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The Maximum Modulus Set of a Polynomial

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Abstract
We study the maximum modulus set, $\mathcal{M}(p)$, of a polynomial p. We are interested in constructing p so that $\mathcal{M}(p)$ has certain exceptional features. Jassim and London gave a cubic polynomial p such that $\mathcal{M}(p)$ has one discontinuity, and Tyler found a quintic polynomial $\tilde{p}$ such that $\mathcal{M}(\tilde{p})$ has one singleton component. These are the only results of this type, and we strengthen them considerably. In particular, given a finite sequence $a_1, a_2, \ldots, a_n$ of distinct positive real numbers, we construct polynomials p and $\tilde{p}$ such that $\mathcal{M}(p)$ has discontinuities of modulus $a_1, a_2, \ldots, a_n$, and $\mathcal{M}(\tilde{p})$ has singleton components at the points $a_1, a_2, \ldots, a_n$. Finally we show that these results are strong, in the sense that it is not possible for a polynomial to have infinitely many discontinuities in its maximum modulus set.

Keywords Polynomials · Maximum modulus

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1 Introduction

Let f be an entire function, and define the *maximum modulus* by

$$M(r, f) := \max_{|z|=r} |f(z)|, \quad \text{for } r \geq 0.$$

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Following [7], denote by $\mathcal{M}(f)$ the set of points where f achieves its maximum modulus; we call this the *maximum modulus set*. In other words

$$\mathcal{M}(f) := \{z \in \mathbb{C}\colon |f(z)| = M(|z|, f)\}.$$

If f is a monomial, then $\mathcal{M}(f) = \mathbb{C}$; clearly this case is not interesting. Otherwise, $\mathcal{M}(f)$ consists of a countable union of closed *maximum curves*, which are analytic except at their endpoints, and may or may not be unbounded; [1]. It is straightforward to check that the maximum modulus set is closed.

Our interest in this paper is in the case that f is a polynomial. In particular, we study two "exceptional" features in the maximum modulus set. The first concerns discontinuities, which we define as follows.

Definition 1.1 Let f be an entire function, and $r > 0$. If there exists a connected component Γ of $\mathcal{M}(f)$ such that $\min\{|z| : z \in \Gamma\} = r$, then we say that $\mathcal{M}(f)$ has a *discontinuity of modulus* r. Note that a maximum modulus set may have more than one discontinuity of the same modulus.

These discontinuities were first studied by Blumenthal [1], see also [2]. Hardy [3] was the first to give an entire function with discontinuities in its maximum modulus set; in fact he constructed a transcendental entire function whose maximum modulus set has infinitely many discontinuities. Such discontinuities were studied further in [6], where it was shown that, given a sequence $(a_k)_{k\in\mathbb{N}}$ of strictly positive real numbers tending to infinity, there is a transcendental entire function whose maximum modulus set has discontinuities of modulus a_k for each $k \in \mathbb{N}$.

Blumenthal did not give any examples of a polynomial whose maximum modulus set has discontinuities, although he conjectured that there is a cubic polynomial with this property. Such a polynomial was given in [5]. Remarkably, this is the only such example in the literature, and seems to have only one discontinuity. Our first result is a significant generalisation of that in [5], and complements the main result in [6] mentioned above.

Theorem 1.2 *Suppose that $a_1, a_2, \ldots, a_n$ is a finite sequence of distinct positive real numbers. Then there exists a polynomial p, of degree $2n + 1$, such that $\mathcal{M}(p)$ has discontinuities of modulus $a_1, a_2, \ldots, a_n$.*

It is possible for some of the analytic curves that make up the maximum modulus set to be degenerate; in other words, to be singletons. The only examples of this behaviour are due to Tyler [8], who gave a transcendental entire function f and a polynomial p such that $\mathcal{M}(f)$ has infinitely many singleton components, and $\mathcal{M}(p)$ has a singleton component. We show that it is possible to significantly strengthen this polynomial case.

Theorem 1.3 *Suppose that $a_1, a_2, \ldots, a_n$ is a finite sequence of distinct positive real numbers. Then there exists a polynomial p, of degree $4n + 1$, such that $\mathcal{M}(p)$ has singleton components at the points $a_1, a_2, \ldots, a_n$.*

Remarks 1. Unlike in [6], where the construction required complicated and delicate approximations, our results here are direct and elementary.

2. Note that we are not claiming in Theorems 1.2 and 1.3 that there might not be additional discontinuities and/or singleton components in the maximum modulus sets; see Fig. 1 which indicates that this indeed may happen.
3. Note also that singleton components of $\mathcal{M}(f)$ are always discontinuities in the sense we have defined them. Thus the conclusion of Theorem 1.2 is already contained in that of Theorem 1.3. However we have retained Theorem 1.2, partly for reasons of historical interest, and partly because the degree of the polynomials is smaller in Theorem 1.2 than in Theorem 1.3.
4. It is natural to ask if these results can be achieved with polynomials of smaller degree. This does not seem possible with the techniques of this paper.

