# A characterization of $p$-automatic sequences as columns of linear cellular automata

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**Abstract**

We show that a sequence over a finite field $\mathbb{F}_q$ of characteristic $p$ is $p$-automatic if and only if it occurs as a column of the spacetime diagram, with eventually periodic initial conditions, of a linear cellular automaton with memory over $\mathbb{F}_q$. As a consequence, the subshift generated by a length $p$-substitution can be realized as a topological factor of a linear cellular automaton.

## 1 Introduction

In a cellular automaton, each cell has a value at each time step, so it is natural to consider the sequence of values taken by a given cell at times 0, 1, 2, etc. For a one-dimensional cellular automaton, such a sequence of states is a column sequence in a two-dimensional spacetime diagram of the cellular automaton. Some column sequences are quite simple, such as the characteristic sequence of powers of 2, which occurs as a column in the space-time diagram of Rule 90 begun from a single 1 cell on background of 0s; this rule adds its two neighbours modulo 2. On the other hand, some sequences are statistically random, such as the center column of Rule 30 [Wol02, page 28].

A relatively unexplored question is the following. Given a sequence on a finite alphabet, does this sequence occur as a column of a cellular automaton spacetime diagram? Without additional restrictions, it is possible to obtain any sequence (for example, the base-10 digits of $\pi$) as a column: simply place the sequence in the

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initial condition, and let the cellular automaton be the shift map $\sigma$. To avoid this trivial case, we impose the restriction that initial conditions be eventually periodic in both directions.

Wolfram found, by brute-force search, spacetime diagrams containing the characteristic sequence of squares and the Thue–Morse sequence [Wol02, page 1186]. It is also possible to construct spacetime diagrams with more exotic column sequences, such as the characteristic sequence of primes [Wol02, page 640].

In this paper we study $p$-automatic sequences occurring as columns of cellular automaton spacetime diagrams. We assume that $p$ is prime and $\mathbb{F}_q$ is a finite field of characteristic $p$ throughout. Litow and Dumas [LD93] gave several examples of cellular automata containing well-known $p$-automatic sequences as columns. They also proved that each column of a linear cellular automaton over $\mathbb{F}_q$, begun from an initial condition with finitely many nonzero entries, is necessarily $p$-automatic. We use Litow and Dumas's approach to establish the following characterization of $p$-automatic sequences. (All relevant definitions are in Section 2.)

**Theorem 1.1.** *A sequence of elements in $\mathbb{F}_q$ is $p$-automatic if and only if it is a column of a spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.*

Furthermore, the proof in each direction is constructive. In particular, there is an algorithm to compute the local cellular automaton rule, given a finite automaton for the $p$-automatic sequence.

In [AvHP+97], the authors study the automaticity of the two-dimensional sequence of entries in the spacetime diagram generated by a linear cellular automaton over the integers modulo $m$, with eventually constant initial conditions. Let such a cellular automaton be generated by the polynomial $C(x) \in (\mathbb{Z}/(m\mathbb{Z}))[x]$. They show that the two-dimensional spacetime diagram is $p$-automatic if and only if the set of prime divisors of $m$ such that the $C(x) \bmod p$ is not a monomial is either $\{p\}$ or the empty set. It follows that a linear cellular automaton over $\mathbb{F}_p$ generates a $p$-automatic spacetime diagram. Moreover, columns of this spacetime diagram, being one-dimensional slices of a two-dimensional $p$-automatic sequence, are also $p$-automatic.

As a consequence of Theorem 1.1 we are able to prove the following.

**Theorem 1.2.** *If $\mathbf{u} = (u_n)_{n\geq 0}$ is $p$-automatic, then for some $q$ and $d \geq 1$, there exists a linear cellular automaton $\Phi : (\mathbb{F}_q^d)^{\mathbb{Z}} \to (\mathbb{F}_q^d)^{\mathbb{Z}}$ and a subsystem $(Y, \Phi)$ of $((\mathbb{F}_q^d)^{\mathbb{Z}}, \Phi)$ such that $(X_{\mathbf{u}}, \sigma)$ is a topological factor of $(Y, \Phi)$.*

We remark that for each $n \geq 1$ a sequence is $p^n$-automatic if and only if it is $p$-automatic [AS03, Theorem 6.6.4], so Theorems 1.1 and 1.2 also apply to $p^n$-automatic sequences. Moreover, by injecting the alphabet of a general $p$-automatic sequence into some $\mathbb{F}_q$, we can find an image of that sequence, under the injection, as a column of a spacetime diagram.

The proof of Theorem 1.1 appears in Section 3 along with some corollaries. In Section 4 we discuss an algorithm that, given a finite automaton for a sequence, generates the desired cellular automaton with memory, and we compute several examples. In Section 5 we prove Theorem 1.2 and give conditions that ensure that the factor mapping of Theorem 1.2 is an embedding.

## 2 Definitions and notation

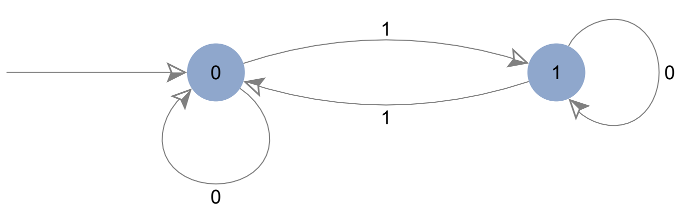
In this section we recall definitions of some terms that we use. Let $\Sigma_k = \{0, 1, \ldots, k-1\}$. If $n = \sum_{i=0}^{l} a_i k^i$ is the standard base-$k$ representation of $n$ with $0 \leq a_i \leq k-1$ and $a_l \neq 0$, define $(n)_k$ to be the word $a_0\, a_1\, \cdots\, a_l$. We start with the cumbersome, but necessary, definition of a finite automaton that generates an automatic sequence:

**Definition 2.1.** A *deterministic finite automaton with output* (DFAO) is a 6-tuple $(\mathcal{S}, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$, where $\mathcal{S}$ is a finite set of "states", $s_0 \in \mathcal{S}$ is the *initial state*, $\mathcal{A}$ is a finite alphabet, $\omega : \mathcal{S} \to \mathcal{A}$ is the *output function*, and $\delta : \mathcal{S} \times \Sigma_k \to \mathcal{S}$ is the *transition function*.

In the symbolic dynamics literature, $\omega$ is also known as a *coding* or a *letter-to-letter projection*. The function $\delta$ extends in a natural way to the domain $\mathcal{S} \times \Sigma_k^*$. Namely, define $\delta(s, a_0 a_1 \cdots a_l) := \delta(\delta(s, a_0), a_1 \cdots a_l)$ recursively. This allows us to feed the automaton with base-$k$ representations of natural numbers:

**Definition 2.2.** A sequence $(u_n)_{n\geq 0}$ of elements in $\mathcal{A}$ is *$k$-automatic* if there is a DFAO $(\mathcal{S}, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ such that $u_n = \omega(\delta(s_0, (n)_k))$ for all $n \geq 0$.

**Example 2.3.** The Thue–Morse sequence is the sequence $(u_n)_{n\geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, \ldots$ where $u_n = 0$ if the number of occurrences of 1 in the binary representation of $n$ is even and $u_n = 1$ otherwise. The Thue–Morse sequence is 2-automatic, and it is generated by the following automaton, where the two states are labeled with their images under $\omega$.




Let $\mathbb{N} := \{0, 1, \ldots\}$. If $\mathbb{M} = \mathbb{Z}$ or $\mathbb{M} = \mathbb{N}$, then the space of all $\mathbb{M}$-indexed sequences from $\mathcal{A}$ is written as $\mathcal{A}^{\mathbb{M}}$, and an element in $\mathcal{A}^{\mathbb{M}}$, a *configuration*, is written $R = (R(m))_{m\in\mathbb{M}}$. (Our configurations will be thought of as rows of a two-dimensional array.) Let $\mathcal{A}$ be endowed with the discrete topology and $\mathcal{A}^{\mathbb{M}}$ with the product topology; then $\mathcal{A}^{\mathbb{M}}$ is a Cantor space, that is, a zero-dimensional perfect compact metric space. If $b \in \mathcal{A}$ and $m \in \mathbb{M}$, the clopen sets $\{R : R(m) = b\}$ generate a countable basis for the topology on $\mathcal{A}^{\mathbb{M}}$. The (left) *shift map* $\sigma : \mathcal{A}^{\mathbb{M}} \to \mathcal{A}^{\mathbb{M}}$ is the map defined as $(\sigma(R))(m) = R(m + 1)$. Given two configuration spaces $\mathcal{A}^{\mathbb{M}}$ and $\mathcal{B}^{\mathbb{M}}$, we shall use $\sigma$ to refer to the shift map on either of these spaces. Recall that if $\Phi : \mathcal{A}^{\mathbb{M}} \to \mathcal{B}^{\mathbb{M}}$, we say that $\Phi$ *commutes with the shift* if $\sigma \circ \Phi = \Phi \circ \sigma$.

