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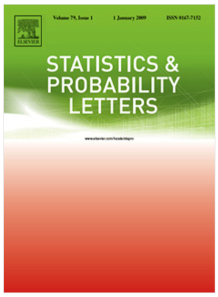

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Duals of Multiplicative Relationships Involving Beta and Gamma Random Variables

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ABSTRACT

By interpreting four known multiplicative relationships involving independent beta and gamma random variables as scale mixtures, their 'dual' mixture relationships are obtained, making links to a variety of other distributional relationships. One consequence of these dual relationships is the provision of a number of first-order Markov processes with beta and gamma marginals.

Keywords: first-order Markov process; inverse Gaussian distribution; Legendre's duplication formula; scale mixture.

1. Introduction

Let G_α, $B_{\alpha,\beta}$ and $B'_{\alpha,\beta}$ denote random variables (r.v.s) following the unit-scale gamma distribution with shape parameter α, Gamma(α), the usual Beta(α, β) distribution (of the first kind) on the unit interval, and the beta-prime distribution or beta distribution of the second kind, Beta$'(\alpha, \beta)$, respectively. Here, $\alpha, \beta > 0$. This article is based on four known multiplicative relationships involving independent r.v.s of these types. These relationships are:

$$B_{\alpha,\beta}\, G_{\alpha+\beta} \stackrel{d}{=} G_\alpha, \tag{1}$$

$$B_{\alpha,\beta}\, B_{\alpha+\beta,\gamma} \stackrel{d}{=} B_{\alpha,\beta+\gamma}, \tag{2}$$

$$B_{\alpha,\gamma}\, B'_{\alpha+\gamma,\beta} \stackrel{d}{=} B'_{\alpha,\beta} \tag{3}$$

and

$$G_\alpha\, G_{\alpha+(1/2)} \stackrel{d}{=} \tfrac{1}{4} G^2_{2\alpha}. \tag{4}$$

Here, the r.v.s on the left-hand sides of the relationships are independent, '$\stackrel{d}{=}$' denotes 'has the same distribution as' and $\gamma > 0$.

These relationships have a number of interesting, and perhaps under-appreciated, links with a variety of other distributional relationships, and it is these that this article will explore. The key to these links is to write each of the relationships as that of a scale mixture and thence to construct – and interpret – a related mixture that I shall call its 'dual'. In general, suppose that r.v.s X, Z and Y are related through

$$XZ \stackrel{d}{=} Y. \tag{5}$$

In mixture terms, (5) can be written:

$$\text{if } Y|X = x \stackrel{d}{=} xZ \text{ and } X \sim F, \text{ then } Y \sim G. \tag{6}$$

Here, '$\sim$' denotes 'is distributed as' and F and G are the marginal distributions of X and Y, respectively. By fundamental properties of the bivariate distribution of X and Y, there is also the dual mixture relationship of the form:

$$\text{if } X|Y = y \sim F_{X|y} \text{ and } Y \sim G, \text{ then } X \sim F \tag{7}$$

where $F_{X|y}$ is the conditional distribution of $X|Y = y$. In fact, each multiplicative relationship has two (scale) duals, the second arising from swapping the roles of X and Z. Note that (6) and (7) hold whether or not X and Z are independent.

It is the dual relationships of (1), (2), (3) and (4) that are the foci of Sections 2, 3, 4 and 5, respectively. In addition to the interesting duals themselves, the bivariate distributions of X and Y are identified and, from them, first-order Markov processes with given marginals are obtained. The latter include two linear gamma processes in Section 2.1, two linear beta processes in Section 3.2, and two linear beta-prime processes in Section 4, each of practical applicability; one of each of the first two pairs of processes is already known. Some concluding remarks setting results in a wider context are given in Section 6.

2. Dual Links of Relationship (1)

Relationship (1) is a rather remarkable, if well known and easy to prove, result, remarkable because although $B_{\alpha,\beta} \stackrel{d}{=} G_\alpha/(G_\alpha+G_\beta)$, where G_α and G_β are independent, and $G_\alpha + G_\beta \stackrel{d}{=} G_{\alpha+\beta}$, the denominator of $B_{\alpha,\beta}$ is independent of the similarly distributed gamma multiplier and would not normally be expected to 'cancel out'. Three special cases of (1) seem noteworthy yet perhaps not widely appreciated; these are described in the Appendix.

