A family of sand automata

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Abstract

We study some dynamical properties of a family of two-dimensional cellular automata: those that arise from an underlying one dimensional sand automaton whose local rule is obtained using a latin square. We identify a simple sand automaton Γ whose local rule is algebraic, and classify this automaton as having equicontinuity points, but not being equicontinuous. We also show it is not surjective. We generalise some of these results to a wider class of sand automata.

Keywords: Sand automata, Higher-dimensional cellular automata, Equicontinuity, Surjectivity.

1 Introduction

In [4], the authors introduce the family of *sand automata*: these are dynamical systems $\Phi : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$, where $\mathcal{A}$ is a countably infinite alphabet, that satisfy certain constraints (see Section 2 for all definitions). In turn, with the appropriate topology $\mathcal{T}$ put on $\mathcal{A}^{\mathbb{Z}^d}$, $(\mathcal{A}^{\mathbb{Z}^d}, \Phi)$ is topologically conjugate to a CA $\Phi_{\binom{1}{0}} : S_{\binom{1}{0}} \to S_{\binom{1}{0}}$ where $S_{\binom{1}{0}} \subset \{0,1\}^{\mathbb{Z}^{d+1}}$ is a subshift of finite type. In [10] the authors ask if any of the cellular automata arising from sand automata are chaotic. This is a particular instance of the more general project of finding chaotic higher dimensional cellular automata. In this article we study a family of sand automata and their dynamical

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properties. We work with a definition of sand automata that is slightly more general than that in [4], decoupling the definition of the radius of the sand automaton into a *spatial* and an *output* radius (see Section 2 for details).

In Section 2.4, we define *linear* sand automata: these are automata whose local rules are built using a group endomorphism $\phi : \mathcal{A}^{2r+1} \to \mathcal{A}$, where $\mathcal{A}$ is a finite cyclic group. We say that local rules are 'built', because, unlike cellular automata, the initial configuration has to be 'relativised' before the local rule is applied, and the output of the local rule is added to the initial configuration. We emphasize here though that unlike the case of linear cellular automata, the relativisation process prevents the resulting sand automaton from having explicit linear properties. Our motivation for studying this family of sand automata was that cellular automata whose local rule is a group endomorphism have been extensively studied and are a rich source of interesting results and examples ([5], [14], [18], [21]). Despite the non-linearity of the sand automata we study, the underlying algebraic structure helps to answer questions, for example whether the resulting sand automaton is surjective, or equicontinuous. These are properties that have been shown in [4, 10, 22] to be undecidable for sand automata in general. In general we work exclusively with one dimensional sand automata (ie those whose corresponding cellular automata act on two dimensional configuration space), whose local rules have spatial radius 1. A natural next step is to study which of our results generalise to linear sand automata of higher spatial radius by applying a standard recoding of the underlying space.

We frequently work with one fixed sand automaton, $\Gamma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, whose local rule is built using the rule $\gamma : \mathbb{Z}_5^3 \to \mathbb{Z}_5$ defined by $\gamma(x_{-1}\, x_0\, x_1) = x_{-1} + x_1$, which is the local rule for the famous XOR cellular automaton (albeit on a different alphabet), and whose dynamical properties have been studied extensively, for example in [5, 7, 15, 16, 17]. Although we are interested in the dynamical properties of the cellular automaton $\Phi_{\binom{1}{0}}$, in practice we often work with Φ, since all of the dynamical properties that we discuss are topological invariants. In Section 3, we show that Γ is not surjective, and therefore cannot be transitive. In Section 3.1, we identify a proper Γ-invariant subspace, $\mathcal{G}$, and show that for points $\mathbf{x}$ in $\mathcal{G}$, computation of $\Gamma^n(\mathbf{x})$ is simple, for all n. We identify other sand automata for which $\mathcal{G}$ acts similarly and demonstrate that this accounts for about 17% of all spatial radius one sand automata, all of which cannot be transitive. In Section 4 we show that Γ is not equicontinuous, by showing the existence of (many) *vertical inducing points*. In fact all spatial radius one sand automata whose local rule tables have, for some non-zero m, the value m appear in each column (or each row), have vertical inducing points, and so are not equicontinuous. This means that at least 99% of all sand automata are not equicontinuous. Despite not being equicontinuous, we show that Γ has equicontinuity points by finding a *blocking word* for Γ. This completes the classification of Γ according to the classification scheme in [10]. As is the case for linear cellular automata, we suspect that the underlying algebraic structure of linear sand automata is what drives many of these results, and that what we prove for Γ holds for a large set of linear sand automata.