**Definition 2.4.** A (one-dimensional) *cellular automaton with memory* $d$ is a continuous, $\sigma$-commuting map $\Phi : (\mathcal{A}^d)^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$.

By memory here we mean "time" memory, and this will become clear after Definition 2.5. To recover the classical definition of a cellular automaton, we let $d = 1$. In this case the cellular automaton only needs to know the current configuration, and nothing of the configuration's past. The Curtis–Hedlund–Lyndon theorem [Hed69] states that $\Phi$ is a cellular automaton if there is some *local rule* $\phi : \mathcal{A}^{l+r+1} \to \mathcal{A}$ for some $l \geq 0$ (the *left radius of* $\phi$) and $r \geq 0$ (the *right radius of* $\phi$), such that for all $R \in \mathcal{A}^{\mathbb{Z}}$ and all $m \in \mathbb{Z}$,

$$(\Phi(R))(m) = \phi(R(m - l), R(m - l + 1), \ldots, R(m + r))\,. \tag{1}$$

Conversely, any local rule $\phi$ defines a cellular automaton $\Phi$ using Identity (1).

The Curtis–Hedlund–Lyndon theorem also holds for a cellular automaton with memory, so that there is a local rule $\phi : (\mathcal{A}^d)^{l+r+1} \to \mathcal{A}$ satisfying (1), and conversely any such local rule defines a cellular automaton with memory. We shall often use the fact that the domain of a cellular automaton with memory $d$ is also $(\mathcal{A}^{\mathbb{Z}})^d$.

**Definition 2.5.** If $\Phi : (\mathcal{A}^d)^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a cellular automaton with memory $d$, then a *spacetime diagram* for $\Phi$ with initial conditions $R_0, \ldots, R_{d-1}$ is the sequence $(R_n)_{n\geq 0}$ where we inductively define $R_n := \Phi(R_{n-d}, \ldots, R_{n-1})$ for $n \geq d$.

We visualize a spacetime diagram by letting time evolve down the page: the $n$th row $R_n$ represents the configuration at time $n$, and $R_n(m)$, the entry on Row $n$ and Column $m$ of the spacetime diagram, is the state of the $m$th cell at time $n$. Whereas each row in an ordinary cellular automaton (with memory 1) is determined by the previous row, in a cellular automaton with memory $d$ each row is determined by the previous $d$ rows. To be brief, we will often use the term 'cellular automaton' to mean 'a spacetime diagram of the cellular automaton'.

Now suppose that $\mathcal{A}$ is the finite field $\mathbb{F}_q$. In this case $\mathbb{F}_q^{\mathbb{Z}}$ and $(\mathbb{F}_q^d)^{\mathbb{Z}}$ are groups, with componentwise addition; they are also $\mathbb{F}_q$-vector spaces.

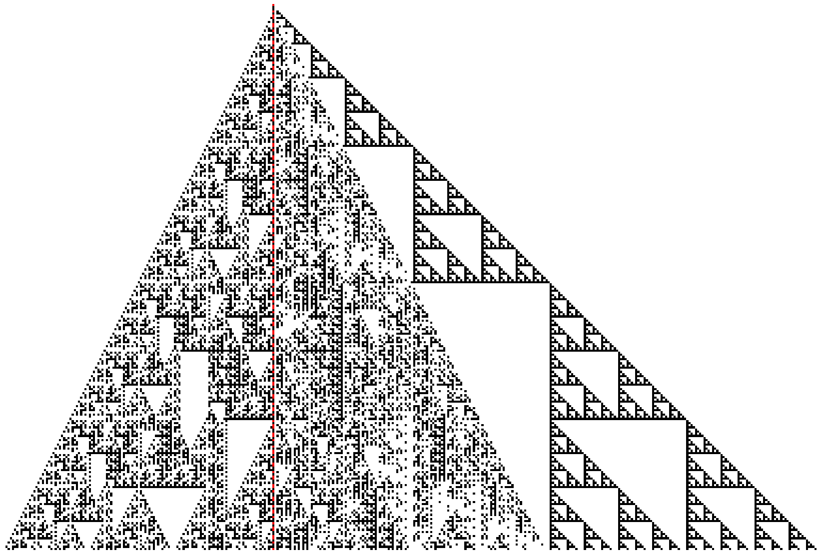


Figure 1: Spacetime diagram of a linear cellular automaton with memory 12 containing the Thue–Morse sequence as a column.

**Definition 2.6.** We say that the cellular automaton $\Phi : (\mathbb{F}_q^d)^{\mathbb{Z}} \to \mathbb{F}_q^{\mathbb{Z}}$ with memory $d$ is *linear* if $\Phi$ is an $\mathbb{F}_q$-linear map.

Thus the Curtis–Hedlund–Lyndon theorem implies that the memory-$d$ cellular automaton $\Phi$ is linear if and only if there exist coefficients $f_{j,i} \in \mathbb{F}_q$ for $-l \leq j \leq r$ and $0 \leq i \leq d-1$ such that $(\Phi(R_0, \ldots, R_{d-1}))(m) = \sum_{i=0}^{d-1} \sum_{j=-l}^{r} f_{j,i}\, R_i(m+j)$ for all $R_0, \ldots, R_{d-1} \in \mathbb{F}_q^{\mathbb{Z}}$ and $m \in \mathbb{Z}$. An example of a linear cellular automaton with memory 1 is Rule 90, whose field is $\mathbb{F}_2$ and whose local rule is $\phi(a, b, c) = a + c$. Begun from the initial condition $R_0$ where $R(0) = 1$ and $R(m) = 0$ for all $m \neq 0$, Rule 90 computes the array of binomial coefficients modulo 2. In fact, Pascal's triangle modulo $p$ is the spacetime diagram of a linear cellular automaton with the corresponding initial condition. These rules have been studied extensively in the literature, as have been linear cellular automata in general; the algebraic properties of the local rule yield much theoretic structure.

**Example 2.7.** Figure 1 shows the first 256 rows of the spacetime diagram of a linear cellular automaton with memory 12 over $\mathbb{F}_2$, where 0 is rendered as a white cell and 1 is rendered as a black cell. The "center" column, containing the top vertex of the triangular region, consists of the Thue–Morse sequence. This column is highlighted by rendering 0 as red. It lies to the left of a column which is identically zero. We compute the local rule for this cellular automaton in Section 4.

## 2.1 Classical results

We recall some results that we shall use. As before, we use $p$ to denote a prime and $q$ to denote a power of $p$. Let $\mathbb{F}_q$ be the field of cardinality $q$. If $(u_n)_{n\geq 0}$ is $p$-automatic generated using the DFAO $(\mathcal{S}, \Sigma_p, \delta, s_0, \mathcal{A}, \omega)$, find $q$ such that $|\mathcal{A}| \leq q$. By injecting $\mathcal{A}$ into $\mathbb{F}_q$, we can, and henceforth do, assume that the output function $\omega$ has range in $\mathbb{F}_q$.

Recall that $\mathbb{F}_q[t]$, $\mathbb{F}_q(t)$, and $\mathbb{F}_q((t))$ are the sets of polynomials, rational functions, and formal Laurent series respectively with coefficients in $\mathbb{F}_q$. Elements of $\mathbb{F}_q((t))$ are expressions of the form $F(t) = \sum_{n\geq n_0} u_n t^n$, where $u_n \in \mathbb{F}_q$ and $n_0 \in \mathbb{Z}$. We can also define polynomials, rational functions and formal Laurent series in several variables. For us, a formal Laurent series in $t, x$ is an element of $\mathbb{F}_q((x))((t))$. The Laurent series $F(t)$ is *algebraic over* $\mathbb{F}_q(t)$ if there exists a nonzero polynomial $P(t, x) \in \mathbb{F}_q[t, x]$ such that $P(t, F(t)) = 0$. Finally we define the *$q$-kernel* of a sequence $(u_n)_{n\geq 0}$ to be the collection of sequences $\{(u_{q^k n+r})_{n\geq 0} : k \geq 0,\ 0 \leq r \leq q^k - 1\}$. The equivalence of Statements 1 and 2 in Theorem 2.8 is known as Christol's theorem; see [Chr74, CKMFR80]. The equivalence of Statements 2 and 3 dates back to Eilenberg [Eil74].

**Theorem 2.8.** *Let $(u_n)_{n\geq 0}$ be a sequence of elements in $\mathbb{F}_q$. The following are equivalent:*

1. *$F(t) = \sum_{n\geq 0} u_n t^n$ is algebraic over $\mathbb{F}_q(t)$.*
2. *The $q$-kernel of $(u_n)_{n\geq 0}$ is finite.*
3. *$(u_n)_{n\geq 0}$ is $q$-automatic.*

We can also define a formal series $E(t, x) = \sum_{n\geq n_0} \sum_{m\in\mathbb{Z}} a_{n,m} t^n x^m$ in two variables $t$ and $x$. In this case the *diagonal* is the formal Laurent series

$$\sum_{n\geq n_0} a_{n,n} t^n$$

in one variable. Similarly we define the $m$th *column* of $E(t, x)$ to be

$$\sum_{n\geq n_0} a_{n,m} t^n .$$

We say that $E(t, x)$ is a *rational* series if there exist polynomials $Q(t, x)$ and $P(t, x)$ such that $P(t, x)E(t, x) = Q(t, x)$. The following result is due to Furstenberg [Fur67].

