Canadian Journal of Mathematics
http://dx.doi.org/10.4153/CJM-2013-041-6

Perfect Orderings on Finite Rank Bratteli Diagrams

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Abstract. Given a Bratteli diagram B, we study the set $\mathcal{O}_B$ of all possible orderings on B and its subset $\mathcal{P}_B$ consisting of perfect orderings that produce Bratteli–Vershik topological dynamical systems (Vershik maps). We give necessary and sufficient conditions for the ordering ω to be perfect. On the other hand, a wide class of non-simple Bratteli diagrams that do not admit Vershik maps is explicitly described. In the case of finite rank Bratteli diagrams, we show that the existence of perfect orderings with a prescribed number of extreme paths constrains significantly the values of the entries of the incidence matrices and the structure of the diagram B. Our proofs are based on the new notions of skeletons and associated graphs, defined and studied in the paper. For a Bratteli diagram B of rank k, we endow the set $\mathcal{O}_B$ with product measure μ and prove that there is some $1 \leq j \leq k$ such that μ-almost all orderings on B have j maximal and j minimal paths. If j is strictly greater than the number of minimal components that B has, then μ-almost all orderings are imperfect.

1 Introduction

Bratteli diagrams (Definition 2.1) originally appeared in the theory of C^*-algebras and have turned out to be a very powerful and productive tool for the study of dynamical systems in the measurable, Borel, and Cantor settings. The importance of Bratteli diagrams in dynamics is based on the remarkable results obtained in the pioneering works by Vershik, Herman, Giordano, Putnam, and Skau [V81], [HPS92], [GPS95]. During the last two decades, diverse aspects of Bratteli diagrams and dynamical systems defined on their path spaces have been extensively studied, such as measures invariant under the tail equivalence relation, measurable and continuous eigenvalues, entropy, and orbit equivalence of these systems. We refer to a recent survey by Durand [D10] where the reader will find more references on this subject.

A Bratteli diagram B can be thought of as a partial recursive set of instructions for building a family of symbolic dynamical systems on X_B, the space of infinite paths on B. The n-th level of the diagram defines a clopen partition ξ_n of X_B so that the diagram gives us a sequence of refining partitions of X_B. The information contained in B also allows us to write ξ_n as a finite collection of unordered "towers" indexed by the vertices of the n-th level of B. At this point, however, we do not know the order of the elements in these towers. The edge set at the $(n+1)$-st level tells us how the partition ξ_{n+1} is built from the partition ξ_n, using a "cutting" method. In particular, if we see k edges from the n-th level vertex v' to the vertex v of the $(n+1)$-st level,

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Received by the editors May 9, 2012; revised August 13, 2013.
Published electronically December 4, 2013.
The third author is partially supported by an NSERC Discovery Grant.
AMS subject classification: **37B10**, 37A20.
Keywords: Bratteli diagrams, Vershik maps.

this tells us that there are exactly k copies of the v'-tower placed somewhere in the v-tower. The set of edges with range v, denoted by $r^{-1}(v)$, thus contains all information about how many copies of towers from ξ_n we use to build the v-tower.

We can define a homeomorphism on X_B by putting a linear order on the edges from $r^{-1}(v)$, which describes how we stack our level n towers to get the level $(n+1)$ towers. We do this for each vertex v and each level n. The resulting partial order ω on B (Definition 2.9) admits a map φ_ω on X_B, where each point x moves up the tower to which it belongs. But what if x lives at the top of a tower for each level? In this case x is called a *maximal path*, and it is on this set of maximal paths that we may not be able to extend the definition of φ_ω so that it is continuous. We call an order ω *perfect* if it admits a homeomorphism φ_ω (called a *Vershik* or *adic* map) on X_B. In this case each maximal path is sent to a *minimal* path: one that lives at the bottom of a tower for each level. The model theorem [HPS92, Thm 4.7] tells us that every minimal[1] dynamical system on a Cantor space can be represented as a Bratteli-Vershik system (X_B, φ_ω), where B is a simple Bratteli diagram (Definition 2.3). In [Me06] the model theorem is extended to aperiodic homeomorphisms of a Cantor set where the corresponding Bratteli diagrams are aperiodic (Definition 2.5).

Different orderings on B generate different dynamical systems. In this article, we fix a Bratteli diagram B and study the set $\mathcal{O}_B$ of all orderings on B and its subset $\mathcal{P}_B$ of all perfect orderings on B. We investigate the following questions. Do there exist simple criteria that would allow us to distinguish perfect and non-perfect orderings? Given a diagram B and a natural number j can one define a perfect order on B with j maximal paths? Which diagrams B "support" no perfect orders, *i.e.,* when is $\mathcal{P}_B$ empty? Given a Bratteli diagram B, the set $\mathcal{O}_B$ can be represented as a product space and the product topology turns it into a Cantor set. It can also be endowed with a measure. Since it is natural to assume that orders on $r^{-1}(v)$ have equal probability, we consider the uniformly distributed product measure μ on $\mathcal{O}_B$. In this context, the following questions are of interest. Given a Bratteli diagram B, what can be said about the set $\mathcal{O}_B$ and its subset $\mathcal{P}_B$ from the topological and measurable points of view? It is worth commenting here that we use in this paper the term "ordering", instead of the more usual "order", to stress the difference between the case of ordered Bratteli diagrams, when an order comes with the diagram, and Bratteli diagrams with variable orderings, which is our context.

In Section 2, we study general topological properties of $\mathcal{O}_B$. How "big" is $\mathcal{P}_B$ for a Bratteli diagram B? An order on B is *proper* if it has a unique maximal path and a unique minimal path in X_B. For a simple Bratteli diagram, the set of proper orderings is a nonempty subset of $\mathcal{P}_B$.[2] The relation $\mathcal{O}_B = \mathcal{P}_B$ holds only for diagrams with one vertex at infinitely many levels (Proposition 2.20). With this exception, we show that in the case of most[3] simple diagrams, the set of perfect orderings $\mathcal{P}_B$ and its complement are both dense in $\mathcal{O}_B$ (Proposition 2.23). The case of non-simple Bratteli diagrams is more complicated. An example of a non-simple diagram B such that

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[1] A minimal system (X, T) is one that has no non-trivial proper subsystems: there is no closed, proper $Y \subset X$ such that $T(Y) \subset Y$.

[2] The family of proper orderings generates strongly orbit equivalent Vershik maps ([GPS95, Theorem 2.1] and [GW95, Proposition 5.1]).

[3] We assume, without loss of generality, that all incidence matrix entries are positive (see Definition 2.2).

$\mathcal{P}_B = \varnothing$ was first found by Medynets [Me06]. In this work, we clarify the essence of Medynets' example and describe a wide class of non-simple Bratteli diagrams which support no perfect ordering in Section 3.3.

Can one decide whether a given order is perfect? We are interested mainly in the case when ω is not proper. Suppose that B has the same vertex set V at each level. When an ordering ω is chosen on B, we can consider the set of all words over the alphabet V, formed by sources of consecutive finite paths[4] in B that have the same range. This set of words[5] defines the *language* of the ordered diagram (B, ω) (Definition 3.1). We use the language of (B, ω) to characterize whether or not ω is perfect (Proposition 3.3), in terms of a permutation σ of a finite set. This permutation encodes the action of φ_ω on the set of maximal paths of ω, in this case a finite set. For *finite rank* Bratteli diagrams the number of vertices at each level is bounded. If (B, ω) is an ordered finite rank diagram, it can be *telescoped* (Definitions 2.6 and 2.12) to an ordered diagram (B', ω') where B' has the same vertex set at each level. Since (B, ω) is perfectly ordered if and only if (B', ω') is perfectly ordered (Lemma 3.8), our described characterization of perfect orders in terms of a language can be used to verify whether any order on a finite rank diagram is perfect. As an example of how to apply these concepts, in Section 3.4 we find sufficient conditions for a Bratteli–Vershik system (X_B, φ_ω) to be topologically conjugate to an *odometer* (Definition 3.29).

Next, we wish to study further the set $\mathcal{P}_B$. Let $\mathcal{O}_B(j)$ denote the set of orders with j maximal and j minimal paths. Given a finite rank diagram B, when is $\mathcal{O}_B(j) \cap \mathcal{P}_B \neq \varnothing$? If B has *rank* d (Definition 2.3), then j must be at most d. This problem is only interesting when $j > 1$. For, if B is simple or if B is aperiodic and generates dynamical systems with one minimal component,[6] then $\mathcal{O}_B(1) \subset \mathcal{P}_B$, and it is simple to construct these orders. On the other hand, if B generates dynamical systems with k minimal components, then $\mathcal{O}_B(j) \cap \mathcal{P}_B = \varnothing$ for $j < k$. We mention a result from [GPS95], first proved in [P89], where it was shown that if $\mathcal{P}_B \cap \mathcal{O}_B(j) \neq \varnothing$, then the *dimension group* of B contains a copy of $\mathbb{Z}^{j-1}$ in its *infinitesimal subgroup* (see [GPS95] for definitions of these terms). However the proof of this result sheds little light on the structure of B. Given a finite rank diagram B, we attempt to construct orders in $\mathcal{P}_B \cap \mathcal{O}_B(j)$ by constraining their languages to behave as we would expect a perfect order's language to. Thus we fix a diagram B with the same vertex set at each level, and given an integer j between 2 and the rank of B, we fix a permutation σ of $\{1, \ldots, j\}$. We then create a framework to build perfect orderings ω such that φ_ω acts on the set of ω-maximal paths according to the instructions given by σ. We build such orderings by first specifying the set of all maximal edges in a certain way. This is the idea behind the notion of a *skeleton* $\mathcal{F}$ (Definition 3.13), which partially defines an order. Given a skeleton and permutation, we define a (directed) associated graph $\mathcal{H}$ (Definition 3.15). The graph $\mathcal{H}$, whose paths will correspond to words in the language of the putative perfect order, is used to take the partial instructions that we have been given by $\mathcal{F}$ and extend them to a perfect order on B. Whether a perfect order exists on B with a specified skeleton depends on whether the *incidence matrices*

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[4] Consecutive finite paths are determined by the given order ω on B

[5] Rather, the subset of this set of words that are "seen" infinitely often.

[6] We use the term "minimal component" as a synonym to "minimal subset". A dynamical system with k minimal components has k proper nontrivial minimal subsystems.

of B (Definition 2.2) are related according to Theorem 4.6. The simplest case is if B a simple rank d diagram and $\mathcal{O}_B(d) \cap \mathcal{P}_B \neq \varnothing$. Then B's incidence matrices (F_n) are almost completely determined, as is the dynamical behaviour of the corresponding φ_ω (Theorem 3.32). A consequence of Theorem 4.6 and Remark 4.7, along with the fact that aperiodic Cantor homeomorphisms can be represented as adic systems, is that non-minimal aperiodic dynamical systems do not exist in abundance. We remark that these notions can be generalized to non-finite rank diagrams; however the corresponding definitions are more technical, especially notationally.

In Section 5, we endow the set $\mathcal{O}_B$ with the uniform product measure, and study questions about the measure of specific subsets of $\mathcal{O}_B$. The results of this section are independent of those in Sections 3 and 4. We show in Theorem 5.1 that for a finite rank d diagram there is some $1 \leq j \leq d$ such that almost all orderings have exactly j maximal and j minimal paths. Whether for diagrams with isomorphic dimension groups the j is the same is an open question. In particular, in this section we cannot freely telescope our diagram: if B' is a telescoping of B, then $\mathcal{O}_B$ is a set of 0 measure in $\mathcal{O}_B'$. We give necessary and sufficient conditions, in terms of the incidence matrices of B, for verifying the value of j, and show that $j = 1$ for a large class of diagrams which include linearly recurrent diagrams. We show in Theorem 5.4 that if B is simple and $j > 1$, then a random ordering is not perfect.

We end with some questions. If B' is a telescoping of B, how do $\mathcal{P}_B$ and $\mathcal{P}_B'$ compare? Do Bratteli diagrams that support non-proper, perfect orders have special spectral properties? Do their dimension groups have any additional structure? Can one identify any interesting topological factors? Do these results generalize in some way to non-finite rank diagrams? If B has finite rank and almost all orders on B have j maximal paths, is j invariant under telescoping?

2 Bratteli Diagrams and Vershik Maps

2.1 Main Definitions on Bratteli Diagrams

In this section, we collect the notation and basic definitions that are used throughout the paper. More information about Bratteli diagrams can be found in the papers [HPS92], [GPS95], [DHS99], [Me06], [BKM09], [BKMS10], [D10] and references therein.

Definition 2.1 A *Bratteli diagram* is an infinite graph $B = (V^*, E)$ such that the vertex set $V^* = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i where

(i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exists a range map r and a source map s, both from E to V^*, such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \varnothing$, $r^{-1}(v') \neq \varnothing$ for all $v \in V^*$ and $v' \in V^* \setminus V_0$.

The pair (V_i, E_i) or just V_i is called the i-th level of the diagram B. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. For $m < n$, $v \in V_m$ and $w \in V_n$, let $E(v, w)$ denote the set of all paths $\overline{e} = (e_1, \ldots, e_p)$ with $s(e_1) = v$ and $r(e_p) = w$. If $m > n$, let $E(n, m)$ denote all paths whose source belongs to V_n and whose range belongs to V_m. For a Bratteli diagram B, let X_B be the set of infinite paths starting at the top vertex v_0. We endow X_B with the topology generated by cylinder sets

$$\{U(e_j, \ldots, e_n) : j, n \in \mathbb{N}, \text{ and } (e_j, \ldots, e_n) \in E(v, w), v \in V_{j-1}, w \in V_n\},$$

where $U(e_j, \ldots, e_n) := \{x \in X_B : x_i = e_i,\ i = j, \ldots, n\}$. With this topology, X_B is a 0-dimensional compact metric space. We will consider such diagrams B for which the path space X_B has no isolated points. Letting $|A|$ denote the cardinality of the set A, this means that for every $(x_1, x_2, \ldots) \in X_B$ and every $n \geq 1$ there exists $m > n$ such that $|s^{-1}(r(x_m))| > 1$.

Definition 2.2 Given a Bratteli diagram B, the n-th *incidence matrix* $F_n = (f^{(n)}_{v,w})$, $n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix whose entry $f^{(n)}_{v,w}$ is equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, *i.e.*,

$$f^{(n)}_{v,w} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

Observe that every vertex $v \in V^*$ is connected to v_0 by a finite path, and the set $E(v_0, v)$ of all such paths is finite. Set $h^{(n)}_v = |E(v_0, v)|$ for $v \in V_n$. Then

$$h^{(n+1)}_v = \sum_{w \in V_n} f^{(n)}_{v,w} h^{(n)}_w \quad \text{or} \quad h^{(n+1)} = F_n h^{(n)},$$

where $h^{(n)} = (h^{(n)}_w)_{w \in V_n}$.

Next we define some popular families of Bratteli diagrams that we work with in this article.

Definition 2.3 Let B be a Bratteli diagram.

(a) We say B has *finite rank* if for some k, $|V_n| \leq k$ for all $n \geq 1$.
(b) Let B have finite rank. We say B has *rank d* if d is the smallest integer such that $|V_n| = d$ infinitely often.
(c) We say that B is *simple* if for any level n there is $m > n$ such that $E(v, w) \neq \varnothing$ for all $v \in V_n$ and $w \in V_m$.
(d) We say B is *stationary* if $F_n = F_1$ for all $n \geq 2$.

Definition 2.4 For a Bratteli diagram B, the *tail (cofinal) equivalence* relation $\mathcal{E}$ on the path space X_B is defined as $x\mathcal{E}y$ if $x_n = y_n$ for all n sufficiently large, where $x = (x_n)$, $y = (y_n)$.

Let $X_{\text{per}} = \{x \in X_B : |[x]_{\mathcal{E}}| < \infty\}$. By definition, we have

$$X_{\text{per}} = \left\{ x \in X_B : \exists n > 0 \text{ such that } \left(|r^{-1}(r(x_i))| = 1\ \forall i \geq n\right) \right\}.$$

Definition 2.5 A Bratteli diagram B is called *aperiodic* if $X_{\mathrm{per}} = \varnothing$; *i.e.,* every $\mathcal{E}$-orbit is countably infinite.

We shall constantly use the following telescoping procedure for a Bratteli diagram.

Definition 2.6 Let B be a Bratteli diagram and $n_0 = 0 < n_1 < n_2 < \cdots$ be a strictly increasing sequence of integers. The *telescoping of B to (n_k)* is the Bratteli diagram B' whose k-level vertex set $V'_k = V_{n_k}$ and whose incidence matrices (F'_k) are defined by

$$F'_k = F_{n_{k+1}-1} \circ \cdots \circ F_{n_k},$$

where (F_n) are the incidence matrices for B.

Roughly speaking, in order to telescope a Bratteli diagram, one takes a subsequence of levels (n_k) and considers the set $E(n_k, n_{k+1})$ of all finite paths between the levels (n_k) and (n_{k+1}) as edges of the new diagram. In particular, a Bratteli diagram B has rank d if and only if there is a telescoping B' of B such that B' has exactly d vertices at each level. When telescoping diagrams, we often do not specify to which levels (n_k) we telescope, because it suffices to know that such a sequence of levels exists.

Lemma 2.7 *Every aperiodic Bratteli diagram B can be telescoped to a diagram B' with the property that $|r^{-1}(v)| \geq 2,\ v \in V^* \setminus V_0$ and $|s^{-1}(v)| \geq 2,\ v \in V^* \setminus V_0$.*

In other words, we can state that, for any aperiodic Bratteli diagram, the properties $|r^{-1}(v)| \geq 2,\ v \in V^* \setminus V_0$, and $|s^{-1}(v)| \geq 2,\ v \in V^* \setminus V_0$ hold for infinitely many levels n.

Proof We shall show that any periodic diagram B can be telescoped so that

$$|r^{-1}(v)| \geq 2,\ v \in V^* \setminus V_0;$$

the proof of the other statement is similar. We need to show that for every $n \in \mathbb{N}$ there exists $m > n$ such that for each vertex $v \in V_m$ there are at least two finite paths $e, f \in E(n,m)$ with $r(e) = r(f) = v$. Assume that the converse is true. Then there exists n such that for all $m > n$ the set

$$U_m = \left\{x = (x_i) \in X_B : |r^{-1}(r(x_i))| = 1,\ i = n+1, \ldots, m\right\}$$

is not empty. Clearly, U_m is a clopen subset of X_B and $U_m \supset U_{m+1}$. It follows that $X_{\mathrm{per}} \supset U = \bigcap_{m>n} U_m \neq \varnothing$. This contradicts the aperiodicity of the diagram. ■

We will assume the convention that *our diagrams are never disjoint unions of two subdiagrams*. Here $B = (V^*, E)$ is a *disjoint union* of $B^1 = (V^{*,1}, E^1)$ and $B^2 = (V^{*,2}, E^2)$ if $V^* = V^{*,1} \cup V^{*,2}$, $V^{*,1} \cap V^{*,2} = \{v_0\}$ and $E = E^1 \sqcup E^2$.

Throughout the paper, we only consider aperiodic Bratteli diagrams B. For these diagrams X_B is a Cantor set and $\mathcal{E}$ is a Borel equivalence relation on X_B with countably infinitely many equivalence classes.

Remark 2.8 Given an aperiodic dynamical system (X, T), a Bratteli diagram is constructed by a sequence of Kakutani–Rokhlin partitions generated by (X, T) (see [HPS92] and [Me06]). The n-th level of the diagram corresponds to the n-th Kakutani-Rokhlin partition and the number $h_w^{(n)}$ is the height of the T-tower labeled by the symbol w from that partition.

2.2 Orderings on a Bratteli Diagram

Let B be a Bratteli diagram whose path space X_B is a Cantor set.

Definition 2.9 A Bratteli diagram $B = (V^*, E)$ is called *ordered* if a linear order "$>$" is defined on every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$. We use ω to denote the corresponding partial order on E and write (B, ω) when we consider B with the ordering ω. Denote by $\mathcal{O}_B$ the set of all orderings on B.

Every $\omega \in \mathcal{O}_B$ defines the *lexicographic* ordering on the set $E(k, l)$ of finite paths between vertices of levels V_k and V_l: $(e_{k+1}, \dots, e_l) > (f_{k+1}, \dots, f_l)$ if and only if there is i with $k+1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It follows that, given $\omega \in \mathcal{O}_B$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by ω. If two infinite paths are tail equivalent and agree from the vertex v onwards, then we can compare them by comparing their initial segments in $E(v_0, v)$. Thus ω defines a partial order on X_B, where two infinite paths are comparable if and only if they are tail equivalent.

Definition 2.10 We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every e_i is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$.

Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}(\omega)$ and $X_{\min}(\omega)$ the sets of all maximal and minimal infinite paths in X_B, respectively. It is not hard to show that $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are *non-empty closed subsets* of X_B; in general, $X_{\max}(\omega)$ and $X_{\min}(\omega)$ may have interior points. For a finite rank Bratteli diagram B, the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are always finite for any ω, and if B has rank d, then each of them have at most d elements [BKM09, Proposition 6.2].

Definition 2.11 An ordered Bratteli diagram (B, ω) is called *properly ordered* if the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are singletons.

We denote by $\mathcal{O}_B(j)$ the set of all orders on B that have j maximal and j minimal paths. Thus $\mathcal{O}_B(1)$ is the set of proper orders.

Definition 2.12 Let (B, ω) be an ordered Bratteli diagram, and suppose that $B' = (V', E')$ is the telescoping of B to levels (n_k). Let $v' \in V'$ and suppose that the two edges e'_1, e'_2, both with range v', correspond to the finite paths e_1, e_2 in B, both with range v. Define the order ω' on B' by $e'_1 < e'_2$ if and only if $e_1 < e_2$. Then ω' is called the *lexicographic order generated by* ω and is denoted by $\omega' = L(\omega)$.

It is not hard to see that if $\omega' = L(\omega)$, then

$$|X_{\max}(\omega)| = |X_{\max}(\omega')|, \ \ |X_{\min}(\omega)| = |X_{\min}(\omega')|.$$

Let (B,ω) be an ordered Bratteli diagram. Then $x \in X_{\max}(\omega) \cap X_{\min}(\omega)$ if and only if $|\mathcal{E}(x)| = 1$. Thus, if B is an aperiodic Bratteli diagram, then $X_{\max}(\omega) \cap X_{\min}(\omega) = \varnothing$.

Definition 2.13 Let B be a stationary diagram. We say an ordering $\omega \in \mathcal{O}_B$ is *stationary* if the partial linear order defined by ω on the set E_n of all edges between levels V_{n-1} and V_n does not depend on n for $n > 1$.

It is well known that for every stationary ordered Bratteli diagram (B,ω) one can define a "substitution τ read on B' by the following rule. For each vertex $i \in V = \{1, 2, \dots, d\}$, we write $r^{-1}(i) = \{e_1, \dots, e_t\}$ where $e_1 < e_2 < \cdots < e_t$ with respect to ω. Then we set $\tau(i) = j_1 j_2 \cdots j_t$ where $j_k = s(e_k)$, $k = 1, \dots, t$; this defines the substitution read on B. Conversely, such a substitution τ describes completely the stationary ordered Bratteli diagram (B,ω) whose vertex set V_n coincides with the alphabet of τ for all $n \geq 1$.

Now we give a useful description of infinite paths in an ordered Bratteli diagram (B,ω) (see also [BDK06]). Take $v \in V_n$ and consider the finite set $E(v_0, v)$, whose cardinality is $h_v^{(n)}$. The lexicographic ordering on $E(v_0, v)$ gives us an enumeration of its elements from 0 to $h_v^{(n)} - 1$, where 0 is assigned to the minimal path and $h_v^{(n)} - 1$ is assigned to the maximal path in $E(v_0, v)$. Note that $h_v^{(1)} = f_{vv_0}^{(0)}$ for $v \in V_1$, and we have by induction for $n > 1$,

$$h_v^{(n)} = \sum_{w \in s(r^{-1}(v))} |E(w, v)| h_w^{(n-1)}, \quad v \in V_n.$$

Let $y = (e_1, e_2, \dots)$ be an infinite path from X_B. Consider a sequence (P_n) of enlarging finite paths defined by y where $P_n = (e_1, \dots, e_n) \in E(v_0, r(e_n))$, $n \in \mathbb{N}$. Then every P_n can be identified with a pair (i_n, v_n) where $v_n = r(e_n)$ and $i_n \in [0, h_{v_n}^{(n)} - 1]$ is the number assigned to P_n in $E(v_0, v_n)$. Thus, every $y = (e_n) \in X_B$ is uniquely represented as the infinite sequence (i_n, v_n) with $v_n = r(e_n)$ and $0 \leq i_n \leq h_{v_n}^{(n)} - 1$. We refer to the sequence (i_n, v_n) as the *associated sequence.*

Proposition 2.14 *Two infinite paths $e = (e_1, e_2, \dots)$ and $e' = (e'_1, e'_2, \dots)$ from the path space X_B are cofinal with respect to $\mathcal{E}$ if and only if the sequences (i_n, v_n) and (i'_n, v'_n) associated with e and e' satisfy the following condition: there exists $m \in \mathbb{N}$ such that $v_n = v'_n$ and $i_n - i'_n = i_m - i'_m$ for all $n \geq m$.*

Proof Suppose e and e' are cofinal. Take m such that $e_n = e'_n$ for all $n \geq m$. Consider the associated sequences (i_n, v_n) and (i'_n, v'_n). Then we see that $v_n = v'_n$ for all $n \geq m$. Without loss of generality, we can assume that $c_m = i_m - i'_m \geq 0$. This means that the finite path $P_m = P(e_1, \dots, e_m)$ is the c_m-th successor of the finite path $P'_m = P(e'_1, \dots, e'_m)$. Let $c_{m+1} = i_{m+1} - i'_{m+1}$. By definition of the lexicographic ordering on $E(v_0, v_{m+1})$, we obtain that $c_{m+1} = c_m$. Thus, by induction, $c_n = c_m$ for all $n \geq m$.

Conversely, suppose that two associated sequences (i_n, v_n) and (i'_n, v'_n) possess the following property. There exists $m \in \mathbb{N}$ such that $v_n = v'_n$ and $i_n - i'_n = i_m - i'_m$

for all $n \geq m$. To see that e and e' are cofinal, notice that e_{m+1} and e'_{m+1} are in $E(v_m, v_{m+1})$. By definition of the lexicographic ordering on $E(v_0, v_{m+1})$, we conclude that $e_{m+1} = e'_{m+1}$. ■

Proposition 2.15 *A Bratteli diagram B admits an ordering $\omega \in \mathcal{O}_B$ on B with* $\mathrm{Int}(X_{\max}(\omega)) \neq \varnothing$ *if and only if there exist $x = (x_i) \in X_B$ and $n > 0$ such that* $U(x_1, \dots, x_n) = \{y \in X_B : y_i = x_i,\ i = 1, \dots, n\}$ *has no cofinal paths;* i.e., $U(x_1, \dots, x_n)$ *meets each $\mathcal{E}$-orbit at most once. A similar result holds for* $\mathrm{Int}(X_{\min}(\omega))$.

Proof Let x be an interior point of $X_{\max(\omega)}$. Then there is an $n > 0$ such that $U(x_1, \dots, x_n) \subset X_{\max}(\omega)$; thus, $U(x_1, \dots, x_n)$ contains no distinct cofinal paths.

Now, suppose that there exist $x = (x_i) \in X_B$ and $n > 0$ such that $U = U(x_1, \dots, x_n)$ meets each $\mathcal{E}$-orbit at most once. Define a linear order ω_v on $r^{-1}(v)$, $v \in V^* \setminus V_0$, as follows. If there exists an $e \in r^{-1}(v)$ that is an edge in an infinite path $y \in U$, then we order $r^{-1}(v)$ such that e is maximal in $r^{-1}(v)$. If such an e does not exist, we order $r^{-1}(v)$ in an arbitrary way. It follows that for this ordering, $U \subset X_{\max}(\omega)$. ■

Definition 2.16 A Bratteli diagram B is called *regular* if for any ordering $\omega \in \mathcal{O}_B$ the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have empty interior.

In particular, finite rank Bratteli diagrams are regular.

Given a Bratteli diagram B, we can describe the set of all orderings $\mathcal{O}_B$ in the following way. Given a vertex $v \in V^* \setminus V_0$, let P_v denote the set of all orders on $r^{-1}(v)$; an element in P_v is denoted by ω_v. Then $\mathcal{O}_B$ can be represented as

$$\mathcal{O}_B = \prod_{v \in V^* \setminus V_0} P_v. \tag{2.1}$$

Giving each set P_v the discrete topology, it follows from (2.1) that $\mathcal{O}_B$ is a Cantor set with respect to the product topology. In other words, two orderings $\omega = (\omega_v)$ and $\omega' = (\omega'_v)$ from $\mathcal{O}_B$ are close if and only if they agree on a sufficiently long initial segment: $\omega_v = \omega'_v$, $v \in \bigcup_{i=0}^{k} V_i$.

It is worth noticing that the order space $\mathcal{O}_B$ is sensitive with respect to a telescoping. Indeed, let B be a Bratteli diagram and B' denote the diagram obtained by telescoping B with respect to a subsequence (n_k) of levels. We see that any ordering ω on B can be extended to the (lexicographic) ordering ω' on B'. Hence the map $L\colon \omega \to \omega' = L(\omega)$ defines a closed proper subset $L(\mathcal{O}_B)$ of $\mathcal{O}_{B'}$.

The set of all orderings $\mathcal{O}_B$ on a Bratteli diagram B can be considered also as a *measure space* whose Borel structure is generated by cylinder sets. On the set $\mathcal{O}_B$ we take the product measure $\mu = \prod_{v \in V^* \setminus V_0} \mu_v$ where μ_v is a measure on the set P_v. The case where each μ_v is the uniformly distributed measure on P_v is of particular interest: $\mu_v(\{i\}) = (|r^{-1}(v)|!)^{-1}$ for every $i \in P_v$ and $v \in V^* \setminus V_0$. Unless $|V_n| = 1$ for almost all n, if B' is a telescoping of B, then in $\mathcal{O}_{B'}$, $L(\mathcal{O}_B)$ is a set of zero measure.

2.3 Vershik Maps

Definition 2.17 Let (B, ω) be an ordered Bratteli diagram. We say that $\varphi = \varphi_\omega\colon X_B \to X_B$ is a *(continuous) Vershik map* if it satisfies the following conditions:

(i) φ is a homeomorphism of the Cantor set X_B;
(ii) $\varphi(X_{\max}(\omega)) = X_{\min}(\omega)$;
(iii) if an infinite path $x = (x_1, x_2, \dots)$ is not in $X_{\max}(\omega)$, then $\varphi(x_1, x_2, \dots) = (x_1^0, \dots, x_{k-1}^0, \overline{x_k}, x_{k+1}, x_{k+2}, \dots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of x_k in $r^{-1}(r(x_k))$, and $(x_1^0, \dots, x_{k-1}^0)$ is the minimal path in $E(v_0, s(\overline{x_k}))$.

If ω is an ordering on B, then one can always define the map φ_0 that maps $X_B \setminus X_{\max}(\omega)$ onto $X_B \setminus X_{\min}(\omega)$ according Definition 2.17(iii). The question about the existence of the Vershik map is equivalent to that of an extension of

$$\varphi_0 \colon X_B \setminus X_{\max}(\omega) \longrightarrow X_B \setminus X_{\min}(\omega)$$

to a homeomorphism of the entire set X_B. If ω is a proper ordering, then φ_ω is a homeomorphism. For a finite rank Bratteli diagram B, the situation is simpler than for a general Bratteli diagram because the sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are finite.

Definition 2.18 Let B be a Bratteli diagram B. We say that an ordering $\omega \in \mathcal{O}_B$ is *perfect* if ω admits a Vershik map φ_ω on X_B. Denote by $\mathcal{P}_B$ the set of all perfect orderings on B. We call an ordering belonging to $\mathcal{P}_B^c$ (the complement of $\mathcal{P}_B$ in $\mathcal{O}_B$) *imperfect*.

We observe that for a regular Bratteli diagram with an ordering ω, the Vershik map φ_ω, if it exists, is defined in a unique way. More precisely, if B is a regular Bratteli diagram such that the set $\mathcal{P}_B$ is not empty, then the map $\Phi \colon \omega \mapsto \varphi_\omega \colon \mathcal{P}_B \to \mathrm{Homeo}(X_B)$ is injective. Also, a necessary condition for $\omega \in \mathcal{P}_B$ is that $|X_{\max}(\omega)| = |X_{\min}(\omega)|$. If B has rank d, then $\mathcal{O}_B \cap \mathcal{P}_B \subset \bigcup_{j=1}^{d} \mathcal{O}_B(j)$.

Remark 2.19 We note that if B is a simple Bratteli diagram with positive entries in all its incidence matrices, then the set $\mathcal{P}_B \neq \varnothing$. Indeed, it is not hard to see that if x and y are two paths in X_B going through disjoint edges at each level, then one can find an ordering ω on B such that $X_{\max}(\omega) = \{x\}$ and $X_{\min}(\omega) = \{y\}$. Simply choose all maximal edges in E_n to go through the same vertex that x goes through at level $n-1$, and all minimal edges in E_n to go through the same vertex that y goes through at level $n-1$, for each n. Then ω is properly ordered, and so $\omega \in \mathcal{P}_B$.

Another example of a family of perfect (indeed proper) orders for a simple Bratteli diagram, all of whose incidence matrices are positive, is the following. For each n, fix a labeling $V_n = \{v(n,1), \dots v(n,k_n)\}$ of V_n. Take $v \in V_{n+1}$ and enumerate the edges from $E(v(n,1), v)$ in an arbitrary order from 0 to $|E(v(n,1), v)| - 1$. Similarly, for $2 \leq i \leq k_n$, we enumerate edges from $E(v(n,i), v)$ by numbers from

$$\sum_{j=1}^{i-1} |E(v(n,j), v)| \quad \text{to} \quad \sum_{j=1}^{i} |E(v(n,j), v)| - 1.$$

Repeating this procedure for each vertex $v \in V^* \setminus V_0$ and each level n, we define an order ω_0 on B called a *natural order*. This is a variation of the well known “left-to-right” order. For ω_0, the unique minimal path runs through $v(n,1)$, and the unique maximal path runs through $v(n,k_n)$.

In the next section, we will describe a class of non-simple Bratteli diagrams that do not admit a perfect ordering.

Proposition 2.20 *Let B be a simple Bratteli diagram, where the entries of the incidence matrices (F_n) are positive. Then $\mathcal{P}_B = \mathcal{O}_B$ holds if and only if B is rank 1.*

Proof The part "if" is obvious because the condition $|V_n| = 1$ for infinitely many levels n implies any ordering is proper.

Conversely, suppose that the rank of B is at least 2. Then for some N, $|V_n| \geq 2$ when $n > N$. We need to show that, in this case, there are imperfect orderings.

First, assume that infinitely often, $|V_n| \geq 3$. Call three distinct vertices at these levels u_n, v_n, and w_n. For the other levels $n > M$, there are at least two distinct vertices u_n and v_n. For levels n such that $|V_n| \geq 3$, choose all maximal edges in E_{n+1} to have source w_n. Let the minimal edges with ranges u_{n+1}, v_{n+1} have source u_n, v_n respectively. For levels n such that $|V_n| = 2$, let the minimal edges with ranges u_{n+1}, v_{n+1} have source u_n, v_n, respectively. Any order that satisfies these constraints has only one maximal path and at least two minimal paths, so cannot be perfect.

Next suppose that B has rank 2, and suppose two sequences of vertices (v_n) and (w_n) can be found such that $v_n \neq w_n$ for each $n > N$, $v_n, w_n \in V_n$ and $|E(w_n, w_{n+1})| > 1$ infinitely often. Let the minimal edge with range v_{n+1} have source v_n. Similarly, let the minimal edge with range w_{n+1} have source w_n. Whenever $|E(w_n, w_{n+1})| > 1$, choose all maximal edges in E_{n+1} to have source w_n. The resulting order has one maximal and two minimal paths.

Finally suppose that B does not satisfy the above conditions. Then for all large n, the matrices F_n are equal to $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and there are orders on B with two maximal and two minimal paths. To see this we just ensure that for all large n, the two minimal edges have distinct sources, as do the two maximal edges. Now Example 3.5 shows that no such ordering is perfect. ■

In contrast, one can find aperiodic diagrams for which any ordering is perfect. Indeed, it suffices to take a rooted tree and turn it into a non-simple Bratteli diagram B by replacing every single edge with a strictly larger number of edges. Then every ordering on B produces a continuous Vershik map.

Remark 2.21 Let (B, ω) be an ordered Bratteli diagram and let ω' be an ordering on B such that ω and ω' are different on $r^{-1}(v)$ only for a finite number of vertices v. Then ω is perfect if and only if ω' is perfect.

Proposition 2.22 *Let B be a regular Bratteli diagram such that the set $\mathcal{P}_B$ is not empty. Let $\mathcal{P}_B$ be equipped with the topology induced from $\mathcal{O}_B$ and let the set $\Phi(\mathcal{P}_B)$ be equipped with the topology of uniform convergence induced from the group* $\mathrm{Homeo}(X_B)$ *where the map $\Phi\colon \omega \mapsto \varphi_\omega$ has been defined above. Then $\Phi\colon \mathcal{P}_B \to \Phi(\mathcal{P}_B)$ is a homeomorphism.*

Proof We need only to show that Φ and Φ^{-1} are continuous, because the injectivity of Φ is obvious.

Fix an ordering $\omega_0 \in \mathcal{P}_B$ and let φ_{ω_0} be the corresponding Vershik map. Consider a neighborhood

$$W = W(\varphi_{\omega_0}; E_1, \ldots, E_k) = \{f \in \mathrm{Homeo}(X_B) : f(E_i) = \varphi_{\omega_0}(E_i),\ i = 1, \ldots, k\}$$

of φ_{ω_0} defined by clopen sets $E_1, \dots, E_k$. It is well known that the uniform topology is generated by the base of neighborhoods $\{W\}$. Take $m \in \mathbb{N}$ such that all clopen sets $E_1, \dots, E_k$ "can be seen" at the first m levels of the diagram B. This means that every set E_i is a finite union of the cylinder sets defined by finite paths of length m.

Suppose $\omega_n \to \omega_0$ where $\omega_n \in \mathcal{P}_B$. By (2.1), the ordering ω_0 is an infinite sequence in the product $\prod_{v \in V^* \setminus V_0} P_v$. Let Q be the neighborhood of ω_0 in $\mathcal{O}_B$ that is defined by the finite part of ω_0 from v_0 to V_{m+1}. Find N such that $\omega_n \in Q$ for all $n \geq N$. This means that the ordering ω_n $(n \geq N)$ agrees with ω_0 on the first $m+1$ levels of the diagram B. Therefore, φ_{ω_n} acts as φ_{ω_0} on all finite paths from v_0 to V_m. Hence, $\varphi_{\omega_n}(E_i) = \varphi_{\omega_0}(E_i)$ and $\varphi_{\omega_n} \in W$.

Conversely, let $\varphi_{\omega_n} \to \varphi_\omega$ in the topology of uniform convergence; we prove that $\omega_n \to \omega$. Take the neighborhood $Q(\omega)$ of ω consisting of all orderings ω' such that ω' agrees with ω on the sets $r^{-1}(v)$, where $v \in \bigcup_{i=1}^{N} V_i$. Let $F_1, \dots, F_p$ denote all cylinder subsets of X_B corresponding to the finite paths between v_0 and the vertices from $\bigcup_{i=1}^{N+1} V_i$. Consider the neighborhood $W = W(\varphi_\omega; F_1, \dots, F_p)$. Then there exists an $m \in \mathbb{N}$ such that $\varphi_{\omega_i} \in W$ for $i \geq m$. This means that $\varphi_{\omega_i}(F_j) = \varphi_\omega(F_j)$ for all $j = 1, \dots, p$. Let us check that $\omega_i \in Q(\omega)$ for $i \geq m$. Indeed, if one assumes that $\omega' \notin Q(\omega)$, then there exists a least k and a vertex $v \in V_k$ such that ω and ω' define different linear orders on $r^{-1}(v)$, but ω and ω' agree for all $v \in \bigcup_{i=1}^{k-1} V_i$. Let e be an edge from $r^{-1}(v)$ such that the ω-successor and ω'-successor of e are different edges. Then take the cylinder set F that corresponds to the finite path (f, e), where f is the maximal path from v_0 to $s(e)$ for both the orders. It follows from the above construction that $\varphi_\omega(F) \neq \varphi_{\omega'}(F)$, a contradiction. ■

Theorem 2.23 *Let B be a simple rank d Bratteli diagram where $d \geq 2$ and all incidence matrix entries are positive. Then both sets $\mathcal{P}_B$ and $\mathcal{P}_B^c$ are dense in $\mathcal{O}_B$.*

Proof By Proposition 2.20, $\mathcal{P}_B^c \neq \varnothing$. Take an ordering $\omega \in \mathcal{O}_B$ and consider its neighborhood

$$U_N(\omega) = \left\{ \omega' \in \mathcal{O}_B : \omega \text{ and } \omega' \text{ coincide on } r^{-1}(v) \text{ for all } v \in \bigcup_{i=1}^{N} V_i \right\}.$$

We have assumed that N is large enough that $|V_n| \geq 2$ for $n > N$.