In Section 4.2, we generalise the definition of a vertical inducing point to that of a *local-rule-constant point*. Both vertical inducing and local-rule-constant points have easily computable Φ^n iterates, and similar to the case of vertical inducing points, the existence of local-rule-constant points for a sand automaton implies the non-equicontinuity of the latter. The space of local-rule-constant points is closed and invariant under Φ. We define a subspace $\mathcal{G}'$ containing the local-rule-constant points, and conjecture that $\mathcal{G}'$ is an attractor for Γ, in that $\lim_{n\to\infty} d(\Gamma^n \mathbf{x}, \mathcal{G}') = 0$ whenever $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$.

2 Preliminaries

2.1 Notation

If $\mathcal{A}$ is a countable alphabet, let $\mathcal{A}^+$ denote the set of finite concatenations of letters from $\mathcal{A}$; elements of $\mathcal{A}^+$ are called *words*. If d is a natural number, elements $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}^d}$ are called *configurations*. If $\mathbf{x} \in \mathcal{A}^{\mathbb{Z}}$ and $m, n \in \mathbb{Z}$, with $m \leq n$, let $\mathbf{x}_{[m,n]} := x_m x_{m+1} \ldots x_n$. We will be working in dimensions 1 and 2, and in dimension 2 only when $\mathcal{A}$ is finite. If $\mathcal{A}$ is given, let $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{-\infty, \infty\}$. If $x_{\mathbf{n}} = \infty$ $(-\infty)$ we say there is a *source* (sink) at location $\mathbf{n}$.

If $\mathcal{A} = \mathbb{Z}$, we define $\mathbb{Z}_{2r+1}$ to be the additive group of integers mod $2r+1$ which we will represent as $\mathbb{Z}_{2r+1} := \{-r, \ldots r\}$. In this paper $\mathcal{A}$ will either be $\mathbb{Z}$ or $\mathbb{Z}_{2r+1}$, or k-tuples from these alphabets.

Let $L \subset \mathbb{Z}^d$ be a finite set. If $\mathbf{c} = (c_l)_{l\in L} \in A^L$ define the *cylinder set* $[\mathbf{c}] := \{\mathbf{x} : x_{\mathbf{l}} = c_{\mathbf{l}}, \text{ for } \mathbf{l} \in L\}$. If $\mathcal{A}$ is finite, the Cantor topology generated by the cylinder sets, as L and $\mathbf{c}$ vary, makes $\mathcal{A}^{\mathbb{Z}^d}$ compact. If $\mathcal{A} = \widetilde{\mathbb{Z}}$ this is no longer the case so we embed $\widetilde{\mathbb{Z}}^{\mathbb{Z}}$ in $\{0,1\}^{\mathbb{Z}^2}$, and equip it with the subspace topology $\mathcal{T}$ that makes it compact, as done in [10]. We define the embedding $e : \widetilde{\mathbb{Z}}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}^2}$ by

$$(e(\mathbf{x}))_{i,j} = \begin{cases} 1 & \textit{if } j \leq x_i \\ 0 & \textit{if } j > x_i \end{cases} .$$

The transformation e is bijective and its image is a *subshift of finite type*: a closed subset of $\{0,1\}^{\mathbb{Z}^2}$ whose elements $\mathbf{x}$ are characterised by the fact that no shift, horizontal or vertical, of $\mathbf{x}$ belongs to a finite number of forbidden cylinder sets. Here the unique forbidden cylinder set is defined by the word $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline\end{array}$.

2.2 Dynamical systems

We recall some basic definitions. Let (Y, F) be a compact, topological dynamical system. We say (Y, F) is *transitive* if for all nonempty open sets $U, V \subseteq Y$, there exists an integer $n \geq 0$ such that $F^{-n}(U) \cap V \neq \emptyset$. (Y, F) is *equicontinuous* at $y \in Y$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x \in B_\delta(y)$, then $d(F^n(x), F^n(y)) < \epsilon$ for all natural n. (Y, F) is *equicontinuous* if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each y in Y, whenever $x \in B_\delta(y)$, then $d(F^n(x), F^n(y)) < \epsilon$ for each natural n. If Y is compact, then (Y, F) is equicontinuous if and only if every point in Y is an equicontinuity point of F. (Y, F) is *sensitive to initial conditions* if for some $\epsilon > 0$, and for all $y \in Y$ and $\delta > 0$, there exist $x \in B_\delta(y)$ and $n > 0$ such that $d(F^n(x), F^n(y)) > \epsilon$. A point $y \in Y$ is periodic if $F^n(y) = y$ for some n; if $n = 1$ then y is a *fixed point.* The point $y \in Y$ is *eventually periodic* if for some p, $F^p(y)$ is periodic. A dynamical system (Y, F) is *ultimately periodic* if there is some nonnegative p such that $F^{n+p} = F^n$ for each natural n .