**Theorem 2.9.** *For a Laurent series $F(t) \in \mathbb{F}_q((t))$ to be algebraic over $\mathbb{F}_q(t)$, it is necessary and sufficient that it is the diagonal of a rational Laurent series $E(t, x)$.*

The necessary direction of Furstenberg's theorem follows from the following two propositions in [Fur67], which we will use in Section 3.

**Proposition 2.10.** *Suppose that the Laurent series $F(t) = \sum_{n \geq n_0} c_n t^n \in \mathbb{F}_q((t))$ is algebraic over $\mathbb{F}_q(t)$. Then there exists $r^* \geq n_0$, $m \geq 0$, and a polynomial*

$$P^*(t,x) = A_0^*(t)x + A_1^*(t)x^p + \cdots + A_m^*(t)x^{p^m} + B^*(t),$$

*with $A_i^*(t), B^*(t) \in \mathbb{F}_q[t]$ and $A_0^*(t)$ not divisible by $t$, such that $F(t) = R^*(t) + t^{r^*} G^*(t)$, $R^*(t) = \sum_{n=n_0}^{r^*-1} c_n t^n$, and $P^*(t, G^*(t)) = 0$.*

Proposition 2.10 is similar to a result known as Ore's lemma [AS03, Lemma 12.2.3].

Let $P^{(0,1)}$ denote the derivative of a function $P$ with respect to its second argument.

**Proposition 2.11.** *Suppose that the series $G(t) = \sum_{n \geq 1} c_n t^n \in \mathbb{F}_q((t))$ with $G(0) = 0$ satisfies $P(t, G(t)) = 0$, where $P(t,x) \in \mathbb{F}_q[t,x]$ and $P^{(0,1)}(0,0) \neq 0$. Then $G(t)$ is the diagonal of the unique series expansion of*

$$\frac{x^2 P^{(0,1)}(tx, x)}{P(tx, x)}.$$

# 3 Columns of linear cellular automata

In this section we prove Theorem 1.1. Theorems 2.8 and 2.9 are main ingredients. To work with spacetime diagrams algebraically, we represent the spacetime diagram of a cellular automaton as a bivariate series, a technique that, to our knowledge, was first used by Martin, Odlyzko, and Wolfram [MOW84]. If $a_{n,m}$ is the entry of the spacetime diagram on row $n \in \mathbb{N}$ and column $m \in \mathbb{Z}$, then the series $E(t,x) = \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} a_{n,m} t^n x^m$ encodes the entire cellular automaton evolution from the initial condition $R_0$. We identify the $n$th row $R_n$ of the spacetime diagram with the series $R_n(x) = \sum_{m \in \mathbb{Z}} a_{n,m} x^m$, which is the coefficient of $t^n$ in $E(t,x) = \sum_{n \geq 0} R_n(x) t^n$.

We begin with the easier direction of Theorem 1.1. Litow and Dumas [LD93] established the case where the initial condition is zero everywhere save the central entry and the memory is 1. The proof of the more general statement is similar.

**Theorem 3.1.** *Let $\Phi$ be a linear cellular automaton rule with memory $d$ over $\mathbb{F}_q$. Let $R_0, \ldots, R_{d-1} \in \mathbb{F}_q^{\mathbb{Z}}$ be rows that are eventually periodic in both directions. For each $m \in \mathbb{Z}$, the column sequence $(R_n(m))_{n \geq 0}$ is $p$-automatic.*

*Proof.* The linearity of $\Phi$ is equivalent to the existence of Laurent polynomials $C_1(x), \ldots, C_d(x)$ such that for all $n \geq d$

$$R_n(x) = \sum_{i=1}^{d} C_i(x) R_{n-i}(x).$$

Let $C_0(x) := -1$; then $\sum_{i=0}^{d} C_i(x) R_{n-i}(x) = 0$ for all $n \geq d$, so

$$\begin{aligned}\left(\sum_{i=0}^{d} C_i(x)t^i\right) E(t,x) &= \left(\sum_{i=0}^{d} C_i(x)t^i\right)\left(\sum_{j\geq 0} R_j(x)t^j\right)\\ &= \sum_{n\geq 0}\left(\sum_{i=0}^{d} C_i(x)R_{n-i}(x)\right)t^n\\ &= \sum_{n=0}^{d-1}\left(\sum_{i=0}^{d} C_i(x)R_{n-i}(x)\right)t^n.\end{aligned}$$

Each $R_{n-i}(x)$ is a rational expression since it is the sum of two one-sided eventually periodic series. Therefore $\left(\sum_{i=0}^{d} C_i(x)t^i\right) E(t,x)$ is a rational expression in $t$ and $x$, and since $\sum_{i=0}^{d} C_i(x)t^i$ is also rational this implies that $E(t,x)$ is also rational. Column $m$ of $E(t,x)$ is the diagonal of $x^{-m}E(tx,x)$ and therefore by Theorem 2.9 is algebraic over $\mathbb{F}_q$. Hence, by Theorem 2.8, the sequence of entries in column $m$ is $p$-automatic. □

For the other direction of Theorem 1.1, we basically reverse the steps of the previous proof. The difficulty arises in obtaining a recurrence $\sum_{i=0}^{d} C_i(x)R_{n-i}(x) = 0$ in which $C_0(x)$ is a (nonzero) monomial. It is necessary that $C_0(x)$ is a monomial so that each $\frac{C_i(x)}{C_0(x)}$ is a Laurent polynomial, and hence the update rule that determines the value of each cell is local. Since the coefficients $C_i(x)$ come from the denominator of $E(t,x)$ and the denominator of $E(t,x)$ comes (by Theorem 2.9) from a polynomial equation satisfied by $F(t)$, we seek a polynomial $P(t,x)$ where the coefficient of $t^0$ is a nonzero monomial in $x$, and where $P(t,F(t)) = 0$. The following proposition allows us to find such a polynomial.

**Proposition 3.2.** *Suppose that $F(t) = \sum_{n\geq 0} u_n t^n \in \mathbb{F}_q[[t]]$ is algebraic over $\mathbb{F}_q(t)$. Then there exist $G(t) \in \mathbb{F}_q[[t]]$ and $P(t,x) \in \mathbb{F}_q[t,x]$ of the form*

$$P(t,x) = A_0(t)x + A_1(t)x^p + \cdots + A_m(t)x^{p^m} + B(t)$$

*with $A_i(t), B(t) \in \mathbb{F}_q[t]$ for $0 \leq i \leq m$ such that*

1. *$F(t) = R(t) + t^r G(t)$ for some $r \geq 0$ and $R(t) \in \mathbb{F}_q[t]$,*

2. *$G(0) = 0$,*

*3.* $A_0(0) \neq 0$,

*4.* $B(0) = A_i(0) = 0$ *for* $1 \leq i \leq m$, *and*

*5.* $P(t, G(t)) = 0$.

*Proof.* Proposition 2.10 tells us that for some $r^* \geq 0$ there exist polynomials $A_i^*(t), B^*(t) \in \mathbb{F}_q[t]$ such that $A_0^*(0) \neq 0$ and $G^*(t) = \sum_{n\geq 0} u_{n+r^*} t^n$ satisfies $P^*(t, G^*(t)) = 0$, where

$$P^*(t,x) = A_0^*(t)x + A_1^*(t)x^p + \cdots + A_m^*(t)x^{p^m} + B^*(t).$$

Let $r = r^* + 1$ and $R(t) = u_0 + u_1 t + \cdots + u_r t^r$. Write $G^*(t) = u_{r-1} + u_r t + tG(t)$ so that $G(t) = \sum_{n\geq 1} u_{n+r} t^n$. Expanding $P^*(t, u_{r-1} + u_r t + tx)$ and using that $(a+b)^p = a^p + b^p$ in characteristic $p$ shows that $x = G(t)$ satisfies

$$A_0^*(t)tx + A_1^*(t)t^p x^p + \cdots + A_m^*(t)t^{p^m} x^{p^m} + B^{**}(t) = 0$$

for some polynomial $B^{**}(t)$. Since each formal power series $A_i^*(t)t^{p^i}G(t)^{p^i} = A_i^*(t)t^{2p^i}(G(t)/t)^{p^i}$ is divisible by $t^2$, $B^{**}(t)$ is also divisible by $t^2$. Dividing by $t$, let $A_i(t) = A_i^*(t)t^{p^i-1}$, $B(t) = B^{**}(t)/t$, and

$$P(t,x) = A_0(t)x + A_1(t)x^p + \cdots + A_m(t)x^{p^m} + B(t).$$

One verifies that the conclusions of the proposition are satisfied. □

We may now prove the other direction of Theorem 1.1.

