xy\rangle$, respectively, and with incidence determined by non-empty intersection. Also $\mathrm{Aut}(M)$ acts by right multiplication on cosets. We can then write $M = (G; a, b, c)$ or $M = (G; x, y, c)$ to indicate a regular map M associated with a group G having a presentation in the form (2) or (3), respectively.

Here the group $G = \mathrm{Aut}(M)$ can be viewed as a quotient of the *full* $(2, d, \infty)$-*triangle group* $\Delta(2, d, \infty) = \langle\, X, Y, C \mid X^2 = Y^d = C^2 = (XC)^2 = (CY)^2 = 1\,\rangle$, which is isomorphic to a split extension of $\Delta^+(2, d, \infty)$ by the group $\langle C\rangle$ of order 2, and then also $G \cong \Delta(2, d, \infty)/N(M)$ where the map subgroup $N(M)$ is the kernel of the epimorphism from $\Delta(2, d, \infty)$ to G taking (X, Y, C) to (x, y, c). Isomorphism between fully regular maps $M_1 = (G_1; x_1, y_1, c_1)$ and $M_2 = (G_2; x_2, y_2, c_2)$ of valency d can be decided algebraically by two means: by coincidence of the map subgroups $N(M_1)$ and $N(M_2)$ in $\Delta(2, d, \infty)$, or by the existence of a group isomorphism from G_1 to G_2 taking (x_1, y_1, c_1) to (x_2, y_2, c_2).

Orientably-regular and fully regular maps exhibit the largest degree of symmetry, restricted to symmetries preserving orientation in the former case. As noted in Section 1, however, such a map may have additional features that enable us to regard it as being 'even more symmetric'. This happens, for example, if a regular map is self-dual, or self-Petrie dual, or 'self-opposite' (as indicated in [25]), but we will focus on the other major source of such 'external' symmetries mentioned earlier, namely the 'holes' described in [12] and the 'hole operators' described in [25].

Let M be an orientably-regular map $(G; x, y)$ or a non-orientable regular map $(G; x, y, c)$ defined via a group G as in (1) or (3), respectively, in both cases having valency $d \geq 3$. Then for every $j \in U_d$, the *j th rotational power* $M^{(j)}$ of M is the orientably-regular map $(G; x, y^j)$ in the orientable case, or the map $(G; x, y^j, c)$ in the non-orientable case. Geometrically, $M^{(j)}$ is obtained by re-embedding the underlying graph of M in such a way that the local cyclic permutation π_v of arcs emanating from every vertex v in M is replaced by its jth power π_v^j. See [22] for further details.

Typically, such a re-embedding will result in $M^{(j)}$ having face length different from that of M (which happens when xy and xy^j have different orders), and hence in M and $M^{(j)}$ lying on different surfaces. Even if the orders of xy and xy^j are the same, the maps M and $M^{(j)}$ need not be isomorphic. If they are isomorphic, then the unit j mod d is called an *exponent* of the map M.

Recalling the way that isomorphisms between a pair of orientably-regular or fully regular maps translate into automorphisms of the relevant group $G = \mathrm{Aut}^+(M)$ or $\mathrm{Aut}(M)$, we arrive at the following. If M is an orientably-regular map given by $(G; x, y)$, then j is an exponent of M if and only if G admits an automorphism fixing x and taking y to y^j. Similarly, if M is a non-orientable regular map given by $(G; x, y, c)$, then j is an exponent of M if and only if there is an automorphism of G fixing both x and c and taking y to y^j. In both cases we call such an automorphism of the group G a *j-rotational automorphism*.

Exponents of orientable maps were introduced and studied in depth in [19], and for non-orientable regular maps in [22]. The product of any pair of exponents of an orientably-regular or fully regular map M of valency d is again an exponent of M, and hence the set of all exponents of M forms a subgroup of U_d called the *exponent group* of M, and denoted by $\mathrm{Exp}(M)$. This group naturally measures the richness in external symmetries of the map M induced by rotational powers.

Having reviewed the existing literature on orientably-regular and regular maps with given exponent group, we can now proceed to introduce the operation of join (or parallel product) of maps in the next section, and use this to give our alternative proof of the theorem from [11] on orientably-regular maps with given exponent group, before applying it in Section 4 to extend that to non-orientable surfaces.

3 Orientably-regular maps with given exponent group

Let J be a finite linearly-ordered set, and let $\mathcal{M} = (M_j : j \in J)$ be a correspondingly ordered family of finite maps, such that either (a) every M_j is orientably-regular and has the form $M_j = (G_j; x_j, y_j)$ where the group G_j has a presentation as in (1) with map subgroup $N_j^+ = N^+(M_j)$, or (b) every M_j is fully regular and has the form $M_j = (G_j; x_j, y_j, c_j)$ where G_j has a presentation as in (3) with map subgroup $N_j = N(M_j)$.

We will assume that all the maps in $\mathcal{M}$ have the same valency d. (For a general theory not assuming any restriction on valency, orientability and regularity, we refer the reader to [26] and [16].)

