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The Nonlinear Dynamics of Order-Up-To Inventory Systems with Lost Sales

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Abstract: We study the dynamic consequences of lost sales when there is insufficient inventory to satisfy demand. Demand is assumed to be independently and identically distributed and drawn from a normal distribution. We consider the industrially popular order-up-to policy with unit lead time is used to make replenishment orders. In this scenario, we obtain expressions for the order and inventory distributions, allowing us to quantify the Bullwhip and Net Stock Amplification ratios. We show that both these metrics are equivalent. We also determine the mean inventory levels held, and the achieved fill rate. We do this when the lost sales are fully observable, and when the lost sales are unobservable.

Keywords: Inventory control, Supply chain dynamics, Bullwhip Effect, Order-up-to policy, Lost sales, Fill rate.

1. INTRODUCTION

Verbeke et al. (1998) investigated the response of 1,750 customers after encountering an out-of-stock situation, finding only 20% of them would postpone the purchase (and buy the same product in the same store later). A decade later, Van Woensel et al. (2007) analyzed the behavior of 3,800 customers of a Dutch grocery retail chain and revealed that only 12% of regular customers and 6% of occasional customers would opt to delay a purchase when faced with an out-of-stock situation. Both studies observed that most customers facing a stock-out would either forfeit the planned purchase and look for the same item in another store or purchase a substitute product. These examples illustrate that many practical supply chain settings are governed by the rules of lost-sales inventory systems. It also seems reasonable to assume that customers' tolerance of lost sales is decreasing over time as more purchasing alternatives become available.

In contrast, the majority of inventory theory is based upon backlog systems, where excess demand is accumulated in an order book and delivered as soon as inventory becomes available, Bijvank and Vis (2011). Zipkin (2008) also argues that while we have a deep understanding of the behavior of backlog systems, we have much less comprehension of lost-sales systems. Note that the backlog assumption keeps the system linear [1] , whereas lost-sales systems have a nonlinear nature, hampering their mathematical analysis. Indeed complex dynamics emerge when the linearity assumption is removed, see Wang et al. (2014), which may even dominate the system behavior, Nagatani and Helbing (2004).

A noteworthy line of research has analytically investigated the optimality of traditional inventory policies in lost-sales environments. Karlin and Scarf (1958) showed the optimal reorder quantity with backlogs can be derived from a single number (the sum of inventory-on-hand and on-order), but under lost sales this quantity is function of the on-hand inventory as well as the timing and quantity of all outstanding orders. Since then, several authors have mined this vain, searching for replenishment policies under different cost models; see e.g. Morton (1969), Johansen (2001), Chen et al. (2006), and Goldberg et al. (2016). We refer interested readers to the comprehensive literature review by Bijvank and Vis (2011) for more details on prior literature in the lost-sales inventory discipline.

Our approach here is different to the established literature. We aim to investigate the impact of the lost-sales non-linearity on the dynamics of the supply chain under the well-known, industrially popular, order-up-to (OUT) replenishment policy. We are concerned with the implications of lost sales compared to backlogging demand. To this end, we explore the inventory and order variances maintained by the inventory system. To the best of our knowledge, no prior study in the field of *supply chain dynamics* provides a clear understanding to these research questions. Indeed, Wang and Disney (2016) flag this as a key area for future research in a recent literature review.

[1] Linear inventory models rely on assumptions that provide a physical meaning for the negative values of variables, see Ponte et al. (2017); in this case, negative inventories indicate backlogged products.

2. LOST-SALES INVENTORY MODEL

To gain a thorough understanding of the lost sales-induced dynamics in the supply chain, we study a single echelon. We focus on the retailer, as this echelon frequently operates within a lost-sales environment, (Verbeke et al., 1998; Van Woensel et al., 2007).

The discrete-time sequence of events that governs this operation is composed of three main stages in each time period t. At the beginning of t, the inventory is received from the upstream supplier. During the course of t, consumer demand is satisfied as long as on-hand inventory is available. Finally, at the end of t, the state of the inventory is reviewed, a forecast of demand is made, and the order is issued. Formally this can be modeled with difference equations, which we describe below, together with a discussion of the implicit assumptions they entail.