**Theorem 3.3.** *If* $(u_n)_{n\geq 0}$ *is a* $p$*-automatic sequence of elements in* $\mathbb{F}_q$*, then* $(u_n)_{n\geq 0}$ *occurs as a column of a linear cellular automaton with memory over* $\mathbb{F}_q$*, whose initial condition rows have finitely many nonzero entries.*

*Proof.* By Proposition 3.2, there exist $G(t)$ and $P(t,x)$ with $A_0(t)$ and $r$ as described. Using the notation of the proposition, we shall show that the shifted sequence $u_{r+1}, u_{r+2}, \ldots$ can be found as a column of a spacetime diagram of a linear cellular automaton with memory, and then we describe how to reinstate the initial terms. Let us write $P(t,x) = \sum_{i=0}^d C_i(x)t^i$ where $C_i(x) \in \mathbb{F}_q[x]$. Conclusion 4 of Proposition 3.2 implies that $C_0(x) = A_0(0)x$, and Conclusion 3 is that $A_0(0) \neq 0$.

Conclusions 2, 5, and 3 of Proposition 3.2 imply that the conditions of Proposition 2.11 are met. Therefore $G(t)$ is the diagonal of $\frac{x^2 P^{(0,1)}(tx,x)}{P(tx,x)}$. It follows that $G(t)$ is Column $-2$ of $\frac{P^{(0,1)}(t,x)}{P(t,x)}$. Since the coefficient of $t^0$ in $P(t,x)$ is the monomial $A_0(0)x$, write $P(t,x) = A_0(0)x + tQ(t,x)$ where $Q(t,x) \in \mathbb{F}_q[t,x]$. Then expand

$$\frac{P^{(0,1)}(t,x)}{P(t,x)} = \frac{P^{(0,1)}(t,x)}{A_0(0)x} \cdot \frac{1}{1 + \frac{tQ(t,x)}{A_0(0)x}} = \frac{P^{(0,1)}(t,x)}{A_0(0)x} \sum_{n\geq 0} \left(-\frac{Q(t,x)}{A_0(0)x}\right)^n t^n = \sum_{n\geq 0} R_n(x)t^n$$

as a series in $t$. Each $R_n(x)$ is a Laurent polynomial.

Create a two-dimensional array where the entry in Row $n \in \mathbb{N}$ and Column $m \in \mathbb{Z}$ is the coefficient of $x^m$ in $R_n(x)$. By Proposition 2.11, Column $-2$ of this array consists of the sequence $0, u_{r+1}, u_{r+2}, \ldots$. It remains to show that the array is the spacetime diagram of a cellular automaton with memory and to then restore the terms $u_0, u_1, \ldots, u_r$.

Since the series $\sum_{n\geq 0} R_n(x)t^n$ is rational, the sequence $(R_n(x))_{n\geq 0}$ satisfies a linear recurrence with coefficients $C_0(x), \ldots, C_d(x)$. Namely, multiplying both sides by $P(t,x)$ gives

$$\begin{aligned} P^{(0,1)}(t,x) &= \sum_{i=0}^{d} C_i(x)t^i \sum_{j\geq 0} R_j(x)t^j = \sum_{n\geq 0}\left(\sum_{i+j=n} C_i(x)R_j(x)\right)t^n \\ &= \sum_{n=0}^{d}\left(\sum_{i=0}^{n} C_i(x)R_{n-i}(x)\right)t^n + \sum_{n\geq d+1}\left(\sum_{i=0}^{d} C_i(x)R_{n-i}(x)\right)t^n, \end{aligned}$$

and since $P^{(0,1)}(t,x)$ is a polynomial with $\deg_t P^{(0,1)}(t,x) \leq d$, we have $\sum_{i=0}^{d} C_i(x)R_{n-i}(x) = 0$ for all $n \geq d+1$. Solving for $R_n(x)$ gives

$$R_n(x) = -\sum_{i=1}^{d} \frac{C_i(x)}{C_0(x)} R_{n-i}(x)$$

for all $n \geq d+1$, where each $\frac{C_i(x)}{C_0(x)}$ is a Laurent polynomial in $x$. Therefore the coefficient of $x^m$ in $R_n(x)$ depends only on the coefficients of $x^{m+1-\max_i \deg C_i(x)}, \ldots, x^{m+1}$ in $R_{n-1}(x), \ldots, R_{n-d}(x)$. In particular, each entry of the two-dimensional array is computed by the same local rule. Therefore the coefficients of $R_1(x), R_2(x), \ldots$ form the rows of a spacetime diagram of a cellular automaton with memory $d$, where the first $d$ rows consist of initial conditions and rows $R_{d+1}, R_{d+2}, \ldots$ are computed by the local rule. Remove row $R_0$, since no other rows depend on it and the coefficient of $x^{-2}$ in $R_0(x)$ is 0 rather than $u_r$. Finally, we restore the initial $r+1$ terms of $(u_n)_{n\geq 0}$. We do this by redefining $R_0(x) := u_r x^{-2}$ and defining $R_{-r}(x) := u_0 x^{-2}, R_{1-r}(x) := u_1 x^{-2}, \ldots, R_{-1}(x) := u_{r-1}x^{-2}$. Then trivially increase the memory of the local rule from $d$ to $d+r+1$ without actually introducing dependence of $R_n$ on rows $R_{n-(d+1)}, \ldots, R_{n-(d+r+1)}$. Then the local cellular automaton rule with memory $d+r+1$, run from initial conditions $R_{-r}, \ldots, R_d$, produces a spacetime diagram where the sequence $(u_n)_{n\geq 0}$ occurs in Column $-2$. □

The construction in the proof of Theorem 3.3 gives us some additional information about the spacetime diagram. For example, since $C_0(x) = A_0(0)x$ has degree 1, the right radius of the local rule is at most 1. Therefore the left boundary of the nonzero triangular region in the spacetime diagram grows with speed at most 1 cell per step.

Additionally, Column $-1$ is identically 0. This column can be seen, for example, immediately to the right of the Thue–Morse column in Figure 1 (and helps the reader identify the location of the desired sequence in the rest of our diagrams). To see that this is the case, factor $P(t,x) = (x - G(t))Q(t,x)$ for some $Q(t,x) \in \mathbb{F}_q((t))[x]$, following the proof of Proposition 2.11 in [Fur67]. Then we have

$$\frac{P^{(0,1)}(t,x)}{P(t,x)} = \frac{1}{x - G(t)} + \frac{Q^{(0,1)}(t,x)}{Q(t,x)}.$$

These two summands contain the entries of the two halves of the spacetime diagram. Since $Q(0,0) \neq 0$, the exponent of $x$ in each nonzero term of the series $\frac{Q^{(0,1)}(t,x)}{Q(t,x)}$ is nonnegative. Moreover, the only nonzero term in the series $\frac{1}{x-G(t)} = \frac{1}{x}\sum_{n\geq 0}(\frac{1}{x}G(t))^n$ whose exponent of $x$ is greater than $-2$ is $\frac{1}{x}$, which appears in $R_0(x)$, which we removed in the proof of Theorem 3.3.

In Section 4 we discuss an algorithm to generate the polynomial $P(t,x)$ of Proposition 3.2 and thus the cellular automaton as constructed in Theorem 3.3. We also work through some examples. First though we mention a few corollaries.

**Corollary 3.4.** *If $(u_n)_{n\geq 0}$ is a $p$-automatic sequence, then $(u_n)_{n\geq 0}$ is the letter-to-letter projection of a sequence $(v_n)_{n\geq 0}$ which occurs as a column of a linear cellular automaton (without memory) whose initial condition is eventually periodic in both directions.*

*Proof.* Theorem 3.3 guarantees the existence of a linear rule $\Phi$ with memory $d+r+1$ such that $(u_n)_{n\geq 0}$ occurs as a column of some spacetime diagram of $\Phi$. Wrap every $(d+r+1)$-tuple of consecutive cells in each column into a single cell. In other words, consider the new alphabet $\mathbb{F}_q^{d+r+1}$. The new cellular automaton $\Phi^* : (\mathbb{F}_q^{d+r+1})^{\mathbb{Z}} \to (\mathbb{F}_q^{d+r+1})^{\mathbb{Z}}$, without memory, has the same left and right radius as the old, and has a local rule which takes the "central" cell, discards the top entry of that $(d+r+1)$-tuple, shifts the most recent $d+r$ entries up, and inserts the output of the old local rule at the bottom. This construction means that there is some sequence $(v_n)_{n\geq 0} \in (\mathbb{F}_q^{d+r+1})^{\mathbb{N}}$ which occurs as a column of some spacetime diagram for $\Phi^*$, and such that if we project each $v_n$ to its first entry, we obtain $(u_n)_{n\geq 0}$. It is straightforward that $\Phi^*$ is linear. □

**Corollary 3.5.** *If $(u_n)_{n\geq 0}$ is a $p$-automatic sequence, then $(u_n)_{n\geq 0}$ occurs as a column of a cellular automaton (without memory) whose initial condition is eventually periodic in both directions.*

*Proof.* Corollary 3.4 provides a cellular automaton on the alphabet $\mathbb{F}_q^{d+r+1}$ such that the sequence in Column $-2$, when projected onto first entries, is $(u_n)_{n\geq 0}$. We modify the cellular automaton to implement this projection and produce a column which is the sequence $(u_n)$. Dilate the existing spacetime diagram spatially by adding a new column between every pair of consecutive existing columns. Adjust the local rule correspondingly so that the retained (now "even-indexed") columns emulate the original spacetime diagram. In each new ("odd-indexed") column, let the state of each cell be the first entry of the cell to its left on the previous step. These two cases combine to form a local rule on the alphabet $\mathbb{F}_q^{d+r+1} \cup \mathbb{F}_q$, since if the value of a cell is in $\mathbb{F}_q$ then the rules "knows" to perform the coding and otherwise to perform a linear rule on tuples. Finally, remove the top row and use the second row as the initial condition, since $u_0$ is the projection of an entry on the top row and therefore is an entry on the second row. $\square$

Note that the cellular automaton constructed in Corollary 3.5 is not linear; in particular, the alphabet on which it is defined is no longer a group.