A dynamical system is *chaotic* if it is sensitive, transitive and has a dense set of periodic points [11]. In [3] it is shown that if (Y, F) is transitive and has a dense set of periodic points then (Y, F) is sensitive. In [6], the authors show that if (Y, F) is a CA (see below for definitions) then transitivity alone implies sensitivity. In [12], the authors show that having dense periodic orbits is equivalent to surjectivity for linear cellular automata, and in [1] the authors give equivalent reformulations of the longstanding conjecture that surjectivity implies dense periodic orbits.

2.3 Cellular and sand automata

Let $X = \mathcal{A}^{\mathbb{Z}^d}$ where $\mathcal{A}$ is some finite alphabet. If $L_r := [-r, r]^d$, and $f : \mathcal{A}^{L_r} \to \mathcal{A}$ is any map then the *cellular automaton* $F : X \to X$ *with local rule* f is defined by $(F(\mathbf{x}))_{\mathbf{n}} := f(\{x_{\mathbf{n}+\mathbf{i}} : \mathbf{i} \in L_r\})$ for each $\mathbf{x}$ in X and $\mathbf{n} \in \mathbb{Z}^d$; r is called the *radius* of F. In this article $d = 2$. The simplest example of a cellular automaton (CA) in one dimension is the *shift* $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, where for each configuration $\mathbf{x}$, $(\sigma(\mathbf{x}))_n = x_{n+1}$. There are two shift maps on two-dimensional lattice spaces $X = \mathcal{A}^{\mathbb{Z}^2}$: let $(\sigma_H(\mathbf{x}))_{\mathbf{n}} = x_{\mathbf{n}+(1,0)}$ and $(\sigma_V(\mathbf{x}))_{\mathbf{n}} = x_{\mathbf{n}+(0,1)}$. In [13], Hedlund showed that F is a CA if and only if F is a continuous map which is shift commuting, i.e. $F \circ \sigma = \sigma \circ F$.

Let $X = \widetilde{\mathbb{Z}}^{\mathbb{Z}}$. Like CA, sand automata (SA) are defined in [4] as shift-commuting maps $\Phi : X \to X$ which have a local rule ϕ, except that the output of the local rule is added to the original entry. First, in [4] the authors define a sequence of "measuring instruments" of precision s. If $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ then the *measuring* *tool of precision s at reference height m* is the function $\beta_s^m : \widetilde{\mathbb{Z}} \to \widetilde{\mathbb{Z}_{2s+1}}$ defined by

$$\beta_s^m(n) = \begin{cases} +\infty & \text{if } n > m + s, \\ -\infty & \text{if } n < m - s, \\ n - m & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $\phi : \widetilde{\mathbb{Z}_{2s+1}}^{2r} \to \mathbb{Z}_{2s+1}$ be given. Define the *sand automaton Φ with local rule ϕ* by

$$\Phi(\mathbf{x})_i = \begin{cases} x_i + \phi(\beta_s^{x_i}(x_{i-r}), \dots \beta_s^{x_i}(x_{i-1}), \beta_s^{x_i}(x_{i+1}), \dots \beta_s^{x_i}(x_{i+r})) & \text{if } x_i \in \mathbb{Z} \\ x_i & \text{if } x_i = \pm\infty \end{cases}$$

Thus $\Phi(\mathbf{x})_i$ differs from x_i by at most $2s + 1$, and is a function of the frame $\mathbf{x}_{[i-r,i+r]}$. Note that ϕ is applied not to $\mathbf{x}_{[i-r,i+r]}$ but to a relativised version of this word. The number r is called the *spatial radius* of Φ and the number s is called the *output radius* of Φ. The spatial radius is the analogue of the radius of a CA. If $s = r$, then this definition coincides with the original definition of a SA in [4]. Indeed any SA with spatial radius r and output radius s is a SA with radius $r^* = \max\{r, s\}$ according to the definition in [4], so all known results on sand automata apply to sand automata as we have defined them. The original definition of a SA in [4] is also more general in that the space on which Φ acts can be $X = \widetilde{\mathbb{Z}}^{\mathbb{Z}^d}$; however as we will only be working with one dimensional sand automata, we restrict our definition. In this article we study only sand automata with spatial radius one, as we often use the fact that their local rule is easily displayed using an array. If Φ has spatial radius r, it would be interesting to define the appropriate measuring instruments, so that by a standard recoding where the underlying alphabet is $\widetilde{\mathbb{Z}}^r$, one can convert Φ to a SA with spatial radius one. Also, in this article we study only sand automata with output radius 2, although there is no impediment to extending our results to sand automata with higher output radius; we stick to $s = 2$ for simplicity.