2.1. Linking (1) with Convolution

Following (6), (1) can be written:

$$\text{if } Y|X = x \stackrel{d}{=} xB_{\alpha,\beta} \text{ and } X \sim \text{Gamma}(\alpha+\beta), \text{ then } Y \sim \text{Gamma}(\alpha). \tag{8}$$

But now $X|Y = y$ has density proportional to the product of those of $Y|X = x$ and X, namely,

$$\frac{(x-y)^{\beta-1}}{x^{\alpha+\beta-1}} \times x^{\alpha+\beta-1}e^{-x} = (x-y)^{\beta-1}e^{-x}, \qquad 0 < y < x,$$

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which is the distribution of $y+G_\beta$. We have therefore shown that, for (1), (7) becomes:

$$\text{if } X|Y = y \overset{d}{=} y + G_\beta \text{ and } Y \sim \text{Gamma}(\alpha), \text{ then } X \sim \text{Gamma}(\alpha + \beta). \tag{9}$$

But this is nothing other than the well known convolution property for unit-scale gamma distributions mentioned above:

$$G_{\alpha+\beta} \overset{d}{=} G_\alpha + G_\beta \tag{10}$$

where G_α and G_β are independent. One might therefore say that the 'beta thinning' operation of (8) 'reverses' the act of convolution in (9).

Combination of (8) and (9) in one order and then in the other leads readily to two attractive first-order linear gamma Markov processes. Suppose that $X_i \sim \text{Gamma}(\alpha)$. First, by (9), $X_i + G_\beta \sim \text{Gamma}(\alpha + \beta)$ so that, by (8), we have:

$$X_{i+1} = B_{\alpha,\beta}\,(X_i + G_\beta) \sim \text{Gamma}(\alpha), \tag{11}$$

where $B_{\alpha,\beta}$ and G_α are independent of X_i and of each other. This simple structure affords autocorrelation function ϕ^k where $0 < \phi \equiv \alpha/(\alpha+\beta) < 1$. To see this, observe that

$$\text{Cov}(X_{i+1}, X_i) = \mathbb{E}(B_{\alpha,\beta})\text{Var}(X_i) = \phi\text{Var}(X_i)$$

and similarly that

$$\begin{aligned} \text{Cov}(X_{i+k}, X_i) &= \mathbb{E}(B_{\alpha,\beta})\text{Cov}(X_{i+k-1}, X_i) = \phi\,\text{Cov}(X_{i+k-1}, X_i) \\ &= \cdots = \phi^{k-1}\text{Cov}(X_{i+1}, X_i), \qquad k = 2, \ldots . \end{aligned}$$

Alternatively, reparametrize from (α, β) to $(\phi\alpha, (1-\phi)\alpha)$. Then, by (8), $X_i B_{\phi\alpha,(1-\phi)\alpha} \sim \text{Gamma}(\phi\alpha)$ so that, by (9):

$$X_{i+1} = B_{\phi\alpha,(1-\phi)\alpha}\,X_i + G_{(1-\phi)\alpha} \sim \text{Gamma}(\alpha) \tag{12}$$

also. Here, $B_{\phi\alpha,(1-\phi)\alpha}$ and $G_{(1-\phi)\alpha}$ are independent of X_i and of each other. A similar argument as above shows that the autocorrelation structure of (12) is the same as that of (11). This gamma process is also constructed in Section 2.2 of Wolpert (2016).

The bivariate gamma distribution underlying (8) and (9), and hence these observations, is a scaled version of that of McKay (1934).