Finally, we show that these constructions are strong, in the sense that a polynomial can have at most finitely many discontinuities in its maximum modulus set.

Theorem 1.4 *Suppose that p is a polynomial. Then $\mathcal{M}(p)$ has at most finitely many discontinuities.*

2 Proofs of Theorems 1.2 and 1.3

We require a few lemmas before proving our main results. The first is well-known, and we omit the proof.

Lemma 2.1 *If $q(z) := \sum_{k=0}^{n} a_k z^k$ is a polynomial, then*

$$\left|q(re^{i\theta})\right|^2 = \sum_{k=0}^{n} |a_k|^2 r^{2k} + \sum_{0\le j<k\le n} 2|a_j||a_k| r^{j+k} \cos((j-k)\theta + \arg(a_j) - \arg(a_k)). \tag{2.1}$$

We use Lemma 2.1 to prove the following. Roughly speaking, this result states that we can force part of the maximum modulus set of a certain class of polynomials to lie on the real line. This result is the crux of our construction.

Lemma 2.2 *Suppose that $\hat{p}$ is a polynomial with only real coefficients, and that $0 < R < R'$. For $a > 0$, set*

$$p(z) := a(z^2 + 1) + \hat{p}(z). \tag{2.2}$$

If a is sufficiently large, then

$$z \in \mathcal{M}(p) \text{ and } R \le |z| \le R' \implies \operatorname{Im} z = 0.$$

Proof Let $\hat{p}$, R and R' be as in the statement of the lemma. Note that since

$$p(-z) = a(z^2 + 1) + \hat{p}(-z),$$

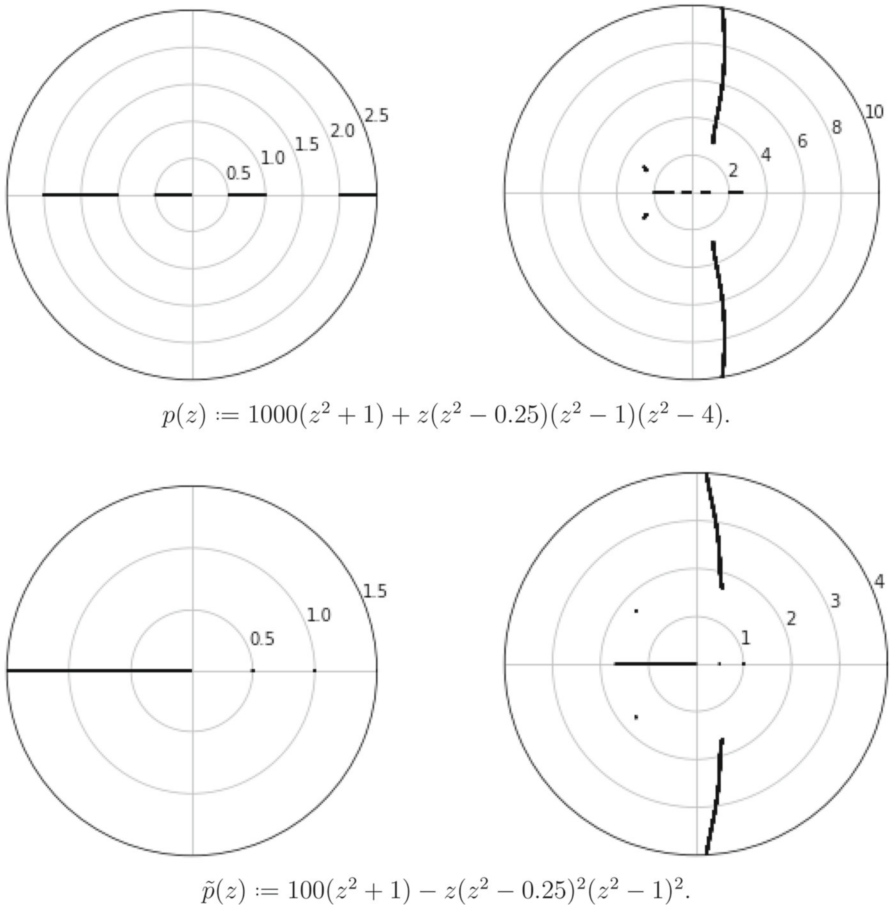

Fig. 1 Computer generated graphics of $\mathcal{M}(p)$ and $\mathcal{M}(\tilde{p})$, where the former has discontinuities of modulus 0.5, 1 and 2, as in Theorem 1.2, and the latter has singleton components at 0.5 and 1 as in Theorem 1.3. Note on the right that, when zoomed out, there appear to be additional discontinuities in these maximum modulus sets

and $\hat{p}$ is arbitrary, we lose no generality in proving this result only in the right half-plane. In other words, we need to prove that if a is sufficiently large, then

$$z \in \mathcal{M}(p) \text{ and } \operatorname{Re} z \geq 0 \text{ and } R \leq |z| \leq R' \implies \operatorname{Im} z = 0.$$