If $\Phi$ and $\Psi$ are two cellular automata with memory $d$ such that

$$\Psi(\Phi(R_0, R_1, \ldots, R_{d-1}), R_{d-1}, \ldots, R_1) = R_0$$

for all $R_0, \ldots, R_{d-1} \in \mathcal{A}^{\mathbb{Z}}$, we say that $\Phi$ is *invertible*. The spacetime diagram of an invertible cellular automaton can be evolved backward in time as well as forward, just as the spacetime diagram of a cellular automaton whose local rule is a bijective function of the leftmost or rightmost dependent cell can be continued up the page [Hed69, Row06]. Figure 3 shows the spacetime diagram of such an automaton.

**Corollary 3.6.** *If $(u_n)_{n\geq 0}$ is a $p$-automatic sequence, then for some $r \geq 0$ the sequence $(u_n)_{n\geq r}$ occurs as a column of an invertible cellular automaton with memory.*

*Proof.* It suffices to arrange that $C_d(x)$ is a nonzero monomial, since solving $\sum_{i=0}^{d} C_i(x) R_{n-i}(x) = 0$ for $R_{n-d}(x)$ then gives a linear local rule for each entry on row $n-d$ in terms of entries on rows $n-d+1, \ldots, n$.

We may assume that $(u_n)_{n\geq 0}$ has infinitely many zero terms, since if not then some permutation of $\mathbb{F}_q$ results in a sequence $(v_n)_{n\geq 0}$ with infinitely many zero terms, and after constructing a spacetime diagram containing $(v_n)_{n\geq r}$ we can apply the inverse permutation to obtain a spacetime diagram containing $(u_n)_{n\geq r}$.

As in the proof of Proposition 3.2, start with the polynomial

$$P^*(t, x) = A_0^*(t)x + A_1^*(t)x^p + \cdots + A_m^*(t)x^{p^m} + B^*(t)$$

where $A_0^*(0) \neq 0$, whose existence is guaranteed by Proposition 2.10. However, instead of letting $r = r^* + 1$ as in the proof of Proposition 3.2, we determine $r$ as follows.

Observe that the polynomial $P^*(t, u_{r^*} + tx)$ has the same form as $P^*(t, x)$ but has the property that, for each $i$ such that $1 \leq i \leq m$, the coefficient of $x^{p^i}$ is more highly divisible by $t$ than $A_i^*(t)$ is. Similarly, $P^*(t, u_{r^*} + t(u_{r^*+1} + tx))$ has the same form again but with coefficients that are even more highly divisible by $t$. Under this iterative substitution $x \mapsto u_n + tx$ for $n = r^*, r^* + 1, \ldots$, the exponent of $t$ grows fastest in the coefficient of $x^{p^m}$, so there exists $r^{**} \geq r^*$ such that the highest power of $t$ in the polynomial

$$P^*(t, u_{r^*} + u_{r^*+1}t + \cdots + u_{r^{**}}t^{r^{**}-r^*} + t^{r^{**}-r^*+1}x)$$

appears only in the coefficient of $x^{p^m}$. Let $r \geq r^{**} + 1$ such that $u_r = 0$, which exists since $(u_n)_{n\geq 0}$ has infinitely many zeros.

Now resume the proof of Proposition 3.2: The series $x = G(t) := \sum_{n\geq 1} u_{n+r}t^n$ is a zero of the polynomial

$$P^*(t, u_{r^*} + u_{r^*+1}t + \cdots + u_{r-1}t^{r-1-r^*} + 0t^{r-r^*} + t^{r-r^*}x),$$

and moreover the highest power of $t$ in this polynomial appears only in the coefficient of $x^{p^m}$ (and not also in the coefficient of $x^0$, as it would have if $u_r \neq 0$). Therefore $C_d(x)$ is a monomial. By dividing by an appropriate power of $t$, the conclusions of Proposition 3.2, and hence Theorem 3.3, are preserved. □

If the polynomials $A_m(t)$ and $B(t)$ are monomials, then we can also think of time as moving to the left and right; we use this idea in Section 5 to prove Corollary 5.8.

We conclude this section with some open questions suggested by the previous results.

- Corollary 3.5 provides an upper bound of $q^{d+r+1} + |\mathcal{A}|$ for the number of states in a cellular automaton spacetime diagram containing a given $p$-automatic sequence as a column, where the alphabet of the sequence is $\mathcal{A} \subset \mathbb{F}_q$. Can this bound be improved?

- Each column in the spacetime diagram constructed in Corollary 3.6 is bi-infinite. Does every letter-to-letter projection of a bi-infinite fixed point of a $p$-uniform substitution (see Section 5) occur as a column of a bi-infinite spacetime diagram?

- Does there exist a 3-automatic sequence $(u_n)_{n\geq 0}$ on a binary alphabet such that $(u_n)$ is not eventually periodic and $(u_n)$ occurs as a column of a (non-linear) 2-state spacetime diagram? Theorem 3.1 rules out the possibility of the rule being linear over $\mathbb{F}_2$ since a sequence which is both 2-automatic and 3-automatic is eventually periodic by Cobham's theorem.

- Which $k$-automatic sequences (if $k$ is not a prime power) occur as columns of cellular automaton spacetime diagrams?

# 4 Examples

Provided that we have an algorithm for generating the required polynomial in Proposition 3.2, the proof of Theorem 3.3 shows us how to build the required cellular automaton.

An inspection of the proof of Theorem 2.8 yields an algorithm for producing an explicit polynomial equation satisfied by $F(t) = \sum_{n\geq 0} u_n t^n$. For, given the $p$-DFAO that defines $(u_n)_{n\geq 0}$, we can build the $q$-DFAO $(\mathcal{S}, \Sigma_q, \delta, s_0, \mathbb{F}_q, \omega)$ that defines $(u_n)_{n\geq 0}$. Next, the $q$-kernel of $(u_n)_{n\geq 0}$ consists of the $q$-automatic sequences built with the DFAO $(\mathcal{S}, \Sigma_q, \delta, s, \mathbb{F}_q, \omega)$, where the initial state $s$ varies over $\mathcal{S}$. If the $q$-kernel of $(u_n)_{n\geq 0}$ contains $d$ elements, let them be generated by initial states $s_1, \dots, s_d$: thus each $s_i$ determines a sequence $(u_n^{(i)})_{n\geq 0}$ in the $q$-kernel of $(u_n)_{n\geq 0}$. We can then write each generating function $F_i(t)$ of $(u_n^{(i)})_{n\geq 0}$ as an explicit linear combination of the functions $F_1(t^q), \dots, F_d(t^q)$. We repeat this procedure $d$ times to explicitly express each $F_j(t^{q^i})$, for $i$ and $j$ satisfying $1 \leq j \leq d$ and $0 \leq i \leq d$, as a linear combination, over the field $\mathbb{F}_q(x)$, of the elements in $\{F_1(t^{q^{d+1}}), \dots, F_d(t^{q^{d+1}})\}$. Now we have a linear relationship between $F_{s_0}(t), \dots, F_{s_0}(t^{q^d})$, and this yields a polynomial that is almost of the form required by Proposition 2.10 — the polynomial $P^*(t, x)$ may need to be modified so that $A_0^*(t)$ is not divisible by $t$. Inspection of the proof of Proposition 2.10 in [Fur67] shows that this modification can be done mechanically. We refer the interested reader to the proof of Theorems 6.6.2 and 12.2.5 in [AS03].

As examples, next we compute the cellular automaton rules and initial conditions that generate three well-known automatic sequences as columns.

**Example 4.1.** First let us compute the cellular automaton for the Thue–Morse sequence shown in Figure 1. Christol's theorem gives $tx + (1+t)x^2 + (1+t^4)x^4 = 0$ satisfied by $x = F(t) = \sum_{n\geq 0} u_n t^n$.

