Compressive behaviour of cellular structures with aperiodic order

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A B S T R A C T

Cellular structures are commonplace in engineering applications, such as aerospace and medical engineering, because material-air composites offer significant mechanical benefits, for example due to improved weight-to-strength ratio. Typically, cellular structures are based on patterns of periodically repeating unit cells, such as squares or hexagons, but the periodic nature and the available symmetries of the patterns give rise to anisotropic performance. This is where patterns with aperiodic order are a viable alternative. Patterns created with rotational symmetry, yet no translational repetition do not possess the orders of symmetry from which mechanical anisotropy originates and therefore have the potential to mitigate this issue. In this study, additive manufacturing was used to create 2.5D, 45% dense, honeycomb cuboids based on the Penrose P3 aperiodic tiling. These were then tested under compression loading. Honeycomb cuboids based on periodic patterns were also manufactured using identical processes for the purpose of comparison. The outcome shows a significant improvement in isotropy and notably different progression of strain localisation for the honeycombs based on Penrose P3 patterns compared to the periodic comparisons during both elastic and plastic deformation.

1. Introduction

In modern materials engineering, a key driving force for innovation is to minimise the weight of components while maintaining sufficient performance. Excess weight results in inefficient designs which cost more to manufacture and operate. One obvious way to reduce weight is to reduce the overall density of a component, and this can be achieved by structural topological optimisation (optimising where material is used within a design) [1], or by creating a material of fractional density [2]. In this study the latter is the focus.

Honeycombs [2,3], lattice structures [4,5] and foams [6] are the most common methods for producing materials with less than 100% material density. Honeycombs are 2.5-dimensional, based on extruded two-dimensional repeating patterns; one of the simplest of these, made famous by honeybees [7], is the beehive honeycomb, which is based on repeated hexagonal cells. Lattice structures are 3-dimensional, based on repeating unit cells, such as cubes, and foams are randomly generated, for example from bubbles introduced into molten material. Honeycombs, lattices and foams are all examples of patterned cellular structures, composed of regions of material and regions of air, essentially material/air composites. They have found applications in a range of engineering disciplines including aerospace [8] and medical engineering [9], because of improved performance in terms of energy absorption, heat transfer and weight-to-strength ratio [2]. However, cellular structures based on repeated tessellation of a unit cell, i.e. honeycombs and lattices, show varying degrees of mechanical anisotropy that coincide with the global symmetries of the underlying pattern [10]. Minimising these anisotropic effects is a common aim for research into cellular structures [3] and while foams [6] and stochastic or randomised patterns [11] alleviate some of the issues of anisotropy their random nature is not well suited to design applications such as aerospace, where the predictability and repeatability of performance is a key requirement.

Aperiodic patterns suggest an alternative area of investigation, enabling the creation of cellular structures without translational symmetry, yet free from the uncertainty of randomness. Aperiodic patterns are those which exhibit order without periodicity, and mathematically they are relatively well understood [12]. They were first physically...
observed in rapidly cooled aluminium-manganese alloys [13], an unexpected and fundamental discovery that was recognised by the award of the 2011 Nobel Prize in Chemistry. As a result of their unusual atomic scale crystal structure, these quasicrystals show remarkable properties. They can be more isotropic than periodic crystals giving nearly uniform mechanical properties in all directions, and they are unusually hard and resemble ceramics rather than metallic materials [14]. Advances in additive manufacturing have made it possible to reproduce the structure of quasicrystals at a larger scale, and this study investigates how such patterns perform mechanically under compression loading, when used as the basis of cellular structures.

The study is concerned with the mechanical properties of honeycombs based on the Penrose P3 tiling [15] as illustrated in Fig. 1. These two-dimensional tilings of the plane are composed of two types of rhombus (thin and thick) of equal side length, with internal angles \( \pi/5 \) and \( 2\pi/5 \), arranged without any translational symmetry, but with localised five-fold rotational and reflective symmetry. The tilings can be generated according to a deflation algorithm, where a rhombus is subdivided according to the deflation rules illustrated in Fig. 1a, with rhombi replaced by tilings of smaller rhombi, scaled by the reciprocal of the golden ratio, \( \phi = \frac{1}{2}(1 + \sqrt{5}) \). Using this method a tiling of any size can be generated.