Choose $\theta_0 \in (0, \pi/4)$. We begin by showing that if $a > 0$ is sufficiently large, then

$$z \in \mathcal{M}(p) \text{ and } \operatorname{Re} z \geq 0 \text{ and } R \leq |z| \leq R' \implies |\arg z| < \theta_0.$$

Consider first the polynomial $q(z) := z^2 + 1$. Note that, by Lemma 2.1,

$$q(r)^2 - |q(re^{i\theta})|^2 = 2r^2(1 - \cos 2\theta).$$

Set $\alpha = 1 - \cos 2\theta_0 > 0$. It follows that if $\theta_0 \le |\theta| \le \pi/2$, then

$$q(r)^2 - |q(re^{i\theta})|^2 \ge 2r^2\alpha.$$

We can deduce that, if, in addition, $r \ge R$, then

$$q(r) - |q(re^{i\theta})| \ge \frac{r^2\alpha}{r^2+1} \ge \frac{R^2\alpha}{R^2+1}.$$

Let $K := M(R', \hat{p})$. Choose $a > \dfrac{2K(R^2+1)}{\alpha R^2}$. We can deduce that if $r \in [R, R']$ and $\theta_0 \le |\theta| \le \pi/2$, then

$$\begin{aligned} p(r) - |p(re^{i\theta})| &\ge (aq(r) - K) - (a|q(re^{i\theta})| + K) \\ &= a(q(r) - |q(re^{i\theta})|) - 2K \\ &> 0, \end{aligned}$$

which establishes our first claim.

We have shown that if $a > 0$ is large enough, then the point(s) of $\mathcal{M}(p)$ of modulus $r \in [R, R']$ are "close" to the real line. It remains to show that, increasing a if necessary, we can ensure that these points are in fact *on* the real line.

Note, by Lemma 2.1, that

$$|p(re^{i\theta})|^2 = a^2r^4 + a^2 + 2a^2r^2\cos 2\theta + \beta(\theta),$$

where $\beta(\theta)$ is a finite sum of terms of the form $b_k \cos k\theta$, where each k is an integer, and the coefficients b_k are all $O(a)$ as $a \to \infty$. Moreover, the constant in the $O(a)$ terms is independent of r when we restrict ourselves to the r values in the bounded set $[R, R']$. Note finally that these coefficients are positive or negative depending on whether the corresponding coefficients in $\hat{p}$ are positive or negative, though we do not use this fact.

We then have that

$$\frac{\partial}{\partial\theta}|p(re^{i\theta})|^2 = -4a^2r^2\sin 2\theta + \beta'(\theta), \tag{2.3}$$

and

$$\frac{\partial^2}{\partial\theta^2}|p(re^{i\theta})|^2 = -8a^2r^2\cos 2\theta + \beta''(\theta). \tag{2.4}$$

Note that β, β' and β'' are all $O(a)$ as $a \to \infty$.

Now, equation (2.3), together with the form of β, implies that $\frac{\partial}{\partial\theta}|p(re^{i\theta})|^2 = 0$ when $\theta = 0$, and also that

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}|p(re^{i\theta})|^2 < -4a^2r^2 + O(a), \quad \text{for } |\theta| < \theta_0,$$

as $a \to \infty$. It follows that, increasing a if necessary, for each $r \in [R, R']$ the value $\theta = 0$ is the only stationary point of the map $\theta \mapsto |p(re^{i\theta})|$ in the range $|\theta| \leq \theta_0$.

Moreover, Eq. (2.4), together with the form of β, implies that

$$\frac{\partial^2}{\partial \theta^2}|p(re^{i\theta})|^2 < -8a^2r^2\cos 2\theta_0 + O(a), \quad \text{for } |\theta| < \theta_0,$$

as $a \to \infty$. Hence, increasing a one final time if necessary, we can deduce that the stationary point above is a local maximum. The result follows, using our first claim. □

We use Lemma 2.2 to deduce the following.

Lemma 2.3 *Suppose that $\hat{p}$ is an odd polynomial with only real coefficients, and that $0 < R < R'$. For $a > 0$, let p be the polynomial defined in (2.2). If a is sufficiently large, then the following holds. Suppose that $R \leq r \leq R'$. Then:*

$$\mathcal{M}(p) \cap \{z \in \mathbb{C} : |z| = r\} = \begin{cases} \{r\} & \text{if } \hat{p}(r) > 0, \\ \{-r\} & \text{if } \hat{p}(r) < 0, \\ \{-r, r\} & \text{if } \hat{p}(r) = 0. \end{cases}$$

Proof We first choose $a > 0$ large enough that the consequence of Lemma 2.2 holds. Since $\hat{p}$ is odd, we have that

$$p(\pm r) = a(r^2 + 1) \pm \hat{p}(r).$$

The result then follows easily. □

We can now prove our two main constructions.