The *join* (or *parallel product*) of the family of maps $\mathcal{M}$, is the map $\vee\mathcal{M}$ that is uniquely defined by taking as the map subgroup the intersection $N^+(\vee\mathcal{M}) = \cap_{j\in J} N_j^+$ in the orientably-regular case, or $N(\vee\mathcal{M}) = \cap_{j\in J} N_j$ in the non-orientable case.

The map subgroup $N^+ = N^+(\vee\mathcal{M})$ or $N = N(\vee\mathcal{M})$ is a normal subgroup of finite index in $\Delta^+(2, d, \infty)$ or $\Delta(2, d, \infty)$, and hence $\vee\mathcal{M}$ is a finite d-valent orientably-regular or fully regular map, respectively. Also $\vee\mathcal{M}$ can be described as the map $(G; x, y)$ or $(G; x, y, c)$

where either x and y are the images of X and Y in the quotient $G = \Delta^+(2, d, \infty)/N^+(\vee\mathcal{M})$, or x, y and c are the images of X, Y and C in the quotient $G = \Delta(2, d, \infty)/N(\vee\mathcal{M})$, respectively. Moreover, $\vee\mathcal{M}$ is a regular cover of each of the maps M_j, via the assignment $gN^+ \mapsto gN_j$ for $g \in \Delta^+(2, d, \infty)$, or $gN \mapsto gN_j$ for $g \in \Delta(2, d, \infty)$, respectively.

In general it might not be easy to determine the group G for the join $\vee\mathcal{M}$ in terms of the constituent groups G_j for $j \in J$, but as observed in [14, 16], this task is manageable when the G_j are simple and non-abelian. Accordingly, we take a restricted approach that is suitable for what we need later.

Also for notational convenience, if $(H_j : j \in J)$ is an ordered family of groups that are not necessarily distinct, then we will represent an element of the direct product $\prod_{j\in J} H_j$ simply in the form $(u_j)_{j\in J}$, with the obvious multiplication rule $(u_j)_{j\in J}(v_j)_{j\in J} = (u_j v_j)_{j\in J}$. Here we reiterate that J is assumed to be linearly-ordered, so that this product notation is unambiguous.

Theorem 1 *Let Γ be the free product of cyclic groups of (not necessarily distinct) orders m and n, with presentation $\Gamma = \langle\, R, S \mid R^m = S^n = 1\,\rangle$, and let $(\Gamma_j : j \in J)$ be a non-empty finite ordered family of epimorphic images of Γ, given by pairwise distinct normal subgroups K_j of Γ such that $\Gamma/K_j \cong \Gamma_j$ for all $j \in J$, and with intersection $K = \cap_{j\in J} K_j$. If each Γ_j is a simple non-abelian group, then the mapping*

$$\theta:\ \Gamma/K \to \prod_{j\in J} \Gamma_j \ \cong\ \prod_{j\in J} (\Gamma/K_j) \quad \text{given by} \quad \theta:\ Kz \mapsto (K_j z)_{j\in J} \tag{4}$$

is a group isomorphism taking the images of R and S in Γ/K to the product of the images of R and S in the groups $\Gamma_j \cong \Gamma/K_j$.

A proof of this result is straightforward group theory, even for more general 'parent' groups than a free product of a pair of cyclic groups, but also follows from Lemma 6.1 and Corollary 6.2 of [14]. We emphasise that the essence of this theorem is the fact that under the given assumptions, the group homomorphism θ given in (4) is *surjective*, which is equivalent to the statement that the θ-images of R and S form a generating pair for the whole product $\prod_{j\in J}(\Gamma/K_j)$, rather than for a proper subgroup.

We may now apply Theorem 1 in the special case where $(m, n) = (2, d)$, $(R, S) = (X, Y)$, $\Gamma = \Delta^+(2, d, \infty) = \langle\, X, Y \mid X^2 = Y^d = 1\,\rangle$, and $\Gamma_j = G_j$ for all $j \in J$, coming from the family $M_j = (G_j; x_j, y_j)$ of orientably-regular maps considered at the beginning of this section, and obtain the following consequence.

Theorem 2 *Let $\mathcal{M} = (M_j : j \in J)$ be a finite ordered family of orientably-regular maps $M_j = (G_j; x_j, y_j)$, where the groups G_j are non-abelian and simple, but not necessarily distinct. If the map subgroups $N^+(M_j)$ are pairwise distinct (as j runs through J), then the direct product $\overline{G} = \prod_{j\in J} G_j$ is generated by $\overline{x} = (x_j)_{j\in J}$ and $\overline{y} = (y_j)_{j\in J}$, and the corresponding orientably-regular map $(\overline{G}; \overline{x}, \overline{y})$ is isomorphic to the join $\vee\mathcal{M}$.* □

As the next step, we apply Theorem 2 to the construction of an orientably-regular map with given valency and given exponent group. Verification of its properties depends on knowing the automorphism group of a direct product of simple groups.