As a book-keeping device we define a projection $\pi : \widetilde{\mathbb{Z}}^{2r+1} \to \widetilde{\mathbb{Z}_{2s+1}}^{2r}$ by $(\pi(x_{-r}, \dots x_r)) = \beta_s^{x_0}(x_{-r}), \dots \beta_s^{x_0}(x_{-1}), \beta_s^{x_0}(x_1), \dots \beta_s^{x_0}(x_r)$ for all $(x_{-r}, \dots x_r) \in \widetilde{\mathbb{Z}}^{2r+1}$. The map π depends on s and r but for simplicity we omit this dependence in the notation. Let $\Pi : \widetilde{\mathbb{Z}}^{\mathbb{Z}} \to (\widetilde{\mathbb{Z}_{2s+1}}^{2r})^{\mathbb{Z}}$ be the map whose local rule is π. Clearly Π is not injective. We refer to $(\Pi(\mathbf{x}))_i$ as a *gradient tuple*, and later in this article we use the notation $(L_r(i), \dots L_1(i), R_1(i), \dots, R_r(i))$ to describe the gradients, with $L_1(i) = \beta_s^{x_i}(x_{i-1})$ and $R_1(i) = \beta_s^{x_i}(x_{i+1})$. If $r = 1$, we drop the subscript, using $(L(i), R(i))$.

Example 2.1. This is Example 12 in [4], which emulates the behaviour of the original model defined in [2]. Define a 1 dimensional SA F whose local rule ϕ :

$\widetilde{\mathbb{Z}_3}^2 \to \mathbb{Z}_3$ is given by:

$$\phi(a,b) = \begin{cases} 1 & \text{if } a = \infty,\ b \neq -\infty \\ -1 & \text{if } a \neq \infty,\ b = -\infty \\ 0 & \text{otherwise} \end{cases}$$

F has the property that a grain of sand falls to the right (and only the right) if the right neighbour is at least 2 smaller. If the number of sand grains in the initial configuration $\mathbf{x}$ is finite, then $F^n(\mathbf{x})$ is eventually fixed.

Define the *vertical map* $\rho : X \to X$ where

$$\rho(\mathbf{x})_i = \begin{cases} x_i + 1 & \text{if } |x_i| < \infty, \\ x_i & \text{if } |x_i| = \infty, \end{cases}$$

and say that Φ is *vertical commuting* if $\Phi(\rho(x)) = \rho(\Phi(x))$. Also, say that Φ is *infiniteness conserving* if $\Phi(\mathbf{x})_i = \pm\infty \Leftrightarrow x_i = \pm\infty$. Note that all sand automata are shift commuting, vertical commuting, and infiniteness conserving; in fact this characterises them, as shown in Theorem 3.8 of [10]:

Theorem 2.2. *$\Phi : X \to X$ is a sand automaton if and only if Φ is continuous, shift commuting, vertical commuting and infiniteness conserving.*

Using the injection $e : \widetilde{\mathbb{Z}}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}^2}$, we can transform a 1-dimensional SA into a 2-dimensional CA as done in [10]. Dynamical properties of higher dimensional cellular automata are discussed in [8, 9]. Letting $S_{\binom{1}{0}}$ denote $e(\widetilde{\mathbb{Z}}^{\mathbb{Z}})$, define $\Phi_{\binom{1}{0}} = e \circ \Phi \circ e^{-1} : S_{\binom{1}{0}} \to S_{\binom{1}{0}}$. With this notation, we have the following lemma, whose proof is straightforward:

Lemma 2.3. *$\Phi_{\binom{1}{0}}$ commutes with both the vertical and horizontal shifts, and if $\widetilde{\mathbb{Z}}^{\mathbb{Z}}$ is endowed with the topology $\mathcal{T}$, then $(\widetilde{\mathbb{Z}}^{\mathbb{Z}}, F)$ is topologically conjugate to $(S_{\binom{1}{0}}, \Phi_{\binom{1}{0}})$.*

2.4 Linear sand automata

We say that $F : \mathbb{Z}_{2k+1}^{\mathbb{Z}} \to \mathbb{Z}_{2k+1}^{\mathbb{Z}}$ is a *linear* cellular automaton if its local rule $f : \mathbb{Z}_{2k+1}^{2r+1} \to \mathbb{Z}_{2k+1}$ is of the form $f(\mathbf{x}_{[i-r,i+r]}) = a_0 x_{i-r} + a_1 x_{i-r+1} + \ldots + a_{2r+1} x_{i+r}$ where a_i and $x_i \in \mathbb{Z}_{2k+1}$ for $0 \leq i \leq 2r+1$. The topological properties of F depend on the coefficients a_i: for example if the $a_i's$ are relatively prime, then F is topologically transitive, so that many linear cellular automata are chaotic (see for example [5, 7, 15, 16, 17]).