2.2. Linking (1) with the Definition of Beta

There is, however, a second dual to (1), corresponding to choosing $G_{\alpha+\beta}$ rather than $B_{\alpha,\beta}$ as the scaling variable. We then have:

$$\text{if } Y|X = x \overset{d}{=} xG_{\alpha+\beta} \text{ and } X \sim \text{Beta}(\alpha, \beta), \text{ then } Y \sim \text{Gamma}(\alpha). \tag{13}$$

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In this case, the density of $X|Y = y$ is proportional to

$$\frac{1}{x^{\alpha+\beta}} e^{-y/x} \times x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{x^2}\left(\frac{1}{x}-1\right)^{\beta-1} e^{-y/x}, \qquad 0 < x < 1.$$

Now, this is the distribution of $y/(y + X)$ where $X \sim \text{Gamma}(\beta)$ and hence this second dual relationship is:

$$\text{if } X|Y = y \stackrel{d}{=} y/(y + G_\beta) \text{ and } Y \sim \text{Gamma}(\alpha), \text{ then } X \sim \text{Beta}(\alpha, \beta). \tag{14}$$

But (14) is nothing other than the defining property of the beta distribution (also mentioned at the start of this section), namely,

$$B_{\alpha,\beta} \stackrel{d}{=} \frac{G_\alpha}{G_\alpha + G_\beta} \tag{15}$$

where G_α and G_β are independent. As McKay's bivariate gamma distribution associated with (8) and (9) is the (scaled) distribution of $G_\alpha, G_\alpha + G_\beta$, so the bivariate distribution corresponding to (13) and (14) is that of $G_\alpha, G_\alpha/(G_\alpha+G_\beta)$, a distribution with gamma and beta marginals, for which see Nadarajah (2009).

Combination of (14) and (13) in one order leads to another first order gamma process, in brief,

$$X_{i+1} = \frac{G_{\alpha+\beta}X_i}{G_\beta + X_i} \sim \text{Gamma}(\alpha)$$

when $X_i \sim \text{Gamma}(\alpha)$, and in the other order yields a first order beta process

$$X_{i+1} = \frac{G_{\alpha+\beta}X_i}{G_{\alpha+\beta}X_i + G_\beta} \sim \text{Beta}(\alpha, \beta)$$

when $X_i \sim \text{Beta}(\alpha, \beta)$. Here, G_β and $G_{\alpha+\beta}$ are independent, and independent of X_i. Unlike the gamma processes in Section 2.1 and two of the beta processes to follow in Section 3.2, these constructions are non-linear in X_i and consequently their autocorrelation properties are more difficult to discern. It is unclear whether or when they might be preferred to linear processes.

3. Dual Links of Relationship (2)

Multiplicative relationship (2) is famously due to Rao (1949).

3.1. Linking (2) with Jones & Balakrishnan (2021)

First, with $B_{\alpha+\beta,\gamma}$ in the role of (scale) mixing variable, (2) is equivalent to:

$$\text{if } Y|X = x \stackrel{d}{=} xB_{\alpha,\beta} \text{ and } X \sim \text{Beta}(\alpha + \beta, \gamma), \text{ then } Y \sim \text{Beta}(\alpha, \beta + \gamma). \tag{16}$$

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The conditional density of $X|Y=y$ is proportional to

$$\frac{(x-y)^{\beta-1}}{x^{\alpha+\beta-1}} \times x^{\alpha+\beta-1}(1-x)^{\gamma-1} = (x-y)^{\beta-1}(1-x)^{\gamma-1}, \qquad 0<y<x<1.$$

This is the distribution of $y+(1-y)B_{\beta,\gamma}$ and so it is also the case that

$$\text{if } X|Y=y \overset{d}{=} y+(1-y)B_{\beta,\gamma} \text{ and } Y\sim \text{Beta}(\alpha,\beta+\gamma), \text{ then } X\sim \text{Beta}(\alpha+\beta,\gamma) \quad (17)$$

or

$$B_{\alpha,\beta+\gamma}+(1-B_{\alpha,\beta+\gamma})B_{\beta,\gamma} \overset{d}{=} B_{\alpha+\beta,\gamma} \quad (18)$$

where $B_{\alpha,\beta+\gamma}$ and $B_{\beta,\gamma}$ are independent. After a small amount of manipulation, this can be seen to be equivalent to (3) of Jones & Balakrishnan (2021).

