# GEORGE BIRKHOFF'S FORGOTTEN MANUSCRIPT AND HIS PROGRAMME FOR DYNAMICS

JUNE BARROW-GREEN

**ABSTRACT**

In 1912 George Birkhoff created a sensation with his proof of Poincaré's so-called "last geometric theorem." He followed it with prize-winning papers on "The restricted problem of three bodies" (1915) and "Dynamical systems with two degrees of freedom" (1917). Many of the essential ideas from these papers can be found in his book *Dynamical Systems* (1927). At the end of the 1920s, Birkhoff began to draw up a programme of research on unsolved problems in dynamics, and in 1941 presented his ideas at the 50th anniversary celebration of the University of Chicago. Soon afterwards a summary of his lecture was published. At the time of his death in 1944, he left unfinished a manuscript of a revised and extended version of his lecture. In this paper I describe Birkhoff's work leading up to this manuscript before describing the contents of the manuscript itself.

**MATHEMATICS SUBJECT CLASSIFICATION 2020**

Primary ?; Secondary ?, ?, ?

**KEYWORDS**

?, ?, ?

INTERNATIONAL CONGRESS OF MATHEMATICIANS
2022 JULY 6–14 SAINT PETERSBURG


Published by EMS Press. DOI 10.4171/ICM2022/119
Proc. Int. Cong. Math. 2022, Vol. ?, pp. 2–24

## 1. BIRKHOFF'S WORK IN DYNAMICS

In 1924 the Russian mathematician Nikolai Krylov described George Birkhoff as "the Poincaré of America."[1] It was an apt description. As a student in Chicago, Birkhoff had been introduced to Poincaré's work by the mathematical astronomer Forest Ray Moulton and he had immersed himself in it, especially Poincaré's great treatise on celestial mechanics—the three volume *Les Méthodes Nouvelles de la Mécanique Céleste*—which had appeared in the last decade of the 19th century. In fact, so closely did Birkhoff's name become linked with that of Poincaré that when Birkhoff died Poincaré's name featured often in the obituaries, an extreme example being the short notice written by Jacques Hadamard in which Poincaré's name appears more often than Birkhoff's **[28]**.[2] Although Birkhoff made significant advances in other fields of mathematics, such as the theory of difference equations and the four-color problem, it is his work in dynamics, notably his proof of Poincaré's "last geometric theorem" and his individual ergodic theorem, on which his fame principally rests.

Indeed, Birkhoff maintained an interest in dynamics throughout his career. His *Collected Mathematical Papers* list 32 papers under the heading, the first published in 1912, when he was aged 28, and the last, posthumously, in 1945. The second was his proof of Poincaré's last geometric theorem which he presented to the American Mathematical Society in October 1912 and which appeared in print in January 1913, with a French translation the following year **[8]**. Poincaré had published the theorem in 1912 shortly before his death, having been working on it for two years previously **[39]**. Despite (correctly) believing it to be true, Poincaré had been unable to prove it except for a few special cases.[3] Birkhoff was not the only mathematician to rise to the challenge but no-one was better prepared—his proof came only a few months after Poincaré's death.[4] Remarkably for an American mathematician at the time, Birkhoff had never been to Europe—he had learnt all his mathematics in the United States. As Norbert Wiener later wrote, "Before 1912 it had been considered indispensable for any young American mathematician of promise to complete his training abroad. Birkhoff marks the beginning of the autonomous maturity of American mathematics" **[42, P. 177]**.

Birkhoff gave Poincaré's theorem in the following form:

> *Let us suppose that a continuous one-to-one transformation $T$ takes the ring [annulus] $R$ formed by concentric circles $C_a$ and $C_b$ of radii $a$ and $b$, respec-*

---

**1** On 9 August 1924, Raymond Archibald, who had just met Krylov at the International Congress of Mathematicians in Toronto, wrote to Birkhoff to tell him that Krylov (whom he described as "a magnificent man") wanted especially to meet him. HUG 4213.2, Birkhoff Papers, Harvard University Archives.

**2** Hadamard and Birkhoff were friends for over 30 years, and Hadamard translated some of Birkhoff's work into French. Birkhoff was a popular speaker at the famous Séminaire Hadamard in Paris, and he was one of the mathematicians interviewed by Hadamard for his famous *Psychology of Invention in the Mathematical Field* (1945).

**3** In 1992 Golé and Hall would show that Poincaré had been closer to success than he had realized **[25]**.

**4** Among those who made a determined but unsuccessful attempt was L. E. J. Brouwer **[40, PP. 147–148]**.

*tively ($a > b > 0$), into itself in such a way as to advance the points of $C_a$ in a positive sense, and the points of $C_b$ in the negative sense, and at the same time to preserve areas. Then there are at least two invariant points* **[8, P. 14]**.

Birkhoff's proof of the theorem would soon come to be considered as "one of the most exciting mathematical events of the era and widely acclaimed" **[20, P. IV]**, although at the time, as Oswald Veblen wrote to Birkhoff from Germany in December 1913, the reaction in Göttingen was only that Birkhoff was someone who "probably [had] to be reckoned with"! **[7, P. 42]**.

There is a close connection between Poincaré's theorem and what is known as "the restricted three-body problem." This is a particular case of the three-body problem in which two large bodies, with masses $\mu$ and $1 - \mu$, respectively, rotate about their center of mass in circular orbits under their mutual gravitational attraction, and a third body of neglible mass, which is attracted by the other two bodies but does not influence their motion, moves in the plane defined by the two revolving bodies. The problem is then to ascertain the motion of the third body. The problem has one integral, which was first obtained by Carl Jacobi in 1836 and hence is known as the Jacobian integral or constant. Although the problem may appear contrived, it turns out to be a reasonable approximation to the Sun–Earth–Moon system. It was first explored by Leonhard Euler in connection with his lunar theory of 1772, but it was Poincaré who brought the problem to prominence in his celebrated memoir of 1890 **[37]**, and who later gave it its name.[5] Poincaré knew that if his theorem could be shown to be true, then it would confirm the existence of an infinite number of periodic motions for the problem for all values of the mass parameter $\mu$. Poincaré also believed that the theorem would eventually be instrumental in establishing whether or not the periodic motions are densely distributed amongst all possible motions. As Aurel Wintner later observed, much of the dynamical work of Birkhoff was either directed towards or influenced by the restricted three-body problem **[44, P. 349]**.

In 1925 Birkhoff extended Poincaré's theorem to a nonmetric form by removing the condition that the outer boundaries of the ring and the transformed ring must coincide, and replacing it instead with the alternative condition that the outer boundary and the transformed outer boundary are met only once by a certain radial line **[11]**. He proved that the revised form held for annular regions with arbitrary boundary curves, and, correcting an earlier omission—he had not taken into account that the first invariant point might have index zero which meant that the existence of a second invariant point does not follow automatically—proved that there are always two distinct invariant points. Since the extension does not involve an invariant area integral it is essentially a topological result. Its importance lies in the fact that it can be used to establish the existence of infinitely many periodic motions near a stable periodic motion in a dynamical system with two degrees of freedom, from which the existence of quasiperiodic motions—that is motions which are not periodic

---

**5** Poincaré's work on the three-body problem is discussed in detail in my book **[4]**.

themselves but which are limits of periodic motions—follows.[6] Three years later Birkhoff explored the relationship between the dynamical system and the area-preserving transformation used in the theorem [14]. Having shown that corresponding to such a dynamical problem there exists an area-preserving transformation in which the important properties of the system for motions near periodic motions correspond to properties of the transformation, he showed that a converse form of this correspondence also exists. In other words, given a particular type of area preserving transformation there exists a corresponding dynamical system. In 1931 he generalized the theorem to higher dimensions [22].