Apply the proof of Proposition 2.10 to this polynomial. The coefficient of $x^1$ is already nonzero. However, since it is divisible by $t$ we find an appropriate $r$ such that replacing[1] $x$ with $\sum_{n=0}^{r-2} u_n t^n + t^{r-1}x$ and dividing by common powers of $t$ leaves the coefficient of $x^1$ not divisible by $t$. In this case $r = 2$ suffices, so we replace $x \mapsto 0 + tx$ and divide by $t^2$. Then $x = G^*(t) := \sum_{n\geq 0} u_{n+1} t^n$ satisfies

$$x + (1+t)x^2 + (t^2 + t^6)x^4 = 0.$$

Now apply Proposition 3.2. Replace $x$ with $u_{r-1} + u_r t + tx = 1 + 1t + tx$ so that $x = G(t) := \sum_{n\geq 1} u_{n+2} t^n$ satisfies $P(t, G(t)) = 0$, where

$$P(t, x) = (t^2 + t^9) + x + (t + t^2)x^2 + (t^5 + t^9)x^4.$$

[1] Formally we are making the substitution $x = \sum_{n=0}^{r-2} u_n t^n + t^{r-1} y$ but as we will be making additional substitutions we prefer to be slightly sloppy than overly complicated.

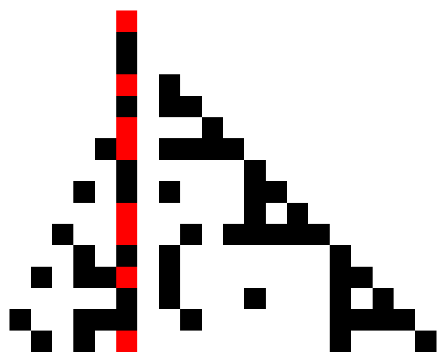
Note that $P^{(0,1)}(t,x) = 1$. By Proposition 2.11, $u_{n+2}$ is the coefficient of $x^{-2}$ in $R_n(x)$ for all $n \geq 1$, where $R_n(x)$ is the coefficient of $t^n$ in the series

$$\frac{P^{(0,1)}(t,x)}{P(t,x)} = \frac{1}{x}\sum_{n\geq 0}\left(\frac{x - P(t,x)}{tx}\right)^n t^n = \frac{1}{x}+t+\left(\frac{1}{x^2}+1+x\right)t^2+\cdots = \sum_{n\geq 0} R_n(x)t^n.$$

By collecting the terms of $P(t,x)$ by common powers of $t$, we see that $R_n(x)$ satisfies the recurrence

$$R_n(x) = xR_{n-1}(x) + \left(\frac{1}{x}+x\right)R_{n-2}(x) + x^3R_{n-5}(x) + \left(\frac{1}{x}+x^3\right)R_{n-9}(x)$$

for all $n \geq 10$. This recurrence determines a linear cellular automaton rule $\Phi$ with memory 9. Extend the memory to $d + r + 1 = 12$ without introducing dependence on the earliest $r + 1 = 3$ rows. Let $R_{-2}(x) = u_0x^{-2} = 0$, $R_{-1}(x) = u_1x^{-2} = x^{-2}$, and $R_0(x) = u_2x^{-2} = x^{-2}$. Then the sequence $(u_n)_{n\geq 0}$ occurs in Column $-2$ of the spacetime diagram of $\Phi$ begun from initial conditions $R_{-2}, \ldots, R_9$. Columns $-7$ through 12 of rows $R_{-2}, \ldots, R_{13}$ appear below, with Column $-2$ highlighted.



Note that the polynomial we computed from Christol's theorem is not the minimal polynomial of $F(t)$, since $x = F(t)$ is also a zero of $t+(1+t^2)x+(1+t+t^2+t^3)x^2$. Using this polynomial instead produces a different polynomial

$$P(t,x) = (t + t^3 + t^4) + (1 + t^2)x + (t + t^2 + t^3 + t^4)x^2$$

and hence a different cellular automaton. In fact $d = 4$ and $r = 1$, so the memory is lowered to 6.

**Example 4.2.** In Example 4.1 the coefficient of $x^1$ in the polynomial obtained from Christol's theorem was nonzero, and this saved some work. Therefore let us work out example where none of the steps in the algorithm are trivial. Additionally, we go through the construction of the polynomial from Christol's theorem. The Rudin–Shapiro sequence is the 2-automatic sequence $(u_n)_{n\geq 0} = 0, 0, 0, 1, 0, 0, 1, 0, \ldots$ where $u_n = 0$ if the number of (possibly overlapping) occurrences of 11 in the binary representation of $n$ is even and $u_n = 1$ otherwise.

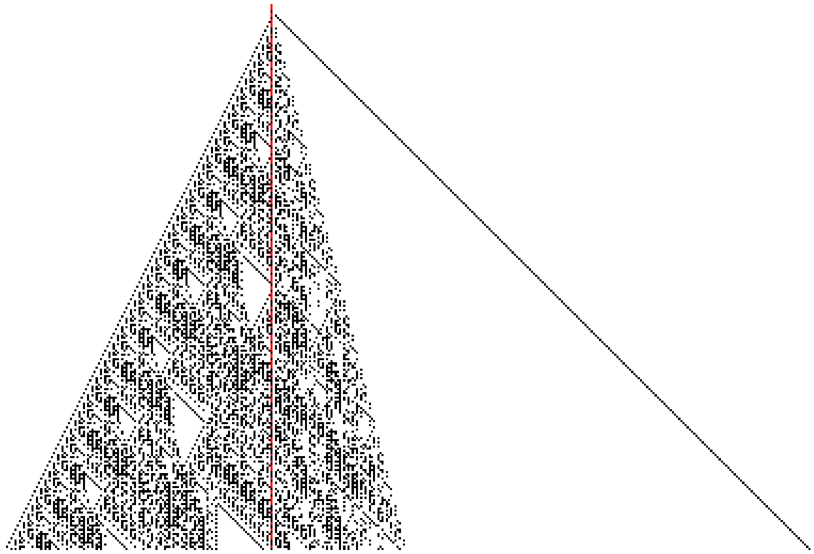


Figure 2: Spacetime diagram of a cellular automaton with memory 20 containing the Rudin–Shapiro sequence.

First we apply Theorem 2.8 as follows to construct a polynomial equation satisfied by $F(t) = \sum_{n \geq 0} u_n t^n = t^3 + t^6 + t^{11} + t^{12} + \cdots$. There are $|\mathcal{S}| = 4$ sequences in the 2-kernel of $(u_n)$: $(u_n) = (u_{2n}) = (u_{4n+1})$, $(u_{2n+1}) = (u_{8n+7})$, $(u_{4n+3}) = (u_{16n+11})$, and $(u_{8n+3}) = (u_{16n+3})$. Each of the generating functions of these four sequences can be broken up into its even- and odd-index terms and rewritten in terms of the original four generating functions:

$$\begin{aligned} F(t) = F_1(t) &:= \sum_{n \geq 0} u_n t^n = F_1(t^2) + t\, F_2(t^2) \\ F_2(t) &:= \sum_{n \geq 0} u_{2n+1} t^n = F_1(t^2) + t\, F_3(t^2) \\ F_3(t) &:= \sum_{n \geq 0} u_{4n+3} t^n = F_4(t^2) + t\, F_2(t^2) \\ F_4(t) &:= \sum_{n \geq 0} u_{8n+3} t^n = F_4(t^2) + t\, F_3(t^2). \end{aligned}$$

Using these four equations iteratively, we may write each $F(t^{2^i})$ for $0 \leq i \leq |\mathcal{S}|$ as a linear combination of $F_j(t^{2^{|\mathcal{S}|+1}})$. The result of doing so is

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 + t^2 + t^4 + t^8 + t^{10} + t^{14} & t^{16} + t^{18} + t^{20} + t^{28} & t^{22} + t^{24} + t^{26} + t^{30} & t^6 + t^{12} \\ 1 + t^4 + t^8 & t^{16} + t^{20} + t^{28} & t^{24} & t^{12} \\ 1 + t^8 & t^{16} & t^{24} & 0 \\ 1 & t^{16} & 0 & 0 \end{pmatrix} \begin{pmatrix} F_1(t^{32}) \\ F_2(t^{32}) \\ F_3(t^{32}) \\ F_4(t^{32}) \end{pmatrix} = \begin{pmatrix} F(t) \\ F(t^2) \\ F(t^4) \\ F(t^8) \\ F(t^{16}) \end{pmatrix}$$

where

$$
\begin{aligned}
a_{11} &= 1 + t + t^2 + t^4 + t^5 + t^7 + t^8 + t^9 + t^{10} + t^{14} \\
a_{12} &= t^{16} + t^{17} + t^{18} + t^{20} + t^{21} + t^{23} + t^{27} + t^{28} + t^{29} + t^{31} \\
a_{13} &= t^{19} + t^{22} + t^{24} + t^{25} + t^{26} + t^{30} \\
a_{14} &= t^3 + t^6 + t^{11} + t^{12} + t^{13} + t^{15}.
\end{aligned}
$$