Second, with $B_{\alpha,\beta}$ in the role of (scale) mixing variable, (2) also corresponds to:

$$\text{if } Y|X=x \overset{d}{=} xB_{\alpha+\beta,\gamma} \text{ and } X\sim \text{Beta}(\alpha,\beta), \text{ then } Y\sim \text{Beta}(\alpha,\beta+\gamma). \quad (19)$$

In this case, the density of $X|Y=y$ is proportional to

$$\frac{(x-y)^{\gamma-1}}{x^{\alpha+\beta+\gamma-1}} \times x^{\alpha-1}(1-x)^{\beta-1} = \frac{(x-y)^{\gamma-1}(1-x)^{\beta-1}}{x^{\beta+\gamma}}, \qquad 0<y<x<1.$$

It can be checked that this distribution is as given in the dual mixture relationship below:

$$\text{if } X|Y=y \overset{d}{=} y/\{1-(1-y)B_{\gamma,\beta}\} \text{ and } Y\sim \text{Beta}(\alpha,\beta+\gamma), \text{ then } X\sim \text{Beta}(\alpha,\beta). \quad (20)$$

That is,

$$\frac{B_{\alpha,\beta+\gamma}}{1-(1-B_{\alpha,\beta+\gamma})B_{\gamma,\beta}} \overset{d}{=} B_{\alpha,\beta} \quad (21)$$

where $B_{\alpha,\beta+\gamma}$ and $B_{\gamma,\beta}$ are independent. After more manipulation involving taking one minus r.v.s three times, this can be shown to be equivalent to (2) of Jones & Balakrishnan (2021).

It is interesting to see that the main result of Jones & Balakrishnan (2021), comprising their (2) and (3), arises from (2) here – which is (1) of Jones & Balakrishnan (2021) – in this new way; Jones & Balakrishnan came up with their (2) and (3) using direct proof. See Jones & Balakrishnan (2021) for numerous special cases of, and results related to, relationships (18) and (21).

The bivariate distribution corresponding to (16) and (17) is the natural extension to real-valued parameters of the joint distribution of two uniform order statistics (Wilks, 1962, Papadatos, 1995). It is also the second bivariate beta distribution considered by Nadarajah & Kotz (2005). The distribution associated with (19) and (20) is the first bivariate beta distribution considered by Nadarajah & Kotz (2005).

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3.2. Beta Processes Arising From (2)

There are no fewer than four beta processes arising from (2), two associated with (16) and (17) in either order, two with (19) and (20) likewise. The first two are linear, and therefore probably to be preferred, the other two non-linear.

First, replacing β by $\phi\beta$ and γ by $(1-\phi)\beta$, $0 < \phi < 1$, (16) and (17) give the following process with a Beta(α, β) marginal distribution:

$$X_{i+1} = B_{\alpha,\phi\beta} \left\{ X_i + B_{\phi\beta,(1-\phi)\beta}(1 - X_i) \right\} \tag{22}$$

where $B_{\alpha,\phi\beta}$, $B_{\phi\beta,(1-\phi)\beta}$ and $X_i \sim$ Beta(α, β) are independent. This, again after some manipulation, turns out to be precisely the PBAR first-order autoregressive model with beta marginals of McKenzie (1985). The autocorrelation function associated with (22) is readily seen to be the kth power of $\alpha(1-\phi)/(\alpha+\phi\beta)$, which takes all values in $(0, 1)$ (McKenzie, 1985).

To reverse the roles of (16) and (17), replace α by $\phi\alpha$, β by $(1-\phi)\alpha$ and γ by β; again, $0 < \phi < 1$. Then, if $X_i \sim$ Beta(α, β) independently of the independent pair $B_{\phi\alpha,(1-\phi)\alpha}$ and $B_{(1-\phi)\alpha,\beta}$,

$$X_{i+1} = B_{\phi\alpha,(1-\phi)\alpha} \left(1 - B_{(1-\phi)\alpha,\beta}\right) X_i + B_{(1-\phi)\alpha,\beta} \tag{23}$$

also follows the Beta(α, β) distribution. This is a new, alternative, first-order linear autoregressive beta process very like PBAR and with just the same range of autocorrelations, readily seen to be $0 < [\phi\beta/\{(1-\phi)\alpha+\beta\}]^k < 1$.