Birkhoff published three papers on the restricted three-body problem itself. The first [21], which appeared in 1915 and for which he won the Quirini Stampalia prize of the Royal Venice Institute of Science, provided the first major qualitative attack on the problem since Poincaré. Unlike Poincaré, Birkhoff, in his treatment of the problem, made little concession to analysis, and his investigation was founded almost entirely on topological ideas. By considering the representation from a topological point of view, he was able to illustrate the problem's dependence on the value of the Jacobian constant. He established a transformation of the variables which enabled him to derive a new form of the equations in which the equations are regular, providing the third body is not rejected to infinity. From this he created a geometric representation in which the manifolds of states of motion are represented by the stream-lines of a three-dimensional flow and are without singularity unless the Jacobian constant takes one of five exceptional values. Having excluded these five values, the totality of the states of motion could then be represented by the stream lines of a three-dimensional flow occupying a nonsingular manifold in a four-dimensional space. But, as Poincaré had shown, providing the mass of the one of the two main bodies is sufficiently small, the representation of the problem as a three-dimensional flow can be reduced to a representation which depends on the transformation of a two-dimensional ring into itself [38, pp. 372–381]. Birkhoff showed that Poincaré's transformation could be considered as the product of two involutory transformations, a result he subsequently used to prove the existence of an infinite number of symmetric periodic motions, as well as results concerning their characteristic properties and distribution.

Twenty years elapsed before Birkhoff next published on the problem. In the interim he had worked extensively on general dynamical systems, the crowning result of which was another prize memoir which appeared in 1935 [16], the prize having been awarded by the Pontifical Academy of Sciences. In two later papers on the restricted problem which derived from lectures given at the Scuola Normale Superiore di Pisa, he combined ideas from the prize memoir of 1915 together with some general results from the one of 1935, notably his development of Poincaré's idea of a surface of section (now often called a Poincaré section).[7] In the first of these two later papers [17], he focused on the analytic properties of the surface

6 A modern and slightly modified account of Birkhoff's proof is given by Brown and Neumann [23].

7 Given an $n$-dimensional phase space, a surface of section is an $(n-1)$-dimensional space embedded in the original space and transversal to the flow of the system.

of section and the transformation he had used in 1935, while in the second [18] he used qualitative methods to explore the results from the first in order to obtain further information about the different types of motion and the relationships existing between them.

In 1923 Birkhoff was awarded the Bôcher Memorial Prize of the American Mathematical Society for a paper in which he provided a general treatment of dynamical systems with two degrees of freedom [9]. Such systems comprise the simplest type of nonintegrable dynamical problems, and, as exemplified in the work of Poincaré, they form the natural starting point for qualitative explorations into questions of dynamics. According to Marston Morse, Birkhoff stated that he thought the Bôcher prize paper was as good a piece of research as he would be likely to do [35, p. 380].

Birkhoff began with the equations of motion in standard Lagrangian form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x}\right) = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y}\right) = 0,$$

where the function $L$, which is quadratic in the velocities, involves six arbitrary functions of $x$ and $y$. By making an appropriate transformation of variables, he reduced the equations to a normal form which involved only two arbitrary functions of $x$ and $y$. In the reversible case, that is, when the equations of motion remain unchanged when $t$ is replaced by $-t$, the transformation was already well known. In this case the equations of motion can be interpreted as those of a particle constrained to move on a smooth surface and the orbits of the particle interpreted as geodesics on the surface. But in the irreversible case, as, for example, in restricted three-body problem, Birkhoff's transformation was new and he gave a dynamical interpretation in which the motions can be regarded as the orbits of a particle constrained to move on a smooth surface which rotates about a fixed axis with uniform angular velocity and carries with it a conservative force field. The central part of the paper concerned various methods by which the existence of periodic motions could be established. These include his "minimum method," and his "minimax method," the latter later providing a starting point for the work of Morse on calculus of variations in the large. Birkhoff also considered Poincaré's method of analytic continuation which is applicable to both reversible and irreversible periodic motions. One of the problems with the method was that it was only valid for a small variation in the value of the parameter. The restriction was due to the possibility that the period of the motion under consideration might become infinite. Thus to increase the interval of the variation it is necessary to show that this possibility cannot arise and Birkhoff did precisely that for a wide range of periodic motions.

It was in the Bôcher prize paper that Birkhoff first began to generalize Poincaré's idea of a surface of section and formally develop a theory attached to it. Poincaré had used the idea specifically to reduce the restricted three-body problem to the transformation of a ring to itself, but if the method was to have a general validity it was important to establish under what circumstances surfaces of section exist. Birkhoff was able to show that not only do they exist in a wide variety of cases but also that they can be of varying genus and have different numbers of boundaries.

In his "Surface transformations and their dynamical applications" of 1920 [10], Birkhoff elaborated and extended some of the ideas he had broached at the end of the Bôcher prize paper. By reducing the dynamical problem to a transformation problem and studying certain transformations and their fixed points, which he did at length, he was able to classify certain different types of motion. For example, whether a periodic motion, which is represented by a fixed point, is stable or unstable can be determined by examining the behavior of a point sufficiently close to the fixed point under repeated iterations of the transformation. Later Birkhoff considered the question of stability in more detail [13].

Invited by Gösta Mittag-Leffler in 1926 to contribute to the 50th volume of *Acta Mathematica*—the journal which Mittag-Leffler had edited since its inception in 1882—Birkhoff chose to tackle Poincaré's conjecture concerning the denseness of periodic motions. It was a particularly fitting choice of subject, given Poincaré's early and consistent support of *Acta*.[8] A feature of Birkhoff's paper [12] is his introduction of the billiard ball problem—that is, to determine the motion of a billiard ball on a convex table—which he used to show how Poincaré's last geometric theorem could be applied to dynamical systems with two degrees of freedom.[9] Having considered certain types of periodic motion, he was able to conclude that if a dynamical system admits one stable periodic motion of nonexceptional type—the exception being when the period of the perturbed motion is independent of the constants of integration—then it admits an infinite number of stable periodic motions within its immediate vicinity, and the totality of these stable periodic motions form a dense set. Although this does not resolve Poincaré's conjecture, it does show that it cannot be true unconditionally. He was able to prove the conjecture in the case of a transitive system—that is a system in which "motions can be found passing from nearly one assigned state to nearly any other arbitrarily assigned state" [12, P. 379]—showing that the periodic motions together with those asymptotic to them are densely distributed.

Birkhoff's influential book, *Dynamical Systems*, which derived from the American Mathematical Society Colloquium Lectures he delivered in Chicago in 1920, was published in 1927, with a new edition appearing in 1966. A Russian translation, which also contained translations of several of Birkhoff's papers including [15], was published in 1941 and reprinted in 1999. Although representing "essentially a continuation of Poincaré's profound and extensive work on Celestial Mechanics" [20, P. III], Birkhoff's book opened a new era in the study of dynamics by detaching the subject from its origins in celestial mechanics and making use of topology [3]. It provides a summary of Birkhoff's research in dynamics during the preceding 15 years, with the final three chapters—on the general theory of dynamical systems, the case of two degrees of freedom, and the three-body problem—bringing together the main strands of his work. As Bernard Koopman, one of Birkhoff's former students, remarked, *Dynamical Systems* is better described as a theory than as a book [31, P. 165]. Birkhoff's goal was clear: "The final aim of the theory of the motions of a dynamical system must be directed

8 Poincaré's contributions to *Acta Mathematica* are discussed in my article [5, PP. 148–150].

9 It is indicative of the paper's status that it was selected by Robert MacKay and James Meiss for reproduction in their book of the most significant writings on Hamiltonian dynamics published since the First World War [33].

toward the qualitative determination of all possible types of motions and of the interrelations of these motions." [20, P. 189].