First, the retailer receives the product. The receipts, r_t, respond to the orders, q_t, placed in the previous period,

$$r_t = q_{t-1}. \tag{1}$$

We assume what was ordered is always delivered. That is, production and transportation constraints are not present. We also assume the supplier is able to satisfy the order in the next period. That is, a unit lead time exists.

The consumer demand, d_t, is assumed to be an independent and identically distributed (i.i.d.) variable, z_t, drawn from a normal distribution with mean μ and variance σ^2,

$$d_t = z_t : z_t \in \mathcal{N}(\mu, \sigma^2). \tag{2}$$

Let γ be the coefficient of variation, $\gamma = \sigma/\mu$. We assume the mean demand is sufficiently high so that the probability of negative demand is negligible. We adopt the normality assumption [2] as this distribution captures the behavior of many real demand patterns (Schneeweiss, 1974; Disney et al., 2016). The demand fluctuates randomly around a constant mean; that is, no trends, seasonal effects, or autocorrelation exist.

The on-hand, end-of-period, inventory level, i_t, is the accumulated difference between receipts and demand. Under lost-sales, inventory is constrained to non-negative values,

$$i_t = [i_{t-1} + r_t - d_t]^+. \tag{3}$$

with $[\cdot]^+ = \max\{\cdot, 0\}$, the maximum operator, truncating negative values. Note, i_t=0 indicates that stock-outs have occurred in t (except in the particular case when demand is exactly equal to the available inventory, i.e., d_t=i_{t-1}+r_t). We do not consider any other nonlinear effects, e.g. defective products, quality loss, or limited storage capacity.

The satisfied demand, s_t, is given by,

$$s_t = \min\{i_{t-1} + r_t, d_t\}. \tag{4}$$

As backlogs are not allowed, the difference between the actual and the satisfied demand, $d_t - s_t$, represents the size of the stock-out [3].

We assume the retailer is aware the demand is i.i.d. and forecasts it with a static model:

$$f_t = \eta. \tag{5}$$

Ideally, the retailer is able to *observe* the lost sales and can forecast the mean demand with $\eta = \mu$, thus resulting in a minimum mean squared error (MMSE) forecast, see Disney et al. (2016). However, in other settings, retailers may be *unable to observe* the whole demand, and may under-estimate the actual mean demand, that is, $\eta < \mu$.

Finally, orders are generated with the OUT replenishment decision (Lalwani et al., 2006), which accounts for (i) the demand forecast; and (ii) the discrepancy between the target and actual net stock. Following common practice, see Lin et al. (2017), we consider the target net stock, the safety stock, to be the product of the safety factor δ and the expected demand f_t. Note, δ is a decision variable. Overall, the order quantity is given by

$$q_t = f_t + (\delta f_t - i_t) = (1 + \delta) f_t - i_t. \tag{6}$$

Here the first addend, $(1 + \delta) f_t$, represents the OUT level (that is, the desired inventory at the beginning of t) and the second addend, i_t, represents the actual inventory.

2.1 Performance indicators

To measure performance, we employ two common and practically relevant approaches. First, we explore the variability of orders and inventories. These are well-known sources of inefficiencies in production and distribution systems, Disney and Lambrecht (2008). Second, we consider the trade-off between the fill rate and the average inventory held, another key business concern.

We measure order variability via the Bullwhip ratio,

$$BW = \text{var}(q_t)/\text{var}(d_t), \tag{7}$$

where var$(\cdot)$ is the variance operator. BW is directly related to capacity costs (Disney and Lambrecht, 2008). To measure the variability in inventories, we employ the Net Stock Amplification ($NSAmp$) metric,

$$NSAmp = \text{var}(i_t)/\text{var}(d_t). \tag{8}$$

$NSAmp$ is directly related to inventory costs (Disney and Lambrecht, 2008). In linear systems, the BW and $NSAmp$ metrics represent a key trade-off for managers, as it is often possible to decrease one at the cost of increasing the other, see e.g. Disney et al. (2004).

We measure customer service through the fill rate, β. This is a popular metric in the fast moving consumer goods industry, representing the proportion of demand satisfied immediately from stock (Disney et al., 2015). In this sense, the fill rate is the ratio of the mean positive satisfied demand to the mean positive demand by

$$\beta = \mathbb{E}[(s_t)^+]/\mathbb{E}[(d_t)^+], \tag{9}$$

Sobel (2004), where $\mathbb{E}[\cdot]$ is the expectation operator. In order to consider the stock required to meet the customer, we consider the average on-hand inventory, τ, by

$$\tau = \mathbb{E}[i_t]. \tag{10}$$

Again there is a key trade-off to consider, as the fill rate can be improved at the cost of an higher average inventory.