Then there exists a perfect ordering ω_1 belonging to $U_N(\omega)$. To see this, choose $(u_n)_{n>N}$, $(v_n)_{n>N}$ where $u_n \neq v_n$ and $u_n, v_n \in V_n$. Choose an ordering all of whose maximal edges in E_{n+1} have source u_n and all of whose minimal edges in E_{n+1} have source v_n, for $n > N$. Let this ordering agree with ω up to level N. This ordering is proper, hence perfect.

Conversely, if ω is perfect, we can construct ω^N by letting ω^N agree with ω on the first N levels. Beyond level N, we work as in the proof of Proposition 2.20 to define ω^N so that it is imperfect. ■

3 Finite Rank Ordered Bratteli Diagrams

In this section, we focus on the study of orderings on a finite rank Bratteli diagram B. To do this, we define new notions related to an unordered finite rank Bratteli diagram that will be used in our considerations. If (B, ω) is ordered and $V_n = V$ for each n, then in Section 3.1 we first define the language generated by ω, and characterize whether (B, ω) is perfect in terms of the language of ω. Our notions of skeleton and associated graph are defined in Section 3.2 for non-ordered diagrams. We note that on one diagram, there exist several skeletons. By telescoping a perfectly ordered diagram in a particular way, we will obtain the (unique, up to labeling) skeleton associated with the lexicographical image of ω under the telescoping. In the associated graph $\mathcal{H}$, paths will correspond to (families of) words in ω's language. Given a skeleton $\mathcal{F}$ on a diagram, we describe how $\mathcal{H}$ constrains us when trying to extend $\mathcal{F}$ to a perfect order.

In Section 3.3 we describe a class of non-simple diagrams that do not admit any perfect ordering, using the poor connectivity properties of any skeleton's associated graph. In Section 3.4 we give descriptions of perfect orderings that yield odometers, in terms of their language, and explicitly describe, in terms of an associated skeleton and associated graph, the class of rank d diagrams that can have a perfect ordering with exactly $k \leq d$ maximal and minimal paths.

3.1 Language of a Finite Rank Diagram

Let ω be an ordering on a Bratteli diagram B where $V_n = V$ for each $n \geq 1$ and $|V| = d$. For each vertex $v \in V_n$ and each m such that $1 \leq m < n$, consider $\bigcup_{w \in V_m} E(w, v)$ as the ω-ordered set $\{e_1, \ldots e_p\}$ where $e_i < e_{i+1}$ for $1 \leq i \leq p-1$. Define the word $w(v, m, n) := s(e_1)s(e_2)\cdots s(e_p)$ over the alphabet V. We use the notation $w' \subseteq w$ to indicate that w' is a subword of w, and, if w and w' are two words, by ww' we mean the word that is the concatenation of w and w'.

Definition 3.1 The set

$$\mathcal{L}_{B,\omega} = \{w : w \subseteq w(v_n, m_n, n), \text{ for infinitely many } n \text{ where } v_n \in V_n, 1 \leq m_n < n\}$$

is called the *language* of B with respect to the ordering ω.

We remark that the notion of the language $\mathcal{L}_{B,\omega}$ is not always robust under telescoping. Let (B', ω') be a telescoping of an ordered Bratteli diagram (B, ω) where $\omega' = L(\omega)$. Then $\mathcal{L}_{B',\omega'} \subset \mathcal{L}_{B,\omega}$ where the inclusion can be strict. For example, consider B where

$$F_{2n} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad F_{2n-1} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, \quad n \geq 1.$$

Let ω be defined by the substitution $\tau_1(a) = aba$, $\tau_1(b) = aaba$ on E_{2n} and by the substitution $\tau_2(a) = bab$, $\tau_2(b) = abba$ on E_{2n-1} for $n \geq 1$. Thus the order of letters in a word $\tau(v)$ determines the order on the sets of edges with range v. Then $\{aa, ab, ba, bb\} \subset \mathcal{L}_{B,\omega}$. Now telescope B to the levels $(2n+1)$ to get the stationary Bratteli diagram B' whose incidence matrix is

$$F'_n = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 10 & 4 \end{pmatrix}$$

for each $n \geq 1$ so that $\omega' := L(\omega)$ is defined by the substitution $\tau := \tau_1 \circ \tau_2$ where $\tau(a) = aaba\,aba\,aaba$ and $\tau(b) = aba\,aaba\,aaba\,aba$, then $bb \notin \mathcal{L}_{B',\omega'}$. Note however that both ω and ω' are perfect (in fact proper).

Also, in the special case where B is stationary and ω is defined by a substitution τ (so that ω is also stationary), we see that $\mathcal{L}_{B,\omega}$ is precisely the language $\mathcal{L}_\tau$ defined by the substitution τ, and in this case, if B' is a telescoping of B to levels (n_k) with $\omega' = L(\omega)$, then $\mathcal{L}_{B,\omega} = \mathcal{L}_{B',\omega'}$. Indeed, any word $w \in \mathcal{L}_{B,\omega}$ is a subword of $\tau^j(a)$ for some $j \in \mathbb{N}$ and letter a. Now the order on the k-th level of B' is generated by $\tau^{n_k - n_{k-1}}$, and as long as $n_k - n_{k-1} > j$, we will see w as a subword of $w(a, n_{k-1}, n_k) \subset \mathcal{L}_{B',\omega'}$. The relationship between $\mathcal{L}_{B,\omega}$ and the continuity of the Vershik map has been studied in [Yas11] in the case where ω is stationary, *i.e.*, generated by a substitution, and also in [HZ01].[7]

Definition 3.2 Suppose B is such that $V_n = V$ for each $n \geq 1$. If ω is an order on B, where a maximal (minimal) path M (m) goes through the same vertex v_M (v_m) for each level $n \geq 1$ of B, we will call this path *vertical*.

We note that for any order ω on a finite rank Bratteli diagram B there exists a telescoping B' of B such that the extremal (maximal and minimal) paths with respect to $\omega' = L(\omega)$ are vertical. The following proposition characterizes when ω is a perfect ordering on such a finite rank Bratteli diagram.

Proposition 3.3 *Let (B, ω) be a finite rank ordered Bratteli diagram where $V_n = V$ for each $n \geq 1$. Suppose that the ω-maximal and ω-minimal paths $M_1, \ldots, M_k$ and $m_1, \ldots, m_{k'}$ are vertical passing through the vertices $v_{M_1}, \ldots, v_{M_k}$ and $v_{m_1}, \ldots, v_{m_{k'}}$, respectively. Then ω is perfect if and only if*

(i) *$k = k'$ and*

(ii) *there is a permutation σ of $\{1, \ldots, k\}$ such that for each $i \in \{1, \ldots, k\}$, $v_{M_i} v_{m_j} \in \mathcal{L}_{B,\omega}$ if and only if $j = \sigma(i)$.*

Proof We first assume that the Vershik map φ_ω exists. Then φ_ω defines a bijection between the finite sets $X_{\max}(\omega)$ and $X_{\min}(\omega)$ by sending each M_i to some m_j. Let $\sigma(i) = j$. Clearly, $k = k'$. We need to check that $v_{M_i} v_{m_j}$ is in the language $\mathcal{L}_{B,\omega}$ if and only if $j = \sigma(i)$. It follows from continuity of φ_ω and the relation $\varphi_\omega(M_i) = m_j$ that if $x_n \to M_i$, then $\varphi_\omega(x_n) = y_n \to m_j$ as $n \to \infty$. We see that for every n the condition $\varphi_\omega(x_n) = y_n$ implies that $v_{M_i} v_{m_j} \in w(v, m, N)$ for some $v \in V_N$ and some $m < N$, because x_n and y_n are taken from neighborhoods generated by finite paths going through v_{M_i} and v_{m_j}, respectively. Furthermore, as $n \to \infty$, so do N and m. Hence $v_{M_i} v_{m_j} \in \mathcal{L}_{B,\omega}$ when $j = \sigma(i)$. By the same argument, if $v_{M_i} v_{m_k} \in \mathcal{L}_{B,\omega}$ for some $k \neq \sigma(i)$, then one can find $x_n \to M_i$ such that $\varphi_\omega(x_n) = y_n \to m_k$, a contradiction.

Conversely, assuming that (i) and (ii) hold, extend φ_ω to $X_{\max}(\omega)$ by defining $\varphi(M_i) := m_{\sigma(i)}$. It is obvious that φ_ω is one-to-one. Fix a pair (M_i, m_j) where $j = \sigma(i)$, and let $x_n \to M_i$ as $n \to \infty$; we show that $y_n = \varphi_\omega(x_n) \to m_j$.

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[7] The relevant formula on page 5 is incorrect in the final version: the correct version is in the preprint, that can be found at http://combinatorics.cis.strath.ac.uk/papers/lucaz.

We can assume that the first n edges of x_n coincide with those of M_i, *i.e.*, $x_n = \overline{e}^{(n)}_{\max}(v_0, v_{M_i})e_{n+1}e_{n+2}\cdots$, where e_{n+1} is not maximal in $r^{-1}(r(e_{n+1}))$. Then

$$y_n = \overline{f}^{(n)}_{\min}\left(v_0, s(e'_{n+1})\right) e'_{n+1}e_{n+2}\cdots ,$$

where e'_{n+1} is the successor of e_{n+1}. Take a subsequence (y'_n) of (y_n) convergent to a point $z \in X_B$. By construction, z must be a minimal path. It follows from the uniqueness of j in condition (ii) that $z = m_j$; this proves the continuity of φ_ω. ■

Example 3.4 Let (B, ω) be a stationary ordered Bratteli diagram whose vertex set $V_n = \{a, b, c, d\}$ for each $n \geq 1$, and where the ordering is defined by the substitution $a \to acbda$, $b \to bdcbdacb$, $c \to acdcb$, $d \to bdacda$. There are two pairs of vertical maximal and minimal paths going through vertices a and b. The words of length two that appear in $\mathcal{L}_{B,\omega}$ are $\{aa, ac, bb, bd, cb, cd, da, dc\}$, and using Proposition 3.3 we conclude that $\omega \in \mathcal{P}_B$ and $\varphi_\omega(M_a) = m_a$, and $\varphi_\omega(M_b) = m_b$.

Example 3.5 Let B be the stationary ordered Bratteli diagram whose vertex set $V_n = \{a, b\}$ for each $n \geq 1$, and whose incidence matrices are $F_n = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ for each n. We claim that any ordering on B with two maximal and two minimal paths cannot be perfect. The only possible choices to ensure that ω has this many extremal paths is, for all large n, to either choose the ordering $w(a, n, n{+}1) = ab$ and $w(b, n, n{+}1) = ba$ or to choose the ordering $w(a, n, n{+}1) = ba$ and $w(b, n, n{+}1) = ab$. Whatever choice one makes at level n and level $n + 1$, all four words $\{aa, ab, ba, bb\}$ occur somewhere in one of the two words $w(a, n, n + 2)$ or $w(b, n, n + 2)$. Thus, ω cannot be perfect.

Remark 3.6 Suppose that (B, ω) satisfies the conditions of Proposition 3.3. This means that there exists an N such that if we see $v_{M_i}v_{m_j}$ appearing in some word $w(v, m, n)$ with $m \geq N$, then $j = \sigma(i)$. We can telescope B to levels $N, N + 1, N + 2, \ldots$, so that if we see $v_{M_i}v_{m_j}$ appearing in some word $w(v, m, n)$ with $m \geq 1$, then $j = \sigma(i)$. Thus, unless otherwise indicated, for the remainder of Section 3, when we have an ordered diagram (B, ω) that satisfies the conditions of Proposition 3.3, *we shall assume that if* $v_{M_i}v_{m_j} \subset w(v, m, n)$ *with* $m \geq 1$, *then* $j = \sigma(i)$.

We now generalize Proposition 3.3 to arbitrary finite rank diagrams where the extremal paths are not necessarily vertical. Although the notion of language is not defined for these diagrams, we can still define and use words $w(v, m, n)$ for $v \in V_n$ and $m < n$. The proof of this lemma is elementary, so we omit it, although Figure 1 is explanatory.

Lemma 3.7 *Let B be a finite rank diagram. Then the following statements are equivalent:*

(i) $\omega \notin \mathcal{P}_B$;

(ii) *For some* ω *maximal path* M *and two* ω *minimal paths* m *and* m^*, *there exist strictly increasing sequences of levels* (n_k), (n^*_k), (N_k) *and* (N^*_k), *vertices* $\{w_k, v_k\} \subset V_{n_k}$, $\{w^*_k, v^*_k\} \subset V_{n^*_k}$, *vertices* $u_k \in V_{N_k}$, $u^*_k \in V_{N^*_k}$ *such that* M *passes through* w_k *and* w^*_k, m *and* m^* *pass through* v_k *and* v^*_k *respectively, and* $w_kv_k \subset w(u_k, n_k, N_k)$, $w^*_kv^*_k \subset w(u^*_k, n^*_k, N^*_k)$.

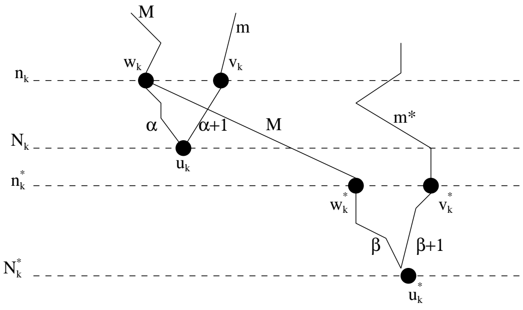

Figure 1: A discontinuous φ_ω.

Lemma 3.8 *Let B be a Bratteli diagram of finite rank and B' a telescoping of B. Then an ordering $\omega \in \mathcal{P}_B$ if and only if the corresponding lexicographic ordering $\omega' = L(\omega) \in \mathcal{P}_{B'}$.*

Proof If ω does not determine a Vershik map, then by Lemma 3.7 there is a maximal path M, two distinct minimal paths m and m^*, infinite sequences of levels (n_k) and (n_k^*), (N_k), and (N_k^*), vertices $\{w_k, v_k\} \subset V_{n_k}$, $\{w_k^*, v_k^*\} \subset V_{n_k^*}$ and vertices $u_k \in V_{N_k}$, $u_k^* \in V_{N_k^*}$ such that M passes through w_k and w_k^*, m (m^*) pass through v_k (v_k^*), and $w_k v_k \subset w(u_k, n_k, N_k)$, $w_k^* v_k^* \subset w(u_k^*, n_k^*, N_k^*)$ (see Figure 1). Note that in B it cannot be the case that for infinitely many levels the minimal paths go through the same vertex; otherwise, they are not distinct. Thus, there is some N such that if $n \geq N$, the level n edge in m has a different source and range from the level n edge in m^*.

Let B' be a telescoping of B to levels (m_k). If the images of M, m, and m^* in B' are denoted by M', m', and $(m^*)'$ respectively, then by the comment above, apart from an initial segment, the paths m' and $(m^*)'$ pass through distinct vertices in B'.

Find the levels m_j and m_J in (m_k) such that $m_{j-1} < n_k \leq m_j$, $m_{J-1} < N_k \leq m_J$, and let E_j' denote the edge set in B' obtained by telescoping between the m_{j-1}-st and m_j-th levels of B, and let E_J' denote the edge set obtained by telescoping between the m_{J-1}-st and m_J-th levels of B. Let the path M go through $w_j' \in V_{m_j}$, and m through $v_j' \in V_{m_j}$.

Let $u_J' \in V_{m_J}$ be any vertex such that there is a path from $u_k \in V_{N_k}$ to u_J'. Then for the corresponding vertices $w_{j-1}', v_{j-1}' \in V_{j-1}'$ and $u_J' \in V_J'$ respectively, it is the case that $w_{j-1}' v_{j-1}' \in w(u_J', j-1, J)$ with M' passing through w_{j-1}' and m' passing through v_{j-1}'. Repeat this procedure for m^*. By Lemma 3.7, the ordering ω' on B' obtained from ω by telescoping does not determine a Vershik map.

The converse is proved similarly. ■

Lemma 3.8 and the compactness of X_B imply the following corollary.

Corollary 3.9 *Suppose that B has rank d. Then $\omega \in \mathcal{P}_B$ if and only if there exists a telescoping (B', ω') of (B, ω) such that $V_n' = V'$ for each $n \geq 1$, the ω'-maximal and ω'-minimal paths $M_1, \ldots, M_k$ and $m_1, \ldots, m_{k'}$ are vertical, and ω' satisfies the conditions of Proposition 3.3.*

Now we give another criterion, which guarantees the existence of a Vershik map on an ordered Bratteli diagram (B,ω) (not necessarily of finite rank). Let $\omega = (\omega_v)_{v\in V^*\setminus V_0}$ be an ordering on a regular Bratteli diagram B. For every $x_{\max} = (x_n) \in X_{\max}(\omega)$, we define the set $\mathrm{Succ}(x_{\max}) \subset X_{\min}(\omega)$ as follows: $y_{\min} = (y_n)$ belongs to the set $\mathrm{Succ}(x_{\max})$ if for infinitely many n there exist edges $y' \in s^{-1}(r(x_n))$ and $y'' \in s^{-1}(r(y_n))$ such that $r(y') = r(y'') = v_{n+1}$ and y'' is the successor of y' in the set $r^{-1}(v_{n+1})$. Given a path $y_{\min} \in X_{\min}(\omega)$, we define the set $\mathrm{Pred}(y_{\min}) \subset X_{\max}(\omega)$ in a similar way. It is not hard to prove that the sets $\mathrm{Succ}(x_{\max})$ and $\mathrm{Pred}(y_{\min})$ are non-empty and closed for any $x_{\max}$ and $y_{\min}$.

Proposition 3.10 *An ordering $\omega = (\omega_v)_{v\in V^*\setminus V_0}$ on a regular Bratteli diagram B is perfect if and only if for every $x_{\max} \in X_{\max}(\omega)$ and $y_{\min} \in X_{\min}(\omega)$ the sets* $\mathrm{Succ}(x_{\max})$ *and* $\mathrm{Pred}(y_{\min})$ *are singletons.*

Proof Let $x_{\max}$ be any path from $X_{\max}(\omega)$. If $\mathrm{Succ}(x_{\max}) = \{y_{\min}\}$, then one can define $\varphi_\omega : x_{\max} \to y_{\min}$. Since $\mathrm{Pred}(y_{\min})$ is also a singleton, we obtain a one-to-one correspondence between the sets of maximal and minimal paths. The fact that φ_ω is continuous can be checked directly.

Conversely, if ω is perfect, then it follows from the existence of the Vershik map φ_ω that either of the sets $\mathrm{Succ}(x_{\max})$ and $\mathrm{Pred}(y_{\min})$ must be singletons. ■

3.2 Skeletons and Associated Graphs

Let B be a finite rank Bratteli diagram. We do not need to assume here that B is simple unless we state this explicitly. If ω is an order on B, and $v \in V^*\setminus V_0$, we denote the minimal edge with range v by $\overline{e}_v$, and we denote the maximal edge with range v by $\widetilde{e}_v$.

