A computer-assisted proof of dynamo growth in the stretch-fold-shear map

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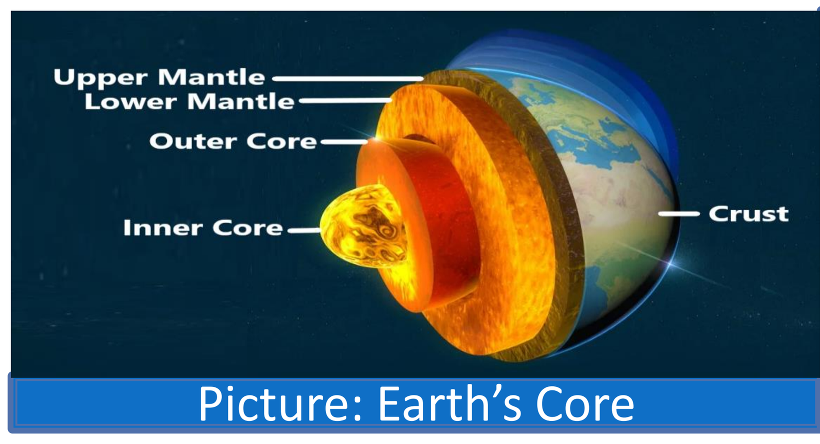
Introduction

- Our research work is based on a functional linear operator called the Stretch-Fold-Shear (SFS) operator S [1] which arises from a model of dynamo growth.
- A dynamo is the source of the magnetic field of a celestial body such as the Earth. Earth's magnetic field is generated by a self-exciting dynamo or geo-dynamo.
- Existing magnetic field in the inner core + fluid iron in the outer core = electrical current
- Electrical current + Earth's movement = new magnetic field.



Picture: Earth's Core

Source: scienceabc.com/nature

- The existence of an eigenvalue of the SFS operator of magnitude greater than one ensures the dynamo growth.
- Aim of this research is to prove such existence, using a computer-assisted proof, which confirms a conjecture of Andrew Gilbert [1].
- We are conducting mathematical study of eigenvalues-eigenfunctions of the SFS operator.

The SFS Operator

- $F = \{c(x)$ complex-valued function of a real variable $x \in [-1,1]\}$
- $S: F \to F$
- $Sc(x) = e^{\frac{i\alpha(x-1)}{2}}\, c\left(\frac{x-1}{2}\right) - e^{\frac{i\alpha(1-x)}{2}}\, c\left(\frac{1-x}{2}\right)$
- $\alpha \geq 0$ (real parameter) is the shear parameter.

When α=0

- We have derived a generating function to generate the eigenvalues and eigenfunctions for α=0:

$$G(x,t) = \sum_{n=0}^{\infty} c_n(x)\frac{t^n}{n!} = \frac{2t\left[e^{(x-1)t} - e^{(1-x)t}\right]}{e^{2t} - e^{-2t}}.$$

- This generating function gives all the eigenfunctions $c_n(x)$ as the coefficients of $\frac{t^n}{n!}$ with eigenvalue $\lambda_n = \frac{1}{2^{n-1}}$.

k	Degree (n)	λ_n	$c_n(x)$
0	1	1	$x-1$
1	3	2^{-2}	$x^3 - 3x^2 - x + 3$
2	5	2^{-4}	$x^5 - 5x^4 - \frac{10x^3}{3} + 30x^2 + \frac{7x}{3} - 25$
k	$2k+1$	2^{-2k}	Polynomial of degree $2k+1$

Table: The eigenvalue-eigenfunction pairs for $\alpha = 0$

When α >0

Using Computer-assisted Proof

- For α >0 , it may not possible to determine the eigenvalues and eigenfunctions of S exactly.
- This generating function method and perturbation method were tried out to obtain the eigenvalues and eigenfunctions. These have not been successful.
- Finally a computer-assisted proof is used to handle the case α >0.

Mathematical Tools

- Interval arithmetic
- Function ball
- Julia Programming Language

Methodology

- At first the eigenvalue-eigenfunction pairs of the SFS operator have been computed for $\alpha = 0$.
- Using these values, numerical approximations of eigenvalue-eigenfunction pairs have been calculated for dynamically increasing α.
- Finally a computer-assisted proof is used to find rigorous bounds around a closed disc function ball of each approximation by implementing interval arithmetic in Julia.