3. ANALYTICAL STUDY

Under our assumptions, the following equations can be obtained by substituting (1) and (5) into (3), (4), and

[2] This is acceptable when demand originates from many independent customers as the central limit theorem can be evoked.

[3] Note, the double-accounting-of-backlogs problem (Cachon and Terwiesch, 2006; Disney et al., 2015) when calculating the fill-rate does not occur in lost sales systems.

(6). Here we focus on the on-hand inventory, satisfied demand, and orders, which we define as the *state variables* of the lost-sales system, as they determine the performance metrics we employ.

$$i_t = [i_{t-1} + q_{t-1} - d_t]^+, \tag{11}$$
$$s_t = \min\{i_{t-1} + q_{t-1}, d_t\}, \tag{12}$$
$$q_t = (1+\delta)\eta - i_t. \tag{13}$$

An important property of the lost-sales system emerges from (13): for all periods, $q_t + i_t = (1+\delta)\eta$. This implies the on-hand inventory position at the beginning of every period will be $(1+\delta)\eta$.

Theorem 1. (Fundamental relations). The following *fundamental relationships* exist in our lost-sales system:

$$i_t = [(1+\delta)\eta - d_t]^+, \tag{14}$$
$$s_t = \min\{(1+\delta)\eta, d_t\}, \tag{15}$$
$$q_t = s_t. \tag{16}$$

Proof. Eqs. (14) and (15) can be easily obtained by substituting $q_t + i_t = (1+\delta)\eta$ into (11) and (12). Using $[a-b]^+ = \max\{a-b, 0\} = a - \min\{a, b\}$, (14) can be expressed as $i_t = (1+\delta)\eta - \min\{(1+\delta)\eta, d_t\}$. Note that the second addend represents the satisfied demand, see (15); thus obtaining $i_t = (1+\delta)\eta - s_t$. Replacing $(1+\delta)\eta$ by $i_t + q_t$ leads to (16). □

Eq. (16) defines another interesting property of the lost-sales system: the equality of satisfied demand and orders. Thus, the OUT model results in the retailer ordering every period exactly what they sold during the period [4].

3.1 Distributions of the state variables

Eqs. (14), (15), and (16) express the state variables of the lost-sales system as functions of constants and the demand. First, we discuss the statistical distributions of these variables, as they will allow us to derive analytical expressions for all our metrics.

From inspection of (14), we can see the inventory is a translated, scaled, and truncated demand distribution. This leads to the following probability density function (pdf) of the on-hand inventory:

$$\phi_i[x] = \frac{h[x]}{\sigma}\phi\left[\frac{x-(1+\delta)\eta+\mu}{\sigma}\right] + \tag{17}$$
$$\Delta[x]\Phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right]. \tag{18}$$

Here $\phi[\cdot]$ and $\Phi[\cdot]$ are respectively the pdf and cumulative distribution function (cdf) of the standard normal distribution $N(0,1)$, $h[x] = \{1,$ if $x \geq 0$; 0, otherwise$\}$ is the Unit Step function, and $\Delta[x] = \{1,$ if $x = 0$; 0, otherwise$\}$ is the Dirac Delta function. To obtain (17), the translation and scaling is dealt with by changing the mean and standard deviation of the standard normal distribution, while the Unit Step function attends to the trunction. Finally, the pdf at $x = 0$ is captured by the product of the Dirac Delta function and the cdf at $x = 0$.

We can use a similar approach to obtain the pdf of the orders and the satisfied demand from (15). Eq. (13) reveals the pdf of the orders (and the pdf of the satisfied demand) is a translated reflection of the pdf of the inventory. Note that this suggests an important implication for our study: the variance of the orders and the variance of on-hand inventories are equal, thus

$$BW = NSAmp. \tag{19}$$

The pdf of the orders and the satisfied demand in the lost-sales system is given by

$$\phi_q[x] = \phi_s[x] = \frac{h[(1+\delta)\eta - x]}{\sigma}\phi\left[\frac{x-\mu}{\sigma}\right] + \tag{20}$$
$$\Delta[(1+\delta)\eta - x]\Phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right]. \tag{21}$$

3.2 Deriving expressions for the performance metrics

From the pdfs of the state variables, we can derive the expressions of the four metrics. First, we investigate the mean of the variables to consider the trade-off between the fill rate β and the average inventory τ.