2. Experimental methods

2.1. Specimen design

Fifteen honeycombs of 36 × 36 × 36mm were additively manufactured out of polylactic acid (PLA) using a fused deposition modelling (FDM) process, for the purpose of compression testing. Three patterns were chosen as the basis of the honeycombs, to allow comparison between cellular structures based on the P3 tiling and structures based on two commonly used periodic tilings. Seven of the honeycombs were based on the P3 tiling, five were based on a hexagon (beehive) tiling and three were based on a square (checkerboard) tiling. The honeycombs were produced from the different patterns under a range of orientations (see Fig. 1b and c, for examples of orientations for the P3 tiling); this was to allow comparison of how the structures perform when compressive loads are applied from different orientations relative to the underlying patterns. The samples printed are summarized in Table 1. The mechanical response of honeycombs can be characterised as either stretch dominated or bending dominated [2]. To ensure comparability between the honeycombs, the Maxwell stability criterion has been calculated for each pattern using

\[ M = b - 2j + 3 \]

where \( M \) is the Maxwell number, \( b \) is the number of structs and \( j \) is the number of joints. For cases where \( M < 0 \), the structures are bending dominated and where \( M \geq 0 \) are stretch dominated. In the case of the structures studied here, they are all bending dominated.

To ensure all the honeycombs were comparable, they were all generated using the same deflation algorithm, with hexagons subdivided into smaller hexagons and squares into smaller squares, scaled by a factor of \( \frac{1}{2} \). The algorithm includes optimisation routines which ensured that all the honeycombs had walls of equal thickness of 0.8 mm, and an overall material density of 45%. For example, generation of the structures based on the P3 tiling started with a large thick rhombus, greater in

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Table 1: Orientations of specimens produced for testing.

Fig. 1. Making and testing a sample based on the P3 tiling: a) deflation rules for P3 tilings; b) pattern generated using repeated deflation and optimisation to achieve goal density of 45%, and patches selected; c) patches exported as vector files; d) top and bottom surface added; e) pattern extruded as a 2.5D honeycomb; f) an additively manufactured PLA honeycomb block; g) a honeycomb under compression testing.
size than the desired final pattern, and the deflation rules in Fig. 1a were applied repeatedly until the material density exceeded 45%. The initial rhombus was then increased in size by a small amount and the process reiterated until the desired density was achieved. Patches of the patterns, of 36 × 36mm and at the required orientations, were extracted, as illustrated in Fig. 1b for P3 patterns at 0° and 45°, and these were converted into vector format for importing into a CAD package, as illustrated in Fig. 1c.

After the pattern was generated, a layer of material was added to the top and bottom surface of the CAD model, to ensure load was transmitted evenly into the structure during compression testing, as illustrated in Fig. 1d. These final patterns were then extruded by 36 mm in the direction normal to the plane of the pattern, resulting in 2.5D honeycomb cuboids, as illustrated in Fig. 1e and exported as STL files.

2.2. 3d printing

STL files generated by the python algorithm were sliced using Cura [16] with a 0.1 mm layer height and additively manufactured using an Anyubic i3 3d printer based on fused deposition modelling (FDM) technology. Samples were fabricated using white generic Polylactic acid (PLA) feed stock. A honeycomb manufactured out of PLA using FDM is illustrated in Fig. 1f.

2.3. Compression testing

Compression tests were performed on an Instron universal tester fitted with 100 mm compression platens. Tests were conducted with a strain rate of 8.3 × 10⁻⁴ s⁻¹ to a strain of at least 25%. For specimens which exhibited large levels of lateral strain the test was stopped before 25% in order to preserve safety. Strain was calculated from the crosshead displacement. Fig. 1g shows a sample during compression testing.