Lemma 3.11 *Let (B',ω') be a rank d ordered diagram. Then there exists a telescoping (B,ω) of (B',ω') such that*

(i) *$|r^{-1}(v)| \geq 2$ for each $v \in V^*\setminus V_0$;*

(ii) *$V_n = V$ for each $n \geq 1$ and $|V| = d$;*

(iii) *all ω-extremal paths are vertical, with $\widetilde{V}$, $\overline{V}$ denoting the sets of vertices through which maximal and minimal paths run respectively;*

(iv) *$s(\widetilde{e}_v) \in \widetilde{V}$ and $s(\overline{e}_v) \in \overline{V}$ for each $v \in V^*\setminus(V_0 \cup V_1)$, and this is independent of n.*

In addition, if $\omega \in \mathcal{P}_B$, we can further telescope so that

(v) *if $\widetilde{v}\,\overline{v}$ appears as a subword of some $w(v,m,n)$ with $m \geq 1$, then $\sigma(\widetilde{v}) = \overline{v}$ defines a one-to-one correspondence between the sets $\widetilde{V}$ and $\overline{V}$.*

Proof Property (i) is guaranteed by Lemma 2.7. To obtain property (ii), we telescope through the levels (n_k) such that $|V_{n_k}| = d$, where d is the rank of B'. To obtain (iii), note that each maximal path M' passes through one vertex $\widetilde{v}_M$ infinitely often. Telescope B to the levels where this occurs; the image M of M' is then a maximal vertical path passing though $\widetilde{v}_M$ at each level. Repeat this procedure for each maximal path M' and each minimal path m'. To see (iv), we assume we have telescoped so that properties (i)–(iii) hold. We denote the vertical maximal path passing through $\widetilde{v} \in \widetilde{V}$ by $M_{\widetilde{v}}$; similarly, the vertical minimal path $m_{\overline{v}}$ passes through $\overline{v}$. We claim the following: for any level n there exist $l_n > n$ such that for every $l \geq l_n$ and every vertex $u \in V_l$, the maximal and minimal finite paths in $E(v_0, u)$ agree with some $M_{\widetilde{v}}$, $m_{\overline{v}}$ respectively on the first n entries, where the vertices $\widetilde{v} \in \widetilde{V}$ and $\overline{v} \in \overline{V}$ depend on u and l. Indeed, if we assumed that the contrary holds, then we would have additional maximal (or minimal) paths not belonging to $\{M_{\widetilde{v}} : \widetilde{v} \in \widetilde{V}\}$ (or $\{m_{\overline{v}} : \overline{v} \in \overline{V}\}$). Thus, after an appropriate telescoping, we can assume that if v is any vertex in V_n, $n \geq 2$, and $\widetilde{e}_v$ and $\overline{e}_v$ are the maximal and minimal edges in the set $r^{-1}(v)$ with respect to ω, then $\widetilde{e}_v \neq \overline{e}_v$ and $s(\widetilde{e}_v) \in \widetilde{V}_{n-1}$, $s(\overline{e}_v) \in \overline{V}_{n-1}$. By further telescoping we can assume that the sources of $\widetilde{e}_v$ and $\overline{e}_v$ do not depend on the level in which v lies. If ω is perfect, Remark 3.6 explains why it is possible to telescope (B, ω) so that (v) is true. ■

Definition 3.12 Let B be a finite rank d Bratteli diagram.

(a) If B satisfies the conditions (i)–(ii) of Lemma 3.11, we say that B is *strictly rank d*.
(b) If (B, ω) satisfies conditions (i)–(iv) of Lemma 3.11, or if (B, ω) is a finite rank perfectly ordered diagram satisfying conditions (i)–(v) of Lemma 3.11, we say that (B, ω) is *well-telescoped.*

For the remainder of Section 3, *we assume that unordered finite rank d Bratteli diagrams are strictly rank d. We assume that finite rank ordered Bratteli diagrams are well-telescoped.*

Thus, any ordering ω determines a collection

$$\{M_{\widetilde{v}}, m_{\overline{v}}, \widetilde{e}_w, \overline{e}_w : w \in V^* \backslash V_0,\ \widetilde{v} \in \widetilde{V} \text{ and } \overline{v} \in \overline{V}\}.$$

This collection of paths and edges contains all information about the extremal edges of ω, though only partial information about ω itself. We now extend this notion to an unordered diagram B.

Let B be a strictly rank d Bratteli diagram. We denote by V the set of vertices of B at each level $n \geq 1$, but if we need to point out that this set is considered at level n, then we write V_n instead of V. For some $k \leq d$, take two subsets $\widetilde{V}$ and $\overline{V}$ of V such that $|\widetilde{V}| = |\overline{V}| = k$. Given any $\widetilde{v} \in \widetilde{V}$, $\overline{v} \in \overline{V}$ choose $M_{\widetilde{v}} = (M_{\widetilde{v}}(1), \dots, M_{\widetilde{v}}(n), \dots)$ and $m_{\overline{v}} = (m_{\overline{v}}(1), \dots, m_{\overline{v}}(n), \dots)$, two vertical paths in B going downwards through the vertices $\widetilde{v} \in \widetilde{V}$ and $\overline{v} \in \overline{V}$. If $v \in \overline{V} \cap \widetilde{V}$, then the paths M_v and m_v are taken such that they do not share common edges. Next, for each vertex $w \in V_n, n \geq 2$, we choose two vertices $\widetilde{v}$ and $\overline{v}$ in $\widetilde{V}$ and $\overline{V}$ respectively, and for each $n \geq 2$ and each $w \in V_n$, distinct edges $\widetilde{e}_w$ and $\overline{e}_w$ with range w such that $s(\widetilde{e}_w) = \widetilde{v}$ and $s(\overline{e}_w) = \overline{v}$. If $w \in \widetilde{V}$ or $w \in \overline{V}$, then the edges $\widetilde{e}_w$ and $\overline{e}_w$ in E_n are chosen such that $\widetilde{e}_w = M_w(n)$ and $\overline{e}_w = m_w(n)$, respectively. We introduce the concept of a skeleton to create a framework for defining a perfect ordering with precisely this extremal edge structure.

Definition 3.13 Given a strict rank d diagram B and two subsets $\widetilde{V}, \overline{V}$ of V of the same cardinality $k \leq d$, a *skeleton* $\mathcal{F} = \mathcal{F}(B)$ of B is a collection

$$\{M_{\widetilde{v}}, m_{\overline{v}}, \widetilde{e}_w, \overline{e}_w : w \in V^* \backslash (V_0 \cup V_1),\ \widetilde{v} \in \widetilde{V} \text{ and } \overline{v} \in \overline{V}\}$$

of paths and edges with the properties described above. The vertices from $\widetilde{V}$ will be called *maximal* and those from $\overline{V}$ *minimal.*

In other words, while not an ordering, a skeleton is a constrained choice of all extremal edges. As an example, when $\widetilde{V}=\overline{V}=V$, the skeleton is simply the set $\{M_{\widetilde{v}}, m_{\overline{v}} : \widetilde{v}, \overline{v} \in V\}$. As discussed in Lemma 3.11, any well telescoped ordered finite rank Bratteli diagram (B,ω) has a natural skeleton $\mathcal{F}_\omega$ (recall that the extremal paths are vertical). Conversely, it is obvious that there are several skeletons that one can define on B, and for any skeleton $\mathcal{F}$ of a Bratteli diagram B there is at least one ordering ω on B such that $\mathcal{F}=\mathcal{F}_\omega$. A skeleton $\mathcal{F}_\omega$ contains no information about whether $\omega \in \mathcal{P}_B$. Note that a skeleton does not contain information about which are the maximal edges in E_1; this will not impact our work.

Next we define a directed graph $\mathcal{H}=(T,P)$ associated with a Bratteli diagram B of strict finite rank and having skeleton $\mathcal{F}$. Implicit in the definition of this directed graph is the assumption that we are working towards constructing perfect orderings ω whose skeleton $\mathcal{F}_\omega=\mathcal{F}$. Thus we suppose that we also have a bijection $\sigma\colon \widetilde{V}\to\overline{V}$ that, in the case when $\mathcal{F}=\mathcal{F}_\omega$ with $\omega\in\mathcal{P}_B$, will be the bijection described in Proposition 3.3, so that $\varphi_\omega(M_{\widetilde{v}})=m_{\sigma(\widetilde{v})}$.

Definition 3.14 For any vertices $\widetilde{v}\in\widetilde{V}$ and $\overline{v}\in\overline{V}$, we set

$$W_{\widetilde{v}}=\{w\in V : s(\widetilde{e}_w)=\widetilde{v}\},\ \ W'_{\overline{v}}=\{w\in V : s(\overline{e}_w)=\overline{v}\}.$$

Then $W=\{W_{\widetilde{v}} : \widetilde{v}\in\widetilde{V}\}$ and $W'=\{W'_{\overline{v}} : \overline{v}\in\overline{V}\}$ are both partitions of V. We call W and W' the *partitions generated by* $\mathcal{F}$.

Let $[\overline{v},\widetilde{v}] := W'_{\overline{v}}\cap W_{\widetilde{v}}$, and define the partition

$$W\cap W' := \left\{ [\overline{v},\widetilde{v}] : \overline{v}\in\overline{V}, \widetilde{v}\in\widetilde{V}\right\}.$$

Definition 3.15 Let B be a strict finite rank diagram, let

$$\mathcal{F}=\left\{M_{\widetilde{v}}, m_{\overline{v}}, \widetilde{e}_w, \overline{e}_w : w\in V^*\backslash(V_0\cup V_1),\ \widetilde{v}\in\widetilde{V} \text{ and } \overline{v}\in\overline{V}\right\}$$

be a skeleton on B, and suppose $\sigma\colon\widetilde{V}\to\overline{V}$ is a bijection. Let the graph $\mathcal{H}=\mathcal{H}(T,P)$, have vertex set

$$T=\{[\overline{v},\widetilde{v}]\in\overline{V}\times\widetilde{V} : [\overline{v},\widetilde{v}]\neq\varnothing\},$$

and edge set P, where there is an edge from $[\overline{v},\widetilde{v}]$ to $[\overline{v}_1,\widetilde{v}_1]$ if and only if $\sigma(\widetilde{v})=\overline{v}_1$. The directed graph $\mathcal{H}$ is called the *graph associated with* $(B,\mathcal{F},\sigma)$.

Note that for a fixed skeleton, different bijections σ will define different graphs $\mathcal{H}$.

Remark 3.16 Suppose (B,ω) is a perfectly ordered, well-telescoped finite rank Bratteli diagram, $\mathcal{F}_\omega$ is the skeleton on B defined by ω and σ is the bijection given by Proposition 3.3. Let $\mathcal{H}=(T,P)$ be the graph associated with $(B,\mathcal{F},\sigma)$. Let $w=v_1\cdots v_p$ be a word in the language $\mathcal{L}_{B,\omega}$ and suppose $v_i\in t_i$ where $t_i\in T$. Then there exists a path in $\mathcal{H}$ starting at t_1 and ending at t_p. Moreover, the following lemma is also true; the proof is straightforward and is omitted.

Lemma 3.17 *Let B be an aperiodic strict finite rank Bratteli diagram, let $\mathcal{F}$ be a skeleton on B, $\sigma\colon \widetilde{V} \to \overline{V}$ be a bijection, and let $\mathcal{H} = (T, P)$ be the graph associated with $(B, \mathcal{F}, \sigma)$. Suppose there exists an ordering ω on B with skeleton $\mathcal{F}$, and there is an M such that whenever $N > n \geq M$, if a word $w = v_1 \cdots v_p \subset w(v, n, N)$ for $v \in V_N$, then w corresponds to a path in $\mathcal{H}$ going through vertices $t_1, \dots t_p$, where $v_i \in V_n$ belong to $t_i \in T$. Then ω is perfect and $\varphi_\omega(M_{\widetilde{v}}) = m_{\sigma(\widetilde{v})}$ for each $\widetilde{v} \in \widetilde{V}$.*

Definition 3.18 We define the family $\mathcal{A}$ of Bratteli diagrams, all of whose incidence matrices are of the form

$$F_n := \begin{pmatrix} A_n^{(1)} & 0 & \dots & 0 & 0 \\ 0 & A_n^{(2)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_n^{(k)} & 0 \\ B_n^{(1)} & B_n^{(2)} & \dots & B_n^{(k)} & C_n \end{pmatrix}, \quad n \geq 1,$$

where

(a) for $1 \leq i \leq k$ there is some d_i such that for each $n \geq 1$, $A_n^{(i)}$ is a $d_i \times d_i$ matrix;
(b) all matrices $A_n^{(i)}$, $B_n^{(i)}$ and C_n are strictly positive;
(c) C_n is a $d \times d$ matrix;
(d) there exists $j \in \{\sum_{i=1}^k d_i + 1, \cdots \sum_{i=1}^k d_i + d\}$ such that for each $n \geq 1$, the j-th row of F_n is strictly positive.

If a Bratteli diagram's incidence matrices are of the form above, we shall say that it has *k minimal components.*

As shown in [BKMS11], the family $\mathcal{A}$ of diagrams corresponds to aperiodic homeomorphisms of a Cantor set that have exactly k minimal components with respect to the tail equivalence relation $\mathcal{E}$.

Recall that a directed graph is *strongly connected* if for any two vertices v, v', there is a path from v to v', and also a path from v' to v. If at least one of these paths exists, then G is *weakly connected*, or just *connected*. We notice that, given $(B, \mathcal{F}, \sigma)$, an associated graph $\mathcal{H} = (T, P)$ is not connected, in general.

Proposition 3.19 *Let (B, ω) be a finite rank, perfectly ordered and well-telescoped Bratteli diagram, and suppose ω has skeleton $\mathcal{F}_\omega$ and permutation σ.*

(i) *If B is simple, then the associated graph $\mathcal{H}$ is strongly connected.*
(ii) *If $B \in \mathcal{A}$, then the associated graph $\mathcal{H}$ is weakly connected.*

Proof We prove (i); the proof of (ii) is similar if we focus on $w(v, n-1, n)$ where v is the vertex that indexes the strictly positive row in F_n. Recall that in addition to assuming that (B, ω) is well-telescoped, since ω is perfect, we assume we have telescoped so that all entries of F_n are positive for each n, and also so that if $\widetilde{v}\,\overline{v}$ is a subword of $w(v, m, n)$ for $1 \leq m < n$, then $\sigma(\widetilde{v}) = \overline{v}$. We need to show that for any two vertices $t = [\overline{v}, \widetilde{v}]$ and $t' = [\overline{v}', \widetilde{v}']$ from the vertex set T of $\mathcal{H}$, there exists a path from t to t'.

Claim 1 Let $n > 2$ and $w(u, n-1, n) = v_1 \cdots v_k$ be a word where $v_i \in [\overline{v}_i, \widetilde{v}_i]$, $i = 1, \ldots, k$. Then there is a path from $[\overline{v}_1, \widetilde{v}_1]$ to $[\overline{v}_k, \widetilde{v}_k]$ going through the vertices $[\overline{v}_i, \widetilde{v}_i]$, $i = 1, \ldots, k$, in that order.

For, given $1 \leq i \leq k-1$, since $v_i v_{i+1}$ is a subword of $w(u, n-1, n)$, then the concatenation of the two words $w(v_i, n-2, n-1) w(v_{i+1}, n-2, n-1)$ is a subword of $w(u, n-2, n)$, so that $\widetilde{v}_i \overline{v}_{i+1}$ is a subword of $w(u, n-2, n)$. By our telescoping assumptions, $\sigma(\widetilde{v}_i) = \overline{v}_{i+1}$.

Now, let T^* be the subset of T of vertices of the form $[\overline{v}, s(\widetilde{e}_{\overline{v}})]$ where $\overline{v} \in \overline{V}$. (Note that $[\overline{v}, s(\widetilde{e}_{\overline{v}})] \neq \varnothing$, since $\overline{v} \in [\overline{v}, s(\widetilde{e}_{\overline{v}})]$.) It is obvious that there is an edge from $t = [\overline{v}, \widetilde{v}]$ to $t^* = [\sigma(\widetilde{v}), s(\widetilde{e}_{\sigma(\widetilde{v})})]$ in $\mathcal{H}$.

Claim 2 For any $t^* \in T^*$ and $t' = [\overline{v}', \widetilde{v}'] \in T$, there is a path from t^* to t'.

To see that this, we will use Claim 1. Let $t^* = [\overline{v}^*, \widetilde{v}^*]$ where $\widetilde{v}^* = s(\widetilde{e}_{\overline{v}^*})$. Let $v \in V_{n-1}$ belong to t' in $\mathcal{H}$. By the simplicity of B, there exists an edge $e \in E(v, \overline{v}^*)$ where $\overline{v}^* \in V_n$. Thus $w(\overline{v}^*, n-1, n) = \overline{v}^* \cdots v \cdots \widetilde{v}^*$. If $n > 2$, then by Claim 1 there is a path from t^* to t'.

To complete the proof of the proposition, we concatenate the paths from t to t^* and from t^* to t' in $\mathcal{H}$. ■

Remark 3.20 It is not hard to see that the converse statement to Proposition 3.19 is not true. There are examples of non-simple perfectly ordered diagrams of finite rank whose associated graphs are strongly connected.

Note also that the assumption that ω is perfect is crucial. Moreover, there are examples of *simple* finite rank Bratteli diagrams and skeletons whose associated graphs are not strongly connected. Indeed, let B be a simple stationary diagram with $V = \{a, b, c\}$ with the skeleton $\mathcal{F} = \{M_a, M_b, m_a, m_b; \widetilde{e}_c, \overline{e}_c\}$, where $s(\widetilde{e}_c) = b, s(\overline{e}_c) = a$. Let $\sigma(a) = a, \sigma(b) = b$. Constructing the associated graph $\mathcal{H}$, we see that there is no path from $[b, b]$ to $[a, a]$. It can be also shown that there is no perfect ordering ω such that $\mathcal{F} = \mathcal{F}_\omega$. This observation complements Proposition 3.19 by stressing the importance of the strong connectedness of $\mathcal{H}$ for the existence of perfect orderings.

We illustrate the definitions of skeletons and associated graphs with several examples that will also be used later.

Example 3.21 Let (B, ω) be an ordered Bratteli diagram of strict rank d, where $V = \{1, \ldots, d\}$ and ω has d vertical maximal and d vertical minimal paths. Then the skeleton $\mathcal{F}_\omega$ is formed by pairs of vertical paths (M_i, m_i) going downward through the vertex $i \in \{1, \ldots, d\}$.

Let σ be a permutation of the set $\{1, 2, \ldots, d\}$. The graph $\mathcal{H}$ is represented as a disjoint union of connected subgraphs generated by cycles of σ. If ω is perfect, then by Proposition 3.19, σ is cyclic. In this case, $[i, i] = \{i\}$, so vertices of $\mathcal{H}$ are $\{[i, i] : 1 \leq i \leq d\}$, and there is an edge from $[i, i]$ to $[j, j]$ if and only if $j = \sigma(i)$. Thus, the structure of $\mathcal{H}$ is represented by the cyclic permutation σ.

Example 3.22 Let $\mathcal{F}$ be a skeleton on a simple strict rank d diagram B such that $V = \{1, \dots, d-1, d\}$ and $\widetilde{V} = \overline{V} = \{1, \dots, d-1\}$. Depending on σ, the associated graph $\mathcal{H}$ that can be associated with $\mathcal{F}$ is one of two kinds:

(a) Suppose $s(\widetilde{e}_d) = s(\overline{e}_d) = j$ where $1 \leq j \leq d-1$. Then $[i,i] = \{i\}$ for $1 \leq i \leq d-1$, $i \neq j$, and $[j,j] = \{j,d\}$. In $\mathcal{H}$ the vertex set is

$$T = \{[i,i] : 1 \leq i \leq d-1\}.$$

For $\mathcal{H}$ to be strongly connected, σ must be a cyclic permutation of $\{1, \dots, d-1\}$, and in this case there is an edge from $[i,i]$ to $[i',i']$ if and only if $i' = \sigma(i)$.

(b) Suppose $s(\widetilde{e}_d) = j \neq s(\overline{e}_d) = i$ where $1 \leq i, j \leq d-1$; we can assume that $i < j$. Here $[l,l] = \{l\}$ for $1 \leq l \leq d-1$ and $[i,j] = \{d\}$, so that

$$T = \left\{ [l,l] : 1 \leq l \leq d-1 \right\} \cup \{[i,j]\} .$$

Here also for $\mathcal{H}$ to be strongly connected, σ must be a cyclic permutation of $\{1, \dots, d-1\}$, and the edges described in (a) form a subset of P. In addition there is an edge from $[\sigma^{-1}(i), \sigma^{-1}(i)]$ to $[i,j]$, and also an edge from $[i,j]$ to $[\sigma(j), \sigma(j)]$. If $\sigma(j) = i$, then there is also a loop at $[i,j]$.

Example 3.23 We continue with Example 3.4. Since $\varphi_\omega(M_a) = m_a$, $\varphi_\omega(M_b) = m_b$, this means that $\sigma(a) = a, \sigma(b) = b$. Noting that $s(\widetilde{e}_c) = b, s(\widetilde{e}_d) = a, s(\overline{e}_c) = a, s(\overline{e}_d) = b$, we have the completely determined skeleton $\mathcal{F}_\omega$. Note that the vertices T of $\mathcal{H}$ are $[a,a] = \{a\}$, $[a,b] = \{c\}$, $[b,a] = \{d\}$ and $[b,b] = \{b\}$. The associated graph $\mathcal{H}$ is shown in Figure 2.

Example 3.24 Let $V = \{v_1, v_2, v_1^*, v_2^*, w_1, w_2\}$ and $\widetilde{V} = \overline{V} = \{v_1, v_1^*\}$; let $\sigma(v_1) = v_1$ and $\sigma(v_1^*) = v_1^*$. Suppose that $W'_{v_1} = \{v_1, v_2, w_1\}$, $W_{v_1} = \{v_1, v_2, w_2\}$, $W'_{v_1^*} = \{v_1^*, v_2^*, w_2\}$ and $W_{v_1^*} = \{v_1^*, v_2^*, w_1\}$. Then the associated graph $\mathcal{H}$ is strongly connected. We remark that this can be the skeleton of an aperiodic diagram with two minimal components living through the vertices $\{v_1, v_2\}$ and $\{v_1^*, v_2^*\}$ respectively.