Result

Theorem

Let $\alpha \in [0,5]$. Then there exists an eigenvalue-eigenfunction pair $(\lambda, c(x))$ of the operator S satisfying the following:

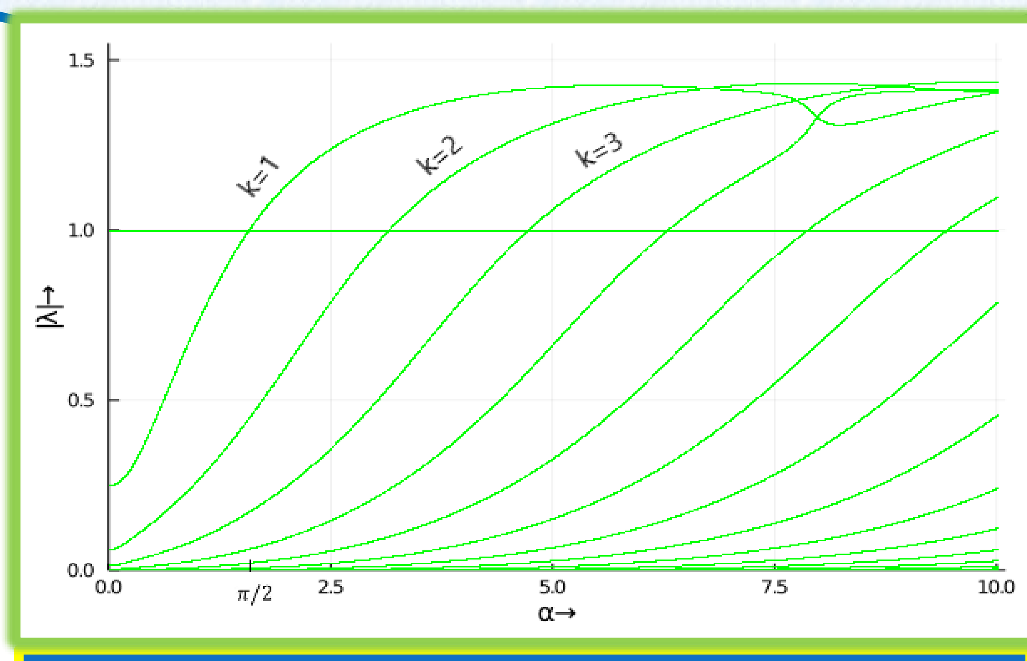
1. For $\alpha \in [0, \pi/2)$, $|\lambda| < 1$.
2. For $\alpha \in (\pi/2, 5]$, $|\lambda| > 1$.
3. For $\alpha = \pi/2, \lambda = -i$.

*It is not claimed that for all λ, (1) and (2) are true.

Graphical Investigation

The graph (reproducing Andrew Gilbert's graph) shows that S has an eigenvalue of modulus greater than 1 for all α satisfying $\alpha > \pi/2$.

$k = 1, 2, 3, \ldots$ represent eigenvalues (modulus) corresponding to degree-$(2k+1)$ eigenfunctions.



Graph: Modulus of Eigenvalues of S

Conclusion

- To guarantee dynamo growth, we need to prove the existence of an eigenvalue of modulus > 1.
- We used a computer-assisted proof to find rigorous bounds on the leading eigenvalue (k=1) for $\alpha \in [0,5]$, showing that S has an eigenvalue of modulus greater than 1 for all α satisfying $\pi/2 < \alpha \leq 5$ (above theorem), thereby proving dynamo growth.
- The methods can be readily adapted to other $k = 2, 3$ ... etc, and for larger values of α.
- It is notable that, although the eigenvalue-eigenfunction has been approximated numerically, the computer-assisted results found in our work are the first rigorous results to be obtained.

Literature Cited

[1] A. D. Gilbert, "Advected fields in maps - I. Magnetic flux growth in the stretch-fold-shear map," Physica D, Elsevier, vol. 166, pp. 167–196, 2002.

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