The limit of elastic behaviour was determined using the difference in tangent modulus and chord modulus technique [17]. The Elastic modulus was calculated from data up to the elastic limit using least squares linear fit.

3. Results

The compressive stress-strain response for each of the honeycombs based on the P3 tilings is shown in Fig. 2a. Each curve is labelled according to the underlying pattern and its orientation in degrees, for example ‘P3_0’ is based on a P3 pattern at 0°, and ‘P3_09’ is based on a P3 pattern at 9°. Despite differences in both the elastic and plastic regions of the data, the overall response across the different orientations of the P3 based honeycombs is similar. This is in the form of a relatively large elastic region, a reduction in stress shortly after yield, finally followed by a plastic response which undulates about a nominally constant stress. These results reflect the findings of other authors who reported plastic deformation in various types of honeycomb, which oscillates about an approximately constant stress until densification occurs [3,11,18,19].

Fig. 2b shows a comparison between the P3 based honeycombs and the hexagon (hex) and square (sq) based honeycombs, under different orientations. The plastic response in the stress strain curves for the P3 based honeycombs are noticeably smoother. Undulations in plastic response for honeycombs have been attributed to points in the plastic loading regime where changes in the localisation of strain occur [18], for instance when a further layer of cells starts to collapse, or a section of collapsing cells reaches contact points. Therefore, these results indicate a difference in the way in which strain localises in the P3 based samples compared to other honeycombs.

In Fig. 2, the variations in mechanical properties of the honeycombs based on different underlying patterns are visually noticeable, particularly the elastic modulus (E) and yield strength (σy). To put these in context, compression tests were conducted at a range of orientations up to the rotational symmetry of the underlying pattern (45° for the square based structure, 60° for the hexagonal based structure and 72° for the P3 tiling based honeycomb). This data is presented in Fig. 3.

The greatest anisotropy is observed in the samples produced from a square pattern where the elastic modulus and the yield strength in the 0° orientation is more than double that in the 45° orientation. The hexagonal honeycombs and those based on the P3 tiling showed relatively low anisotropy in comparison, with the lowest observed in the P3 based samples. This suggests that for this specific tiling, anisotropy comes at the expensive of stiffness, since all orientations of P3 tilings produce honeycombs with lower stiffness than all the orientations of the hexagon based structures. It should be noted that simulation efforts on the anisotropy of hexagonal based honeycombs, for instance the work of Schumacher et al. [20], shows hexagons to be close to anisotropic. However, although significantly less anisotropic than the square based honeycombs, they are not observed as being completely isotropic. This could be as a result of the manufacturing technique, but this observation is not the main focus of this study.

The largest elastic modulus was found in the honeycomb based on the 0° orientated square pattern, as was the largest yield stress (Fig. 3). And it should further be noted that this structure also exhibited the greatest undulations in stress-strain response, as seen in Fig. 2b. All orientations of the hexagonal honeycombs were considerably lower in both elastic modulus and yield strength at approximately half the value for the honeycomb based on the 0° orientated square pattern. The elastic modulus and yield strength of the P3 based honeycomb was lower still with a value of approximately one third of the honeycomb based on the 0° orientated square pattern.

Fig. 4 compares the compressive stress strain responses for honeycombs based on the three patterns at specific strain values. The

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Fig. 2. a) Compressive stress vs. strain response for the aperiodic patterns loaded at various angles with respect to the initial pattern orientation. Angles range from 0 to 72 b) Compressive stress vs. strain for P3, square and Hex structures in a range of pattern orientations.
Fig. 3. Spread of elastic modulus (E) and yield strength (σ_y) for aperiodic, hexagonal and square honeycombs with patterns at a range of orientations with respect to the loading direction tested in compression. (Lines of best fit are added to aid with data interpretation and not based on any physical or mathematical model).

Fig. 4. a) showing the spread of compressive stress vs strain response for different pattern orientations for the P3 based honeycombs compared with hexagon and square based honeycombs and b) showing the elastic modulus for aperiodic, hexagon and square based honeycombs with respect to orientation of loading.