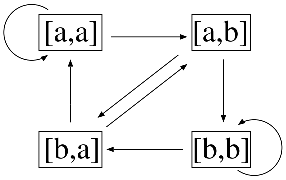

Figure 2: The graph associated with $\mathcal{F}_\omega$ in Example 3.23

We illustrate the utility of the notions of skeleton and accompanying directed graphs in the following results, which give sufficient conditions for an ordering ω to belong to $\mathcal{P}_B^c$. Even though these are conditions on ω, some diagrams B force this condition on all orderings in $\mathcal{O}_B$; this is the content of Proposition 3.26.

Proposition 3.25 *Let (B, ω) be a perfectly ordered, well-telescoped rank d Bratteli diagram. Suppose that ω has k maximal and k minimal paths, where $1 < k \leq d$. Then for some $v \in V$, $vv \notin \mathcal{L}_{B,\omega}$.*

Proof Let ω have skeleton

$$\mathcal{F}_\omega = \left\{ M_{\widetilde{v}}, m_{\overline{v}}, \widetilde{e}_w, \overline{e}_w : w \in V^* \backslash (V_0 \cup V_1),\ \widetilde{v} \in \widetilde{V} \text{ and } \overline{v} \in \overline{V} \right\},$$

and suppose that ω is perfect. Then there exists a bijection σ of $\{1, \ldots, k\}$ such that $\sigma(i) = j$ if and only if $v_{M_i} v_{m_j} \in \mathcal{L}_{B,\omega}$. Suppose that for each v there is some v^* such that $vv \in w(v^*, n, n+1)$ for infinitely many n. We claim that $V = \bigcup_{i=1}^k [v_{m_{\sigma(i)}}, v_{M_i}]$. For if $s(\overline{e}_v) = v_{m_j}$ and $s(\widetilde{e}_v) = v_{M_i}$, then $vv \in w(v^*, n, n+1)$ implies that $v_{M_i} v_{m_j} \in w(v^*, n-1, n+1)$. Since this occurs for infinitely many n, Proposition 3.3 tells us that $j = \sigma(i)$.

Since W and W' are both partitions of V, the relation $V = \bigcup_{i=1}^k [v_{m_{\sigma(i)}}, v_{M_i}]$ actually means that $W_{v_{M_i}} = W'_{v_{m_{\sigma(i)}}}$ for every i. It follows that the associated graph $\mathcal{H}$ has the following simple form: the vertices of $\mathcal{H}$ are $[v_{m_{\sigma(i)}}, v_{M_i}], i = 1, \ldots, k$, and the edges are given by k loops, one around each vertex. Since $k > 1$, this means $\mathcal{H}$ is not connected, contradicting Proposition 3.19. ■

3.3 Bratteli Diagrams that Support no Perfect Orders

The next proposition describes how for some aperiodic diagrams B that belong to the special class $\mathcal{A}$ (see Definition 3.18), there are structural obstacles to the existence of perfect orders on B. This is a generalization of an example in [Me06].

Proposition 3.26 *Let $B \in \mathcal{A}$ have k minimal components, and such that for each $n \geq 1$, C_n is an $s \times s$ matrix where $1 \leq s \leq k-1$. If $k = 2$, there are perfect orderings on B only if $C_n = (1)$ for all but finitely many n. If $k > 2$, then there is no perfect ordering on B.*

Proof We use the notation of Definition 3.18 in this proof. Let V^i be the subset of vertices corresponding to the subdiagram defined by the matrices $A_n^{(i)}$ for $i = 1, \ldots k$, and let V^{k+1} be the subset of vertices corresponding to the subdiagram defined by the matrices C_n. Suppose that ω is a perfect ordering on B, and we assume that (B, ω) is well telescoped and has skeleton $\mathcal{F}_\omega$. (Otherwise we work with the diagram B' on which $L(\omega)$ is well telescoped. Note that if B has incidence matrices of the given form, then so does any telescoping.) Note that $|\overline{V}| = |\widetilde{V}| \geq k$, since each minimal component has at least one maximal and one minimal path. Also, if $\widetilde{v} \in V^i$, then $\sigma(\widetilde{v}) \in V^i$. There are k connected components of vertices $T_1, \ldots T_k$ such that there are no edges from vertices in T_i to vertices in T_j if $i \neq j$. To see this, if $1 \leq i \leq k$, let $T_i = \{[\overline{v}, \widetilde{v}] : \overline{v} \in V^i, \widetilde{v} \in V^i\}$.

If $k = 2$, there are no extremal paths going through c, the unique vertex in V^3; otherwise, there would be disjoint components in $\mathcal{H}$, and since ω is perfect, this would contradict Proposition 3.19. So $c \in [\overline{v}, \widetilde{v}]$ where $\overline{v} \in V^i$ and $\widetilde{v} \in V^j$ for some $i \neq j$. Thus in H there are paths from vertices in T_i to vertices in T_j through c, but not back again. The only way this can occur validly is if $C_n = (1)$ for all large n.

If $k > 2$, then there are at most $k-1$ vertices remaining in $\mathcal{H}$, outside of the components $T_1, \ldots, T_k$. We shall argue that even in the extreme case, where there are $k-1$ such vertices, there would not be sufficient connectivity in $\mathcal{H}$ to support an $\omega \in \mathcal{P}_B$. Call these $k-1$ vertices $t_1, \ldots t_{k-1}$, where $t_i = [\overline{v}_i, \widetilde{v}_i]$. If $V^{k+1} = \{v_1, \ldots, v_{k-1}\}$, we have labeled so that $v_i \in t_i$. For each one of these vertices t_i there are incoming edges from vertices in at most one of the components T_j for $1 \le j \le k$, and also outgoing edges to vertices in at most one of the components $T_{j'}$ for $1 \le j' \le k$. So at least one of the components, say T_1, has no incoming edges with source outside T_1.

Suppose first that each $t_i = [\overline{v}_i, \widetilde{v}_i]$ satisfies $\overline{v}_i \in V^1$, in which case all other $T_i's$ have no outgoing edges. But then for $T_i \neq T_j$, $i \neq j$, $i, j \neq 1$, there is neither a path from T_i to T_j, nor from T_j to T_i. This contradicts the second part of Proposition 3.19.

Suppose next that for some i, $t_i = [\overline{v}_i, \widetilde{v}_i]$ and $\overline{v}_i \notin V^1$. Since $\widetilde{v}_i \notin V^1$, there is no edge between t_i and V_1. Since $B_n^{(1)}$ has strictly positive entries, $w(v_i, n, n+1)$ must contain occurrences from vertices in V^1, and these occurrences have to occur somewhere in the interior of the word. But this contradicts the fact that T_1 has no incoming edges from outside T_1. ∎

In the above proposition, the extreme case, when there are k extremal pairs and the vertex set of $\mathcal{H}$ has size $2k-1$, still does not produce perfect orderings, but only just, as the next proposition demonstrates. First we define the family $\mathcal{M}$ of matrices whose relevance will become apparent in Theorem 3.32.

Definition 3.27 Let $\mathcal{M}$ be the family of matrices whose entries take values in $\mathbb{N}$ and are of the form

$$F = \begin{pmatrix} f_1+1 & f_1 & \cdots & f_1 \\ f_2 & f_2+1 & \cdots & f_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_d & f_d & \cdots & f_d+1 \end{pmatrix} \tag{3.1}$$

for some $d \in \mathbb{N}$.

Proposition 3.28 *Let $B \in \mathcal{A}$ be a Bratteli diagram with k minimal subcomponents, and where for each $n \ge 1$, C_n is a $k \times k$ matrix. If (B, ω) is a perfectly ordered, well-telescoped Bratteli diagram with skeleton $\mathcal{F}_\omega$, then $C_n \in \mathcal{M}$ for all n.*

Proof We use the notation of Proposition 3.26. The proof of this last proposition showed us that for a perfect order to be supported by B, each component T_i has to have an incoming edge from outside T_i. Similarly, each component T_i has to have an outgoing edge with range outside of T_i. Label $V^{k+1} = \{v_1, \ldots v_k\}$ so that $v_i \in [\overline{v}_i, \widetilde{v}_{h(i)}]$ where $\overline{v}_i \in T_i$ and $h\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a bijection. Thus in $\mathcal{H}$, from each T_i there is an edge from T_i to $[\overline{v}_i, \widetilde{v}_{h(i)}]$, and there is an edge from $[\overline{v}_{h^{-2}(i)}, \widetilde{v}_{h^{-1}(i)}]$ to T_i. In addition, for each i there is (possibly) an edge from $[\overline{v}_i, \widetilde{v}_{h(i)}]$ to $[\overline{v}_{h(i)}, \widetilde{v}_{h^2(i)}]$. See Figure 3 for an example of such a graph.

If h is not a cyclic permutation, then the graph $\mathcal{H}$ is disconnected, in which case there are no perfect orders on B that have the skeleton $\mathcal{F}_\omega$. Thus h must be cyclic,

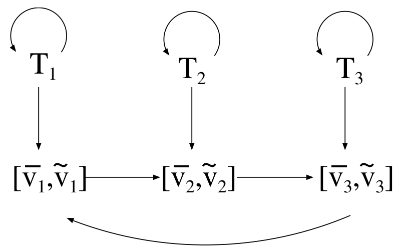

Figure 3: An example of $\mathcal{H}$ when B has 3 minimal subcomponents and $h = (123)$.

and inspection of the graph $\mathcal{H}$ tells us that for each $v_i \in V^{k+1}$, and for each n, $v_i \in [\overline{v}_i, \widetilde{v}_{h(i)}]$ and

$$w(v_i, n-1, n) = \Big(\prod_{j=1}^{k_n} W_i^{(j)} v_i W_{h(i)}^{(j)} v_{h(i)} \cdots W_{h^{-1}(i)}^{(j)} v_{h^{-1}(i)} \Big) W_i v_i W_{h(i)},$$

where $\prod$ refers to concatenation of words, each $W_i^{(j)}$ is a (possibly empty) word with letters in V^i, and W_i, $W_{h(i)}$ are non-empty words. The result follows. ■

3.4 Perfect Orderings that Generate Odometers

Definition 3.29 If a minimal Cantor dynamical system (Y, T) admits an adic representation by a Bratteli diagram B with $|V_n| = 1$ for all levels n, then T is called an *odometer*.

Let $\mathcal{L} \subset \mathcal{A}^{\mathbb{N}}$. A word $W \in \mathcal{L}$ is *periodic* if it can be written as a concatenation $W = U^k$ of k copies of a word U where $k > 1$. Given a word $W = w_1 \cdots w_p$, we define $\sigma^i(W) := w_{i+1}w_{i+2} \cdots w_p w_1 \cdots w_i$. We say that $\mathcal{L}$ is *periodic* if there is some word $V \in \mathcal{L}$ such that any word $W \in \mathcal{L}$ is of the form SV^kP for some suffix (prefix) $S = S(W)$ $(P = P(W))$ of V. Finally if $\mathcal{Q} = \{q_1, q_2, \ldots, q_n\}$ is a partition of a set X and $T\colon X \to X$ is a bijection, then we say that $\mathcal{Q}$ is *periodic* for T if $T(q_i) = q_{i+1}$ for $1 \leq i < n$ and $T(q_n) = q_1$.

Next we state and prove a result that Fabien Durand communicated to us as a known result; the proof below is a direct generalisation of the proof of [DHS99, Proposition 16(ii)].

Proposition 3.30 *Let ω be a perfect ordering on the simple strict finite rank diagram B. If $\mathcal{L}_{B,\omega}$ is periodic, then (X_B, φ_ω) is topologically conjugate to an odometer.*

Proof Suppose $\mathcal{L}_{B,\omega}$ is periodic. Let V denote the vertex set of B at each level. Fix $\overline{v}$ such that there is a vertical minimal path going through the vertex $\overline{v}$. Then for all k, $\lim_{n\to\infty} w(\overline{v}, k, n)$ exists. In particular, $\lim_{n\to\infty} w(\overline{v}, 1, n) = WWW\ldots$ where $W = w_1w_2 \cdots w_p$ is of length p and is not periodic.

We define a sequence of partitions $(\mathcal{Q}_n)$ that will be refining, clopen, generating periodic partitions of (X_B, φ_ω), and such that $|\mathcal{Q}_{n+1}|$ is a multiple of $|\mathcal{Q}_n|$. The existence of this sequence implies that (X_B, φ_ω) is an odometer. For $x = x_1x_2 \cdots \in X_B$

(where $s(x_1) = v_0$), $j \in \mathbb{N}$, and $0 \le i \le p-1$, let

$$[i]_j = \left\{ x : s(x_{j+1})\, s\big((\varphi_\omega(x))_{j+1} \big) \cdots s\big((\varphi_\omega^{p-1}(x)) \big)_{j+1} = \sigma^i(W) \right\}.$$

Let

$$\mathcal{Q}_1 := \left\{ [i]_1 : 0 \le i \le p-1 \right\}.$$

Since W is not periodic, each x lives in only one $[i]_1$, and $\mathcal{Q}_1$ is of period p for φ_ω.

Given a vertex $v \in V_n$, recall that $h_v^{(n)} = |E(v_0, v)|$ for $v \in V_n$. Define for $n > 1$,

$$\mathcal{Q}_n := \left\{ [i_1, i_2] : 0 \le i_2 \le p-1,\ \ 0 \le i_1 \le h^{(n)}_{w_{i_2+1}} - 1 \right\},$$

where

$$[i_1, i_2] := [i_2]_n \cap \left\{ x : x_1 x_2 \cdots x_n \in E(v_0, w_{i_2+1}) \text{ and has } \omega\text{-label } i_1 \right\}.$$

Then for each $n \ge 1$, $\mathcal{Q}_n$ is a clopen partition, $\mathcal{Q}_{n+1}$ refines $\mathcal{Q}_n$, and it is clear that $(\mathcal{Q}_n)$ is a generating sequence of partitions. We claim that $\mathcal{Q}_n$ is φ_ω periodic. For, if $i_1 < h^{(n)}_{w_{i_2+1}} - 1$, then $\varphi_\omega([i_1, i_2]) = ([i_1+1, i_2])$. If $i_1 = h^{(n)}_{w_{i_2+1}} - 1$ and $i_2 < p-1$, then $\varphi_\omega([i_1, i_2]) = [(0, i_2+1)]$. Finally $\varphi_\omega([h^{(n)}_{w_{i_2+1}} - 1, p-1]) = [0, 0]$.

It remains to show that $|\mathcal{Q}_{n+1}|$ is a multiple of $|\mathcal{Q}_n|$. Note that for each $v \in V$ and each $n \ge 2$, $w(v, n-1, n) = S_v^{(n)} W^{\alpha_v^{(n)}} P_v^{(n)}$ with $S_v^{(n)}$ a proper suffix of W, $P_v^{(n)}$ a proper prefix of W, and whenever $vu \in \mathcal{L}(B, \omega)$, $P_v^{(n)} S_u^{(n)}$ is either empty or equal to W. Note that $w_p \overline{w}_1 \in \mathcal{L}(B, \omega)$, so that for each n, $P^{(n)}_{w_p} S^{(n)}_{w_1} = W$ or is the empty word. We assume that $P^{(n)}_{w_p} S^{(n)}_{w_1} = W$ in the computation below, otherwise simply remove the “1”. If $W' \subset W$, let $\#_{W'}(W)$ denote the distinct number of occurrences of W' in W. Then

$$\begin{aligned} |\mathcal{Q}_{n+1}| &= p \sum_{v \in W} \#_v(W) h_v^{(n+1)} \\ &= p \sum_{v \in W} \#_v(W) \Big[\alpha_v^{(n+1)} + \sum_{v_1 w_1 : P^{(n+1)}_{v_1} S^{(n+1)}_{w_1} = W} \#_{v_1 w_1}(W) + 1 \Big] |\mathcal{Q}_n|. \qquad \blacksquare \end{aligned}$$

We will now consider in detail the class of finite rank diagrams described in Example 3.21. Let the Bratteli diagram B have strict rank $d > 1$. We show that if B is to support a perfect ordering with d maximal and d minimal paths, then a certain structure is imposed on the incidence matrices of B.

Definition 3.31 Denote by $\mathcal{D}$ the set of rank d simple Bratteli diagrams B where $V_n = \{v_1, \ldots, v_d\}$ for each $n \ge 1$, whose incidence matrices (F_n) eventually belong to the class $\mathcal{M}$ (see Definition 3.27), and where all entries are non-zero.

It is not hard to check that the set $\mathcal{D}$ is invariant under telescoping of diagrams.

Proposition 3.32 *Let B be a simple strict rank d Bratteli diagram.*

(i) *Suppose $B \in \mathcal{D}$, and σ is a cyclic permutation of the set $\{1, 2, \dots, d\}$. Then there exists an ordering $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(d)$ on B such that*

$$X_{\max}(\omega) = \{M_1, \dots, M_d\}, \quad X_{\min}(\omega) = \{m_1, \dots, m_d\},$$

where M_i (m_j) is an eventually vertical path through the vertex v_i (v_j, respectively), $i, j = 1, \dots, d$. Moreover, the corresponding Vershik map φ_ω satisfies the condition

$$\varphi_\omega(M_i) = m_{\sigma(i)}. \tag{3.2}$$

(ii) *Suppose there exists an ordering $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(d)$ such that all maximal and minimal paths are eventually vertical. Then the Vershik map φ_ω determines a cyclic permutation on the set $\{1, \dots, d\}$ and B belongs to $\mathcal{D}$.*

Proof (i) We need to construct a perfect ordering ω on B such that (3.2) holds. For every $v_j \in \{v_1, \dots, v_d\} = V_n$ and every n large enough, we take d subsets $E(v_i, v_j)$ of $r^{-1}(v_j)$ where $v_i \in V_{n-1}$. Then $|E(v_i, v_j)| = f_j^{(n)}$ if $i \neq j$ and $|E(v_j, v_j)| = f_j^{(n)} + 1$. Hence $|r^{-1}(v_j)| = df_j^{(n)} + 1$. For each $n \geq 1$ and each $v_j \in V_n$ define the order on $r^{-1}(v_j)$ as follows:

$$w(v_j, n-1, n) = \left(v_j\, v_{\sigma(j)}\, v_{\sigma^2(j)} \cdots v_{\sigma^{d-1}(j)}\right)^{f_j^{(n)}} v_j. \tag{3.3}$$

Clearly, relation (3.3) defines explicitly a linear order on $r^{-1}(v_j)$. To see that φ_ω is continuous, it suffices to note that for each j there is a unique $i := \sigma(j)$ such that $v_j v_i \in \mathcal{L}_{B,\omega}$. By Proposition 3.3 we are done.