Next we describe how to define a SA using a linear cellular automaton. As we only consider sand automata of spatial radius 1, we can display the local rule in terms of a *local rule table*. This table has a row for each possible left gradient

L and a column for each possible right gradient R; entry (L, R) of the table is $\phi(L, R)$. For example, Figure 1 is the local rule table for the SA Ω defined in Example 2.1. The local rule table is applied after each gradient pair is calculated. For example, if $\mathbf{x} = \ldots, 4, \cdot 2, 1, \ldots$ then $(\Pi(x))_0 = \binom{\infty}{-1}$, so $L = \infty$ and $R = -1$, and $(\Omega(\mathbf{x}))_0 = x_0 + 1 = 3$.

$L \backslash R$	$-\infty$	-1	0	1	∞
$-\infty$	-1	0	0	0	0
-1	-1	0	0	0	0
0	-1	0	0	0	0
1	-1	0	0	0	0
∞	0	1	1	1	1

Figure 1: The local rule table for the sand automaton Ω.

We now define *linear* sand automata of spatial radius 1 and output radius 2. Recall that to define $(F(\mathbf{x}))_0$ first we relativise with Π (whose local rule is π). Let $\iota : \widetilde{\mathbb{Z}_5}^2 \to \mathbb{Z}_5^2$ be the bijection defined by

$$(\iota(x))_i = \begin{cases} 2 & \text{if } x_i \geq 2 \\ 1 & \text{if } x_i = 1 \\ 0 & \text{if } x_i = 0 \\ -1 & \text{if } x_i = -1 \\ -2 & \text{if } x_i \leq -2 \end{cases}$$

The (radius 0) local rule ι defines a global map I defined on $(\widetilde{\mathbb{Z}_5}^2)^{\mathbb{Z}}$. Given a group homomorphism $\phi^* : \mathbb{Z}_5^2 \to \mathbb{Z}_5$, we say that a SA Φ is a *linear* SA if the local rule $\phi = \phi^* \circ \iota \circ \pi$.

Remark 2.4. We emphasize that a linear SA does *not* satisfy $\Phi(\mathbf{x} + \mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{y})$: the map $I \circ \Pi$ prevents this.

Remark 2.5. The definition of a spatial radius one, output radius s SA, where $s \geq 2$ is similar. Namely, the map ι must have domain $\widetilde{\mathbb{Z}}_{2s+1}^2$, range $\mathbb{Z}_{2s+1}^2$. All results proved in this article for $s = 2$ extend to $s \geq 2$, when $2s + 1$ is prime.

Example 2.6. Let $\gamma_* : \mathbb{Z}_5^2 \to \mathbb{Z}_5$ be defined by $\gamma_*(x, y) = x \oplus y$.

Let Γ be the linear SA with local rule $\gamma = \gamma_* \circ \iota \circ \pi$. Then Γ is a spatial radius 1 linear SA. The local rule table in Figure 2 corresponds to the group homomorphism γ_*. Note that here the rows and columns are indexed by $\mathbb{Z}_5$.

In this article we will often be working with Γ, even though we are interested primarily in $\Gamma_{\binom{1}{0}}$ (see Lemma 2.3). For, $\Gamma_{\binom{1}{0}}$ has radius $r = 7$ (in fact the local

$L\backslash R$	-2	-1	0	1	2
-2	1	2	-2	-1	0
-1	2	-2	-1	0	1
0	-2	-1	0	1	2
1	-1	0	1	2	-2
2	0	1	2	-2	-1

Figure 2: The local rule table for Γ

neighbourhood can be a 7x3 rectangle) grid, so it is often more practical to work with the spatial radius one Γ. Figure 6 contains the local rule table for $\Gamma_{\binom{1}{0}}$.

3 Non-surjectivity of Γ

A non-surjective sand automaton Φ has *Garden of Eden states* (see [19, 20]): these are configurations that have no Φ-preimage. A non-surjective CA cannot be chaotic, since it cannot be transitive. The surjectivity of sand automata is shown to be undecidable in [22]; see also [4]. In this section we show that the SA Γ is not surjective, and generalise this result to some other one dimensional sand automata.