Theorem 3 *Let $M = (G; x, y)$ be an orientably-regular map with valency $d \geq 3$, and let U be a subgroup of the group U_d of units modulo d. For every $j \in U$, let $M^{(j)} = (G; x, y^j)$ be the jth rotational power of M, and let $\mathcal{M}$ be the family $(M^{(j)} \colon j \in U)$, linearly-ordered by elements of U. If G is a non-abelian simple group and M admits only the trivial exponent 1, then the exponent group of the orientably-regular map $\vee\mathcal{M}$ is equal to U.*

Proof. As $\mathrm{Exp}(M)$ is trivial, the maps $M^{(j)} = (G; x, y^j)$ are pairwise non-isomorphic, which by the theory presented in Section 2 means there is no automorphism of the group G that takes (x, y^j) to (x, y^ℓ) for two different $j, \ell \in U$, or equivalently, that the map subgroups $N^+(M^{(j)})$ for $j \in U$ are pairwise distinct. Then since G is simple, we may apply Theorem 2 to the family $\mathcal{M} = (M^{(j)} \colon j \in U)$, and conclude that the group $\overline{G} = \prod_{j\in U} G_j$ of orientation-preserving automorphisms of the join $\overline{M} = \vee_{j\in U} M^{(j)}$ is a direct product of $|U|$ copies of G, and is generated by the two elements $\overline{x} = (x)_{j\in U}$ and $\overline{y} = (y^j)_{j\in U}$.

We now show that U is the exponent group of the orientably-regular map $\overline{M} = (\overline{G}; \overline{x}, \overline{y})$. It is easy to see that $\mathrm{Exp}(\overline{M})$ contains U, because multiplication by any element of U induces a permutation of the constituents $M^{(j)} = (G; x, y^j)$ of $\overline{M}$, and an automorphism of the group $\overline{G}$ permuting its $|U|$ direct factors. Conversely, let k be any exponent of $\overline{M}$, and let ψ be an automorphism of $\overline{G}$ that fixes $\overline{x}$ and takes $\overline{y} = (y^j)_{j\in U}$ to $\overline{y}^k = (y^{jk})_{j\in U}$. Then since $\overline{G}$ is a direct product of $|U|$ copies of the same simple group G, Theorem 3.1 of [5] tells us that $\mathrm{Aut}(\overline{M})$ is isomorphic to the wreath product $\mathrm{Aut}(G) \wr \mathrm{Sym}(n)$ where $n = |U|$. It follows that we can multiply ψ by a suitable element of $\mathrm{Aut}(\overline{M})$ to obtain an automorphism τ of $\overline{G}$ that preserves G_1 (the first copy of G in the expansion $\overline{G} = \prod_{j\in U} G_j$), and then τ induces an automorphism of G_1 ($\cong G$) taking (x, y) to $(x, y^{\ell k})$ for some $\ell \in U$. But once again we can use the fact that $\mathrm{Exp}(M)$ is trivial, to conclude that $\ell k = 1$ in U, and hence that $k = \ell^{-1} \in U$. Thus $\mathrm{Exp}(\overline{M}) = U$, completing the proof. □

Fortunately, there is a plentiful supply of orientably-regular maps with a non-abelian simple automorphism group and trivial exponent group, for use as ingredients in the application of Theorem 3. For example, several constructions were given in [9] for chiral orientably-regular maps of given valency $d \geq 4$ with automorphism group isomorphic to some alternating group (even for infinitely many choices of face lengths, which we do not need here), and in [3] all of these maps were shown to have trivial exponent group. Other such maps can be found with the help of Theorem 6.3 of [7], which showed that for every $\ell \geq 7$ there are infinitely many chiral orientably-regular maps with type $(3, \ell)$ and with automorphism group isomorphic to an alternating group; the duals of these maps provide orientably-regular maps with an alternating group as automorphism group and with trivial exponent group, for any valency $d \geq 7$. For completeness, we note that another construction of orientably-regular maps with any given hyperbolic type with a symmetric or alternating automorphism group was recently given in [4].

For some more explicit examples with given valency $d \geq 5$, let G be the alternating group A_{2d} on the set $\{1, 2, \ldots, 2d\}$, and define two permutations x and y in G by $x = (d-2, d-1)(d, d+1)(2d-3, 2d-2)(2d-1, 2d)$ and $y = (1, 2, \ldots, d)(d+1, d+2, \ldots, 2d)$. It was shown in Subsection 3.4 of [9] that (x, y) is a generating pair for A_{2d} and that the map $M = (G; x, y)$ of valency d is chiral, and then in the follow-up [3] that the exponent group of M is trivial. For orientably-regular maps of valency 3 or 4 the exponent group is trivial (for chiral maps) or has order 2 (for fully regular maps), and examples of both kinds were given in [9].

Combining this plentiful supply of examples with Theorem 3 gives our alternative proof of the following, which was first proved in [11] using a different approach and in a stronger form, furnishing infinitely many d-valent orientably-regular maps with given exponent group, for every $d \geq 3$.