(ii) Conversely, suppose that ω is a perfect ordering on B with d maximal and d minimal eventually vertical paths, so that each vertex has to support both a maximal and a minimal path M_i and m_i. Thus for each i and each n large enough, the word $\omega(v_i, n-1, n)$ starts and ends with v_i. Since ω is perfect, by Proposition 3.3 there is a permutation σ such that for each $j \in \{1, \dots d\}$ only $v_j v_{\sigma(j)} \in \mathcal{L}_{B,\omega}$. So, for each j and all but finitely many n, there is an $f_j^{(n)}$ such that

$$w(v_j, n-1, n) = \left(v_j\, v_{\sigma(j)}\, v_{\sigma^2(j)} \cdots v_{\sigma^{d-1}(j)}\right)^{f_j^{(n)}} v_j. \tag{3.4}$$

Since B is simple, σ has to be cyclic, so that all vertices occur in the right hand side of (3.4). From (3.4) it also follows that all but finitely many of the incidence matrices of B are of the form (3.1). ■

Corollary 3.33 *Let B be a simple Bratteli diagram of rank $d \geq 2$ and let $\omega \in \mathcal{P}_B \cap \mathcal{O}_B(d)$. Then (X_B, φ_ω) is conjugate to an odometer.*

Proof We can assume that (B, ω) is well telescoped (conjugacy of two adic systems is invariant under telescoping of either of them). Note that the proof of Proposition 3.32 tells us that $\mathcal{L}(B, \omega)$ is periodic. Lemma 3.30 tells us that (X_B, φ_ω) is conjugate to an odometer; however in this specific case there is a simpler sequence of periodic, refining, generating partitions (Q_n). Let Q_n be the clopen partition defined by the first n levels of B, and write $Q_n = \coprod_{i=1}^{d} Q_n(v_i)$, where $Q_n(v_i)$ is the set of all paths from v_0 to $v_i \in V_n$. Each non-maximal path in $Q_n(v_i)$ is mapped by φ_ω to its successor in $Q_n(v_i)$. For $i \in \{1, \dots, d\}$, let M_i^n denote the maximal path in $Q_n(v_i)$. Since the ordering ω is perfect, $\varphi_\omega(M_i^n) = m_{\sigma(i)}^n$, where $m_{\sigma(i)}^n$ is the minimal path in $Q_n(v_{\sigma(i)})$. Thus the partition Q_n is φ_ω-periodic. We will also compute the sequence (k_n) such that $|Q_{n+1}| = k_n|Q_n|$. By Proposition 3.32, the incidence matrices of B are of the form (3.1). All columns of F_n sum to the same constant $k_n = (1 + \sum_{i=1}^{d} f_i^{(n)})$. Let $F_n = (f_{i,j}^{(n)})$ and $h_i^{(n)} := |Q_n(v_i)|$; then $h_i^{(n+1)} = \sum_{j=1}^{d} f_{i,j}^{(n)} h_j^{(n)}$ and

$$\begin{aligned}|Q_{n+1}| &= \sum_{i=1}^{d} h_i^{(n+1)} = \sum_{i=1}^{d} \Big[h_i^{(n)} + \sum_{j=1}^{d} h_j^{(n)} f_i^{(n)} \Big] \\ &= |Q_n| + \sum_{i=1}^{d} f_i^{(n)} \sum_{j=1}^{d} h_j^{(n)} = |Q_n| \Big(1 + \sum_{i=1}^{d} f_i^{(n)} \Big). \qquad \blacksquare\end{aligned}$$

Next we consider conditions for a Bratteli diagram B of strict rank d to support a perfect ordering ω such that (X_B, φ_ω) is an odometer. Suppose that we are given a skeleton $\mathcal{F}$ on B. We have subsets $\widetilde{V}$ and $\overline{V}$ of V, both of cardinality $k \le d$, a bijection $\sigma\colon \widetilde{V} \to \overline{V}$, and partitions $W' = \{W'_{\overline{v}} : \overline{v} \in \overline{V}\}$ and $W = \{W_{\widetilde{v}} : \widetilde{v} \in \widetilde{V}\}$ of V. Let $\mathcal{H} = (T, P)$ be the directed graph associated with $\mathcal{F}$. We assume that $\mathcal{H}$ is strongly connected. Let p be a finite path in $\mathcal{H}$. Then p can correspond to several words in $V^+ = \{v_1, \dots, v_d\}^+$. For example, if p starts at vertex $[\overline{v}, \widetilde{v}]$, then it corresponds to words starting with v whenever $v \in [\overline{v}, \widetilde{v}]$. If w is a word in V^+, then we write $w = \cdots v$ to mean that w ends with v and $w = v \cdots$ to mean that w starts with v. It is not difficult to find words $w \in V^+$ corresponding to a path in $\mathcal{H}$ such that

(a) w contains all v_i's;
(b) w^2 corresponds to a legitimate path in $\mathcal{H}$;
(c) for each $\widetilde{v} \in \widetilde{V}$, if $\sigma(\widetilde{v}) = \overline{v}$, there exist words $p(\widetilde{v}) = \cdots \widetilde{v}$ and $s(\overline{v}) = \overline{v} \cdots$ such that $w = p(\widetilde{v})\, s(\overline{v})$.

Call a word that satisfies (a)–(c) *σ-decomposable*. If w is a word, let $\overrightarrow{w}$ be the d-dimensional vector whose i-th entry is the number of occurrences of $v_i \in V$.

The following result generalizes Proposition 3.32 and gives the constraints on the sequence (F_n) of transition matrices that a diagram B has in order for B to support an odometer with a periodic language.

Proposition 3.34 *Let B be a simple, strict rank d Bratteli diagram. Suppose that $\mathcal{F}$ is a skeleton such that the associated graph $\mathcal{H}$ is strongly connected, and let w be a σ-decomposable word that corresponds to a path in $\mathcal{H}$. Let $\{p_v^{(n)}\}_{v \in V, n \ge 1}$ be a set of nonnegative integers. If the incidence matrices (F_n) of B are such that the v-th row of F_n is*

$$\overrightarrow{s(\overline{v})} + p_v^{(n)} \overrightarrow{w} + \overrightarrow{p(\widetilde{v})} \tag{3.5}$$

whenever $v \in [\overline{v}, \widetilde{v}]$, then (X_B, φ_ω) is topologically conjugate to an odometer.

Proof Define, for $v \in [\overline{v}, \widetilde{v}]$, $w(v, n-1, n) := s(\overline{v}) w^{p_v^{(n)}} p(\widetilde{v})$. Note that the v-th row of $F^{(n)}$ is (3.5), and (B, ω) has skeleton $\mathcal{F}$. Now $\mathcal{H}$ tells us what words of length 2 are allowed in $\mathcal{L}_{B,\omega}$: $vv' \in \mathcal{L}_{B,\omega}$ only if $v \in [\overline{v}, \widetilde{v}]$, $v' \in [\overline{v}', \widetilde{v}']$, and $\sigma(\widetilde{v}) = \overline{v}'$. Thus

$$w(v, n-1, n) w(v', n-1, n) = s(\overline{v}) w^{p_v^{(n)}} p(\widetilde{v}) \;\; s(\overline{v}') w^{p_{v'}^{(n)}} p(\widetilde{v}') = s(\overline{v}) w^{p_v^{(n)}} w w^{p_{v'}^{(n)}} P_{\widetilde{v}'}$$

by property (c) of a σ-decomposable word. Since $w(v, n-1, n+1)$ (and more generally, $w(v, n-1, N)$) is a concatenation of words $w(v, n-1, n)$, this implies that $\mathcal{L}_{B,\omega}$ is periodic. Proposition 3.30 implies the desired result. ■

There is a converse to this result, namely that if a perfect order ω on a simple diagram B has a periodic language, then there is some σ-decomposable word that generates $\mathcal{L}(B, \omega)$, so that by Lemma 3.30, (X_B, φ_ω) is an odometer.

If $V = \{v_1, v_2, \ldots, v_d\}$ and a perfect ω is to have d maximal paths, then Proposition 3.32 tells us that $v_1 v_2 \cdots v_d$ is, up to rotation, the only σ-decomposable word. The next example shows that in general σ-decomposable words are easy to find.

Example 3.35 Let $V = \{a_1, a_2, \ldots, a_{n+1}\}$, $\overline{V} = \widetilde{V} = \{a_1, a_2, \ldots, a_n\}$, $\sigma(a_i) = a_{i+1}$ for $i < n$, and $\sigma(a_n) = a_1$, where $[a_i, a_i] = \{a_i\}$ for each i and $a_{n+1} \in [a_i, a_j]$ for some $j \neq i$. Then any word starting with a_i (for $1 \leq i \leq n$), ending with $\sigma^{-1}(a_i)$, and containing all a_i's is σ-decomposable.

4 A Characterization of Finite Rank Diagrams that Support Perfect, Non-proper Orders

In this section, which is built on the results of Section 3, we discuss under what conditions a simple rank d Bratteli diagram B can have a perfect ordering ω belonging to $\mathcal{O}_B(k)$ for $1 < k \leq d$. It turns out that the incidence matrices must satisfy certain conditions, which in turn depend on the skeleton that one is considering.

Let (B, ω) be a perfectly ordered simple Bratteli diagram. We continue to assume that (B, ω) is well telescoped. Let $\mathcal{F} = \mathcal{F}_\omega$ be the skeleton generated by ω and let $\varphi = \varphi_\omega$ be the corresponding Vershik map. We have $|\widetilde{V}| = |\overline{V}|$, and φ_ω defines a one-to-one map $\sigma\colon \widetilde{V} \to \overline{V}$ such that $\varphi_\omega(M_v) = m_{\sigma(v)}$ for $v \in \widetilde{V}$. Recall also the two partitions $W = \{W_{\widetilde{v}} : v \in \widetilde{V}\}$ and $W' = \{W'_{\overline{v}} : v \in \overline{V}\}$ of V generated by $\mathcal{F}$.

We need some new notation. Recall that we write $\widetilde{V}_n$ ($\overline{V}_n$) instead of just $\widetilde{V}$ ($\overline{V}$) if we need to specify in which level $\widetilde{V}$ ($\overline{V}$) lies. Let $E(V_n, u)$ be the set of all finite paths between vertices of level n and a vertex $u \in V_m$ where $m > n$. The symbols $\widetilde{e}(V_n, u)$ and $\overline{e}(V_n, u)$ are used to denote the maximal and minimal finite paths in $E(V_n, u)$, respectively. By $\widetilde{V}_n$ we mean that we are looking at the set $\widetilde{V}$ of vertices at level n. Fix maximal and minimal vertices $\widetilde{v}$ and $\overline{v}$ in $\widetilde{V}_{n-1}$ and $\overline{V}_{n-1}$ respectively. Denote

$$E(W_{\widetilde{v}}, u) = \left\{ e \in E(V_n, u) : s(e) \in W_{\widetilde{v}}, r(e) = u \right\},$$

$$\widetilde{E}(W_{\widetilde{v}}, u) = E(W_{\widetilde{v}}, u) \setminus \left\{ \widetilde{e}(V_n, u) \right\}.$$

Similarly, $\overline{E}(W'_{\overline{v}}, u) = E(W'_{\overline{v}}, u) \setminus \{\overline{e}(V_n, u)\}$. Clearly, the sets $\{E(W_{\widetilde{v}}, u) : \widetilde{v} \in \widetilde{V}\}$ form a partition of $E(V_n, u)$. Let e be a non-maximal finite path, with $r(e) = v$ and

$s(e) \in V_m$, which determines the cylinder set $U(e)$. By $\varphi_\omega(e)$ we mean $\varphi_\omega(U(e))$, the image under $\varphi_\omega(e)$ of the cylinder set $U(e)$, which also has range v and source in V_m.

Lemma 4.1 *Let (B, ω) be a perfectly ordered, well-telescoped finite rank simple diagram, where ω has skeleton $\mathcal{F}_\omega$ and permutation $\sigma\colon \widetilde{V} \to \overline{V}$. If $n > 1$, $\widetilde{v} \in \widetilde{V}_{n-1}$, and $u \in V_m$ ($m > n$), then for any finite path $e \in \widetilde{E}(W_{\widetilde{v}}, u)$ we have $\varphi_\omega(e) \in \overline{E}(W'_{\sigma(\widetilde{v})}, u)$.*

Proof Note that $s(e)s(\varphi_\omega(e))$ is a subword of $w(u, n, m)$. Now $s(e) \in W_{\widetilde{v}}$ by assumption and $s(\varphi_\omega(e)) \in W_{\overline{v}}$ for some $\overline{v}$. This implies that $\widetilde{v}\overline{v}$ is a sub-word of $w(u, n-1, m)$. Recalling that (B, ω) is telescoped, the result follows. ■

We immediately deduce from the previous lemma that the following result on entries of incidence matrices is true.

Corollary 4.2 *In the notation of Lemma 4.1, the following condition holds for the perfectly ordered, well-telescoped finite rank simple diagram (B, ω): for any $n \geq 2$, any vertex $\widetilde{v} \in \widetilde{V}_{n-1}$, $m > n$, and any $u \in V_m$,*

$$|\widetilde{E}(W_{\widetilde{v}}, u)| = |\overline{E}(W'_{\sigma(\widetilde{v})}, u)|.$$

In particular, if B is as above and $(F_n) = ((f^{(n)}_{v,w}))$ denotes the sequence of positive incidence matrices for B, then we can apply Corollary 4.2 to obtain the following property on F_n. Define two sequences of matrices $\widetilde{F}_n = (\widetilde{f}^{(n)}_{w,v})$ and $\overline{F}_n = (\overline{f}^{(n)}_{w,v})$ by the following rule (here $w \in V_{n+1}$, $v \in V_n$ and $n \geq 1$):

$$\widetilde{f}^{(n)}_{w,v} = \begin{cases} f^{(n)}_{w,v} - 1 & \text{if } \widetilde{e}_w \in E(v, w), \\ f^{(n)}_{w,v} & \text{otherwise,} \end{cases}$$

$$\overline{f}^{(n)}_{w,v} = \begin{cases} f^{(n)}_{w,v} - 1 & \text{if } \overline{e}_w \in E(v, w), \\ f^{(n)}_{w,v} & \text{otherwise.} \end{cases}$$

Then for any $u \in V_{n+1}$ and $\widetilde{v} \in \widetilde{V}_{n-1}$, we obtain that under the conditions of Corollary 4.2 the entries of incidence matrices have the property

$$\sum_{w \in W_{\widetilde{v}}} \widetilde{f}^{(n)}_{u,w} = \sum_{w' \in W'_{\sigma(\widetilde{v})}} \overline{f}^{(n)}_{u,w'}, \quad n \geq 2. \tag{4.1}$$

We call relations (4.1) the *balance relations.*

Given $(\mathcal{F}, \sigma)$ on B, is it sufficient for B to satisfy the balance relations so that there is a perfect order on B with associated skeleton and permutation $(\mathcal{F}, \sigma)$? Almost. We need one extra condition on B. First we need finer notation for $\mathcal{H}$. We replace it with a sequence $(\mathcal{H}_n)$ where each $\mathcal{H}_n$ looks exactly the same as $\mathcal{H}$, except that the vertices T_n of $\mathcal{H}_n$ are labeled $[\overline{v}, \widetilde{v}, n]$. Paths in $\mathcal{H}_n$ will correspond to words from V_n, in particular, the word $w(u, n, n+1)$ will correspond to a path in $\mathcal{H}_n$. (In the case where B is a stationary diagram, there is no need to replace $\mathcal{H}$ with $(\mathcal{H}_n)$.)

Definition 4.3 Fix $n \in \mathbb{N}$ and $u \in V_{n+1}$. If $[\overline{v}, \widetilde{v}, n] \in \mathcal{H}_n$, we define the *crossing number* $P_u([\overline{v}, \widetilde{v}, n])$ for the vertex $[\overline{v}, \widetilde{v}, n]$ as

$$P_u([\overline{v}, \widetilde{v}, n]) := \sum_{w \in [\overline{v}, \widetilde{v}, n]} \widetilde{f}^{(n)}_{uw}.$$

This crossing number represents the number of times that we will have to pass through the vertex $[\overline{v}, \widetilde{v}, n]$ when we define an order on $r^{-1}(u)$, for $u \in V_{n+1}$; and here we emphasize that if we terminate at $[\overline{v}, \widetilde{v}, n]$, we do not consider this final visit as contributing to the crossing number; this is why we use the terms $\widetilde{f}^{(n)}_{u,w}$, and not $f^{(n)}_{u,w}$.

Definition 4.4 We say that $\mathcal{H}_n$ is *positively strongly connected* if for each $u \in V_{n+1}$, the set of vertices $\{[\overline{v}, \widetilde{v}, n] : P_u([\overline{v}, \widetilde{v}, n]) > 0\}$, along with all the relevant edges of $\mathcal{H}_n$, form a strongly connected subgraph of $\mathcal{H}_n$.

If $s(\widetilde{e}_u) \in [\overline{v}, \widetilde{v}, n]$, we shall call this vertex in $\mathcal{H}_n$ the *terminal vertex* (for u), as when defining the order on $r^{-1}(u)$, we need a path that ends at this vertex (although it can previously go through this vertex several times, in fact precisely $P_u([\overline{v}, \widetilde{v}, n])$ times).

Example 4.5 In this example we have a stationary diagram, so we drop the dependence on n. Suppose that $V = \{a, b, c, d\}$, $\overline{V} = \widetilde{V} = \{a, b, c\}$, with $a \in [a, a]$, $b \in [b, b]$, $c \in [c, c]$ and $d \in [b, a]$. Let $\sigma(a) = b$, $\sigma(b) = c$, and $\sigma(c) = a$. Suppose that for each $n \geq 1$ the incidence matrix $F = F_n$ is

$$F := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

Then if $u = d$, $P_d([a, a]) = 0$, and the remaining three vertices $[b, b]$, $[c, c]$, and $[b, a]$ do not form a strongly connected subgraph of $\mathcal{H}$, then there is no path from $[c, c]$ to $[b, a]$.

Note also that although the rows of this incidence matrix satisfy the balance relations (4.1), there is no way to define an order on $r^{-1}(d)$ so that the resulting global order is perfect. The lack of positive strong connectivity of the graph $\mathcal{H}$ is precisely the impediment.

The following result shows that, given a skeleton $\mathcal{F}$ on B, as long as the associated graphs $(\mathcal{H}_n)$ are eventually positively strongly connected, the balance relations are sufficient to define a perfect ordering ω on a simple Bratteli diagram.

Theorem 4.6 *Let B be a simple strict rank d Bratteli diagram, let*

$$\mathcal{F} = \{M_{\widetilde{v}}, m_{\overline{v}}, \widetilde{e}_w, \overline{e}_w : w \in V^* \backslash V_0,\ \widetilde{v} \in \widetilde{V} \text{ and } \overline{v} \in \overline{V}\}$$

be a skeleton on B, and let $\sigma\colon \widetilde V \to \overline V$ *be a bijection. Suppose that eventually all associated graphs* $\mathcal{H}_n$ *are positively strongly connected and that the entries of incidence matrices* (F_n) *eventually satisfy the balance relations* (4.1)*. Then there is a perfect ordering* ω *on B such that* $\mathcal{F} = \mathcal{F}_\omega$ *and the Vershik map* φ_ω *satisfies the relation* $\varphi_\omega(M_{\widetilde v}) = m_{\sigma(\widetilde v)}$.

Proof Fix n large enough so that $\mathcal{H}_n$ is positively strongly connected and the balance relations hold. Our goal is to define a linear order ω_u on $r^{-1}(u)$ for each $u \in V_{n+1}$. Once this is done for all large n, the corresponding partial ordering ω on B will be perfect. Recall that each set $r^{-1}(u)$ contains two pre-selected edges $\widetilde e_u, \overline e_u$. and they should be the maximal and minimal edges after defining ω_u.

Fix $u \in V_{n+1}$. The proof is based on an recursive procedure that is applied to the u-th row of the incidence matrix F_n. We describe in detail the first step of the algorithm that will be applied repeatedly. At the end of each step in the algorithm, one entry in the u-th row of F_n will have been reduced by one, and a path in $\mathcal{H}_n$ will have been extended by one edge. At the end of the algorithm, the u-th row will have been reduced to the zero row, and a path will have been constructed in $\mathcal{H}_n$, starting at the vertex in $\mathcal{H}_n$ to which $s(\overline e_u)$ belongs, and ending at the vertex in $\mathcal{H}_n$ to which $s(\widetilde e_u)$ belongs. This path will determine the word $w(u, n, n+1)$, *i.e.,* the order ω_u on $r^{-1}(u)$. It will be seen from the proof of the theorem that for given $\mathcal{F}$ and σ, the order ω_u that is defined is not unique.

We will first consider the particular case when the associated graph $\mathcal{H}_n$ defined by $(\mathcal{F}, \sigma)$ does not have any loops. After that, we will modify the construction to include possible loops in the algorithm. We also include Examples 4.8 and 4.9 to illustrate why it is necessary to consider these cases.

Case I, there is no loop in $\mathcal{H}_n$: Consider the u-th rows of matrices $\overline F_n$ and $\widetilde F_n$. They coincide with the row $(f^{(n)}_{u,v_1}, \dots, f^{(n)}_{u,v_d})$ of the matrix F_n except only one entry corresponding to $|E(s(\overline e_u), u)|$ and one entry corresponding to $|E(s(\widetilde e_u), u)|$. To simplify our notation, since n is fixed we omit it as an index, so that $F = F_n$, $f_{u,w} = f^{(n)}_{u,w}$, $[\overline v, \widetilde v] = [\overline v, \widetilde v, n]$, $\mathcal{H} = \mathcal{H}_n$, etc.

Take $\overline e_u$ and assign the number 0 to it; *i.e.,* $\overline e_u$ is the minimal edge in ω_u. Let $[\overline v_0, \widetilde v_0]$ be the vertex[8] of $\mathcal{H}$ such that $s(\overline e_u) \in [\overline v_0, \widetilde v_0]$. Consider the set

$$\left\{ \widetilde v \in \widetilde V : [\sigma(\widetilde v_0), \widetilde v] \in \mathcal{H} \right\}$$

(this set is formed by ranges of arrows in $\mathcal{H}$ coming out from $[\overline v_0, \widetilde v_0]$). Find w' such that $\widetilde f_{u,w'} \geq \widetilde f_{u,w}$ for all entries $f_{u,w}, w \in W'_{\sigma(\widetilde v_0)}$. If there are several entries that are the maximal value, then $f_{u,w'}$ is chosen arbitrarily amongst them. Take any edge $e_1 \in E(w', u)$. In the case where $\widetilde e_u \in E(w', u)$, we choose $e_1 \neq \widetilde e_u$. Assign the number 1 to e_1 so that e_1 becomes the successor of $e_0 = \overline e_u$. We note also that the choice of w' from $W'_{\sigma(v_0)}$ actually means that we take some $\widetilde v_1 \in \widetilde V$ such that $s(e_1) \in [\sigma(\widetilde v_0), \widetilde v_1]$. In other words, we take the edge from $[\overline v_0, \widetilde v_0]$ to $[\sigma(\widetilde v_0), \widetilde v_1]$ in the associated graph $\mathcal{H}$.