Recall that a SA $\Phi : X \to X$ is *surjective on a set* $Y \subset X$ if for each $\mathbf{y} \in Y$, $\Phi(\mathbf{y}') = \mathbf{y}$ for some $\mathbf{y}' \in Y$. A configuration is *finite* if all configuration entries are finite, and only finitely many entries are non-zero; let $\mathcal{F}$ denote all such points. Similarly, let $\mathcal{P}$ denote the set of all σ-periodic configurations, all of whose entries are finite. In [4] (Proposition 3.14), the following was shown:

Lemma 3.1. *Let $\Phi : X \to X$ be a one-dimensional sand automaton. Then*

1. *Φ is surjective on $\mathcal{P}$ if and only if Φ is surjective, and*
2. *If Φ is surjective on $\mathcal{F}$, then Φ is surjective.*

We show that Γ is not surjective by first showing that there is a word which has no predecessor word under Γ. In the following proof the notation $[n]_5$ denotes the projection of $n \in \mathbb{Z}$ to $\mathbb{Z}_5$.

Lemma 3.2. *Let $\mathbf{w} = (100, 3, 2, 100)$, and let $m \in \mathbb{Z}$. Then there is no configuration $\mathbf{y}$ such that $\Gamma(\mathbf{y})_{[m,m+3]} = \mathbf{w}$.*

Proof. Without loss of generality we assume $m = 1$. Suppose $\Gamma(\mathbf{y})_{[1,4]} = \mathbf{w}$. Then

$$98 \leq y_1 \leq 102, \;\; 1 \leq y_2 \leq 5, \;\; 0 \leq y_3 \leq 4, \text{ and } 98 \leq y_4 \leq 102\,.$$

This implies that $\beta_2^{y_2}(y_1) = \beta_2^{y_3}(y_4) = \infty$, ie $\iota(\beta_2^{y_2}(y_1)) = \iota(\beta_2^{y_3}(y_4)) = \iota(\infty) = 2$. Next under the action of Γ we have:

$$y_2 + \iota(\beta_2^{y_2}(y_1)) + \iota(\beta_2^{y_2}(y_3)) = y_2 + (2 \oplus \iota(\beta_2^{y_2}(y_3))) = 3 \tag{3.2}$$

$$y_3 + \iota(\beta_2^{y_3}(y_4)) + \iota(\beta_2^{y_3}(y_2)) = y_3 + (2 \oplus \iota(\beta_2^{y_3}(y_2))) = 2 \tag{3.3}$$

Thus the only possibilities for $[y_2]_5$ and $[y_3]_5$ are given by:

$[y_2]_5$ / $[y_3]_5$	-2	-1	0	1	2
-2	x				
-1					x
0				x	
1			x		
2		x			

Each of these cases implies a contradiction to one of Equations (3.2) or (3.3).

$\square$

Proposition 3.3. Γ *is not* $\mathcal{P}$*-surjective, so that* Γ *is not surjective.*

Proof. Let $\mathbf{x}$ be the periodic configuration $\mathbf{x} = \overline{100,\ 3,\ 2,\ 100,}\cdot\overline{100,\ 3,\ 2,\ 100}$. Lemma 3.2 implies that there does not exist $\mathbf{y}$ such that $\Gamma(\mathbf{y}) = \mathbf{x}$. Lemma 3.1 implies that Γ is not surjective. $\square$

3.1 Surjective subsets

While Γ is not surjective, in this section we identify a proper, closed, Γ-invariant subspace, $\mathcal{G}$. We then identify sand automata Φ for which $\mathcal{G}$ is also Φ-invariant. Let

$$\mathcal{G} := \{\mathbf{x} : |x_i - x_{i-1}| \geq 2,\ \forall i \in \mathbb{Z}\}\,.$$

We shall show that each member of $\mathcal{G}$ has a Γ-predecessor (though not necessarily in $\mathcal{G}$). We will explain in geometric terms how Γ acts on $\mathcal{G}$, identify subsets of $\mathcal{G}$ that are a possible attractor for Γ, and generalize these properties to other sand automata.