He started with a general class of dynamical systems, that is systems defined by the differential equations,

$$\frac{dx_i}{X_i} = dt \ldots \quad (i = 1, \ldots, n),$$

where the $X_i$ are $n$ real analytic functions, and a state of motion can be represented by a point in a closed $n$-dimensional manifold. A motion can then be represented by a trajectory in the manifold, and its domain is its closed set of limit points. The trajectories composed entirely of limit points are those Birkhoff called "recurrent motions." More generally, recurrent motions are those which trace out with uniform closeness, in any sufficiently large period of their entire history, all their states. Since, by definition, every point on the trajectory of a recurrent motion is a limit point, the motion must approach every point on the trajectory infinitely often and arbitrarily closely. Thus the simplest types of recurrent motions are the stationary motions and the periodic motions. As Birkhoff showed, the idea of recurrent motion is a particularly useful one with regard to the general problem of determining all possible motions in a particular dynamical system. For example, he proved that the set of limit motions of any motion contains at least one recurrent motion; and that any point either generates a recurrent motion or generates a motion which approaches with uniform frequency arbitrarily close to a set of recurrent motions. Furthermore, the concept of recurrent motion can be used to derive definite results about the motion in an arbitrary dynamical system; a significant feature of the theory being that it is valid for systems with any degree of freedom. This is in contrast to Poincaré's theory of periodic motion which is known to be valid only for systems with two degrees of freedom.

The theory developed in Birkhoff's papers and further expounded in *Dynamical Systems* formed the bedrock on which Birkhoff's Chicago lecture and its related manuscript were built, and it is to these we now turn.

## 2. BIRKHOFF'S FORGOTTEN MANUSCRIPT

In September 1941 the University of Chicago celebrated its 50th anniversary. It was a celebration that had been two years in the planning. Honorary degrees were awarded and a symposium was held in conjunction with the American Association for the Advancement of Science. According to an account in the university magazine, the celebration was sufficiently "significant that, in a world at war, it attracted national and even world wide attention" [30, P. 6].

As one of the leading figures in American mathematics and a former student of the university, Birkhoff was a natural choice for an honorary degree and symposium speaker, the citation describing him as the "leading contributor to the fundamentals of dynamics." The only other mathematician amongst the 34 others on the rostrum was Birkhoff's close friend and long-standing colleague Oswald Veblen, also a Chicago protégé.

For the subject of his lecture, Birkhoff chose "Some unsolved problems of theoretical dynamics," a topic well in keeping with the anniversary theme of "New Frontiers in Education and Research." The symposium was well advertised prior to the celebrations and before Birkhoff delivered his lecture he was asked by *Nature* if he could provide the journal with a summary. However, the summary did not appear in *Nature* but in *Science* and it appeared some three months after the lecture had been delivered **[19]**. In fact, the lecture was ready only about a week before it was due to be delivered, as Birkhoff admitted to Eric O'Connor, one of his former doctoral students:

> *During the last few weeks I have been extremely occupied with the address which I have to give next week at Chicago. In it I take a look at Classical Dynamics from the abstract point of view and suggest about a dozen problems, many of them new, which seem to be most directly in the line of further advance. In one or two instances I indicate a partial answer to these. It now looks as though the paper will be in good shape for the 24th September, when I have to deliver it, but it has been a very close squeak!*[10]

The idea of presenting a programme for research in dynamics was not new for Birkhoff. Some 13 years earlier, in 1928, he had given a series of lectures at the University of Berlin on "Some Problems of Dynamics" and the lectures were published in German in a condensed form **[15]**. In these lectures, having emphasized the importance of qualitative dynamical ideas for the exact sciences, he discussed various examples including the billiard ball problem, the motion of a particle on a smooth convex surface and on a smooth closed surface of negative curvature, and the three-body problem. On that occasion, he listed six problems:

- I To construct a dynamical system on a three-dimensional closed phase space, in which the ordinal $r$ of central motion is $> 3$.
- II To prove that in the case of the Hamiltonian problem with two degrees of freedom, with closed phase space and with at least one stable periodic motion, the periodic motions are everywhere dense.
- III To prove that in the case of all Hamiltonian problems with closed phase space the recurrent motions are everywhere dense.
- IV To prove, for a given conservative transformation $T$, the existence of corresponding Hamiltonian systems in particular of geodesic type.
- V If $T$ is any conservative transformation with a fixed point $P$ of stable type, then determine the necessary conditions so that there are infinitely many points $P_n$ existing in the neighborhood of $P$ which are fixed points of $T^m$.
- VI To prove, in the case with two degrees of freedom, the existence of a dynamical system that has a periodic motion of stable type, which is not truly stable.

---

**10** Letter from Birkhoff to O'Connor, 18 September 1941. HUG 4213.2.2, Birkhoff Papers, Harvard University Archives.

Of these, only the first three relate to problems Birkhoff discussed in Chicago. The first was solved in 1946 by A. G. Maier **[34]**. In 1941 the problems were republished in Russian to accompany the Russian edition of *Dynamical Systems*, where they are described as "important, unsolved problems." Further work remains to be done to establish the extent of interest generated by these problems subsequent to both the German and the Russian publications.

Nine years after his lectures in Berlin, Birkhoff returned to the same theme but this time in Paris. In 1937 he gave a lecture at the Institut Henri Poincaré entitled "Quelques problèmes de la Dynamique théorique." Birkhoff referred to this lecture in a footnote of the manuscript where he said that in Paris he had made reference to "one or two of the problems listed in the present paper" but without identifying which ones, and no further information on this lecture has so far come to light.

Birkhoff received the Chicago invitation in November 1940, and in April 1941 he was invited by Otto Schmidt and Anisim Bermant to contribute to the celebratory 50th volume of *Matematicheskii Sbornik*, the prestigious Russian mathematical journal founded in 1866.[11] For some time Russian mathematicians had been closely following Birkhoff's work, especially in dynamics, as is evident from Krylov's remark of 1924 given above. Also in the 1920s, a group in Pavel Aleksandrov's topology seminar in Moscow had specialized in studying Birkhoff's publications;[12] and in 1936 Birkhoff had been invited by A. A. Markov to speak on the ergodic theorem and related topics at an international conference due to take place in Leningrad in 1937, although in the event the conference was canceled.[13] Birkhoff cannot have taken long to decide that an article laying out his programme for dynamics would make a fitting contribution to the journal, knowing that the Chicago meeting would provide him with an excellent opportunity to test out his ideas before committing them to print.

In May 1943, Birkhoff wrote to his Russian colleagues to let them know that he had "written out an extensive article not wholly completed as yet on 'Some Unsolved Problems of Theoretical Dynamics'," mentioning that he had spoken on the subject "in a preliminary way" in Chicago (Figure 1), but that he had decided to delay sending the article to Russia until after the cessation of hostilities.[14] But it was not to be. On 12 November 1944, Birkhoff, aged only 60, died unexpectedly.[15] Thus the manuscript, which runs to some 40 pages, was never submitted. It remains as a hand-annotated typescript, with additional handwritten leaves, among Birkhoff's papers in the Harvard University Archives.[16] In a footnote appended to

---

**11** Letter from Schmidt and Bermant to Birkhoff, 2 April 1941. HUG 4213.2.2, Birkhoff Papers, Harvard University Archives.

**12** Letter from Aleksandrov to Birkhoff, 19 October 1926. HUG 4213.2, Birkhoff Papers, Harvard University Archives.

**13** Letter from Birkhoff to Markov, 26 February 1936; letter from Markov to Birkhoff, 7 May 1936. HUG 4213.2, Birkhoff Papers, Harvard University Archives.