We note that in the collection of relations (4.1), enumerated by vertices from $\widetilde V$, we have worked with the equation defined by u and $\widetilde v_0$. Two edges were labeled in

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[8] The same word "vertex" is used in two meanings: for elements of the set T of the graph $\mathcal{H}$ and for elements of the set V of the Bratteli diagram B. To avoid any possible confusion, we point out explicitly which vertex is meant in that context.

the above procedure, e_0 and e_1. We may think of this step as if these edges were "removed" from the set of all edges in $r^{-1}(u)$. We claim that the remaining non-enumerated edges satisfy the equation

$$\left(\sum_{w\in W_{\widetilde{v}_0}} \widetilde{f}_{u,w}\right) - 1 = \left(\sum_{w\in W'_{\sigma(\widetilde{v}_0)}} \overline{f}_{u,w}\right) - 1. \tag{4.2}$$

To see this, note that $\widetilde{v}_1 \neq \widetilde{v}_0$, for if not, then $\sigma(\widetilde{v}_1) = \sigma(\widetilde{v}_0)$, but this implies that there would be a loop at $[\sigma(\widetilde{v}_0), \widetilde{v}_1]$, a contradiction to our assumption. Thus $\widetilde{v}_1 \neq \widetilde{v}_0$ and this is why there is exactly one edge removed from each side of (4.2). Note that we now have a "new", reduced u-th row of F. Namely, the entry $f_{u,\widetilde{v}_0}$ has been reduced by one. Thus the crossing numbers of the vertices of $\mathcal{H}$ change (one crossing number is reduced by one). Also note that in $\mathcal{H}$, we have arrived at the vertex $[\sigma(\widetilde{v}_0), \widetilde{v}_1]$ to which w' belongs. Thus for this reduced u-th row, $\overline{f}_{u,w'} = f_{u,w'} - 1$. In other words, with each step of this algorithm the row we are working with changes, and the vertex w such that $\overline{f}_{u,w} = f_{u,w} - 1$ changes (in fact, has to change, because there are no loops in $\mathcal{H}$). For the vertex such that $\overline{f}_{u,w} = f_{u,w} - 1$ belongs to the vertex in $\mathcal{H}$ where we are currently, and this changes at every step of the algorithm. Thus the new reduced u-th row of F still satisfies the balance relations (4.1) as $\widetilde{v} \in \widetilde{V}$ varies. This completes the first step of the construction.

We apply the described procedure again to show how we should proceed to complete the next step. Let us assume that all crossing numbers ares still positive for the time being to describe the second step of the algorithm.

Consider the set $\{f_{u,w} : w \in W'_{\sigma(\widetilde{v}_1)}\}$ and find some w'' such that $\widetilde{f}_{u,w''} \geq \widetilde{f}_{u,w}$ for any $w \in W'_{\sigma(\widetilde{v}_1)}$. In the corresponding set of edges $E(w'', u)$ we choose $e_2 \neq \widetilde{e}_u$, and assign the number 2 to the edge e_2, so that e_2 is the successor of e_1.

Observe that now we are dealing with the relation of (4.1) that is determined by $\widetilde{v}_1 \in \widetilde{V}$. If we again "remove" the enumerated edges e_1 and e_2, then this relation remains true with both sides reduced by 1 as we saw in (4.2).

We remark also that the choice that we made of w'' (or e_2) allows us to continue the existing path (in fact, the edge) in $\mathcal{H}$ from $[\overline{v}_0, \widetilde{v}_0]$ to $[\sigma(\widetilde{v}_0), \widetilde{v}_1]$ with the edge from $[\sigma(\widetilde{v}_0), \widetilde{v}_1]$ to $[\sigma(\widetilde{v}_1), \widetilde{v}_2]$, where $\widetilde{v}_2$ is defined by the property that $s(e_2) \in [\sigma(\widetilde{v}_1), \widetilde{v}_2]$.

This process can be continued. At each step we apply the following rules:

(1) the edge e_i, that must be chosen next after e_{i-1}, is taken from the set $E(w^*, u)$ where w^* corresponds to a maximal entry amongst $\widetilde{f}_{u,w}$ as w runs over $W'_{\sigma(\widetilde{v}_{i-1})}$;

(2) the edge e_i is always taken not equal to $\widetilde{e}_u$ unless no more edges except $\widetilde{e}_u$ are left.

After every step of the construction, we see that the following statements hold.

(i) Relations (4.1) remain true when we treat them as the number of non-enumerated edges left in $r^{-1}(u)$. In other words, when a pair of vertices $\widetilde{v}$ and $\sigma(\widetilde{v})$ is considered, we reduce by 1 each side of the equation defined by $\widetilde{v}$.

(ii) The procedure used allows us to build a path p from the starting vertex $[\overline{v}_0, \widetilde{v}_0]$ going through other vertices of the graph $\mathcal{H}$ according to the choice we make at each step. We need to guarantee that at each step we are able to move to a vertex in $\mathcal{H}$ whose crossing number is still positive (unless we are at the terminal stage). As long as the crossing numbers of vertices in $\mathcal{H}$ are positive, there is no concern. Suppose though that we land at a (non-terminal) vertex $[\overline{v}, \widetilde{v}]$ in $\mathcal{H}$ whose crossing number is one (and this is the first time this happens). When we leave this vertex to go to $[\sigma(\widetilde{v}), \widetilde{v}']$, the crossing number for $[\overline{v}, \widetilde{v}]$ will become 0 and therefore it will no longer be a vertex of $\mathcal{H}$ that we can "cross" through, maybe only arriving at it terminally. *Thus at this point, with each step, the graph $\mathcal{H}$ is also changing (being reduced).* We need to ensure that there is a way to continue the path out of $[\sigma(\widetilde{v}), \widetilde{v}']$. Since $\sum_{w\in W_{\widetilde{v}}} \widetilde{f}_{u,w} \geq P_u[\sigma(\widetilde{v}), \widetilde{v}] = 1$, by the balance relations, $\sum_{w'\in W_{\sigma(\widetilde{v}')}} \overline{f}_{u,w'} \geq 1$. If the crossing number of all the vertices $[\sigma(\widetilde{v}'), *]$ have been reduced to 0, then this means that for a unique w', $\overline{f}_{u,w'} = 1$ (the rest of the summands in $\sum_{w'\in W_{\sigma(\widetilde{v}')}} \overline{f}_{u,w'}$ equal 0), and $\widetilde{f}_{u,w'} = 1$. This tells us that we have to move into this terminal vertex for the last time. Then the balance equations, which continue to be respected, ensure we are done. Otherwise, the balance equations guarantee that $\sum_{w'\in W_{\sigma(\widetilde{v}')}} \overline{f}_{u,w'} > 1$, which means there is a valid continuation of our path out of $[\sigma(\widetilde{v}), \widetilde{v}']$ and to a new vertex in $\mathcal{H}$, and we are not at the end of the path. It is these balance equations that always ensure that the path can be continued until it reaches its terminal vertex.

(iii) In accordance with (i), the u-th row of F is transformed by a sequence of steps in such a way that entries of the rows obtained form decreasing sequences. These entries show the number of non-enumerated edges remaining after the completed steps. It is clear that, by the rule used above, we decrease the largest entries first. It follows from the simplicity of the diagram that, for sufficiently many steps, the set $\{s(e_i)\}$ will contain all vertices $v_1, \ldots, v_d$ from V. This means that the transformed u-th row consists of entries that are strictly less than those of the very initial u-th row F. After a number of steps the u-th row will have a form where the difference between any two entries is ± 1. After that, this property will remain true.

(iv) It follows from (iii) that we eventually obtain that all entries of the resulting u-th row are zeros or ones. We apply the same procedure to enumerate the remaining edges from $r^{-1}(u)$ such that the number $|r^{-1}(u)| - 1$ is assigned to the edge $\widetilde{e}_u$. This means that we have constructed the word $W_u = s(\overline{e}_u)s(e_1)\cdots s(\widetilde{e}_u)$; *i.e.,* we have ordered $r^{-1}(u)$.

Looking at the path p that is simultaneously built in $\mathcal{H}$, we see that the number of times this path comes into and leaves a vertex $[\overline{v}, \widetilde{v}]$ of the graph is precisely the crossing number of $[\overline{v}, \widetilde{v}]$. In other words, p is an Eulerian path of $\mathcal{H}$ that finally arrives at the vertex of $\mathcal{H}$ defined by $s(\widetilde{e}_u)$.

Case II, there is a loop in $\mathcal{H} = \mathcal{H}_n$: To deal with this case, we have to refine the described procedure to avoid a possible situation when the algorithm cannot be finished properly.

We start as in Case I and continue until we have arrived at a vertex $[\overline{v}_1, \widetilde{v}]$, where, for the first time, $[\sigma(\widetilde{v}), \widetilde{v}] \in \mathcal{H}$. In other words, this is the first time that our path reaches a vertex that has a successor with a loop. If $[\sigma(\widetilde{v}), \widetilde{v}]$ has crossing number zero; *i.e.,* if it is the terminal vertex and we are not at the terminal stage of defining the order, we ignore this vertex and continue as in Case (I). If $[\sigma(\widetilde{v}), \widetilde{v}]$ has a positive crossing number, *i.e.,* $P_u([\sigma(\widetilde{v}), \widetilde{v}]) > 0$, then at this point, we continue the path to

$[\sigma(\widetilde{v}), \widetilde{v}]$, and then traverse this loop $P_u([\sigma(\widetilde{v}), \widetilde{v}]) - 1$ times. If

$$P_u([\sigma(\widetilde{v}), \widetilde{v}]) = \sum_{w \in [\sigma(\widetilde{v}), \widetilde{v}]} \widetilde{f}_{u,w} = \sum_{w \in [\sigma(\widetilde{v}), \widetilde{v}]} f_{u,w},$$

this means we are traversing this vertex enough times that it is no longer part of the resulting $\mathcal{H}$ that we have at the end of this step; we are removing the looped vertex. If

$$P_u([\sigma(\widetilde{v}), \widetilde{v}]) = \sum_{w \in [\sigma(\widetilde{v}), \widetilde{v}]} \widetilde{f}_{u,w} = \Big(\sum_{w \in [\sigma(\widetilde{v}), \widetilde{v}]} f_{u,w} \Big) - 1,$$

then we are reducing this vertex to a one whose crossing number is 0, and we will only return to this vertex at the very end of our path. Looking at the relation

$$\sum_{w \in W_{\widetilde{v}}} \widetilde{f}_{u,w} = \sum_{w' \in W'_{\sigma(\widetilde{v})}} \overline{f}_{u,w'}, \tag{4.3}$$

we see that we have removed all the values $\widetilde{f}_{u,w}$, where $w \in [\sigma(\widetilde{v}), \widetilde{v}]$ on the left-hand side and also this same number of values from the right-hand side. We consequently enumerate all edges whose source lies in $[\sigma(\widetilde{v}), \widetilde{v}]$ in *arbitrary order*.

We also need to ensure that once we have traversed this loop the required number of times, we can actually leave this vertex $[\sigma(\widetilde{v}), \widetilde{v}]$. To see this, we first make a remark about the graph $\mathcal{H}$. Suppose that there is a loop in $\mathcal{H}$ at $[\overline{v}, \widetilde{v}]$ whose crossing number is positive. If $[\overline{v}_1, \widetilde{v}]$ is a (non-looped) vertex with a positive crossing number that has $[\overline{v}, \widetilde{v}]$ (the vertex with the loop) as a successor, then for some $\widetilde{v}' \neq \widetilde{v}$, the vertex $[\overline{v}, \widetilde{v}']$ will satisfy $\sum_{w' \in [\overline{v}, \widetilde{v}']} \overline{f}_{u,w'} > 0$. This is because of our discussion above concerning (4.3): the crossing number at the looped vertex appears on both sides and cancels. So if $[\overline{v}_1, \widetilde{v}]$ has a positive crossing number, this contributes positive values to the left-hand side of (4.3); thus, there is some vertex $[\overline{v}, \widetilde{v}']$ with a positive value $\sum_{w' \in [\overline{v}, \widetilde{v}']} \overline{f}_{u,w'}$ contributing to the right-hand side of (4.3). All this means that we are able to continue our path out of the looped vertex $[\sigma(\widetilde{v}), \widetilde{v}]$.

Then we return to the procedure from Case I until we reach a vertex with a looped vertex as a successor and revert to the procedure from Case I when we have removed the looped vertex.

To summarize Cases I and II, we notice that in constructing the Eulerian path p, the following rule is used. As soon as p arrives before a loop around a vertex t in $\mathcal{H}$, p traverses this vertex $P_u(t) - 1$ times. Then p leaves t and goes to the vertex t' according to the procedure in Case I.

As noted above, the fact that all edges e from $r^{-1}(u)$ are enumerated is equivalent to defining a word formed by the sources of e. In our construction, we obtain the word $w(u, n, n+1) = s(\overline{e}_u)s(e_1) \cdots s(e_j) \cdots s(\widetilde{e}_u)$.

Applying these arguments to every vertex u at every sufficiently advanced level of the diagram, we define an ordering ω on B. That ω is perfect follows from Lemma 3.17. We chose ω to have skeleton $\mathcal{F}$, and for each $n \geq 1$ constructed all words $w(v, n, n+1)$ to correspond to paths in $\mathcal{H}_n$. The result follows. ■

***Remark* 4.7** We observe that the assumption about simplicity of the Bratteli diagram in the above theorem is redundant. It was used only when we worked with strictly positive rows of incidence matrices. But for a non-simple finite rank diagram B we can use the following result, proved in [BKMS11].

> *Any Bratteli diagram of finite rank is isomorphic to a diagram whose incidence matrices (F_n) are of the form*
>
> $$F_n = \begin{pmatrix} F_1^{(n)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & F_2^{(n)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_s^{(n)} & 0 & \cdots & 0 \\ X_{s+1,1}^{(n)} & X_{s+1,2}^{(n)} & \cdots & X_{s+1,s}^{(n)} & F_{s+1}^{(n)} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ X_{m,1}^{(n)} & X_{m,2}^{(n)} & \cdots & X_{m,s}^{(n)} & X_{m,s+1}^{(n)} & \cdots & F_m^{(n)} \end{pmatrix}. \tag{4.4}$$
>
> *For every $n \geq 1$, the matrices $F_i^{(n)}$, $i = 1, \ldots, s$, have strictly positive entries, and matrices $F_i^{(n)}$, $i = s+1, \ldots, m$, have either strictly positive or zero entries. For every fixed $j = s+1, \ldots, m$, there is at least one non-zero matrix $X_{j,k}^{(n)}$.*

It follows from (4.4) that for $u \in V_{n+1}$ the u-th row of F_n consists of several parts such that the proof of Theorem 4.6 can be applied to each of these parts independently. Indeed, it is obvious that if u belongs to any subdiagram defined by $(F_i^{(n)})$, $i = 1, \ldots, s$, then we have a simple subdiagram. If u is taken from $(F_i^{(n)})$, $i = s+1, \ldots, m$, then by (4.4) we may have some zeros in a row, but they do not affect the procedure in the proof of Theorem 4.6.

We illustrate the proof of Theorem 4.6 with the following examples.

***Example* 4.8** Suppose B is a rank 6 Bratteli diagram defined on the vertices $\{a, b, c, d, e, f\}$. Let $\overline{V} = \widetilde{V} = \{a, b, c\}$ and $\sigma(a) = b, \sigma(b) = c, \sigma(c) = a$. Take the skeleton $\mathcal{F} = \{M_a, M_b, M_c, m_a, m_b, m_c; \overline{e}_d, \widetilde{e}_d, \overline{e}_e, \widetilde{e}_e, \overline{e}_f, \widetilde{e}_f\}$ where $s(\overline{e}_d) = b$, $s(\overline{e}_e) = b$, $s(\overline{e}_f) = c$ and $s(\widetilde{e}_d) = a$, $s(\widetilde{e}_e) = a$, $s(\widetilde{e}_f) = c$. For simplicity of notation, we suppose that B is stationary. For such a choice of data, we see that non-empty intersections of partitions W and W' give the following sets: $[a, a] = \{a\}$, $[b, a] = \{d, e\}$, $[b, b] = \{b\}$, $[c, c] = \{c, f\}$. The graph $\mathcal{H}$ is illustrated in Figure 4.

We see that $\mathcal{H}$ has four vertices and one loop around the vertex $[b, a]$. The directed edges are shown on the figure and defined by σ.

We consider, for example, the case $u = a$ and construct an order on $r^{-1}(a)$ according to Theorem 4.6. In this case, the balance relations have the form $f_{a,a} - 1 = f_{a,b} = f_{a,c} + f_{a,f}$, and the entries $f_{a,d}$, $f_{a,e}$ can be taken arbitrarily, because they correspond to the loop in $\mathcal{H}$. For instance, the row $(3, 2, 1, 3, 2, 1)$ satisfies the above condition. Applying the algorithm in the proof of Theorem 4.6, we can order the edges from $r^{-1}(a)$ such that their sources form the word

$$w(a, n-1, n) = addeedbfabca.$$

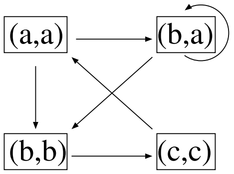

Figure 4: The graph associated with $\mathcal{F}_\omega$ in Example 4.8

To define an order on $r^{-1}(v), v = b, c, d, e, f$, we apply similar arguments (details are left to the reader). By Theorem 4.6, we conclude that if the entries of incidence matrices satisfy (4.1), then B admits a perfect ordering ω such that $\mathcal{F} = \mathcal{F}_\omega$ and the Vershik map agrees with σ.

In the next example, we will show how one can describe the structure of Bratteli diagrams of rank d for which there exists a perfect ordering with exactly $d-1$ maximal and minimal paths. The following example deals with a finite rank 3 diagram.

Example 4.9 Suppose B is a rank 3 diagram defined on the vertices $\{a, b, c\}$ with $\overline{V} = \widetilde{V} = \{a, b\}$ and $\sigma(a) = b, \sigma(b) = a$. Take the skeleton

$$\mathcal{F} = \{M_a, M_b, m_a, m_b; \widetilde{e}_c, \overline{e}_c\},$$

where $s(\overline{e}_c) = b$, $s(\widetilde{e}_c) = a$. For such a choice of the data, we see that $[a, a] = \{a\}, [a, b] = \varnothing, [b, a] = \{c\}, [b, b] = \{b\}$ and $\mathcal{H}$ is illustrated in Figure 4.9.

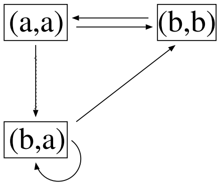

Figure 5: The graph associated to $\mathcal{F}$ in Example 4.9

To satisfy the condition of Theorem 4.6, we have to take the incidence matrix

$$F = \begin{pmatrix} f+1 & f & p \\ g & g+1 & q \\ t & t & s \end{pmatrix},$$

where the entries f, g, p, q, and t are any positive integers. We note that the form of F depends on the given skeleton. In order to see how Theorem 4.6 works, one can choose some specific values for the entries of F and repeat the proof of the theorem. For example, if the incidence matrix is of the form

$$F = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 4 & 4 & 2 \end{pmatrix},$$

then one possibility for a valid ordering is $w(a, n-1, n) = acbaba$, $w(b, n-1, n) = bacbab$, and $w(c, n-1, n) = baccbababa$. Note that there are other valid orderings that do not comply with our algorithm, for example $w(a, n-1, n) = abacba$.

Finally we show how looped vertices can cause trouble. Take the vector

$$(f+1, f, p) = (2, 1, 1)$$

for the a-th row of F. Note that the only possible way to order $r^{-1}(a)$ is $r^{-1}(a) = acba$. In other words, the initial letter a must be followed by the letter c. In our graph $\mathcal{H}$, we must go from the vertex $[a, a]$ to the looped vertex $[b, a]$; otherwise, we cannot complete the ordering on $r^{-1}(a)$.