Proof of Theorem 1.2 Let $a_1, a_2, \ldots, a_n$ be distinct positive real numbers as in the statement of the Theorem. Let $\hat{p}$ be the odd polynomial

$$\hat{p}(z) := z(z^2 - {a_1}^2)(z^2 - {a_2}^2)\cdots(z^2 - {a_n}^2),$$

and set

$$R := \frac{1}{2}\min\{a_1, a_2, \ldots, a_n\} \quad \text{and} \quad R' := 2\max\{a_1, a_2, \ldots, a_n\}. \tag{2.5}$$

Note that $\hat{p}(r)$ changes sign, as r increases from zero, every time we pass through one of the a_k. Let $a > 0$, and let p be the polynomial in (2.2). The result then follows from the comment above, together with Lemma 2.3. □

Proof of Theorem 1.3 Let $a_1, a_2, \ldots, a_n$ be distinct positive real numbers as in the statement of the Theorem. Let $\hat{p}$ be the odd polynomial

$$\hat{p}(z) := -z(z^2 - {a_1}^2)^2(z^2 - {a_2}^2)^2\cdots(z^2 - {a_n}^2)^2,$$

and set R and R' as in (2.5). Note that if $r > 0$, then $\hat{p}(r)$ is strictly negative, except when $r = a_k$, for some k, in which case $\hat{p}(r) = 0$. Let $a > 0$, and let p be the polynomial in (2.2). The result then follows from the comment above, together with Lemma 2.3. □

3 Proof of Theorem 1.4

In this section, we make use of the pioneering work of Blumenthal on $\mathcal{M}(f)$ for any entire map f. The results that we require are summarized in the following theorem.

Theorem 3.1 [1] *Let f be an entire function, and let $S \subset \mathbb{C}$ be a compact set. Then $\mathcal{M}(f) \cap S$ is either empty, or consists of a finite number of closed curves, analytic except at their endpoints, and which can intersect in at most finitely many points.*

Remark Proofs of the results in [1] can also be found in [9, II.3]. We note that they are based on the study of the set of points where local maxima of the map $\theta \mapsto |f(re^{i\theta})|$ occur. The local structure of these points consists of a (finite) collection of analytic arcs. See also the work of Hayman [4].

Corollary 3.2 *Let f be an entire function, and let $S \subset \mathbb{C}$ be any compact set. Then $\mathcal{M}(f) \cap S$ has at most finitely many discontinuities.*

Proof Since $\mathcal{M}(f)$ is a collection of closed curves, by definition of discontinuity, there is a bijection from the set of discontinuities of $\mathcal{M}(f)$ to the set of all connected components of $\mathcal{M}(f)$ that do not contain the point zero. By this, and since by Theorem 3.1 $\mathcal{M}(f) \cap S$ has finitely many components, the result follows. □

Remark Note that it follows easily from Corollary 3.2 that the maximum modulus set of a transcendental entire function can have at most countably many discontinuities.

Proposition 3.3 *Suppose that p is a polynomial of degree n, and define its* reciprocal polynomial*, q, by $q(z) := z^n p(1/z)$. Then $w \in \mathcal{M}(q)\backslash\{0\}$ if and only if $1/w \in \mathcal{M}(p)\backslash\{0\}$.*

Proof Since the reciprocal of the reciprocal of a polynomial equals the original polynomial, it suffices to prove one direction. Suppose that $w \in \mathcal{M}(q)\backslash\{0\}$. Then

$$|q(w)| = \max_{|z|=|w|} |q(z)|,$$

and so

$$\left|p\left(\frac{1}{w}\right)\right| = \max_{|z|=|w|} \left|p\left(\frac{1}{z}\right)\right|.$$

Hence $1/w \in \mathcal{M}(p)\backslash\{0\}$, as required. □

Proof of Theorem 1.4 Let p be a polynomial of degree n and let q be its reciprocal polynomial. Denote by $\overline{\mathbb{D}}$ the closure of the unit disk centred at the origin. Then, by Corollary 3.2, both $\mathcal{M}(p)\cap\overline{\mathbb{D}}$ and $\mathcal{M}(q)\cap\overline{\mathbb{D}}$ have at most finitely many discontinuities. Thus, by Proposition 3.3, $\mathcal{M}(p)\backslash\mathbb{D}$ also has at most finitely many discontinuities, and the result follows. □

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