As signposted above, there are also two non-linear beta processes associated with (19) and (20). Briefly, if $X_i \sim$ Beta(α, β) and the distinct subscripted B r.v.s below are independent, then also Beta(α, β) distributed are

$$X_{i+1} = \frac{B_{\alpha+\beta,\gamma} X_i}{1 - (1 - B_{\alpha+\beta,\gamma} X_i) B_{\gamma,\beta}}$$

and

$$X_{i+1} = \frac{B_{\alpha+\phi\beta,(1-\phi)\beta} X_i}{1 - (1 - X_i) B_{(1-\phi)\beta,\phi\beta}}$$

(with $\gamma > 0$ and $0 < \phi < 1$). Again, autocorrelation properties associated with these non-linear processes are difficult to obtain.

4. Dual Links of Relationship (3)

Relationship (3) involving the beta-prime distribution is a simple consequence of (1) and can first be found in Jambunathan (1954). Note that an $F_{p,q}$ r.v. is simply related to a beta-prime r.v. via $F_{p,q} \stackrel{d}{=} qB'_{p/2,q/2}/p$ from which the following relationships concerning the beta-prime distribution could readily be extended to the F distribution.

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One way of writing (3) in mixture terms is as:

$$\text{if } Y|X = x \stackrel{d}{=} x \times B_{\alpha,\gamma} \text{ and } X \sim \text{Beta}'(\alpha+\gamma,\beta), \text{ then } Y \sim \text{Beta}'(\alpha,\beta). \quad (24)$$

It follows that $X|Y = y$ has density proportional to

$$\frac{(x-y)^{\gamma-1}}{x^{\alpha+\gamma-1}} \times \frac{x^{\alpha+\gamma-1}}{(1+x)^{\alpha+\beta+\gamma}} = \frac{(x-y)^{\gamma-1}}{(1+x)^{\alpha+\beta+\gamma}}, \qquad 0 < y < x.$$

This distribution can be seen to be that of $y + (1+y)\, B'_{\gamma,\alpha+\beta}$; the resulting dual mixture relationship is therefore that

$$\text{if } X|Y = y \stackrel{d}{=} y+(1+y)\, B'_{\gamma,\alpha+\beta} \text{ and } Y \sim \text{Beta}'(\alpha,\beta), \text{ then } X \sim \text{Beta}'(\alpha+\gamma,\beta). \quad (25)$$

This can be interpreted as an analogue for the beta-prime distribution of (18) for the beta distribution:

$$B'_{\alpha,\beta} + (1 + B'_{\alpha,\beta})\, B'_{\gamma,\alpha+\beta} \stackrel{d}{=} B'_{\alpha+\gamma,\beta} \quad (26)$$

where $B'_{\alpha,\beta}$ and $B'_{\gamma,\alpha+\beta}$ are independent.

There follow two novel first-order linear processes with $\text{Beta}'(\alpha,\beta)$ marginals. Let X_i follow such a distribution. Then X_{i+1} also follows this distribution in each of the following cases (where, again, $\gamma > 0$, $0 < \phi < 1$ and the distinct subscripted B and B' r.v.s are independent of each other and of X_i). First,

$$X_{i+1} = B_{\alpha,\gamma}\{X_i + (1 + X_i)B'_{\gamma,\alpha+\beta}\}; \quad (27)$$

second,

$$X_{i+1} = B_{\phi\alpha,(1-\phi)\alpha}X_i + (1 + B_{\phi\alpha,(1-\phi)\alpha}X_i)B'_{(1-\phi)\alpha,\phi\alpha+\beta}. \quad (28)$$

The variance of $B'_{\alpha,\beta}$ exists if and only if $\beta > 2$. Under this constraint, the respective autocorrelation functions are the k'th powers of $\alpha(\alpha+\beta+\gamma-1)/\{(\alpha+\gamma)(\alpha+\beta-1)\}$ and $\phi(\alpha+\beta-1)/(\phi\alpha+\beta-1)$. Both autocorrelations vary over $(0,1)$.

The bivariate distribution underlying relationships (24) and (25) is that of $\{Y, Z + Y\}$ where $\{Y, Z\}$ follow an appropriately scaled version of the usual bivariate F distribution.

It was recognised in Jones & Balakrishnan (2021) that (3) is equivalent to (21) via simple transformation. The dual to the alternative mixture representation of (3) has, therefore, already been obtained in Section 3.1: it is (19).