Fig. 5. Images of samples deformed compressively to a compressive strain of approximately 20%.
maximum and minimum values of stress measured for each level of strain across all the specimens within a pattern class were calculated and then plotted as a band in Fig. 4a and the same data, normalised for the mean stress at each strain value, is plotted in Fig. 4b. Unsurprisingly, this shows the greatest spread is found with the square based honeycomb both during elastic and plastic deformation. However, these plots also clearly demonstrate the improved anisotropy in P3 based honeycombs compared to those based on hexagonal patterns.

In addition to the differences in bulk response, the local response to compressive stress is also markedly different between honeycombs based on periodic and aperiodic patterns. The images of deformed material in Fig. 5, shows the 0° and 45° for honeycombs based on each of the three pattern types at approximately 20% compressive strain. From these images the localisation of strain can be observed by the closing of the cells. The square based honeycombs show a very clear correlation between the orientation of the bands of strain localisation and the orientation of the underlying square pattern, with bands of high strain being approximately perpendicular to the struts. Bands of strain localisation are also observed in the hexagonal honeycomb, yet with a less marked visual correlation with the orientation. Conversely the P3 based honeycombs show several regions of disconnected strain localisation, focused around the thin rhombi which have their long axis orientated perpendicular to the loading direction.

The images presented in Fig. 6 demonstrate the manner in which the deformation progresses for two P3 based honeycombs at 0° and 45° orientation, compared to the hexagon based honeycomb at 0°, and highlights the significance of the thin rhombus orientation in the P3 based structures. In both the 0° and 45° orientated P3 based

![Images of P3 0°, P3 45° and Hex 0° at progressively larger levels of strain illustrating the initiation and growth of regions of strain localisation. New regions of strain denoted by a different colour marker, while persistent or growing regions maintain the same colour marker as in previous steps.](https://example.com/fig6.png)
honeycombs, distinct new regions of strain localisation are seen to initiate as the applied strain increases, while existing regions are seen to increase in size. All the new initiation sites are associated with the collapse of thin rhombi orientated perpendicular to the loading direction. This behaviour is observed to varying extent across all orientations of P3 based honeycombs, in addition to those presented in Fig. 6.

Contrastingly, the hexagon based honeycomb shows a single, widespread region of strain localisation which gradually increases in severity. This is shown in Fig. 5 and again, across several strain levels, in Fig. 6. Periodic honeycombs tested in other orientations in addition to those presented in Fig. 6 also show the same large regions of strain localisation.

4. Discussion

The results of this first study into the mechanical properties of 45% dense honeycombs based on aperiodic tilings confirms there can be reduced anisotropy in these structures compared to periodic based honeycombs. Some degree of scatter in terms of the mechanical properties may be attributed to the manufacturing technique and the size of the samples. Tool path in particular could have influence of the mechanical properties between the periodic and aperiodic samples as a result of the differing number of continuous paths in the two sample types. Furthermore, the number of cells in the aperiodic samples may have a different impact on the properties of P3 samples compared to periodic samples because of the different arrangements possible in the P3 tilings.

However, the origin of the observed elastic behaviour is believed to stem from the characteristics of the patterns used. Consider the periodic patterns; both the square and hexagon tilings contain a single shape (unit cell), in a single orientation, translated across the plane. These unit cells have inherent elastic mechanical properties and anisotropy. Therefore, the elastic properties of the whole structure are governed by the properties of these unit cells. Indeed, this assumption that the overall elastic mechanical properties can be obtained from a single unit cell is the basis of many honeycomb modelling techniques [2]. The P3 based tilings on the other hand each contain two shapes, in five different orientations, arranged in a multitude of ways. Again, both the shapes have inherent elastic mechanical properties and anisotropy, and as a result the overall properties cannot be derived from a single unit cell because this doesn’t exist. Instead, the properties must be a blend of the properties of the two shapes, in their five orientations, along with the influence from the arrangement of cells and the interactions between the shapes. Considering the structural features in this way, it is logical that squares would have the greatest anisotropy, followed by hexagons and then the P3 tiling.