**14** Letter from Birkhoff to Schmidt and Bermant, 18 May 1943. HUG 4213.2.2, Birkhoff Papers, Harvard University Archives.

**15** As described by his Harvard colleague, Edwin B. Wilson, Birkhoff had some time in hand before a lunchtime visit to his son, Garrett, and had taken the occasion to rest but when his wife went to find him he had passed away **[43, P. 578]**.

**16** HUG 4213.52, Birkhoff Papers, Harvard University Archives.

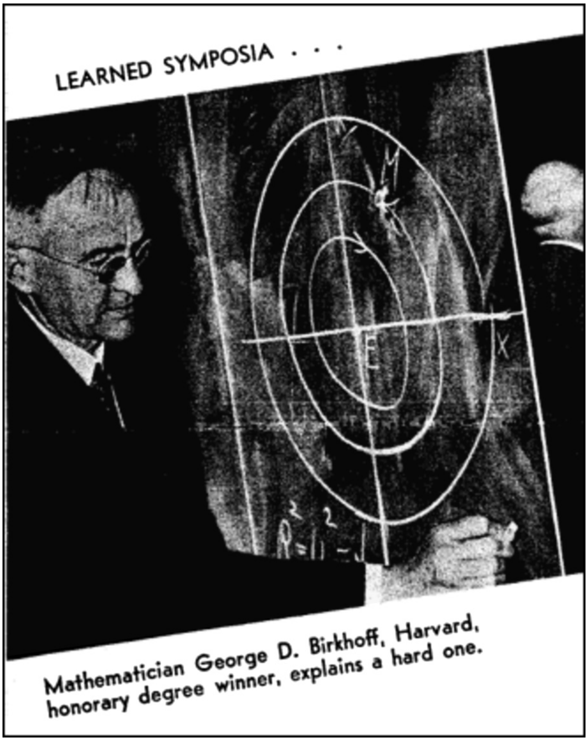


**FIGURE 1** Birkhoff delivering his symposium lecture [30, P. 6]. Courtesy of the University of Chicago Library.

the title page of the manuscript, Birkhoff stated that he had written the paper with the dual purpose of reading it in Chicago and publishing it in the anniversary volume of the Russian journal. Comparing the manuscript with the summary, and taking into account the delay of the publication of the latter, it seems likely that Birkhoff, having lectured from the manuscript then used it to prepare the summary and in the course of the latter's preparation further annotated the manuscript.

The manuscript opens as follows:

> *It scarcely seems too much to say that all the basic problems of point-set theory, topology, and the theory of functions of real variables present themselves naturally in purely dynamical contexts. Some of these dynamical problems are best formulated and solved in terms of an underlying abstract space, as important recent Russian and American work has shown. Others are inherently of more special character.*
> *In the present paper I venture to set forth certain unsolved problems of this type which seem to me worthy of further study. The problems are arranged as much as possible in order of decreasing abstractness. They are formulated in terms of*

*positive conjectures in the belief that this procedure is most likely to stimulate further research. In each case indications of the underlying reasons for these conjectures are made. Some new definitions are given, as for instance that of a "dynamical" flow in an abstract metric space; and some partial results are deduced, as for instance the brief proof in the concluding section that the non-existence of other periodic lunar orbits beside the fundamental variational orbit and the allied retrograde orbit of G. W. Hill's theory of the motion of the Moon about the Earth would imply that all possible lunar orbits with the same constant of Jacobi have the same mean angular advance of perigee per synodic revolution.*

The summary opens rather differently. There Birkhoff gives a pathway for the development of his ideas—he traces them from Poincaré, who first realized that the study of dynamical systems led directly to problems in topology, on through the abstract ideas of E. H. Moore—describing how these ideas fed into his own work.[17] Although Moore deserved a high billing, it was also a diplomatic move on Birkhoff's part to be explicit about the contribution of Moore, his former thesis advisor and first head of the University's mathematics department, who had died in 1932. In the manuscript, the reference to Moore, although laudatory, is considerably abbreviated and consigned to a footnote.

Altogether there are 17 problems, the first ten are formulated in terms of abstract spaces, the 11th is concerned with extensions of results of Karl Sundman on the three-body problem to the motion of a gas. And the last six, which are concerned with $n$-dimensional spaces, are of a topological nature. The paper is also divided up into sections which imposes a useful classification on the problems. The manuscript also includes a "provocative form of conclusion." In the prelecture press release, Birkhoff referred to only ten problems without listing them, so it is possible that he had originally intended to present only ten problems and it was expanding the paper that led to the "close squeak" referred to in the letter to O'Connor mentioned above.[18] In what follows, the section headings and the problems themselves are taken directly from the manuscript. Other material from the manuscript will be given in quotation marks followed by a page number.

The first problem, a conjecture about the interrelationship between continuous and discrete flows in an abstract space $R$, is precursored by three sections on continuous and discrete flows, including an explanation of geodesic flow. As Birkhoff observed in a footnote, the idea of using "this kind of abstract setting for a dynamical problem" did not originate with him but in an article of 1933 by Hassler Whitney, one of Birkhoff's research students **[41]**.

**17** Birkhoff felt especially grateful to Moore for impressing him "with the importance of the abstract domain and for stimulating [him] on the abstract side." Letter from Birkhoff to Raymond Archibald, 5 April 1938. HUG 4213.4.5, Birkhoff Papers, Harvard University Archives.

**18** Another difference between the press release and the summary is that in the former Hassler Whitney and Norbert Wiener are identified as American authors of recent work on abstract dynamics while the latter refers simply to "American mathematicians." University of Chicago Development Campaigns and Anniversaries Records, Box 12, Folder 11.

The idea is that in $R$, which is a compact metric space, a type of reduction of a continuous flow to a discrete one may be effected by showing that there exists a surface of section in $R$ on which the flow can be studied. Each point in $R$ represents a state of motion and as time passes there is a steady flow of $R$ into itself, with each point tracing out a "curve of motion," each curve representing a complete motion of the dynamical system [19, p. 598]. As Birkhoff noted, he had already shown "in the $n$-dimensional case, and very recently Ambrose and Kakutani had established in the abstract case, a kind of converse reduction of a continuous flow to a discrete flow may be made, providing one is content to introduce discontinuous flows" (p. 5). His hope was for a more complete result, and he felt certain that the conjecture would be shown to hold. Since he did not give a citation for the Ambrose and Kakutani paper which had been submitted for publication in 1941 and appeared in 1942 [2], it would appear that he did not return to the manuscript in the years following the lecture apart from reporting on its existence to Schmidt and Bermant.

**Problem 1.** Any (continuous) flow without equilibrium points in a compact metric space $R$ admits of a complete open surface of section $\Sigma$ in $R$, on which the flow defines an extensibly-discrete flow $Q = \phi(P)$ obtained by following any point $P$ of $\phi$ to the first subsequent point $Q$ of $\Sigma$ on the same stream line. Conversely, given any metric space $\Sigma$ on which an extensibly-discrete flow, $Q = \phi(P)$, is defined, then it is possible to imbed $\Sigma$ in an isometric compact metric space $R$ and to define a continuous flow in $R$, so that $\Sigma$ forms a complete open surface of section for this flow, for which $Q = \phi(P)$ in the related extensibly-discrete flow.

Birkhoff next discussed recurrent motions and central motions, central motions being those which recur infinitely often close to any particular state of the motion, or at least have such motions in the infinitesimal vicinity of any state. Having observed that "all the motions of a dynamical system will be central if and only if every molecule of the system overlaps itself as time increases or decreases" (p. 9), he noted that in the classical case there are many examples in which all the motions are central. And it was this that led him to ask the analogous question of recurrent motions, i.e., "what are the circumstances such that *all* the motions of a dynamical system will be recurrent?" (p. 10).