5. Dual Links of Relationship (4)

For $\alpha > 0$, Legendre's duplication formula for the gamma function, $\Gamma(\cdot)$, states that

$$\Gamma(2\alpha) = \frac{1}{\sqrt{\pi}} 2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha + \frac{1}{2}\right) \quad (29)$$

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(e.g. NIST, 2022, 5.5.5). Associated with this duplication formula is (4), an apparently less well known corresponding relationship between gamma r.v.s.

Duals of (4) involve the inverse Gaussian distribution which in its most usual parameterization has density function

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \qquad x > 0,$$

with $\mu, \lambda > 0$. The mean of this distribution is μ and its variance is μ^3/λ. Interest here will concentrate on the one-parameter inverse Gaussian distribution, InvGauss(μ), with variance equal to its mean, that is with density

$$f(x;\mu) = \frac{\mu e^{\mu}}{\sqrt{2\pi x^3}} \exp\left\{-\frac{1}{2}\left(x + \frac{\mu^2}{x}\right)\right\}, \qquad x > 0.$$

With $G_{\alpha+(1/2)}$ as (scale) mixing variable, (4) can be written:

$$\text{if } Y|X = x \stackrel{d}{=} xG_{\alpha+(1/2)} \text{ and } X \sim \text{Gamma}(\alpha), \text{ then } Y \stackrel{d}{=} \tfrac{1}{4}\, G_{2\alpha}^2 \tag{30}$$

which has as its dual:

$$\text{if } \tfrac{1}{2}X|Y = y \sim \text{InvGauss}(y) \text{ and } Y \stackrel{d}{=} G_{2\alpha}, \text{ then } X \sim \text{Gamma}(\alpha). \tag{31}$$

This dual arises from the p.d.f. of $X|Y = y$ being proportional to

$$x^{-\alpha-(1/2)} e^{-y/x} x^{\alpha-1} e^{-x} = x^{-3/2} \exp\left\{-\left(x + \frac{y}{x}\right)\right\}, \qquad x > 0, \tag{32}$$

which is the distribution of $X/2$ when $X \sim \text{InvGauss}(2\sqrt{y})$.

Similarly, with G_α as (scale) mixing variable, (4) can be written:

$$\text{if } Y|X = x \stackrel{d}{=} x \times G_\alpha \text{ and } X \sim \text{Gamma}(\alpha + (1/2)), \text{ then } Y \stackrel{d}{=} \tfrac{1}{4}\, G_{2\alpha}^2. \tag{33}$$

The p.d.f. of $X|Y = y$ is proportional to

$$x^{-\alpha} e^{-y/x} x^{\alpha-(1/2)} e^{-x} = x^{-1/2} \exp\left\{-\left(x + \frac{y}{x}\right)\right\}, \qquad x > 0.$$

This is both the length-biased version of distribution (32) and the distribution of $2y/X$ when $X \sim \text{InvGauss}(2\sqrt{y})$. The dual of (33) can therefore be written as:

$$\text{if } \frac{2Y}{X}|Y = y \sim \text{InvGauss}(y) \text{ and } Y \stackrel{d}{=} G_{2\alpha}, \text{ then } X \sim \text{Gamma}(\alpha + (1/2)). \tag{34}$$

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Relationships (31) and (34) appear to give novel connections between gamma and inverse Gaussian distributions. The corresponding bivariate distributions and first-order gamma processes are not pursued here because they depend on only a single parameter involved in both marginal and dependence properties.

By the way, the duplication formula (29) can also be written in terms of the beta function, $B(\cdot,\cdot)$, as

$$2^{2\alpha-1}B(\alpha,\alpha) = B(\alpha,\tfrac{1}{2}),$$

$\alpha > 0$. This corresponds to the following, known, relationship between beta distributions:

$$C \equiv\; 4B(1-B) \sim \text{Beta}(\alpha,\tfrac{1}{2}) \quad \text{when} \quad B \sim \text{Beta}(\alpha,\alpha). \tag{35}$$

The left-hand side here is a non-linear transformation of a beta r.v. rather than a product of independent r.v.s as considered earlier in this article.