5 The Measurable Space of Orderings on a Diagram

In this section we study $\mathcal{O}_B$ as a measure space. Recall that $\mu = \prod_{v \in V^* \setminus V_0} \mu_v$ has been defined as the product measure on the set $\mathcal{O}_B = \prod_{v \in V^* \setminus V_0} P_v$, where each μ_v is the uniformly distributed measure on P_v. Also recall that $\mathcal{O}_B(j)$ is the set of orders on B with j maximal and j minimal paths. Let $\mathcal{O}_B^*(j)$ be the set of orders on B with j maximal paths.

Theorem 5.1 *Let B be a finite rank d aperiodic Bratteli diagram. Then there exists $j \in \{1, \ldots, d\}$ such that μ-almost all orderings have j maximal and j minimal elements.*

Proof We shall first show that there exists a j such that μ-almost all orderings have j maximal elements. Similarly, there will exist a j' such that μ-almost all orderings have j' minimal elements. To see that $j = j'$, note that the automorphism on $\mathcal{O}_B$ that takes an order ω to its reverse $\overline{\omega}$ (*i.e.*, if $|r^{-1}(v)| = k$, $r(e) = v$ and ω gives the edge e label j, then $\overline{\omega}$ gives e the label $k-1-j$) is an automorphism that preserves μ and maps $\mathcal{O}_B^*(j)$ to the set of orders with j minimal paths.[9]

If B has rank d, then for $k \in \mathbb{N}$, $1 \le i \le d$ and $n > k$, define the event

$$G_k^{n,i} = \{\omega : \text{the maximal paths from level } k \text{ to level } n \text{ have exactly } i \text{ distinct sources}\}$$

and

$$H_k^i := \bigcup_{n>k} G_k^{n,i}.$$

We claim that $\mathcal{O}_B^*(1) = \limsup H_k^1$. For if $\omega \in \limsup H_k^1$, then for some subsequence (n_k), $\omega \in H_{n_k}^1 = \bigcup_{n>n_k} G_{n_k}^{n,1}$ for each k. For each n_k, there is some $n > n_k$ such that the

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[9] We thank the referee for this simplifying remark.

maximal paths from level n_k to level n have only one source. This means there is only one maximal path from level 1 to level n_k that is extended to an infinite maximal path. Letting $n_k \to \infty$, we have that $\omega \in \mathcal{O}_B^*(1)$. Conversely, suppose that $\omega \notin \limsup H_k^1$. Then for some K and all $k > K$,

$$\omega \in \big(\bigcup_{n>k} G_k^{n,1}\big)^c = \bigcap_{n>k} \bigcup_{i=2}^{d} G_k^{n,i}.$$

Fix $k > K$. For some j, and some $\{v_1, \dots, v_j\} \subset V_k$, we have $\omega \in G_k^{n_p,j}$ for infinitely many $n_p > k$, where the sources of the maximal paths from level k to level n_p are $\{v_1, \dots, v_j\}$ for each of these n_p's. Fix n_1; for some set $\{v_1^1, \dots, v_j^1\} \subset V_{n_1}$, and for some subsequence $(n_{p^{(1)}})$ of (n_p) there are j maximal paths from level k to level $n_{p^{(1)}}$ whose sources are $\{v_1, \dots, v_j\}$ and which pass through $\{v_1^1, \dots, v_j^1\} \subset V_{n_1}$, for any $n_{p^{(1)}}$. Let $\{M_1^{(i)} : 1 \leq i \leq d\}$ be the maximal paths from level k to level n_1 with $r(M_1^{(i)}) = v_i^1$ for $1 \leq i \leq j$. Fix one n_2 from $(n_{p^{(1)}})$. There exist $\{v_1^2, \dots, v_j^2\} \subset V_{n_2}$ and $(n_{p^{(2)}})$, a subsequence of $(n_{p^{(1)}})$, such that for each $n_{p^{(2)}}$, there are j maximal paths from level k to level $n_{p^{(2)}}$ with range $\{v_1^2, \dots, v_j^2\} \subset V_{n_2}$. Let $\{M_2^{(i)} : 1 \leq i \leq d\}$ be the set of these maximal paths. Each $M_2^{(i)}$ is a refinement of $M_1^{(i)}$. Continue in this fashion to get, for each $1 \leq i \leq j$, a sequence $(M_j^{(i)})$ of paths converging to j distinct maximal paths, so that $\omega \notin \mathcal{O}_B^*(1)$.

Similarly we can show that for $1 < j \leq d$,

$$\mathcal{O}_B^*(j) = \big(\limsup_{k\to\infty} H_k^j\big) \setminus \bigcup_{i=1}^{j-1} \mathcal{O}_B^*(i).$$

Now order the vertices in $V = \bigcup_{n\geq 1} V_n$ as $\{v_1, v_2, \dots\}$ starting from level 2 and moving to levels V_n, $n = 3, 4, \dots$. For each $n \geq 1$ define the random variable X_n on $\mathcal{O}_B$, where $X_n(w) = i$ if the source of the maximal edge with range v_n is the vertex i. The sequence (X_n) is a sequence of mutually independent variables and if Σ_n is the σ-field generated by $\{X_n, X_{n+1}, \dots\}$ and $\Sigma := \bigcap_n \Sigma_n$, then for each $1 \leq i \leq d$, $\mathcal{O}_B^*(j) \in \Sigma$, and by Kolmogorov's zero-one law, for each $1 \leq j \leq d$, $\mu(\mathcal{O}_B^*(j))$ is either 0 or 1. Note now that one can repeat the definitions of all the above sets replacing the word "maximal" with "minimal". The result follows. ■

In the next result we use our notation from the proof of Theorem 5.1.

Theorem 5.2 *Let B be an aperiodic Bratteli diagram of rank d.*

(i) $\mu(\mathcal{O}_B(1)) = 1$ *if and only if there exists a sequence* $(n_k)_{k=1}^{\infty}$ *such that*

$$\sum_{k=1}^{\infty} \mu(G_{n_k}^{n_{k+1},1}) = \infty.$$

(ii) *Let* $1 < j \leq d$. *Then* $\mu(\mathcal{O}_B(j)) = 1$ *if and only if there exists a sequence* (n_k) *where* $\sum_k \mu(G_{n_k}^{n_{k+1},j}) = \infty$, *and for each* $1 \leq i < j$, *and all sequences* (m_k), $\sum_k \mu(G_{m_k}^{m_{k+1},i}) < \infty$.

Proof (i) Note that for each j and n with $n > j$,

$$G_j^{n,1} \subset G_j^{n+1,1}, \tag{5.1}$$

and, similarly, for each j, n with $n > j+1$, $G_{j+1}^{n,1} \subset G_j^{n,1}$. This implies that

$$H_{j+1}^1 = \bigcup_{n>j+1} G_{j+1}^{n,1} \subset \bigcup_{n>j+1} G_j^{n,1} \subset \bigcup_{n>j} G_j^{n,1} = H_j^1\,. \tag{5.2}$$

If $\mu(\mathcal{O}_B(1)) = 1$, then since from the proof of Theorem 5.1 $\mathcal{O}_B(1) = \limsup H_k^1$, we have

$$1 = \mu(\mathcal{O}_B(1)) = \mu\Big(\bigcap_{k=1}^{\infty} \bigcup_{j\geq k} H_j^1\Big) \overset{(5.2)}{=} \mu\Big(\bigcap_{k=1}^{\infty} H_k^1\Big),$$

which implies that for each k, $\mu(H_k^1) = 1$, and now inclusion (5.1) implies that for each k,

$$1 = \mu(H_k^1) = \mu\Big(\bigcup_{n>k} G_k^{n,1}\Big) = \lim_{n\to\infty} \mu(G_k^{n,1}),$$

and this implies the existence of a sequence (n_k) such that $\sum_{k=0}^{\infty} \mu(G_{n_k}^{n_{k+1},1}) = \infty$.

Conversely, suppose there is some (n_k) such that $\sum_k \mu(G_{n_k}^{n_{k+1},1}) = \infty$. The converse of the Borel–Cantelli lemma implies that for μ-almost all orderings, there is a subsequence (j_k) such that all maximal edges in E_{j_k}, have the same source. This implies that for almost all ω there is at most one and thus exactly one maximal path in X_B.

(ii) We will prove statement (ii) for $j = 2$; the other cases follow similarly. If $\mu(\mathcal{O}_B(2)) = 1$, then $\mu(\mathcal{O}_B(1)) = 0$, and by the proof of Theorem 5.1, this means that

$$\mu(\limsup H_k^2) = 1 \quad \text{and} \quad \mu(\limsup H_k^1) = 0.$$

Using (i), we conclude that for all sequences (m_k), $\sum_k \mu(G_{m_k}^{m_{k+1},1}) < \infty$. Also, as in the proof of (1), we will have that for each k,

$$\lim_{n\to\infty} \mu(G_k^{n,1}) = 0.$$

Note that for all $n > j$,

$$G_j^{n,2} \subset G_j^{n+1,2} \cup G_j^{n+1,1}, \tag{5.3}$$

and for all $n > j+1$, $G_{j+1}^{n,2} \subset G_j^{n,2} \cup G_j^{n,1}$. This implies that

$$H_{j+1}^2 = \bigcup_{n>j+1} G_{j+1}^{n,2} \subset \bigcup_{n>j+1} (G_j^{n,2} \cup G_j^{n,1}) \subset \bigcup_{n>j} (G_j^{n,2} \cup G_j^{n,1}) = H_j^2 \cup H_j^1. \tag{5.4}$$

It follows that $H_n^2 \subset H_j^2 \cup H_j^1$ whenever $n > j$. As in Part (i) we have

$$1 = \mu(\limsup H_k^2) \overset{(5.4)}{\leq} \mu(\bigcap_{k=1}^{\infty} (H_k^2 \cup H_k^1)),$$

so that for all k, $\mu(H_k^2 \cap H_k^1) = 1$, and using inclusion (5.3), this implies that $\lim_{n\to\infty} \mu(G_k^{n,2} \cup G_k^{n,1}) = 1$, so that $\lim_{n\to\infty} \mu(G_k^{n,2}) = 1$. Now one can construct a suitable sequence (n_k) as was done in (i).

Conversely, if for some (n_k), $\sum_k \mu(G_{n_k}^{n_{k+1},2})$ diverges, then the converse of the Borel–Cantelli lemma implies that almost all orders ω have at most 2 maximal paths. Since for each sequence (m_k), $\sum_k \mu(G_{m_k}^{m_{k+1},1}) < \infty$, (i) tells us that $\mu(\mathcal{O}_B(1)) = 0$. The result follows. ■

If (F_n), where $F_n = (f_{v,w}^{(n)})$, is the sequence of incidence matrices for B, consider the Markov matrices $M_n := (m_{v,w}^{(n)})$ where

$$m_{v,w}^{(n)} := \frac{f_{v,w}^{(n)}}{\sum_w f_{v,w}^{(n)}}.$$

Here $m_{v,w}^{(n)}$ represents the proportion of edges with range $v \in V_{n+1}$ that have source $w \in V_n$. Similarly, if (n_k) is a given sequence, consider for $j \geq 1$

$$F'_j := F_{n_{j+1}-1} \circ F_{n_{j+1}-2} \circ \cdots \circ F_{n_j+1} \tag{5.5}$$

and define the Markov matrices $M'_j = (m'^{(j)}_{v,w})$ as before. Proposition 5.2 tells us that the integer j such that $\mu(\mathcal{O}_B(j)) = 1$ depends only on the masses of the sets $G_{n_k}^{n_{k+1},j}$, as j and (n_k) vary. In turn, $\mu(G_{n_k}^{n_{k+1},j})$ depends only on the matrices M'_k where F'_k is defined as in (5.5).

The following corollary gives a sufficient condition for diagrams B satisfying $\mu(\mathcal{O}_B(1)) = 1$. Note that this case includes all simple B with a bounded number of edges at each level. We use the notation of relation (5.5).

Corollary 5.3 *Let B be a Bratteli diagram with incidence matrices (M_n). Suppose there is some $\varepsilon > 0$, sequences (n_k) of levels and (w_k) of vertices (where $w_k \in V_{n_k}$), such that $m'^{(k)}_{v,w_k} \geq \varepsilon$ for all $k \in \mathbb{N}$ and $v \in V_{n_{k+1}}$. Then $\mu(\mathcal{O}_B(1)) = 1$.*

Proof The satisfied condition implies that $\mu(G_{n_k}^{n_{k+1},1}) \geq \varepsilon^d$. Now apply Proposition 5.2. ■

Thus, while in general there is no algorithm that, given a simple diagram B, finds the number of maximal paths that μ almost all orderings on B have, nevertheless, Theorem 5.2 and Corollary 5.3 tell us that one can, in principle, find this number for a large class of diagrams.

Next we want to make measure theoretic statements about perfect subsets in $(\mathcal{O}_B, \mu)$. Recall that if B' is a nontrivial telescoping of B, then the set $L(\mathcal{P}_B)$ is a set of measure 0 in $\mathcal{P}_{B'}$; for this reason we cannot telescope, and we will use the characterization of perfect orders given by Lemma 3.7. Theorem 5.4 implies the following observation for simple diagrams. If B is a diagram for which $\mu(\mathcal{O}_B(j)) = 1$ with $j > 1$, then there is a meagreness of perfect orderings on B and hence dynamical systems defined on X_B. Theorem 5.4(ii) implies an analogous statement for aperiodic diagrams.

Theorem 5.4 *Let B be a finite rank Bratteli diagram.*

(i) *Suppose B is simple. If $\mu(\mathcal{O}_B(1)) = 1$, then $\mu(\mathcal{P}_B) = 1$. If $\mu(\mathcal{O}_B(j)) = 1$ for some $j > 1$, then $\mu(\mathcal{P}_B) = 0$.*

(ii) *Suppose that B is aperiodic with q minimal components that its incidence matrices (F_n) have a strictly positive row R_n for each n and at least one entry in R_n tends to ∞ as $n \to \infty$. If $\mu(\mathcal{O}_B(q)) = 1$, then $\mu(\mathcal{P}_B) = 1$. If $\mu(\mathcal{O}_B(j)) = 1$ for some $j > q$, then $\mu(\mathcal{P}_B) = 0$.*

Proof We remark that if $j = 1$, then clearly μ-almost all orderings are perfect.

Suppose that B is simple, where there are at most d vertices at each level, and $\mu(\mathcal{O}_B(j)) = 1$ for some $j > 1$. Fix $0 < \delta < 1/d$. Define, for $w \in V_{n-1}$,

$$P_n(w) := \{v \in V_n : m_{v,w}^{(n)} \geq \delta\}\,;$$

then $V_n = \bigcup_{w:P_n(w)\neq\varnothing} P_n(w)$, and, if for infinitely many n, fewer than j of the $P_n(w)$'s are non-empty, then for some $j' < j$ and some (n_k) there is some ϵ such that $\mu(G_{n_k}^{n_{k+1},j'}) \geq \epsilon$, and Theorem 5.2 implies $\mu(\mathcal{O}_B(j'')) = 1$ for $j'' \leq j' < j$, a contradiction. There is no harm in assuming that for fixed n, the sets $\{P_n(w) : P_n(w) \neq \varnothing\}$ are disjoint (if not we put $v \in P_n(w)$, for some w where $m_{v,w}^{(n)}$ is maximal) and that there is some set $\{w_1, \ldots, w_j\}$ of vertices such that $P_n(w_i) \neq \varnothing$ for each natural n and each $i = 1, \ldots, j$. If all but finitely many vertices of the diagram are the range of a bounded number of edges, then Lemma 5.3 implies that $\mu(\mathcal{O}_B(1)) = 1$, a contradiction. So we can pick a $v_n^* \in V_n$ that has a maximal number of incoming edges. For ease of notation $v_n^* = v^*$. By the comment just made, we can assume that as n increases, v^* is the range of increasingly many edges.

Let $\mathcal{E}_n$ be the event that

(a) for each $v \in V_n$, the maximal and minimal edge with range v has source w_i whenever $v \in P_n(w_i)$;

(b) for each $n \geq 2$, there is a pair of consecutive edges with range $v^* \in V_n$, both having source w_i when $v^* \in P_n(w_i)$;

(c) for each $n \geq 2$, there is a pair of consecutive edges with range $v^* \in V_n$, the first having source w_i when $v^* \in P_n(w_i)$, the second having source $w_{i'}$ for some $i' \neq i$.

Then there is some δ^* such that $\mu(\mathcal{E}_n) \geq \delta^*$ for all large n. So for a set $\mathcal{O}_B(j)' \subset \mathcal{O}_B(j)$ of full measure, infinitely many of the events $\mathcal{E}_n$ occur. For $\omega \in \mathcal{O}_B(j)'$, if $\omega \in \mathcal{E}_n$ for such n, then the extremal paths go through the vertices $w_1, \ldots w_j$ at level n. Now an application of Lemma 3.7 implies that $\mathcal{O}_B(j)' \subset \mathcal{O}_B\backslash\mathcal{P}_B$.

To prove (ii), first note that if B has q minimal components, then any ordering has at least q extremal pairs of paths. We assume that extremal paths come in pairs; otherwise, the ordering is not perfect. If μ-almost all orderings have q maximal paths, then necessarily each pair of extremal paths lives in a distinct minimal component of B, and μ almost all orderings belong to $\mathcal{P}_B$. Suppose that $\mu(\mathcal{O}_B(j)) = 1$ where $j > q$. Write

$$\mathcal{O}_B(j) = \bigcup_{\{(k_1,\ldots k_q):\sum_{i=1}^q k_i \leq j\}} \mathcal{O}_B\big(j, \{(k_1, \ldots k_q)\}\big)\,,$$

where $\mathcal{O}_B(j,\{(k_1,\ldots,k_q)\})$ is the set of orderings with k_i extremal pairs in the i-th minimal component. If $k_i > 1$ for some i, then by the argument in (i),

$$\mu\big(\mathcal{O}_B\big(j,\{(k_1,\ldots,k_q)\}\big)\big) = 0.$$

If $(k_1,\ldots,k_q) = (1,\ldots,1)$, this means that there is at least one extremal pair of paths that lives outside the minimal components of B. Repeat the argument in (i), except that v^* must be chosen outside the union of the minimal components of B, and also such that at least one of the entries in $\{m^{(n)}_{v^*,v} : v \in V_n\}$ gets large as $n \to \infty$. ■

Example 5.5 It is not difficult to find a simple Bratteli diagram B where almost all orderings are not perfect. Let $V_n = V = \{v_1, v_2\}$ for $n \geq 1$, and let $\sum_{n=1}^{\infty} m^{(n)}_{v_i,v_j} < \infty$ for $i \neq j$. Then for μ-almost all orderings, there is some K such that for $k > K$, the sources of the two maximal/minimal edges at level n are distinct, *i.e.*, $\mu(\mathcal{O}_B(2)) = 1$. Note that here $\mu(\mathcal{O}_B(2)) = 1$ if and only if there are two probability measures on X_B that are invariant with respect to the tail equivalence relation. This is not true in general as the next example shows.

Example 5.6 This example appears in [FFT09, Section 4]. Let

$$F_k := \begin{pmatrix} m_k & n_k & 1 \\ 0 & n_k - 1 & 1 \\ m_k - 1 & n_k & 1 \end{pmatrix},$$

where the sequences (m_k) and (n_k) satisfy $3n_k + 1 \leq 2m_k \leq n_{k+1}$, which implies that they get large. The corresponding stochastic matrix satisfies

$$M_k \approx \begin{pmatrix} \frac{m_k}{m_k+n_k} & \frac{n_k}{m_k+n_k} & 0 \\ 0 & 1 & 0 \\ \frac{m_k}{m_k+n_k} & \frac{n_k}{m_k+n_k} & 0 \end{pmatrix},$$

and if we further require that $n_{k+1} \leq C n_k$ for some $C \geq 4$, then $\frac{n_k}{m_k+n_k} \geq \frac{2}{2+C}$, so that by Corollary 5.3, $\mu(\mathcal{O}_B(1)) = 1$, while in [FFT09] it is shown that (a telescoping of) B has two probability measures that are invariant under the tail equivalence relation.

Example 5.7 Let

$$F_n := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

for n non-prime and

$$F_n := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

if n is prime. Then if n is prime, given any vertex w, $m_{v,w}^{(n)} \geq 1/7$ either for $v = v_1$ or $v = v_5$. So $\mu(G_n^{n+1,2}) \geq (1/7)^7$. Also $\mu(G_n^{n+1,1}) = 0$ for each $n \geq 1$. Theorem 5.2 implies that $j = 2$.

Acknowledgements Most of this work was done during our mutual visits to McGill University, Torun University, The University of Ottawa, and The Institute for Low Temperature Physics. We are thankful to these institutions for their hospitality and support. We would like also to thank M. Cortez, F. Durand, N. Madras, and K. Medynets for useful and stimulating discussions.

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