This last question provides the basis for Problems 2 and 3 in which Birkhoff conjectured that all the motions of a continuous flow would be recurrent if and only if the flow may be decomposed into a set of irreducible constituent flows which are "homogeneous," i.e., such that the stream lines are topologically indistinguishable from one another. As an example, he cited the two-body problem—two particles interacting gravitationally with no other forces acting—as being of this type, providing the value of the energy constant is sufficiently small, with the irreducible constituents being the individual periodic motions.

**Problem 2.** All the motions of a regionally transitive (discrete or continuous) flow in a compact metric space $R$ will be recurrent if and only if the flow is "homogeneous," in the sense that an automorphism of the flow exists (with possible modification of the definition of the "time") which takes an arbitrary point $P$ into a second arbitrary point $Q$.

**Problem 3.** All the motions of regionally transitive flows in a compact metric space $R$ will be recurrent if and only if the closest set of motions formed by any motion $M$ and its limit motions is homogeneous in the subspace $R_M$ of these motions [and] every minimal closed component of the flow is homogeneous.

As contrasting illustrations, Birkhoff gave as examples the two-body problem in which all the motions are periodic and the billiard ball problem which, "although 'integrable', has a family of non-recurrent motions, namely those which pass infinitely often through the two foci and are doubly asymptotic (homoclinic in the sense of Poincaré) to the major axis" (p. 13).

**Conservative flows.** In ordinary dynamical systems, conservative flows are those with an invariant volume integral, e.g., the flow of an incompressible liquid. Here Birkhoff considered the extension of conservative flows to the abstract case. By 1941 this had become an active area of research and in the summary he named several Russian mathematicians (Beboutov, Bogolyubov, Krylov, Stepanov) and American mathematicians (Halmos, Oxtoby, Ulam, von Neumann, Wiener, Wintner) who had made important studies of such flows [19, p. 599], although rather curiously he did not mention them in the manuscript.

In the fourth problem, which was preceded by a four-page introduction, Birkhoff conjectured that if the abstract flow is so regular as to be "geodesic" then it will be conservative if all the motions are central, while in the fifth he conjectured that the recurrent motions are necessarily everywhere densely distributed in the abstract space of a geodesic conservative flow. As he pointed out, Poincaré's recurrence theorem makes the latter conjecture a very natural one.[19] However, he did not "expect the periodic motions to be always everywhere dense in the conservative case or even in the case of a dynamical flow." (p. 17).

**Problem 4.** A geodesic flow all of whose motions are central always admits an invariant positive volume integral.

**Problem 5.** The recurrent motions are everywhere dense in any conservative flow, at least if it be geodesic.

**Ergodic theory and conservative flow.** Birkhoff opened this section with a short discussion relating to his own "individual ergodic theorem," observing that the theorem implies "that for conservative systems almost all motions have definite habits of recurrence with regard to any measurable type of behaviour." (p. 18). He also noted the priority of von Neumann's "mean ergodic theorem."[20] In the summary, he avoided any mention of ergodic theory but instead used features of the billiard ball problem, such as the fact that in the long run the ball will be

---

19 Roughly speaking, Poincaré's recurrence theorem says that if the flow is volume-preserving then, at some point in the future, the system will return arbitrarily close to its initial state. For a discussion of the theorem, see [4, pp. 86–87].

20 An account of the relationship between Birkhoff's individual ergodic theorem and von Neumann's mean ergodic theorem, which also explains the confusing chronology of publication, is given by J. D. Zund [45].

on any designated part of the table a definite proportion of the time, in order to demonstrate the significance of conservative flow. Problem 6 proposes a topological characterization of conservative flows based on this fact of recurrence.

**Problem 6.** If a continuous flow in a compact metric space has the property that for any open region of $R$, the exceptional sets for which a positive mean sojourn time $\tau$ (the same in both senses of the time) fails to exist are always of measure 0 with respect to some measure $\int dP$, then there exists necessarily at least one invariant integral $\int \mu dP$.

Birkhoff also noted that "an important paper" by Oxtoby and Ulam containing questions "closely related" to Problem 6 was in the pipeline **[36]**. This paper, which Koopman summarised as a "thorough and detailed study of the group of measure-preserving and measurability preserving automorphisms (homeomorphisms into itself) in polyhedra, their metrical transitivity, equivalence, and the whole bearing of such questions on ergodic theory,"[21] appeared in 1941, the absence of its publication details providing a further indication that Birkhoff did not edit the manuscript in the years after it was written.

**Discontinuous conservative flows.** This section and its accompanying problem are on two handwritten pages. These pages open with the words "Very recently" (p. 19′) and a footnote gives a full citation for a paper published by Ambrose in July 1941 **[1]**, showing that these pages were written either shortly before, or possibly soon after, the lecture was given. The flows now considered are "measure-preserving flows which are 1–1 except over sets of measure 0 and carry measurable sets into measurable sets (in particular, sets of measure 0 into sets of measure 0) and conserve a positive volume integral" (p. 19′), and such flows are, as Birkhoff noted, of particular interest from the point of view of probability, and in this context he mentioned that they had recently been studied by von Neumann, Kakutani, Ambrose, and Halmos. In the summary Wiener and Wintner are exchanged for Ambrose and Kakutani, and there is no mention of probability.

Having established that the underlying space can be taken as a line segment of unit length, and relaxed the condition of continuity on a conservative flow, Birkhoff proposed a characterization of the invariants of the flow based on what he termed "packing coefficients." He explained the latter as follows: "Make the total $\mu$-measure 1 by choosing the total measure as a unit. Select any $n \geq 1$ and consider all ways of decomposing a minimal metrically transitive constituent into a measurable set $\Sigma$ and its first $n-1$ images under [a discrete flow] $T$, say $S^1, \ldots, S^{(n-1)}$ in such a way that these sets are disjoint. To each such decomposition there will be a measure of the complementary point set. We will call the lower bound of these quantities the '$n$th packing fraction' and denoted it by $\sigma_n$, and it is easy to prove that the inequality $\sigma_n \leq 1/n$ always holds." (p. 19″).

**Problem 7.** Any such discontinuous conservative transformation $T$ is completely characterized by its "spectrum," determining the nature of the metrically transitive constituents, and by the packing coefficients $\sigma_1, \sigma_2, \ldots$, for every such constituent. These packing coefficients may be taken arbitrarily except for the fact that $n\sigma_n$ forms a decreasing sequence.

---

21 *Mathematical Reviews* M0005803.

**Dynamical flows.** The next three problems, Problems 8–10, derive from Birkhoff's attempt to define abstractly a "dynamical flow" where he takes as his model Pfaffian systems,[22] rather than Hamiltonian systems of classical dynamics. However, this part of the manuscript is a little tricky to follow as there are six handwritten pages inserted between two typescript pages (pp. 21–22/23). Unlike the other handwritten pages, there is nothing to show exactly where the text from these pages should be inserted. It is evident that wherever they are inserted the typed text would need to be adjusted for the narrative to flow. The first five of these pages provide the justification for a result he had deduced from the properties of his abstract definition of a line integral, a result he needed for his definition of a continuous dynamical flow which involves the existence of a line integral (in an abstract sense which he made precise). The final handwritten page contains only Problem 8 (after which all subsequent problems in the typescript were renumbered). It is notable that in the summary he remarked, that "the crucial part of the characterization of a dynamical flow lay in the suitable definition of a line integral in any abstract 'geodesic space' $R$", and a few lines later observed that "the question of an adequate characterization of a dynamical flow beyond the obvious properties of conservativeness and continuity has been especially baffling" [19, P. 599], which suggests that he returned to this part of the manuscript after he gave the lecture.