We have a system of equations in 9 variables (the series $F_j(t^{32})$ for $1 \leq j \leq 4$ and $F(t^{2^i})$ for $0 \leq i \leq 4$), and performing Gaussian elimination on the corresponding $5 \times 9$ matrix gives, in the bottom row, the coefficients of a polynomial equation satisfied by $x = F(t)$, namely

$$t^6 x^2 + (1 + t^6)x^4 + (1 + t^4 + t^8 + t^{12})x^8 = 0.$$

Next we apply Proposition 2.10. Since $x^1$ does not appear in the polynomial equation we have found, extract terms whose power of $t$ is even (which in this case is all terms) and raise both sides to the power $1/2$. The resulting equation

$$t^3 x + (1 + t^3)x^2 + (1 + t^2 + t^4 + t^6)x^4 = 0$$

has a nonzero coefficient of $x^1$, as desired. To obtain a coefficient of $x^1$ that is not divisible by $t$, let $r = 4$, replace $x \mapsto 0 + 0t + 0t^2 + t^3 x$, and divide by $t^6$. Then $x = G^*(t) := \sum_{n \geq 0} u_{n+3} t^n$ satisfies

$$x + (1 + t^3)x^2 + (t^6 + t^8 + t^{10} + t^{12})x^4 = 0.$$

Finally, we apply Proposition 3.2. Replace $x$ with $1 + 0t + tx$ so that $x = G(t) := \sum_{n \geq 1} u_{n+4} t^n$ satisfies $P(t, G(t)) = 0$, where

$$P(t, x) = (t^2 + t^5 + t^7 + t^9 + t^{11}) + x + (t + t^4)x^2 + (t^9 + t^{11} + t^{13} + t^{15})x^4.$$

By Proposition 2.11, $u_{n+4}$ is the coefficient of $x^{-2}$ in $R_n(x)$ for all $n \geq 1$, where

$$\sum_{n \geq 0} R_n(x) t^n = \frac{1}{x} + t + \left(\frac{1}{x^2} + x\right) t^2 + x^2 t^3 + \left(\frac{1}{x^3} + x^3\right) t^4 + x^4 t^5 + \cdots.$$

Moreover, $R_n(x)$ satisfies the recurrence

$$
\begin{aligned}
R_n(x) = {} & x R_{n-1}(x) + \frac{1}{x} R_{n-2}(x) + x R_{n-4}(x) + \frac{1}{x} R_{n-5}(x) + \frac{1}{x} R_{n-7}(x) \\
& + \left(\frac{1}{x} + x^3\right) R_{n-9}(x) + \left(\frac{1}{x} + x^3\right) R_{n-11}(x) + x^3 R_{n-13}(x) + x^3 R_{n-15}(x)
\end{aligned}
$$

for all $n \geq 16$. Therefore the cellular automaton rule $\Phi$ has memory 15, which we increase to 20 to reinstate the initial rows. The first 256 rows of the resulting spacetime diagram appear in Figure 2.

Note that since $u_r$ happened to be 0, the highest power of $t$ in $P(t, x)$ appears only in the coefficient of $x^4$, and therefore the cellular automaton is invertible in accordance with Corollary 3.6. Figure 3 shows the spacetime diagram for rows $R_{-215}$ through $R_{40}$.

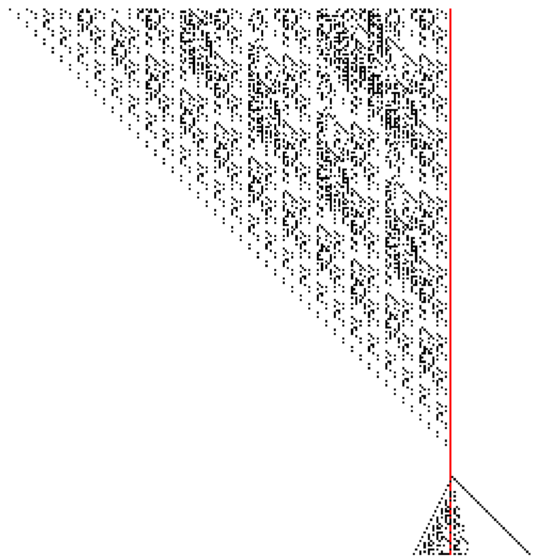


Figure 3: Spacetime diagram showing the beginning of the infinite history of an invertible cellular automaton containing the Rudin–Shapiro sequence.

**Example 4.3.** The Baum–Sweet sequence is the 2-automatic sequence $(u_n)_{n\geq 0} = 1, 1, 0, 1, 1, 0, 0, 1, \ldots$ where $u_n = 0$ if the binary representation of $n$ contains a block of 0s of odd length and $u_n = 1$ if not. (Note we consider the binary representation of 0 to be the empty word.) Christol's theorem gives $t^2x + (1 + t^3 + t^4)x^2 + t^6x^4 + (1 + t^4)x^8 = 0$ satisfied by $x = \sum_{n\geq 0} u_n t^n$. The output of Proposition 3.2 is the polynomial

$$P(t, x) = (t + t^3 + t^4 + t^7 + t^{13} + t^{19} + t^{23}) + x + (t + t^4 + t^5)x^2 + t^{13}x^4 + (t^{19} + t^{23})x^8.$$

Therefore we have a cellular automaton with memory $d + r + 1 = 23 + 3 + 1 = 27$. The first 192 rows appear in Figure 4.

# 5 Substitution dynamical systems as factors of cellular automata

In this section we apply Theorem 1.1 to conclude that certain dynamical systems arise as factors of cellular automata. First we define some terms.

**Definition 5.1.** Let $(X, S)$ and $(Y, T)$ be two dynamical systems.

1. If $X$ is a closed subset of $Y$ and $T(X) \subset X$, then we say that $(X, T)$ is a *subsystem of* $(Y, T)$.

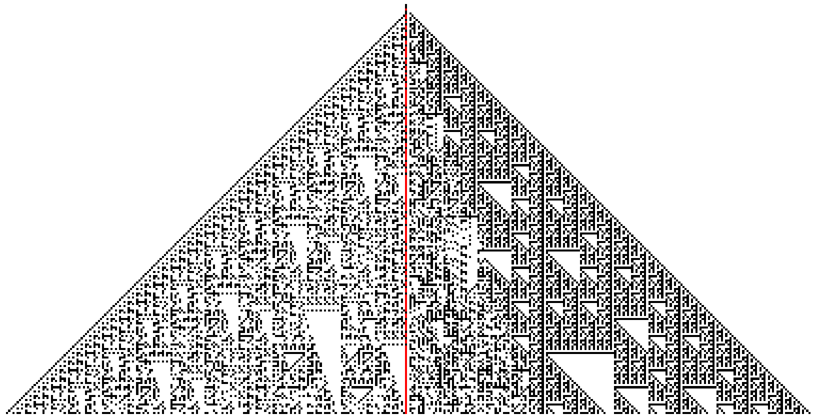


Figure 4: Spacetime diagram of a cellular automaton with memory 27 containing the Baum–Sweet sequence.

2. If there exists a homeomorphism $\Psi : Y \to X$ with $\Psi \circ T = S \circ \Psi$, we say the dynamical systems $(X, S)$ and $(Y, T)$ are *topologically conjugate*.

3. If $(X, S)$ is conjugate to a subsystem of $(Y, T)$, then we say that $(Y, T)$ *embeds* $(X, S)$.

4. If there exists a continuous surjective mapping $\Psi : Y \to X$ such that $S \circ \Phi = \Phi \circ T$, we say $(X, S)$ is a (topological) *factor* of $(Y, T)$.

**Definition 5.2.** If $\mathbf{u} \in \mathcal{A}^{\mathbb{N}}$, define $X_{\mathbf{u}} := \overline{\{\sigma^n(\mathbf{u}) : n \in \mathbb{N}\}}$. The dynamical system $(X_{\mathbf{u}}, \sigma)$ is called the (one-sided) subshift associated with $\mathbf{u}$.

**Theorem 1.2.** *Let* $\mathbf{u}$ *be* $p$*-automatic. Then* $(X_{\mathbf{u}}, \sigma)$ *is a factor of a subsystem of some linear cellular automaton* $((\mathbb{F}_q^d)^{\mathbb{Z}}, \Phi)$.

*Proof.* By Corollary 3.4, $\mathbf{u}$ is the image, under a letter-to-letter projection, of a sequence $\mathbf{v}$ which appears as a column in the space-time diagram, with initial condition $R_0$, of a linear cellular automaton $\Phi$. We shall show that $(X_{\mathbf{v}}, \sigma)$ is a factor of a subsystem of $((\mathbb{F}_q^{d+r+1})^{\mathbb{Z}}, \Phi)$; the fact that $\mathbf{u}$ is a letter-to-letter projection of $\mathbf{v}$ implies that $(X_{\mathbf{u}}, \sigma)$ is a factor of $(X_{\mathbf{v}}, \sigma)$.