It is clear that $I \circ \Pi(\mathcal{G}) \subset \{\binom{2}{2}, \binom{-2}{2}, \binom{2}{-2}, \binom{-2}{-2}\}^{\mathbb{Z}}$, and that $I \circ \Pi(\mathcal{G})$ is the subshift of finite type whose transition graph G is shown in Figure 3. Re-label $\binom{2}{2} = -1$, $\binom{-2}{2} = 0^-$, $\binom{2}{-2} = 0^+$, $\binom{-2}{-2} = 1$, and let Y_G denote the image of $I \circ \Pi(\mathcal{G})$ under this labelling. Let $\{-1, 0, 1\}^{\mathbb{Z}}$ be the full shift on three letters and let $p : \{0^-, 0^+, 1, -1\} \to \{-1, 0, 1\}$ be defined by $p(0^-) = p(0^+) = 0$, $p(1) = 1$, $p(-1) = -1$; then p is a radius 0 local rule for the CA $P : Y_G \to \{-1, 0, 1\}^{\mathbb{Z}}$. With this notation, the following lemma is straightforward.

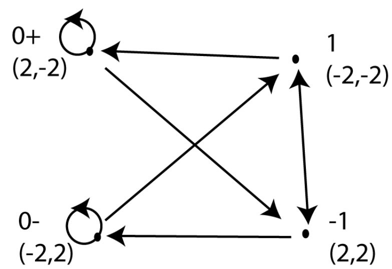

Figure 3: The graph G defines a subshift on its bi-infinite paths.

$L\backslash R$	-2	2
-2	1	0
2	0	-1

Table 1: The local rule table for Γ restricted to $\mathcal{G}$

Lemma 3.4. *If* $\mathbf{g} \in \mathcal{G}$ *then* $\exists \mathbf{y} \in Y_G$ *such that* $\Gamma(\mathbf{g}) = \mathbf{g} + P(\mathbf{y})$.

On $\mathcal{G}$ the local rule table for Γ can be compressed to Table 1. Call 3-tuples in $\widetilde{\mathbb{Z}}^3$ such that $\iota \circ \pi(a,b,c) = \binom{-2}{-2}$ *peaks* (centred at b) and similarly $\iota \circ \pi(a,b,c) = \binom{2}{2}$ *valleys* (centred at b). Also label 3-tuples such that $\iota \circ \pi(a,b,c) = \binom{-2}{2}$ *up-slopes* (centred at b) and $\iota \circ \pi(a,b,c) = \binom{2}{-2}$ *down-slopes* (centred at b). With this labelling of gradient tuples as *geographical features*, the action of Γ^n is easily described, once we know how Γ acts.

Proposition 3.5. *If* $\mathbf{g} \in \mathcal{G}$ *and* $\mathbf{y} \in Y_G$ *are such that* $\Gamma(\mathbf{g}) = \mathbf{g} + \mathbf{y}$*, then* $\Gamma^n(\mathbf{g}) = \mathbf{g} + n\mathbf{y}$ *for all* n *in* $\mathbb{N}$.

Proof. Here we claim that all of the geographical features are preserved under Γ. If we show this then the proposition follows. First note that according to Table 1, $|(\Gamma(\mathbf{g}))_n - g_n| \leq 1$ for each n, whenever $\mathbf{g} \in \mathcal{G}$. Let $(g_{-1}, g_0, g_1) = (a,b,c)$; we show that geographical features are preserved at $\Gamma(\mathbf{g})_0$; the cases at $(\Gamma(\mathbf{g}))_n$ for $n \neq 0$ are identical.

1. If (a,b,c) is a valley centred at b, then $(\Gamma(\mathbf{g}))_0 = g_0 - 1$ (see Table 1). Since for $n = \pm 1$, $(\Gamma(\mathbf{g}))_n - g_n$ is at least -1, then the valley at b is mapped to another valley centred at $(\Gamma(\mathbf{g}))_0$. The case where (a,b,c) is a peak is similar.

2. If (a,b,c) is a down-slope centred at b, then we either have a peak or a down-slope centred at a. Thus $(\Gamma(\mathbf{g}))_{-1} \geq g_{-1}$. Similarly there is either a valley or a down-slope centred at c, so $(\Gamma(\mathbf{g}))_1 \leq g_1$. This means that a down-slope centred at g_0 is mapped to a down-slope centred at $(\Gamma(\mathbf{g}))_0$. The case where (a,b,c) is an up-slope is similar.