Theorem 4 *For every integer $d \geq 3$ and for every subgroup U of units modulo d, there exists an orientably-regular map of valency d with exponent group equal to U.* □

4 Non-orientable regular maps with given exponent group

To extend the above theorem to non-orientable regular maps, we adopt the terminology and notation introduced in Section 2, adjust the machinery developed in Section 3, and address the shortage of well-known examples of non-orientable regular maps of arbitrary valency greater than two having the smallest possible exponent group $\{\pm 1\}$. We achieve the latter by turning to linear fractional groups.

Theorem 5 *For every integer $d \geq 4$ and every prime p congruent to $2d + 1$ modulo $4d$, there exists a non-orientable regular map $M_{d,p} = (G; x, y, c)$ of type $(d, 2d)$ for the group $G = \mathrm{PGL}(2, p)$ with $\mathrm{Exp}(M_{d,p}) = \{\pm 1\}$, such that the generator y of order d lies in the subgroup $\mathrm{PSL}(2, p)$ of G, while the generator x of order 2 lies in $\mathrm{PGL}(2, p) \backslash \mathrm{PSL}(2, p)$.*

Proof. For each prime $p \equiv 2d + 1 \bmod 4d$, it is known that the group $\mathrm{PGL}(2, p)$ is generated by two elements y and z of orders d and $2d$, with product $y^{-1}z$ of order 2. This is a consequence of Theorems 1 and 2 of [21], inspired by the pioneering work by Macbeath on generators for the groups $\mathrm{PSL}(2, q)$ in [18]. Since the order of every element in the index 2 subgroup $K \cong \mathrm{PSL}(2, p)$ is either p or a divisor of $(p \pm 1)/2$, and our assumption on p is equivalent to $(p-1)/2 \equiv d \bmod 2d$, no element of K can have order $2d$ and hence the generator z lies in $G \backslash K$. On the other hand, by [21, Theorem 1 item (2a)] we may suppose that y lies in K. (Indeed y can be represented up to multiplication by ± 1 by the 2×2 diagonal matrix with entries ξ and ξ^{-1}, where ξ is a primitive $(2d)$th root of unity in $\mathrm{GF}(p)$.) Hence also the involution $x = y^{-1}z$ lies in $G \backslash K$, and obviously $\langle x, y \rangle = G$.

Another well-known property of the group $\mathrm{PGL}(2,p)$ is that for each of its generating pairs, there is an inner automorphism that conjugates each of the two generators to its inverse. This is provable by a trace argument that extends the same observation made for $\mathrm{PSL}(2,q)$ by Singerman in [20, Theorem 3]. Moreover, the element inducing this automorphism is a unique involution, because $\mathrm{PGL}(2,p)$ has trivial centre. Hence our group G contains an involution c that inverts each of x and y by conjugation, and it follows that $\langle x,y,c\rangle = \langle x,y\rangle = G$, and so by the theory presented in Section 2, we find that $M_{d,p} = (G;x,y,c)$ is a non-orientable regular map of type $(d,2d)$.

Finally, we show that this map has exponent group $\{\pm 1\}$. Let j be a unit mod d for which $M_{d,p}$ is isomorphic to $M_{d,p}^{(j)} = (G;x,y^j,c)$, so that there exists an automorphism of G taking y to y^j. The automorphism group of $G = \mathrm{PGL}(2,p)$ is known to be isomorphic to G, acting on itself by conjugation, and therefore y and y^j must be conjugate in G and hence must be represented by 2×2 matrices having the same trace, up to multiplication by ± 1. Again representing y by the diagonal matrix with entries ξ and ξ^{-1}, we find that $\xi+\xi^{-1} = \pm(\xi^j+\xi^{-j})$, which was shown in [10] to occur if and only if $j\in\{\pm 1\}$. □

By Dirichlet's theorem on primes in arithmetic progression, we know that for every integer $d\geq 4$ there are infinitely many primes p satisfying $p\equiv 2d+1 \bmod 4d$, and hence for each such d we have an infinite number of non-orientable regular maps $M_{d,p}$ of valency d with properties guaranteed by Theorem 5, and in particular, with exponent group $\{\pm 1\}$. We can now extend this to an arbitrary subgroup $U\leq U_d$ containing $\{\pm 1\}$.

For any such U, let V be a set of coset representatives for its subgroup $\{1,-1\}$, that is, chosen so that $|V| = |U|/2$ and $V\cup(-V) = U$, and consider the join $\vee\mathcal{M}_{d,p,V}$ of the family $\mathcal{M}_{d,p,V} = (M_{d,p}^{(j)} : j\in V)$ of jth rotational powers of the map $M_{d,p} = (G;x,y,c)$ constructed in Theorem 5 for the given prime p.