Define the map $\Psi : \{\Phi^n(R_0)\}_{n\geq 0} \to \{\sigma^n(\mathbf{v})\}_{n\geq 0}$ as $\Psi(\Phi^n(R_0)) := \sigma^n(\mathbf{v})$. Since $\mathbf{v}$ is a column of the space-time diagram of $\Phi$ with initial condition $R_0$, and $\Phi$ is defined by a local rule, then it is straightforward to see that $\Psi$ is uniformly continuous. We claim that the map $\Psi : \{\Phi^n(R_0)\} \to \{\sigma^n(\mathbf{v})\}$ extends to a continuous surjection $\Psi : \overline{\{\Phi^n(R_0)\}} \to X_{\mathbf{v}}$ satisfying $\Psi \circ \Phi = \sigma \circ \Psi$. The proof is standard but we include it. Fix $R \in \overline{\{\Phi^n(R_0)\}}$ and suppose that $\Phi^{n_k}(R_0) \to R$. We will show that the set $\{\sigma^{n_k}(\mathbf{v})\}$ has a unique limit point $\mathbf{y}$, which is independent of the sequence $(n_k)$. Given $\epsilon$, the uniform continuity of $\Psi$ on the $\Phi$-orbit of $R_0$ implies that we can find a $\delta$ such that $d(\sigma^n(\mathbf{v}), \sigma^m(\mathbf{v})) < \epsilon/3$ whenever $d(\Phi^n(R_0), \Phi^m(R_0)) < \delta$ (where $d$ is the metric generated by the topology on the relevant Cantor space). Thus if $\sigma^{n_{k_l}}(\mathbf{v}) \to \mathbf{y}$ and $\sigma^{n_{k'_l}}(\mathbf{v}) \to \mathbf{y}'$, then there is an $L$ such that if $l \geq L$ then $d(\sigma^{n_{k_l}}(\mathbf{v}), \sigma^{n_{k'_l}}(\mathbf{v})) < \epsilon/3$. If $L$ is also large enough so that $\sigma^{n_{k_l}}(\mathbf{v})$, $\sigma^{n_{k'_l}}(\mathbf{v})$ are $\epsilon/3$-close to $\mathbf{y}$, $\mathbf{y}'$ respectively, then $d(\mathbf{y}, \mathbf{y}') < \epsilon$. Hence $\mathbf{y} = \mathbf{y}'$.

Now suppose that $\Phi^{m_k}(R_0) \to R$. The proximity of $\Phi^{n_k}(R_0)$ and $\Phi^{m_k}(R_0)$, for large $k$, implies the proximity of $\sigma^{n_k}(\mathbf{v})$ and $\sigma^{m_k}(\mathbf{v})$ for large $k$, which implies that the limit point of each of the sets $\{\sigma^{n_k}(\mathbf{v})\}$ and $\{\sigma^{m_k}(\mathbf{v})\}$ is the same; let this limit point be $\mathbf{y}$. We can now define $\Psi(R) = \mathbf{y}$. To see that $\Psi$ is continuous, note that if $R$ and $R'$ are close, and $\Phi^{n_k}(R_0) \to R$, $\Phi^{n'_k}(R_0) \to R'$, then for large $k$ $\Phi^{n_k}(R_0)$ and $\Phi^{n'_k}(R_0)$ are close, which implies that $\mathbf{y}$ and $\mathbf{y}'$ are close. To see that $\Psi$ is surjective: if $\sigma^{n_k}(\mathbf{v}) \to \mathbf{y}$, then let $R$ be a limit point of $\Phi^{n_k}(R_0)$: then $\Psi(R) = \mathbf{y}$. Finally if $\Phi^{n_k}(R_0) \to R$ and $\sigma^{n_k}(\mathbf{u}) \to \mathbf{y} = \Psi(R)$ then

$$\Psi \circ \Phi(R) = \Psi \circ \Phi(\lim \Phi^{n_k}(R_0)) = \lim_k \sigma^{n_k+1}(\mathbf{u}) = \sigma \lim_k \sigma^{n_k}(\mathbf{u}) = \sigma(\Psi(R)) . \quad \square$$

We now define a class of well-studied subshifts, and state Cobham's theorem, which tells us that these subshifts arise from $p$-automatic sequences.

**Definition 5.3.** Let $\mathcal{S}$ be a finite alphabet. A *substitution* (or *morphism*) is a map $\tau : \mathcal{S} \to \mathcal{S}^+$. The map $\tau$ extends to a map $\tau : \mathcal{S}^+ \cup \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^+ \cup \mathcal{S}^{\mathbb{N}}$ by concatenation: if $\mathbf{a} = a_1 \cdots a_k \cdots$, then $\tau(\mathbf{a}) := \tau(a_1) \cdots \tau(a_k) \cdots$.

**Definition 5.4.** Suppose the substitution $\tau$ is such that there is some $k$ with $|\tau(a)| = k$ for each $a \in \mathcal{S}$. In this case we say that $\tau$ is a *length-$k$* substitution (or *$k$-uniform morphism*).

**Definition 5.5.** A *fixed point* of $\tau$ is a sequence $\mathbf{v} = (v_n)_{n\geq 0} \in \mathcal{S}^{\mathbb{N}}$ such that $\tau(\mathbf{v}) = \mathbf{v}$.

Cobham's theorem [Cob72] gives us the relationship between $k$-automatic sequences and fixed points of length-$k$ substitutions:

**Theorem 5.6.** *A sequence is $k$-automatic if and only if it is the image, under a letter-to-letter projection, of a fixed point of a length-$k$ substitution.*

Dynamicists have extensively studied substitution subshifts — references detailing some of this work include [Fog02] and [Que10]. Combining Cobham's theorem with Theorem 1.2, we obtain the following.

**Corollary 5.7.** *Let $\mathbf{v}$ be a substitution sequence generated by a length-$p$ substitution. Then $(X_{\mathbf{v}}, \sigma)$ is a factor of a subsystem of some linear cellular automaton $((\mathbb{F}_q^d)^{\mathbb{Z}}, \Phi)$.*

It would be interesting to know whether the factor mapping in Theorem 1.2 is, in some or all cases, an embedding. This is in principle possible: in [PY10], substitution systems are embedded in subsystems of cellular automata; however the cellular automata are tailored for the specific substitution and have no nice algebraic or combinatorial structure. We end with an extra condition on the polynomial given by Proposition 3.2 which would give an embedding of the substitution subshift into a cellular automaton, and leave as an open question whether such a polynomial always exists.

**Corollary 5.8.** *Suppose, using the notation of Proposition 3.2 and Theorem 3.3, that both $A_m(t) = \alpha t^d$ and $B(t) = \beta t^d$ are monomials of degree d. Then $(X_{\mathbf{u}}, \sigma)$ is the letter-to-letter projection of a subshift that can be embedded in a linear cellular automaton.*

*Proof.* Recall that $R_n(x) = -\sum_{i=1}^{d} \frac{C_i(x)}{C_0(x)} R_{n-i}(x)$. If $A_m(t)$ is a monomial, then only one of the polynomials $C_i(x)$, say $i_R$, has the $x^{p^m}$ term. This means that if the left radius of $\Phi$ is $p^m - 1$, and $\Phi = \sum_{i=1}^{d} \Phi_i$ where $\Phi_i$ is the cellular automaton defined by $\frac{C_i(x)}{C_0(x)}$, then other than $\Phi_d$, all $\Phi_i$'s have radius strictly less than $p^m - 1$. Rotating our original spacetime diagram $\mathcal{S}_\Phi$ counter-clockwise by 90 degrees, we see a new spacetime diagram for another cellular automaton with memory $l + r + 1$. Similarly, only one of the cellular automata $\Phi_d$ will have right radius $l = 1$, so that rotating $\mathcal{S}_\Phi$ by 90 degrees clockwise, we see another spacetime diagram for another cellular automaton with memory $l + r + 1$. This means that if in $\mathcal{S}_\Phi$, the central $l+r+1$ columns $C_{-l}, \ldots, C_r$ have the same entries in a large enough block of length $L$ starting at locations $m_1, m_2$ respectively, then the entries in two rows $R_{m_1}, R_{m_2}$ will agree in a large central block. Thus if in the proof of Theorem 1.2, we consider $X_{\mathbf{V}}$ where $\mathbf{V} \in \mathcal{S}^{l+r+1}$ is the infinite word defined by the columns $C_{-l}, \ldots, C_r$, then the map $\Psi$ defined in Theorem 1.2 is a topological conjugacy between $(\overline{\{\Phi^n(R_0)\}}, \Phi)$ and $(X_{\mathbf{V}}, \sigma)$. Projecting $X_{\mathbf{V}}$ to the appropriate column containing $\mathbf{u}$, the result follows. $\square$

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