Looking at the pdf of inventories, the expected inventory held in any single period, τ, is given by

$$\tau = \mathbb{E}[i_t] = \int_0^\infty x\phi_i[x]dx = \sigma\phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right] + \left((1+\delta)\eta - \mu\right)\Phi\left[\frac{(1+\delta)\eta-\mu}{\sigma}\right]. \tag{22}$$

The fill rate is given by the ratio of the expected positive satisfied demand to the expected positive demand,

$$\beta = \frac{\mathbb{E}[(s_t)^+]}{\mathbb{E}[(d_t)^+]} = \int_0^\infty x\phi_s[x]dx \Big/ \int_0^\infty x\phi_d[x]dx, \tag{23}$$

where $\phi_d[x]$ is the pdf of the demand.

Second, consider the variances. As discussed, the variance of the three variables, that is, i_t, o_t, and s_t, is the same. The variance of the inventories is given by

$$\begin{aligned}
\sigma_i^2 &= \int_0^\infty (x-\tau)^2\phi_i[x]dx \\
&= \int_0^\infty x^2\phi_i[x]dx - \tau^2 \\
&= \sigma((1+\delta)\eta-\mu)\phi\left[\frac{(1+\delta)\eta-\mu}{\sigma}\right] + \\
&\quad \left(((1+\delta)\eta-\mu)^2+\sigma^2\right)\Phi\left[\frac{(1+\delta)\eta-\mu}{\sigma}\right] - \\
&\quad \left(\sigma\phi\left[\frac{\mu-(1+\delta)\eta}{\sigma}\right] + \right. \\
&\quad \left.((1+\delta)\eta-\mu)\Phi\left[\frac{(1+\delta)\eta-\mu}{\sigma}\right]\right)^2
\end{aligned} \tag{24}$$

This allows us to obtain the $NSAmp$ metric, and by (19) also the BW metric,

$$BW = NSAmp = \frac{\sigma_i^2}{\sigma^2}. \tag{25}$$

Exposition of the four relevant metrics can be greatly simplified by defining the *relative safety margin*,

$$\lambda = \frac{(1+\delta)\eta - \mu}{\sigma}. \tag{26}$$

[4] Notice the difference with the linear model, in which backlog is allowed, where the considered assumptions results in a *pass-on-orders* strategy, $q_t = d_t$; see e.g. Disney et al. (2016).

Physically λ can be interpreted the protection of the on-hand inventory against shortages in relative terms to the standard deviation of the demand [5].

Using λ, the ratio of the average inventory to the mean demand is

$$\tau/\mu = \gamma\left(\phi\left[\lambda\right] + \lambda\Phi\left[\lambda\right]\right). \tag{27}$$

The fill rate is given by

$$\beta = h[1+\gamma\lambda]\left(1 - \frac{\phi[\lambda] + \lambda(\Phi[\lambda]-1)}{\phi\left[\gamma^{-1}\right] + \gamma^{-1}\Phi\left[\gamma^{-1}\right]}\right), \tag{28}$$

Finally, the BW and $NSAmp$ metrics simplify to

$$BW = NSAmp = \lambda\phi\left[\lambda\right] + (\lambda^2+1)\Phi\left[\lambda\right] - \left(\phi\left[\lambda\right] + \lambda\Phi\left[\lambda\right]\right)^2. \tag{29}$$

4. INSIGHTS FROM THE ANALYSIS

In this section we first study the BW and $NSAmp$ ratios before analyzing the trade-off between the fill rate and average inventory holding. We consider two different cases:

(1) *Full demand observation* (FDO), the ideal case, where the retailer is able to observe the whole customer demand, despite only satisfying a portion of it (due to the lost-sales condition). For example, this could be relevant for an internet based retailer who could track customers browsing history. This scenario is described by $\eta = \mu$, which results in a simplified variant of the relative safety margin, $\lambda = \delta/\gamma$.