**Problem 8.** Any dynamical flow is necessarily conservative with reference to a completely additive measure with positive measure on any open set.

In Problems 9 and 10, Birkhoff returned to the question of the denseness of periodic motions, the question he had addressed in his *Acta Mathematica* paper of 1925. Now he reformulated the question in an abstract setting with the added condition of stability. He defined a periodic motion to be stable (topologically) "if there are other complete motions in its $\varepsilon$-neighborhood", adding that a similar definition can be made for 'stable' recurrent motions', providing neighboring recurrent motions of the same minimal set are excluded from consideration (p. 25). He defined a completely unstable flow as one in which there are no stable periodic or recurrent motions, for example, geodesics on a closed surface of negative curvature, and here he cited the well-known work of Hadamard (1898) and Morse (1921, 1924).

**Problem 9.** In any regionally-transitive nonhomogeneous flow of dynamical type the periodic motions are everywhere dense.

**Problem 10.**

(a) In a regionally transitive dynamical flow not of completely unstable type, the stable periodic motions are everywhere dense, and the set of such motions is dense on itself (i.e., in the infinitesimal neighborhood of any stable periodic motion there exist infinitely many other stable periodic motions). Furthermore,

22 Birkhoff had first defined Pfaffian systems in his Colloquium Lectures of 1920, and later considered them in [13] and in *Dynamical Systems*. They were brought to further prominence by Lucien Feraud in an explanatory paper of 1930 [24].

in the neighborhood of any stable recurrent motion there are similarly infinitely many other stable periodic motions.

(b) In any regionally transitive dynamical flow of completely unstable type, which is furthermore not homogeneous, the unstable periodic motions are everywhere dense.

**On a possible extension of some work of Sundman.** Next Birkhoff asked for the generalization to a gas of certain remarkable results on the three-body problem produced in the early years of the 20th century by the Finnish mathematical astronomer Karl Sundman. Birkhoff was a strong advocate for Sundman's theoretical "solution" to the three-body problem which, due to its practical limitations, had met with a mixed reception.[23] He had even gone as far as to say that "the recent work of Sundman is one of the most remarkable contributions to the problem of three bodies which has ever been made" [13, p. 260]. Of particular relevance here are Sundman's results concerning triple and binary collisions, namely that a triple collision can occur only if all three integrals of angular momentum are simultaneously zero, and that the singularity at binary collision is of removable type. Birkhoff had already shown in *Dynamical Systems* how the essence of Sundman's argument can be used under other laws of force and for a system of more than three bodies to establish that, with similar initial conditions, a simultaneous near approach of the bodies cannot occur, hence the generalization to a gas was a natural next step. The problem was formulated rather vaguely—indeed, in the summary he admitted it was incomplete [19, p. 599]—but he chose to include it because it provided "an interesting illustration of a dynamical flow in a kind of Euclidean space $R$ of infinitely many dimensions, intermediate in type between the flows in abstract metric space and in $n$-dimensional Euclidean space" (p. 27).

**Problem 11.** To determine equations of state and initial conditions of a free bounded gas such that the diameter of the gas can never be less than a specifiable $d > 0$ despite the fact that such configurations are compatible with the known integrals.

Following on, Birkhoff now turned to problems relating to motions in $n$-dimensional space.

**A problem concerning central motions in *n*-dimensional space.** Problem 12 is essentially the first problem he presented in Berlin, and which was solved by Maier in 1946, now extended to the $n$-dimensional case. On this occasion, Birkhoff used the notion of "wandering motions" $W_0$ of a space $R$, a notion he had introduced in *Dynamical Systems*, and which here he described (none too clearly) as "those which can be embedded in a molecule which never overlaps itself as time increases or decreases" (p. 27). When the wandering motions are removed from $R$, there remains a closed subspace, $M_1 = R - W_0$, of lower dimension, which can then be considered from the same point of view. Using this idea, Birkhoff formed "a well-ordered set $M = M_0, M_1, \ldots,$ which is enumerable and terminates in the set of cen-

23 A detailed discussion of Sundman's work on the three-body problem and its reception is given in my article [6].

tral motions $M_c$." Emphasizing the fact that in the known cases the series contain at most $n$ terms, he proposed the following form of the problem:

**Problem 12.** To construct a continuous flow in a closed manifold of $n > 2$ dimensions for which the well-ordered series $M = M_0, M_1, M_2, \ldots$ leading to the central motions $M_c$ contains more than $n$ and if possible an infinite number of terms.

**A problem in the 3-dimensional case.** Problem 13 was suggested by recent work of the Hungarian mathematician Béla Kerékjártó on "regular" or nearly regular transformations of 2-dimensional closed surfaces of arbitrary genus.[24] Here Birkhoff conjectured that 3-dimensional flows that are "regular" have one of only three different forms.

**Problem 13.** For an ordinary 3-dimensional manifold $R_3$, to show that the only regular discrete flows are topologically equivalent to one of the following:

(1) $R_3$, a 3-dimensional torus with a transformation

$$\theta_1 = \theta_1 + \alpha_1, \quad \theta_2 = \theta_2 + \alpha_2, \quad \theta_3 = \theta_3 + \alpha_3,$$

with $\theta_1, \theta_2, \theta_3$ being angular coordinates for the torus.

(2) $R_3$, the product of a surface of sphere and circle, and the transformation of each of these a pure rotation.

(3) $R_3$, a 3-dimensional hypersphere and the transformation of a rigid rotation of this sphere.

**A problem in the 2-dimensional case.** Birkhoff now moved to problems connected with analytic transformations, the ideas emerging from the first of his papers on the restricted three-body problem [21]. In the first of these problems, he conjectured that a particular transformation of the surface of a sphere into itself with two fixed points, which is such that all iterations of the transformation produce no other fixed points, is a pure rotation when considered topologically.

**Problem 14.** A 1–1 direct analytic, conservative transformation $T$ of the surface of a sphere into itself with two and only two fixpoints $P$, $Q$ for $T$ and all its iterations is topologically equivalent to a pure rotation of the sphere about an axis through an angle incommensurable with $2\pi$.

From this he was led to propose the following analogous problem for a plane circular ring:

**Problem 15.** A 1–1 direct analytic conservative transformation of a circular ring into itself, in which two boundaries are invariant and which possess no periodic points is topologically equivalent to a rotation of the ring through an angle $\alpha$ incommensurable with $2\pi$.

---

24 Birkhoff was well acquainted with Kerékjártó. In 1925 he had supported his promotion in Szeged, and in 1928 he had visited Szeged to lecture on Poincaré's last geometric theorem.

**A conjectural supplement to Poincaré's last geometric theorem.** In the final two problems Birkhoff returned again to Poincaré's last geometric theorem. Unsurprisingly, he thought these two problems, since they express conjectures which in a sense represent a complement to the theorem, were the ones likely to generate the most interest.

In Problem 16, Birkhoff conjectured that the theorem would hold in the case when the points on the two concentric circles $C_a$ and $C_b$, are advanced by the same angular distance (in contrast to the original theorem where the points are advanced by distinct distances), that is, their rotation numbers $\alpha$ are equal, provided that some nearby points of the ring become separated widely in an angular sense when the transformation $T$ is repeated sufficiently often, as happens when the rotation numbers are unequal.

**Problem 16.** Let $T$ be a 1–1 continuous discrete conservative transformation $T$ of a circular ring into itself which leaves the two circular boundaries individually invariant, with equal rotation numbers $\alpha$ along these boundaries. Then if nearby pairs of points exist which separate indefinitely in an angular sense under indefinite iteration of $T$, there will necessarily exist periodic points.