By the theory outlined in Section 3, we know that $\vee\mathcal{M}_{d,p,V}$ is a regular map of valency d. Specifically, $\vee\mathcal{M}_{d,p,V} = (\overline{G},\overline{x},\overline{y},\overline{c})$, where $\overline{G} = \prod_{j\in V} G_j$ with $G_j = \langle x,y^j,c\rangle$ for all $j\in V$, and $\overline{x}$ and $\overline{c}$ are the $|V|$-tuples $(x,x,\ldots,x)$ and $(c,c,\ldots,c)$ while $\overline{y} = (y^j)_{j\in V}$. Also it is easy to see that $\vee\mathcal{M}_{d,p,V}$ is non-orientable, with $\overline{c}\in\langle\overline{x},\overline{y}\rangle = \overline{G}$.

We proceed by determining the structure of $\overline{G}$, noting that G is no longer simple in the current context.

Theorem 6 *The map $\vee\mathcal{M}_{d,p,V}$ is non-orientable, with automorphism group $\overline{G}$ isomorphic to a semidirect product $(\prod_{j\in V} K_j)\rtimes\langle\overline{x}\rangle$, where $K_j = \langle y^j, xy^jx\rangle \cong \mathrm{PSL}(2,p)$ for all $j\in V$.*

Proof. First, we note that by non-orientability, $G_j = \langle x,y^j\rangle = G \cong \mathrm{PGL}(2,p)$ for every $j\in V$. Moreover, for every $j\in V$ the subgroup $K_j = \langle y^j, xy^jx\rangle$ has index 2 in G_j and is isomorphic to $\mathrm{PSL}(2,p)$, and indeed is the image of the index 2 subgroup Γ of $\Delta^+(2,d,\infty) = \langle X,Y \mid X^2 = Y^d = 1\rangle$ generated by Y and XYX, under an epimorphism $f_j\colon\Gamma\to G_j$ taking (Y,XYX) to (y^j,xy^jx). Note also that Γ is isomorphic to $C_d * C_d$.

By Theorem 5, we know that $\mathrm{Exp}(M_{d,p}) = \{\pm 1\}$, and then it follows from the way in which the index-set V was introduced above that the rotational powers $M_{d,p}^{(j)}$ and $M_{d,p}^{(\ell)}$ are not isomorphic whenever j and ℓ are distinct members of V. Accordingly, the epimorphisms $F_j : \Delta^+(2, d, \infty) \to G_j$ given by $(X, Y) \mapsto (x, y^j)$ for each $j \in V$ have pairwise distinct kernels, as do their restrictions $f_j : \Gamma \to K_j$. (Indeed $\ker F_j = \ker f_j$ for all $j \in V$.) This means we can apply Theorem 1 to $\Gamma \cong C_d * C_d$, and find that with $\overline{x} = (x, x, \ldots, x)$ and $\overline{y} = (y^j)_{j \in V}$ as above, the elements $\overline{y}$ and $\overline{x}\,\overline{y}\,\overline{x}$ make up a generating pair for $\overline{K} = \prod_{j \in V} K_j$.

Also conjugation by the involution $\overline{x}$ swaps the two generators $\overline{y}$ and $\overline{x}\,\overline{y}\,\overline{x}$ of $\overline{K}$, and as x lies in $\mathrm{PGL}(2, p) \backslash \mathrm{PSL}(2, p)$, it follows that $\overline{x} \notin \overline{K}$, and so $\overline{K} = \prod_{j \in V} K_j$ has index 2 in $\langle \overline{x}, \overline{y} \rangle = \langle \overline{x}, \overline{y}, \overline{c} \rangle = \overline{G}$, making $\overline{G}$ isomorphic to a semidirect product $(\prod_{j \in V} K_j) \rtimes \langle \overline{x} \rangle$. □

We now have all we need to prove the main theorem of this paper, for the non-orientable regular maps $\vee\mathcal{M}_{d,p,V} = (\overline{G}; \overline{x}, \overline{y}, \overline{c})$ constructed above.

Theorem 7 *For every integer $d \geq 3$, and every subgroup U of the group of units modulo d containing -1, there exist infinitely many non-orientable regular maps with valency d and exponent group equal to U.*

Proof. As above, let V be a set of coset representatives for $\{1, -1\}$ in U with $1 \in V$, and let p be any prime chosen from the infinite set of primes congruent to $2d + 1$ mod $4d$. Also let $M_{d,p} = (G; x, y, c)$ be a non-orientable regular map for $G = \langle x, y \rangle \cong \mathrm{PGL}(2, p)$ with exponent group $\{\pm 1\}$, where $y \in K \cong \mathrm{PSL}(2, p)$ and $x \in G \backslash K$, as given by Theorem 5, and let $\overline{M} = \vee\mathcal{M}_{d,p,V} = (\overline{G}, \overline{x}, \overline{y}, \overline{c})$ be the non-orientable regular map of valency d constructed from the family $\mathcal{M}_{d,p,V} = (M_{d,p}^{(j)} : j \in V)$ before the statement of Theorem 6.

All we need to do is show that the exponent group of $\overline{M}$ is equal to U. For this we will let $[u]$ be the representative in V of any given element $u \in U$, namely either u or $-u$.