(2) *Partial demand observation* (PDO), where excess demand becomes unobserved lost sales, leading the retailer to under-estimate the mean demand. This case, characterized by $\eta < \mu$, might be representative of a customer who upon experiencing a stock-out in store, departs without leaving a trace of their disappointment. In such cases, employing high safety stock factors would help the retailer approximate μ.

As discussed by Lariviere and Porteus (1999), the conditions of lost sales ($vs.$ backlog) and unobserved demand ($vs.$ observed demand) lead to different scenarios; all are practically relevant in different settings.

4.1 BW and NSAmp: A study of system volatility

In the linear backlog system, under our assumptions, $BW = 1$ and $NSAmp = 1$ (Disney et al., 2006). In the lost-sales system, however, (29) illustrates that BW and $NSAmp$ depend upon λ, which in turn is a function of $\{\mu, \sigma, \delta, \eta\}$. Fig. 1 represents the S-shaped relationship between the variance ratios and λ. Both BW and $NSAmp$ are increasing in λ; when λ is sufficiently high, the retailer rarely experiences lost sales and operates as a linear system would. Small values of λ reduce supply chain volatility.

FDO case. As $\lambda = \delta/\gamma$, BW and $NSAmp$ only depend on δ and γ. Lines without markers in Fig. 2 represent these metrics in the lost-sales system as a function of the safety stock factor δ for three different demand's coefficients of variation γ, 15%, 30%, and 45%, which are typical of retail time series, (Dejonckheere et al., 2004). When δ is sufficiently high, the lost sales are marginal, and the nonlinear lost-sales system behaves as the linear backlog system does. However, for low and medium values of δ, the variability of orders and inventory decreases significantly. Note, larger γ require greater δ to ensure linear operation. Nonetheless, γ does not impact on the value of the ratios for $\delta = 0$ (which is the value for $\lambda = 0$, see Fig. 1).

[5] Note that $\lambda > 0$ indicates that $(1+\delta)\eta$ is higher than the mean demand; hence the system will be able to satisfy the whole demand more than 50% of the periods. However, under $\lambda < 0$, the system would only satisfy the whole demand in less than 50% of the periods.

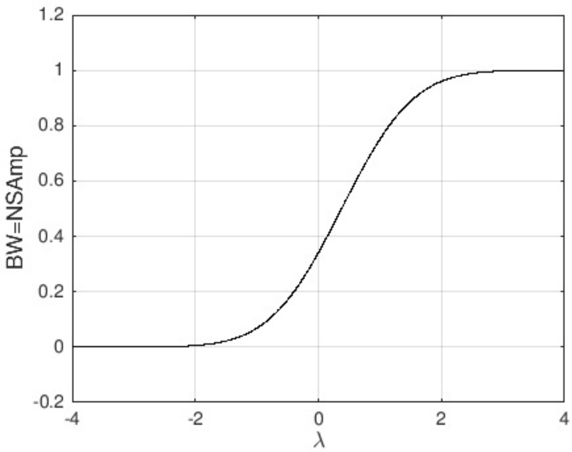

Fig. 1. BW and $NSAmp$ as a function of λ.

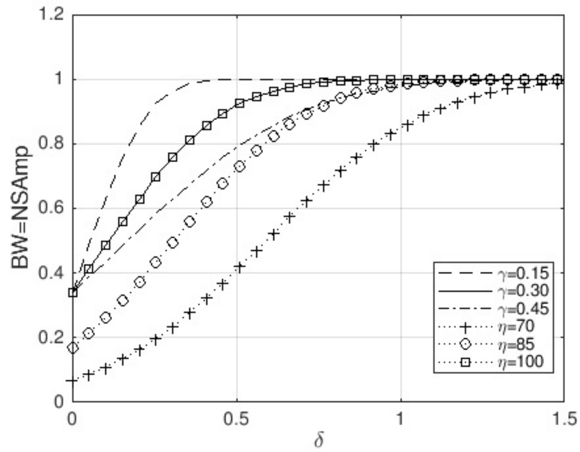

Fig. 2. BW and $NSAmp$ as a function of δ under FDO (lines without markers) and PDO (with markers).