This was followed by a conjecture on the partial converse, the case when the common rotation number $\alpha$ is not a rational multiple of $2\pi$.

**Problem 17.** Under the same hypotheses concerning $T$ as in the first part of Problem 16, let us further require only that for no preliminary deformation of the ring in itself can the angular deviation of all pairs of points less than $2\pi$ apart in angular sense be made to remain less than $2\pi + \varepsilon$ under all iterations of $T$ ($\varepsilon$ arbitrary). There will then exist periodic points on the ring. Furthermore, if $\underline{\alpha}$ and $\overline{\alpha}$ denote the lower and upper bounds of the rates of angular advance for such periodic point groups then we have $\underline{\alpha} < \overline{\alpha}$ and $\underline{\alpha} \le \alpha \le \overline{\alpha}$ and, for any relatively prime integers $m$ and $n > 0$ such that

$$\underline{\alpha} < 2\frac{m}{n}\pi < \overline{\alpha},$$

there exist at least two periodic point groups of $n$ points whose angular coordinates increase by $2m\pi$ under the $n$th power of $T$.

Birkhoff then considered the particular case when the given transformation can be expressed as the product of two involutory transformations, showing that in this case the first part of the conjecture is true.

**Application to the restricted problem of three bodies.** In the final part of the paper Birkhoff applied the above result to the planar restricted three-body problem, the version of the problem treated in the 1870s by the American mathematical astronomer George William Hill in his work on the lunar theory—work which had famously inspired Poincaré—giving the differential equations as Hill had done (p. 33):

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = \frac{\partial\Omega}{\partial x}, \quad \frac{d^2y}{dt^2} - 2\frac{dx}{dt} = \frac{\partial\Omega}{\partial y}, \quad \Omega = \frac{3}{2}x^2 + \frac{1}{\sqrt{x^2+y^2}},$$

together with the equation for the Jacobian constant $C$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C.$$

He described the "actual case" of the Sun–Earth–Moon problem in which the Earth is considered to lie at the origin in the $x, y$ plane, the Sun is at infinity in the direction of the positive $x$-axis and the "infinitesimal" Moon is rotating in the $x, y$ plane at unit angular velocity with the Earth and the Sun, where the positive constant $C_0$ is such that the Moon can never escape from the closed region $2\Omega = C$ about the Earth, symmetric in $x = 0$ and $y = 0$. By considering values of $C$ greater than or equal to $C_0$, and applying the result from the previous section, Birkhoff was led to the result he mentioned in the introduction to the manuscript and here described as a "provocative form of conclusion" (p. 35):

> *Assuming that a surface of section of the type stated exists for $C = C_0$, the non-existence for $C = C_0$ of doubly symmetric periodic orbits other than the fundamental variational periodic orbit of Hill and the corresponding retrograde orbit would imply that all possible lunar orbits whatsoever with $C = C_0$ have exactly the same mean rate of angular advance of perigee per synodic revolution.*

He further remarked in a footnote that, although the figures of the computed orbits show the initial assumption is valid, "a rigorous and mathematical proof might be a complicated and tedious matter!" In the summary he mentioned that he had "pointed out how the absence of infinitely many periodic orbits would indicate that a new *qualitative* integral exists, in addition to the usual analytic integral of Jacobi" [19, p. 600], but this remark was omitted from the manuscript.

**Epilogue.** The manuscript ended with a very short epilogue in which Birkhoff expressed the hope that his problems would "accelerate further advances," but admitted that he thought most of them were likely to "present difficulties which may be difficult to surmount" (p. 35).

## 3. CONCLUSION

So far little evidence has come to light of mathematicians responding directly to the summary of Birkhoff's lecture. Stanislaw Ulam, the Polish mathematician and emigré to the United States,[25] wrote to Birkhoff in November of 1941 to say that he had heard various reports of Birkhoff's "extremely interesting talk" and asked him for a copy of the summary.[26] And in the following January, Shizo Kakutani thanked Birkhoff for a reprint of

25 On Birkhoff's suggestion, Ulam had spent time at Harvard during 1936–1939. Later in 1939 Ulam left Poland for good in advance of the German invasion, and in 1940 was appointed to one of Birkhoff's former institutions, the University of Wisconsin-Madison, with the support of Birkhoff.

26 Letter from Ulam to Birkhoff, 25 November 1941. HUG 4213.2.2, Birkhoff Papers, Harvard University Archives.

the summary and said that he was "hoping to solve one of the problems."[27] But Kakutani did not say which one and he does not appear to have published on any of them. Had Birkhoff's manuscript been published, the situation might have been rather different. For it is only in the manuscript that the problems are set out in full and put formally into their mathematical context. The summary, being meant for a general audience, focusses on the historical rather than the mathematical detail. Indeed, the editor of *Science*, the psychologist James McKeen Cattell, told Birkhoff that he was "anxious to obtain papers on mathematical subjects" but that there were difficulties due to "the fact that the English used by mathematicians is not always understood by other scientific men," and so "complicated mathematical equations that only mathematicians can understand" must be avoided.[28] Furthermore, the fact that the summary was published in *Science* and not in a mathematical journal, and that it appeared during the War, meant it was unlikely to have had high visibility amongst mathematicians, particularly in Europe.

The manuscript is not an easy read and although Birkhoff makes several references to material in *Dynamical Systems* for purposes of clarification not everyone found the latter easy reading either. Walter Gottschalk, who became one of the leading exponents of topological dynamics, had this to say:

> *Somewhere I read that G. D. Birkhoff once said that if he thought mathematics exposition to be important, he would be the world's best expositor. Birkhoff was certainly not the world's best expositor and indeed he came close to the extremum in the other direction. I think this attitude had an important delaying effect on the initial development of topological dynamics. In his American Mathematical Society Colloquium volume* **[20]**, *Birkhoff included a discussion of the topological properties of continuous flows determined by a system of first order ordinary differential equations. … The style of writing he adopted was so inadequate in clarity and precision that almost any beginning reader had to be discouraged from continuing. It was not at all clear what the theorems were and the offered proofs were largely suggestive intuitive discussions* **[26]**.

It must also be said that Gottschalk himself was not always an easy read either.[29] Nevertheless, Gottschalk's criticisms did chime with the Russian view. In 2002 George Lorentz

**27** Letter from Kakutani to Birkhoff, 26 January 1942. HUG 4213.2.2, Birkhoff Papers, Harvard University Archives.

**28** Letters from McKeen Cattell to Birkhoff, 5 September 1941 and 8 October 1941. HUG 4213.2, Birkhoff Papers, Harvard University Archives.

**29** Paul Halmos, when reviewing *Topological Dynamics* **[27]**, the book Gottschalk wrote together with his thesis supervisor Gustav Hedlund, remarked: "The chief fault of the book is its style. The presentation is in the brutal Landau manner, definition, theorem, proof, and remark following each other in relentless succession. The omission of unnecessary verbiage is carried to the extent that no motivation is given for the concepts and the theorems, and there is a paucity of illuminating examples." And he ended his review: "Conclusion: the book is a mine of information, but you sure have to dig for it." **[29]**.

recalled that Andrey Markov Jr., one of the editors of the original Russian edition of *Dynamical Systems*, "made sarcastic corrections of some of its errors" [32, p. 196].[30] Although in their preface the Russian editors urge a critical reading of the proofs—they don't think they have found all the mistakes—they do acknowledge the correctness of the theorems. Even Jürgen Moser in the introduction to the 1966 English edition conceded that "to the modern reader the style of [the] book may appear less formal and rigorous than it is now customary" while fully acknowledging its inspirational role [20, p. iii]. Thus had Birkhoff's manuscript been published when he had hoped, it still may have taken some time before mathematicians were able to rise to the challenges laid down by his problems. Whether Birkhoff was right in his assessment of the direction of travel has yet to be ascertained and further research remains to be done in order to see the extent to which his problems have been tackled, if indeed they have, and to what effect.