It is easy to see that $\mathrm{Exp}(\overline{M})$ contains U, because multiplication by any element of U induces a permutation of the constituents $M_{d,p}^{(j)} = (G; x, y^j, c)$ of $\overline{M}$. Indeed if $\ell \in U$ then multiplication by ℓ takes $M_{d,p}^{(j)} = (G; x, y^j, c)$ to $M_{d,p}^{([j\ell])} = (G; x, y^{[j\ell]}, c)$ for all $j \in V$, as the fact that $\mathrm{Exp}(M_{d,p}) = \{1, -1\}$ implies that each of $(G; x, y^{j\ell}, c)$ and $(G; x, y^{-j\ell}, c)$ is isomorphic to $M_{d,p}^{([j\ell])}$ because of the automorphism of G taking (x, y, c) to (x, y^{-1}, c).

Conversely, let k be any exponent of $\overline{M}$, and let ψ be an automorphism of $\overline{G}$ that fixes $\overline{x}$ and takes $\overline{y} = (y^j)_{j \in V}$ to $\overline{y}^k = (y^{[jk]})_{j \in V}$. Then the restriction of ψ to $\overline{K}$ is an automorphism of $\overline{K}$ taking $\overline{y}$ to $\overline{y}^{[k]}$, and $\overline{x}\,\overline{y}\,\overline{x}$ to $\overline{x}\,\overline{y}^{[k]}\,\overline{x}$. Next, as $\overline{K} = \prod_{j \in V} K_j$ is a direct product of $|V|$ copies of the simple group $K \cong \mathrm{PSL}(2, p)$, we may use [5, Theorem 3.1] again to conclude that $\mathrm{Aut}(\overline{K})$ is isomorphic to the wreath product $\mathrm{Aut}(K) \wr \mathrm{Sym}(n)$ where $n = |V| = |U|/2$. Hence we can multiply $\psi\restriction_{\overline{K}}$ by a suitable element of $\mathrm{Aut}(\overline{K})$ to obtain an automorphism τ of $\overline{K}$ that preserves K_1 (the first copy of K in the expansion $\overline{K} = \prod_{j \in V} K_j$), and then τ induces an automorphism of K_1 $(\cong K)$ taking (x, y) to $(x, xy^{\ell k}x)$ for some $\ell \in U$. But now since $\mathrm{Exp}(M_{d,p}) = \{1, -1\}$, we conclude that $\ell k = \pm 1$ in U, and hence that $k = \pm \ell^{-1} \in U$. Thus $\mathrm{Exp}(\overline{M}) = U$, completing the proof. □

5 Further observations

The method used here can be summed up as follows: To prove existence of orientably-regular or regular maps with given valency and with given exponent group U, we first produce corresponding maps with trivial exponent group (in the orientable case) or 'almost-trivial' exponent group $\{\pm 1\}$ (in the non-orientable case), and then form rotational powers of these maps using the elements of the given group U, and finally, take the parallel product of all these maps. The required properties happen to be controllable if the automorphism groups of the constituent maps are either simple or 'nearly simple'.

Here we note that further infinite families of orientably-regular and non-orientable regular maps with given valency and given exponent group may be obtained from the maps arising from Theorems 3 and 7, with the help of a method known as the 'Macbeath trick' to construct regular covers, as shown by the following theorem (in which the *characteristic* of a map is the Euler characteristic of its carrier surface).

Theorem 8 *Let M be an orientably-regular or a non-orientable regular map with characteristic $\chi < 0$. Then for any prime p not dividing $|\mathrm{Aut}(M)|$ there is (respectively) an orientably-regular or a non-orientable regular map $M^{[p]}$ covering M, with covering group a non-trivial elementary abelian p-group, such that $M^{[p]}$ has the same exponent group as M.*

Proof. For orientably-regular maps, the proof is almost verbatim identical to that of Proposition 2 of [9], despite the arguments there being in a context where the exponent group is trivial. Accordingly, we give details only for the non-orientable case.

So let M be a finite non-orientable regular map with characteristic $\chi = 2 - h < 0$ and hyperbolic type (d, ℓ), and with automorphism group $A = \mathrm{Aut}(M)$ having a presentation of the form $A = \langle\, x, y, c \mid x^2 = y^d = (xy)^\ell = c^2 = (xc)^2 = (cy)^2 = \cdots = 1 \,\rangle$, as in (3). Then by (full) regularity of M, the number of faces of M is $f = |A|/(2\ell)$.

Now consider the epimorphism $\varphi\colon \Delta \to A$ from the infinite full triangle group $\Delta = \Delta(2, d, \infty) = \langle\, X, Y, C \mid X^2 = Y^d = C^2 = (XC)^2 = (CY)^2 = 1 \rangle$ to A that takes (X, Y, C) to (x, y, c), and let K be its kernel. Then the pre-image $\varphi^{-1}(y) = \langle K, Y \rangle$ is infinite.

To explain the topological counterpart of φ, let M° be the 'punctured' map obtained from M by removing a point from the interior of each of the f faces of M. Then the epimorphism φ induces a smooth covering of the punctured map M° via a 'universal' map U of type (d, ∞) on the hyperbolic plane, with full automorphism group $\mathrm{Aut}(U) = \Delta$.