PDO case. To analyze the impact of η, we assume $\gamma = 30\%$, using $\mu = 100$, $\sigma = 30$. Lines with markers in Fig. 2 display the BW and $NSAmp$ metrics as a function of δ for three different values of η: 70, 85, and 100. Evidently, the $\eta = 100$ case is equivalent to the FDO model (for $\gamma = 30\%$); while $\eta = 85$ and $\eta = 70$ represent the case when unobserved demand leads to biased demand forecasts. Fig. 2 shows that both BW and $NSAmp$ are very sensitive to the static forecast η. In this sense, the system benefits from lower BW and lower $NSAmp$ in the case of partial demand observation in the lost-sales system. Note, unlike δ before, η does impact on the value of the ratios for $\delta = 0$ (under PDO, $\delta = 0$ does not result in $\lambda = 0$). Again, lower η require larger δ to ensure linear operation.

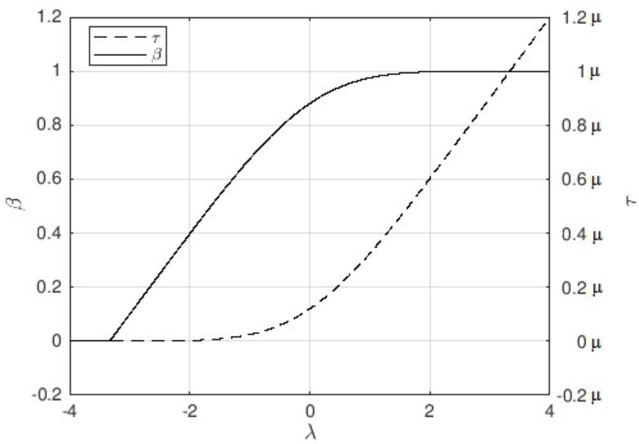

Fig. 3. τ and β as a function of λ.

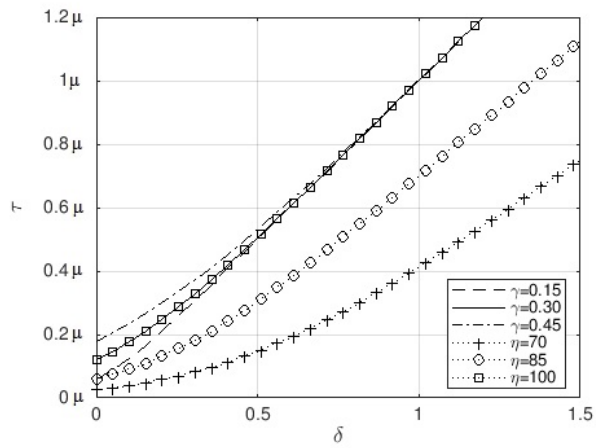

Fig. 4. Average inventory, τ, as a function of δ under FDO (lines without markers) and PDO (with markers).

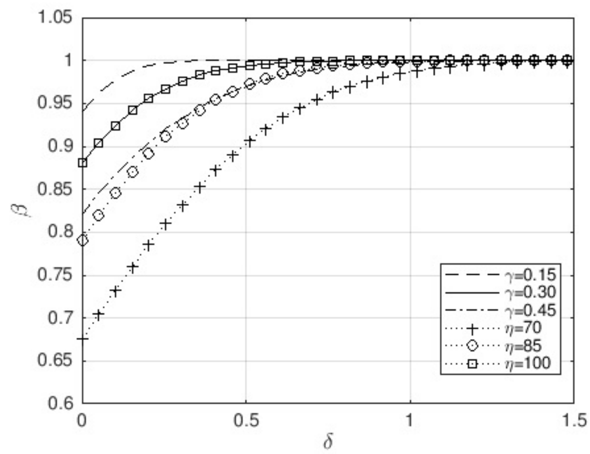

Fig. 5. Fill rate, β, as a function of δ under FDO (lines without markers) and PDO (with markers).

4.2 τ and β: The fill rate–inventory trade-off

The previous variance analysis suggests the dynamics of the lost-sales system benefits from the non-linearity. However, the lost-sales condition introduces behavioral differences that should be carefully considered. Note, while in the linear system the mean demand matches the mean order, lost sales create a gap between them [6] . This means that in some cases BW may be decreased in the nonlinear system as a consequence of an increase in the lost sales (via a reduction in the mean order quantity). Similarly, $NSAmp$ may be reduced in the lost-sales systems by negative forecast errors due unobserved demand. These perspectives explain why $BW = NSAmp = 0$ when $\lambda << 0$ (see Fig. 1). Importantly, while the simultaneous consideration of BW and $NSAmp$ provides a complete picture of the performance of replenishment policies in linear systems (Disney and Lambrecht, 2008), these metrics need to be considered alongside others which account for the difference between mean demand and mean orders in the nonlinear lost-sales system.