Plastic deformation also occurs differently in the periodic tilings compared to the P3 based tilings. In the periodic structures, the regions where plasticity initiates appear to be governed by the sample geometry. For instance, the 0° orientated square specimen (Fig. 5, top left), shows a buckling failure across the entire sample cross section. Assuming stress is constant across the sample cross-section, this behaviour is expected, with all struts across the sample having the same stress acting on them. Therefore, as soon as one strut is compromised and is no longer holding load, the rest will buckle together under the increased stress. The other periodic samples show deformation in large bands at angles close to 45° to the loading direction. This suggests that the plasticity in these structures is governed by shear stress rather than buckling, with maximum shear stress occurring at 45° to the load. Because the mechanical properties of each cell in the periodic samples is nominally identical, all the cells within the region of maximum shear will deform to a similar degree, causing the bands of deforming cells seen in Figs. 5 and 6.

The distribution of plastic strain across the P3 based samples does not appear in bands aligned with either of the above geometric features, instead, plastic strain appears to occur at multiple disconnected locations. This suggests that the deformation occurs at the cell with the lowest resistance to plastic deformation. This is seen to be the thin rhombus with least misalignment between the short dimension and the loading axis. This type of behaviour shows potential in obtaining extended uniform deformation and geometric stability to large plastic strains by distributing the deformation across the structure. Further work is essential to assess the full extent of mechanical properties using much larger structures with many more cells per sample, under different loading regimes and indeed at different densities.

Cellular structures in general offer improved performance in terms of energy absorption, heat transfer and weight-to-strength ratio [2]. The potential application of elastically isotropic honeycombs based on P3 tilings are numerous, for weight reduction and potential for use in thermal barrier applications [21]. Isotropic thermal expansion and mechanical resistance to dimensional changes resulting from temperature change are highly desirable, and the low modulus of honeycombs based on the Penrose P3 tilings potentially offers a lower deformation for the same thermal strain. In metallic systems, reduction in stiffness is also a key feature, e.g. for orthopaedic implants, and the low modulus of P3 based honeycombs could also be of great interest. In an energy absorption and damage tolerance sense, the greatest potential of P3 based honeycombs likely lies in the lack of sample induced geometric strain localisation, thus, encouraging plastic strain to distribute in multiple locations without forming large bands of shear induced deformation. Maintaining overall shape (i.e. no necking or shearing) to a large applied strain could result in lower interference strains between honeycomb and other regions of material.

The scope of this study was limited to a single type of aperiodic pattern, the Penrose P3 aperiodic tiling, which is based on two sizes of rhombus arranged in the plane, with a material density of 45%. However, there are many different aperiodic patterns, based on different shaped cells, organised according to different symmetries, in two- and three-dimensional tilings [12]. It cannot be assumed that all cellular structures based on aperiodic patterns will exhibit similar mechanical properties as honeycombs based on the Penrose P3 tiling investigated here, but there is clearly potential for developing honeycombs with mechanical response very different to that exhibited by traditional honeycombs. Therefore, the potential for other aperiodic patterns to give rise to cellular structures with superior properties is significant.

5. Conclusions

In this manuscript the first physical characterisation of the mechanical properties of honeycombs based on extrusion of an aperiodic tiling pattern are presented. From these results the following conclusions can be drawn:

- Honeycombs based on the Penrose P3 tiling, with a 45% material density, performs better than equally dense square and hexagon based honeycombs in terms of reduction of anisotropy in elastic and plastic deformation.
- At the levels of plastic deformation studied, the localisation of strain in P3 samples is less dependent on regions of maximum shear stress and more dependent on the strength of individual features.
- The potential applications for these structures could include any area where isotropic elastic properties or dimensional stability under plastic deformation are desirable.

Data availability statement

The raw and processed data required to reproduce these findings cannot be shared at this time as the data also forms part of an ongoing study.
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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