In this situation, the kernel K is known to be a non-Euclidean crystallographic (NEC) group, uniquely determined by its Macbeath signature $(h, -, [\infty^{(f)}], \{\ \})$. As an aside, one may check that the hyperbolic areas $\mu(\Delta)$ and $\mu(K)$ of fundamental regions of Δ and K satisfy $\mu(K) = |A|\mu(\Delta)$, which is the Riemann-Hurwitz formula for NEC groups [8]. More precisely, K is generated by f parabolic generators x_i $(1 \le i \le f)$, each corresponding to a puncture in M°, and by further h generators a_j $(1 \le j \le h)$, where h is the non-orientable genus of the carrier surface of M, and these $f + h$ generators satisfy a single defining relation of the form $x_1x_2\dots x_f\,a_1^2\dots a_h^2 = 1$. Eliminating any one of the generators x_i shows that K is (isomorphic to) a free group of rank $f+h-1$, and hence its abelianisation K/K' is isomorphic to the free abelian group $\mathbb{Z}^{f+h-1}$ of rank $f+h-1$.

Next, let p be any (odd) prime that does not divide $|A| = |\mathrm{Aut}(M)|$, and let $K^{[p]}$ be the subgroup of K generated by all the pth powers of elements in K, and let $L = K'K^{[p]}$. Then clearly L is a characteristic subgroup of K and hence a normal subgroup of Δ, with K/L isomorphic to $(C_p)^{f+h-1}$, and then since $\Delta/K \cong (\Delta/L)/(K/L)$, the quotient $A^{[p]} = \Delta/L$ is an extension of $K/L \cong (C_p)^{f+h-1}$ by $\Delta/K \cong A$. Moreover, by choice of p the order of $K/L \cong A$ is coprime to the order of $K/L \cong (C_p)^{f+h-1}$, and hence by the Schur-Zassenhaus theorem we conclude that $A^{[p]} = \Delta/L$ is isomorphic to a semi-direct product $(C_p)^{f+h-1} \rtimes A$.

The group $A^{[p]}$ is the automorphism group of a fully regular map $M^{[p]}$ of valency d, with face length either ℓ or ℓp (and we need not be any more specific about that). The subgroup $K/L \cong (C_p)^{f+h-1}$ is a normal Sylow p-subgroup of $A^{[p]}$ and hence is characteristic in $A^{[p]}$. Topologically, the natural epimorphism $\psi:\ A^{[p]} \cong \Delta/L \to \Delta/K \cong A$ with kernel K/L makes $M^{[p]}$ a (p^{f+h-1})-fold cover of M, which is possibly branched at faces, but as p is odd the carrier surface of $M^{[p]}$ must be non-orientable. Applying the Riemann-Hurwitz formula to the covering ψ one sees that $M^{[p]}$ has characteristic $p^{f+h-1}\chi = p^{f+h-1}(2-h)$, irrespective of possible branch points at face centres.

It remains for us to show that the d-valent non-orientable regular maps M and $M^{[p]}$ have the same exponent group, or equivalently, that for every $j \in U_d$, the group $\mathrm{Aut}(M) \cong \Delta/K$ admits a j-rotational automorphism if and only if $\mathrm{Aut}(M^{[p]}) \cong \Delta/L$ does.

The 'if' part of this is easy: because K/L is characteristic in Δ/L, every j-rotational automorphism of Δ/L preserves K/L and hence induces a j-rotational automorphism $(\Delta/L)/(K/L) \cong \Delta/K$.

Conversely, if Δ/K admits a j-rotational automorphism, then this lifts to the unique j-rotational automorphism α of the infinite full triangle group $\Delta = \Delta(2,d,\infty)$, and then α preserves K and hence also preserves its characteristic subgroup L, so α induces a j-rotational automorphism of Δ/L. This completes the proof. □

In terms of further ingredients for Theorem 7, it is also of interest to ask about *non-orientable* regular maps of given valency, with trivial exponent group and with alternating or symmetric automorphism group. These would be non-orientable analogues of the input maps for Theorem 3 discussed at the end of Section 3.

We conclude by elaborating on the fact mentioned in the Introduction about the impossibility of extending our findings to orientably-regular or regular maps of *all* hyperbolic types (d,ℓ) and arbitrary subgroups $U \le U_d$ as exponent groups (containing -1 in the non-orientable case). For example, for face length $\ell = 3$ it was shown in [24] that for valency $d \equiv \pm 1 \bmod 6$, an orientably-regular map of type $(d,3)$ cannot have more than $\varphi(d)/2$ exponents, where φ is the Euler totient function.