In light of this, we analyze the trade-off between the fill rate, β, and average inventory, τ, in lost-sales systems. Assuming $\gamma = 30\%$ (again $\mu = 100$, $\sigma = 30$), Fig. 3 represents β and τ/μ in the nonlinear system under consideration. For $\lambda << 0$, $\tau = 0$ and $\beta = 0$. For larger values of λ, τ is increasing and convex in λ. In contrast, β is increasing and concave in λ, with $\beta = 1$ for $\lambda >> 0$.

FDO case. Employing the same rationale as before, Fig. 4 and 5 display respectively τ/μ and β as a function of δ for the three values of δ. Generally, similar to a linear system, lower δ reduces the average inventory held, but also reduces the fill rate achieved. In Fig. 4, we observe that for higher γ, there is a more noticeable gap between $\tau\mu$ and the slope $\tau\mu = \delta$ (which characterizes the linear system) for small δ. The fill rate, β, is the same as for the linear backlog version. Despite the difference between linear and nonlinear orders (see footnote 4), the unit lead time ensures all backlogs are cleared in the next period.

[6] Here, orders define the physical system input, while the satisfied, rather than the actual, demand defines the physical system output. The mean order should then be equal to the mean satisfied demand to avoid the lost-sales system suffer from a long-term inventory drift.

PDO case. We now look at the lines with markers in Fig. 4 and 5, representing the relation between τ/μ and β with δ for three different values of η. Similar to the study of variances, we observe the impact of η is meaningful. That is, not being able to observe the whole demand causes a large reduction in the fill rate β, even with relatively large safety stock factors δ. For example, if $\eta = \mu$, $\delta \approx 0.2$ is required to achieve 95% fill rate, while if $\eta = 0.7\mu$, $\delta \approx 0.7$ is required. Our PDO analysis highlights that using inventory configurations derived from linear models in lost-sales environments with unobserved lost sales is risky, as this may result in a dramatically decreased service level.

5. CONCLUSIONS

With most inventory control literature investigating linear models, the dynamics of nonlinear inventory systems is not yet well understood. We focused on the non-negative inventory assumption in order to investigate the behavioral differences between backlog and lost-sales systems. We observe the lost-sales condition influences the dynamics of supply chains. Deriving the expression of the popular BW and $NSAmp$ metrics, we show the linear approximation only works under strict circumstances; thus ignoring the lost-sales condition in such settings may result in a radically different system behaviour. Remarkably, the impact of such non-linearity is the same on BW and $NSAmp$, depending on a relative safety margin λ. Both order and inventory variances are mitigated as λ decreases.

By exploring the trade-off between fill rate and holding requirements, we noticed that λ can be used to optimize inventory-related costs. As $NSAmp$ is minimized for $\lambda = -\infty$, the above has an important theoretical and practical implication: the interpretation of $NSAmp$ needs to be revisited in nonlinear lost-sales settings. While in linear systems it determines the nodes ability to meet a defined fill rate in a cost-effective manner; constraining the inventory distorts the $NSAmp$ metric –lowering $NSAmp$ does not necessarily result in an improved inventory performance. Our results also show unobserved demand in lost-sales systems, causes poor fill rates and high safety stock requirements.

We restricted our study to unit lead times and the use of static forecasting in an OUT policy. Future research could be directed to understanding the general lead time case as well as the dynamics induced by other forecasting techniques and/or replenishment policies. Similarly, we only considered one source of non-linearity. Investigating the interactions of the lost-sales condition with other non-linearities, such as capacity constraints or forbidden returns, is a important area for future research.

REFERENCES

Bijvank, M. and Vis, I. (2011). Lost-sales inventory theory: A review. *European Journal of Operational Research*, 215(1), 1–13.

Cachon, G. and Terwiesch, C. (2006). *Matching Supply with Demand: An Introduction to Operations Management*. McGraw-Hill.