## ACKNOWLEDGMENTS

I am very grateful to the staff at Harvard University Archives for their help in negotiating the many metres of Birkhoff's archive. I also thank Jeremy Gray and Reinhard Siegmund-Schultze for their valuable comments and suggestions on an earlier version of this paper.

## REFERENCES

[1] W. Ambrose, Representation of ergodic flows. *Ann. of Math. (2)* **42** (1941), 723–739.

[2] W. Ambrose and S. Kakutani, Structure and continuity of regular flows. *Duke Math. J.* **91** (1942), 25–42.

[3] D. Aubin and G. D. Birkhoff, Dynamical systems (1927). In *Landmark writings in Western mathematics, 1640–1940*, edited by I. Grattan-Guinness, pp. 871–881, Elsevier, Amsterdam, 2005.

[4] J. E. Barrow-Green, *Poincaré and the three body problem*. Amer. Math. Soc./Lond. Math. Soc., Providence, 1997.

[5] J. E. Barrow-Green, Gösta Mittag-Leffler and the foundation and administration of Acta Mathematica. In *Mathematics unbound: the evolution of an International Mathematical Research Community, 1800–1945*, edited by K. H. Parshall and A. C. Rice, pp. 265–378, Amer. Math. Soc./Lond. Math. Soc., Providence, RI, 2002.

[6] J. E. Barrow-Green, The dramatic episode of Sundman. *Historia Math.* **37** (2010), 164–203.

[7] J. E. Barrow-Green, An American goes to Europe: Three letters from Oswald Veblen to George Birkhoff in 1913/14. *Math. Intelligencer* **33** (2011), 37–47.

30 The 1966 English edition also includes corrections.

[8] G. D. Birkhoff, Proof of Poincaré's geometric theorem. *Trans. Amer. Math. Soc.* **14** (1913), 14–22; Démonstration du dernier théorème de géométrie de Poincaré, *Bull. Soc. Math. France* **42** (1914), 1–12.

[9] G. D. Birkhoff, Dynamical systems with two degrees of freedom. *Trans. Amer. Math. Soc.* **5** (1917), 199–300.

[10] G. D. Birkhoff, Surface transformations and their dynamical applications. *Acta Math.* **43** (1920), 1–119.

[11] G. D. Birkhoff, An extension of Poincaré's last geometric theorem. *Acta Math.* **47** (1925), 297–311.

[12] G. D. Birkhoff, On the periodic motions of dynamical systems. *Acta Math.* **50** (1927), 359–379.

[13] G. D. Birkhoff, Stability and the equations of dynamics. *Amer. J. Math.* **49** (1927), 1–38.

[14] G. D. Birkhoff, A remark on the dynamical rôle of Poincaré's last geometric theorem. *Acta Litt. Sci. Sect. Sci. Math., Szeged* **4** (1928), 6–11.

[15] G. D. Birkhoff, Einige Probleme der Dynamik. *Jahresber. Dtsch. Math.-Ver.* **38** (1929), 1–16.

[16] G. D. Birkhoff, Nouvelles recherches sur les systemes dynamiques. *Mem. Pontif. Acad. Sci. Novi Lyncaei* **1** (1935), 85–216.

[17] G. D. Birkhoff, Sur le problème restreint des trois corps. *Ann. Sc. Norm. Super. Pisa* **4** (1935), 267–306.

[18] G. D. Birkhoff, Sur le problème restreint des trois corps. *Ann. Sc. Norm. Super. Pisa* **5** (1936), 1–42.

[19] G. D. Birkhoff, Some unsolved problems of theoretical dynamics. *Science* **94** (1941), 598–600.

[20] G. D. Birkhoff, *Dynamical systems*. American Mathematical Society, Providence, 1966.

[21] G. D. Birkhoff, The restricted problem of three bodies. *Rend. Circ. Mat. Palermo* **39** (1915), 265–334.

[22] G. D. Birkhoff, Une généralisation à $n$ dimensions du dernier théorème de géométrie de Poincaré. *C. R. Acad. Sci.* **192** (1931), 196–198.

[23] M. Brown and W. D. Neumann, Proof of the Poincaré–Birkhoff fixed point theorem. *Michigan Math. J.* **24** (1977), 21–31.

[24] L. Feraud, On Birkhoff's Pfaffian systems. *Trans. Amer. Math. Soc.* **32** (1930), 817–831.

[25] C. Golé and G. R. Hall, Poincaré's proof of Poincaré's last geometric theorem. Twist mappings and their applications. *IMA Math. Appl.* **44** (1992), 135–151.

[26] W. Gottschalk, The early history of general topological dynamics. In *Gottschalk's Gestalts* #12, Infinite Vistas Press, 2000.

[27] W. Gottschalk and G. Hedlund, *Topological dynamics*. American Mathematical Society, Providence, 1955.

[28] J. Hadamard, Notice nécrologique sur George David Birkhoff. *C. R. Acad. Sci.* **220** (1945), 719–721.

[29] P. Halmos, Topological dynamics. *Bull. Amer. Math. Soc.* **61** (1955), 584–588.

[30] H. P. Hudson, “Life begins …”. *Univ. Chic. Mag.* (1941), 6–11.

[31] B. Koopman, Birkhoff on dynamical systems. *Bull. Amer. Math. Soc.* **36** (1930), 162–166.

[32] G. G. Lorentz, Mathematics and Politics in the Soviet Union from 1928 to 1953. *J. Approx. Theory* **116** (2002), 169–223.31.

[33] R. S. MacKay and J. D. Meiss, *Hamiltonian dynamical systems*. Adam Hilger, Bristol and Philadelphia, 1987.

[34] A. G. Maier, Sur un problème de Birkhoff. *C. R. Acad. Sci. URSS* **55** (1947), 473–475.

[35] M. Morse and G. David, Birkhoff and his mathematical work. *Bull. Amer. Math. Soc.* **52** (1946), 357–391.

[36] J. C. Oxtoby and S. M. Ulam, Measure-preserving homeomorphisms and metrical transitivity. *Ann. of Math. (2)* **42** (1941), 874–920.

[37] H. Poincaré, Sur le problème des trois corps et les équations de la dynamique. *Acta Math.* **13** (1890), 1–270.

[38] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste, vol. III*. Gauthier-Villars, Paris, 1899.

[39] H. Poincaré, Sur un théorème de géométrie. *Rend. Circ. Mat. Palermo* **33** (1912), 375–407.

[40] D. van Dalen, *L. E. J. Brouwer. Topologist, Intuititionist, Philosopher*. Springer, London, 2013.

[41] H. Whitney, Regular families of curves. *Ann. of Math. (2)* **34** (1933), 241–279.

[42] N. Wiener, *Ex-prodigy: my childhood and youth*. The MIT Press, Cambridge, 1953.

[43] E. B. Wilson, George David Birkhoff. *Science* **102** (1945), 578–580.

[44] A. Wintner, *The analytical foundations of celestial mechanics*. Princeton University Press, Princeton, 1941.

[45] J. D. Zund, George David Birkhoff and John von Neumann: a question of priority and ergodic theorems, 1931–1932. *Historia Math.* **29** (2002), 138–156.

**JUNE BARROW-GREEN**

School of Mathematics and Statistics, Faculty of STEM, The Open University, Walton Hall, Milton Keynes MK7 6AA, UK, june.barrow-green@open.ac.uk