An even more striking situation occurs by observations made in [24]: if $p \equiv -1 \bmod 4$

is a Sophie Germain prime (meaning that $2p+1$ is also prime), then no orientably-regular map of type $(2p+1, 3)$ has an exponent other than ± 1. It is not known whether there are infinitely many such primes, but it is easy to check that among primes smaller than 1000 there are 19 Sophie Germain primes p such that $p \equiv -1 \bmod 4$, and then 19 corresponding primes $q = 2p+1 < 2000$ with the property that no orientably-regular map M of type $(q, 3)$ has exponents distinct from ± 1. (Hence for any such q, every orientably-regular map M of type $(q, 3)$ has $\varphi(q)/2$ or $\varphi(q)$ pairwise non-isomorphic rotational powers $M^{(j)}$.)

Investigation of the pairs (d, ℓ) admitting any given subgroup of units modulo d as the exponent group remains an interesting direction of future study.

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References

[1] D.S. Archdeacon, M.D.E. Conder and J. Širáň, Trinity symmetry and kaleidoscopic regular maps, *Trans. Amer. Math. Soc.* 366 (2014) 8, 4491–4512.

[2] D.S. Archdeacon, P. Gvozdjak and J. Širáň, Constructing and forbidding automorphisms in lifted maps, *Math. Slovaca* 47 (1997), 113–129.

[3] K. Asciak and J. Širáň, Orientably regular and regular maps of given hyperbolic type with no non-trivial exponents, submitted.

[4] V. Bachratá and M. Bachratý, Orientably-regular maps of given hyperbolic type with no non-trivial exponents, submitted.

[5] J.N.S. Bidwell, Automorphisms of direct products of finite groups II, *Arch. Math.* 91 (2008), 111–121.

[6] R.P. Bryant and D. Singerman, Foundations of the theory of maps on surfaces with boundary, *Quart. J. Math. Oxford Ser. (2)* 36 (1985) 141, 17–41.

[7] E. Bujalance, M.D.E. Conder and A.F. Costa, Pseudo-real Riemann surfaces and chiral regular maps, *Trans. Amer. Math. Soc.* 362 (2010), 3365–3376.

[8] E. Bujalance, J.J. Etayo, J.M. Gamboa and G. Gromadzki, *Automorphism groups of compact bordered Klein surfaces. A combinatorial approach.* Lecture Notes in Mathematics 1439, Springer, 1990.

[9] M.D.E. Conder, V. Hucíko'vá, R. Nedela and J. Širáň, Chiral maps of given hyperbolic type, *Bull. London Math. Soc.* 48 (2016), 38–52.

[10] M.D.E. Conder, P. Potočnik and J. Širáň, Regular hypermaps over projective linear groups, *J. Australian Math. Soc.* 85 (2008), 155–175.

[11] M.D.E. Conder and J. Širáň, Orientably regular maps with given exponent group, *Bull. London Math. Soc.* 48 (2016), 1013–1017.

[12] H.S.M. Coxeter and W.0 J. Moser, Generators and Relations for Discrete Groups, 4th Ed., Springer-Verlag, Berlin, 1984.

[13] J. Fraser, O. Jeans and J. Širáň, Regular self-dual and self-Petrie-dual maps of arbitrary valency, *Ars Math. Contemp.* 16 (2019), 403–410.

[14] G.A. Jones, Regular dessins with a given automorphism group, in: *Riemann and Klein surfaces, automorphisms, symmetries and moduli spaces*, Contemp. Math. vol. 629, Amer. Math. Soc., Providence, RI, 2014, pp 245–260.

[15] G.A. Jones, Chiral covers of hypermaps, *Ars Math. Contemp.* 8 (2015), 425–431.

[16] G.A. Jones, Combinatorial categories and permutation groups, *Ars Math. Contemp.* 10 (2016), 237–254.

[17] G.A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. Lond. Math. Soc.* 37 (1978), 273–307.

[18] A.M. Macbeath, Generators of the fractional linear groups, *Proc. Sympos. Pure Math.* vol. XII, Amer. Math. Soc., (1969), 14–32.

[19] R. Nedela and M. Škoviera, Exponents of orientable maps, *Proc. Lond. Math. Soc.* 75 (1997), 1–31.

[20] D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* 210 (1974), 17–32.

[21] J. Širáň, Non-orientable regular maps of a given type over linear fractional groups, *Graphs Combin.* 26 (2010), 597–602.

[22] J. Širáň, How symmetric can maps on surfaces be? in: *Surveys in Combinatorics*, London Math. Soc. Lect. Note Ser. vol. 409, Cambridge University Press, Cambridge, 2013, 161–238.

[23] J. Širáň, Ľ. Staneková and M. Olejár, Reflexible regular maps with no non-trivial exponents from residual finiteness, *Glasgow Math. J.* 53 (2011), 437–441.

[24] J. Širáň and Y. Wang, Maps with highest level of symmetry that are even more symmetric than other such maps: regular maps with largest exponent groups, in *Combinatorics and graphs*, Contemp. Math. vol. 531, Amer. Math. Soc., Providence, RI, 2010, pp. 95–102.

[25] S.E. Wilson, Operators over regular maps, *Pacific J. Math.* 81 (1979) 2, 559–568.

[26] S.E. Wilson, Parallel products in groups and maps, *J. Algebra* 167 (1994), 539–546.