Chen, Y., Ray, S., and Song, Y. (2006). Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. *Naval Research Logistics*, 53(2), 117–136.

Dejonckheere, J., Disney, S., Lambrecht, M., and Towill, D. (2004). The impact of information enrichment on the bullwhip effect in supply chains: A control engineering perspective. *European Journal of Operational Research*, 153(3), 727–750.

Disney, S., Farasyn, I., Lambrecht, M., Towill, D., and de Velde, W.V. (2006). Taming the bullwhip effect whilst watching customer service in a single supply chain echelon. *European Journal of Operational Research*, 173(1), 151–172.

Disney, S., Gaalman, G., Hedenstierna, C., and Hosoda, T. (2015). Fill rate in a periodic review order-up-to policy under auto-correlated normally distributed, possibly negative, demand. *International Journal of Production Economics*, 170, 501–512.

Disney, S. and Lambrecht, M. (2008). On replenishment rules, forecasting, and the bullwhip effect in supply chains. *Foundations and Trends in Technology, Information and Operations Management*, 2(1), 1–80.

Disney, S., Maltz, A., Wang, X., and Warburton, R. (2016). Inventory management for stochastic lead times with order crossovers. *European Journal of Operational Research*, 248(2), 473–486.

Disney, S., Towill, D., and Van de Velde, W. (2004). Variance amplification and the golden ratio in production and inventory control. *International Journal of Production Economics*, 90(3), 295–309.

Goldberg, D., Katz-Rogozhnikov, D., Lu, Y., Sharma, M., and Squillante, M. (2016). Asymptotic optimality of constant-order policies for lost sales inventory models with large lead times. *Mathematics of Operations Research*, 41(3), 898–913.

Johansen, S. (2001). Pure and modified base-stock policies for the lost sales inventory system with negligible set-up costs and constant lead times. *International Journal of Production Economics*, 71(1-3), 391–399.

Karlin, S. and Scarf, H. (1958). Inventory models and related stochastic processes. *Studies in the Mathematical Theory of Inventory and Production*, 1, 319.

Lalwani, C., Disney, S., and Towill, D. (2006). Controllable, observable and stable state space representations of a generalized order-up-to policy. *International Journal of Production Economics*, 101(1), 172–184.

Lariviere, M. and Porteus, E. (1999). Stalking information: Bayesian inventory management with unobserved lost sales. *Management Science*, 45(3), 346–363.

Lin, J., Naim, M., Purvis, L., and Gosling, J. (2017). The extension and exploitation of the inventory and order based production control system archetype from 1982 to 2015. *International Journal of Production Economics*, 194, 135–152.

Morton, T. (1969). Bounds on the solution of the lagged optimal inventory equation with no demand backlogging and proportional costs. *SIAM Review*, 11(4), 572–596.

Nagatani, T. and Helbing, D. (2004). Stability analysis and stabilization strategies for linear supply chains. *Physica A: Statistical Mechanics and its Applications*, 335(3-4), 644–660.

Ponte, B., Wang, X., de la Fuente, D., and Disney, S. (2017). Exploring nonlinear supply chains: the dynamics of capacity constraints. *International Journal of Production Research*, 55(14), 4053–4067.

Schneeweiss, C. (1974). Optimal production smoothing and safety inventory. *Management Science*, 20(7), 1122–1130.

Sobel, M. (2004). Fill rates of single-stage and multistage supply systems. *Manufacturing & Service Operations Management*, 6(1), 41–52.

Van Woensel, T., Van Donselaar, K., Broekmeulen, R., and Fransoo, J. (2007). Consumer responses to shelf out-of-stocks of perishable products. *International Journal of Physical Distribution & Logistics Management*, 37(9), 704–718.

Verbeke, W., Farris, P., and Thurik, R. (1998). Consumer response to the preferred brand out-of-stock situation. *European Journal of Marketing*, 32(11/12), 1008–1028.

Wang, X. and Disney, S. (2016). The bullwhip effect: Progress, trends and directions. *European Journal of Operational Research*, 250(3), 691–701.

Wang, X., Disney, S., and Wang, J. (2014). Exploring the oscillatory dynamics of a forbidden returns inventory system. *International Journal of Production Economics*, 147, 3–12.

Zipkin, P. (2008). Old and new methods for lost-sales inventory systems. *Operations Research*, 56(5), 1256–1263.