6. The Wider Context

The dual relationships in this article are special cases of the fundamental properties of any bivariate distribution, namely

$$\text{if } Y|X=x \stackrel{d}{=} G_{Y|x} \text{ and } X \sim F, \text{ then } Y \sim G, \tag{36}$$

where $G_{Y|x}$ is the conditional distribution of $Y|X=x$, together with (7). In particular, (36) has been specialized to (6) for the special cases of (1)–(4), and (6) and (36) have been given their mixture interpretations. Any mixture relationship therefore has a dual relationship in the sense of (36) and (7).

An example based on a bivariate distribution quite different from those associated with (5) – indeed, a degenerate bivariate distribution – is the dual of (35). This can be seen to be that

$$\text{if } B|C=c = \begin{cases} \frac{1}{2}(1-\sqrt{1-c}) & \text{with probability } \frac{1}{2}, \\ \frac{1}{2}(1+\sqrt{1-c}) & \text{with probability } \frac{1}{2}, \end{cases}$$
$$\text{and } C \sim \text{Beta}(\alpha,\tfrac{1}{2}), \text{ then } B \sim \text{Beta}(\alpha,\alpha). \tag{37}$$

This appears to be a novel result. As well as occasional novelty, mixture relationships thought of as duals can sometimes yield insights, as previously in this article, and have the advantage of being straightforward to understand and to manipulate, yielding a role in pedagogy. As just one pedagogical example from above, by reversing the development of Section 2.2, it can be seen how the very definition of the beta distribution in terms of gamma distributions, (15), interpreted in mixture terms, leads to the less obvious (1) as a dual, the latter being a result otherwise "pulled out of thin air".

The main practical consequence of the considerations treated in this article is the various first-order Markov processes proffered throughout. Quite generally, the reader is

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reminded that such a process can be generated by starting with a random variate X_1 from the desired marginal distribution, F say, and then alternating between generating Y_i from the conditional distribution of $Y|X_{i-1} = x_{i-1}$ and the next member of the process, X_i, from the conditional distribution of $X|Y_i = y_i$, $i = 2, ..., n$. Here, $\{X, Y\}$ follow any bivariate distribution with X-marginal F. Such a construction adds to the armoury of the modeller seeking to construct first-order Markov processes with a given marginal distribution.

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Appendix: Special cases of (1)

First, the case $\beta = 1$ gives proof that the Gamma(α) distribution is α-decreasing in the sense that the density of $(G_\alpha)^\alpha$ is decreasing (Olshen & Savage, 1970, Dharmadhikari & Joag-Dev, 1988). Such a property is characterised by a relationship of the form $G_\alpha \stackrel{d}{=} U^{1/\alpha} S_\alpha$ where $U \sim \text{Uniform}(0,1)$ and S_α are independent, S_α being some scaling r.v., in this case turning out to have distribution $G_{\alpha+1}$.

Second, and somewhat similarly, the case $\alpha = 1, \beta = n$ where n is a positive integer shows that the exponential (Gamma(1)) distribution is n-monotone (for all n) in the sense that its n'th density derivative is positive (negative) when n is even (odd) (Williamson, 1956, McNeil & Nešlehová, 2009). This property corresponds to $E \stackrel{d}{=} (1 - U^{1/n})T_n$ where $U \sim \text{Uniform}(0,1)$ and T_n are independent, T_n turning out to follow the G_{n+1} distribution.

Third, when $\alpha = \beta = 1/2$, (1) reads $2B_{1/2,1/2}\, E \sim \chi_1^2$, the chi-squared distribution on 1 degree of freedom. The right-hand distribution arises because $2G_{\nu/2} \sim \chi^2_\nu$ for any $\nu > 0$; in particular, $Z^2 \sim \chi_1^2$ where $Z \sim \text{N}(0,1)$, the standard normal distribution. On the left, $E \stackrel{d}{=} G_1$ is a unit exponential r.v., which can be written $E = -\log U_1$ where $U_1 \sim \text{Uniform}(0,1)$; $B_{1/2,1/2}$ follows the arc-sine distribution, that of $\sin^2(2\pi U_2)$ where $U_2 \sim \text{Uniform}(0,1)$, independently of U_1. Taking square roots, we therefore have a non-standard demonstration of the fact that $\sqrt{-2\log(U_1)}\sin(2\pi U_2) \sim \text{N}(0,1)$, a version of the Box-Muller transform for generating a standard normal random variable (Box & Muller, 1958).