$\square$

Note that the proof of Proposition 3.5 shows that $\Gamma(\mathcal{G}) \subset \mathcal{G}$: each geographical feature under the action of the SA can only become more pronounced or stay the same. However $\Gamma : \mathcal{G} \to \mathcal{G}$ is not surjective. Consider $\mathbf{g} = \overline{0,3}, \cdot\overline{0,3}$, so that $\Gamma(\mathbf{g}) = \mathbf{g} + \overline{-1,1}, \cdot\overline{-1,1}$. If $\Gamma^{-1}(\mathbf{g})$ contained an element $\mathbf{g'}$ in $\mathcal{G}$, then $\mathbf{g} = \mathbf{g'} + \overline{-1,1}, \cdot\overline{-1,1}$ by Lemma 3.5. So $\mathbf{g'} = \overline{1,2}, \cdot\overline{1,2} \notin \mathcal{G}$, a contradiction. The next lemma tells us that although Γ is not surjective on $\mathcal{G}$, all configurations in $P(Y_{\mathcal{G}})$ are used when determining $\Gamma(\mathcal{G})$.

Lemma 3.6. *If* $\mathbf{y} \in P(Y_G)$ *then* $\exists \mathbf{g} \in \mathcal{G}$ *such that* $\Gamma(\mathbf{g}) = \mathbf{g} + \mathbf{y}$.

Proof. First find an element $\mathbf{y}^* \in \{\binom{2}{2}, \binom{-2}{2}, \binom{2}{-2}, \binom{-2}{-2}\}^{\mathbb{Z}}$ with $P(\mathbf{y}^*) = \mathbf{y}$. Then choose g_0 arbitrarily and follow the instructions given by $\mathbf{y}^*$ to specify g_{-1} and g_1. For example, if $y_0^* = \binom{2}{-2}$ then choose $g_1 \leq g_0 - 2$ and $g_{-1} \geq y_0 + 2$. Continue this process, using the gradients at locations -1 and 1 to specify g_{-2} and g_2, moving outwards from the central cell. The result follows by induction. $\square$

The proofs in this section for the SA Γ relied on the preservation of certain geographical features when restricted to the set of configurations $\mathcal{G}$. We now show that knowledge of the entries α, β, δ and λ noted in Table 2, is sufficient to extend these results.

$L\backslash R$	$-\infty$	-1	0	1	∞
$-\infty$	α				β
-1					
0					
1					
∞	δ				λ

Table 2: The positions in the local rule table that are used to create peak/valley preserving sand automata.

Theorem 3.7. *Let the spatial radius one sand automaton* Φ *have local rule table as in Table 2. Then* Φ *is peak preserving if and only if* $\alpha \geq \beta, \delta, \lambda$ *and* Φ *is valley preserving if and only if* $\lambda \leq \beta, \delta, \alpha$.

Proof. Assume that Φ is the SA described above. We use the notation $\mathbf{x}_{[n-1,n+1]} = (a, b, c)$ and $(\Phi(\mathbf{x}))_{[n-1,n+1]} = (a^*, b^*, c^*)$. We prove the statement concerning peak preservation and the proof of the statement for valleys is similar.

Suppose that $\alpha \geq \beta, \delta, \lambda$. Let (a, b, c) be a peak centred at b. Then $b^* = b + \alpha$. (see Table 2). Since α is the largest increase when restricted to $\mathcal{G}$, then

$a^*,\ c^* \leq a+\alpha,\, c+\alpha$. Thus peaks are mapped to peaks. Conversely suppose that Φ is peak preserving, and suppose that $\alpha < \beta$. Then in the configuration $\mathbf{g} = \ldots -2,\, 0 \cdot 2,\, 0 \ldots$, that has a peak at the central position and an up-slope at the -1-st position, ϕ does not map the central peak to another peak. This is due to the fact that $(\Phi(\mathbf{x}))_0 = 2+\alpha$ and $(\Phi(\mathbf{x}))_{-1} = 0+\beta$, where $\alpha < \beta$. Therefore $(\Phi(\mathbf{x}))_0 - (\Phi(\mathbf{x}))_{-1} > -2$. Similar configurations can be constructed when $\alpha < \delta$ or $\alpha < \lambda$ using a valley or a down-slope at the 1 position.

□

For the results of Proposition 3.5 to hold, which would also imply that $\mathcal{G}$ is Φ invariant, it must be both peak and valley preserving as well as being up-slope and down-slope preserving. This implies that we are assuming $\alpha \geq \beta, \delta \geq \lambda$. The next lemma demonstrates that peak and valley preservation implies up-slope/ down-slope preservation.

Proposition 3.8. *Let Φ be a spatial radius one sand automaton with local rule table as in Table 2. If Φ is both peak and valley preserving then it is also up-slope and down-slope preserving, so that $\mathcal{G}$ is Φ invariant and Φ is not surjective.*

Proof. Let Φ be both valley and peak preserving. Then $\alpha \geq \beta, \delta \geq \lambda$. We show that down-slopes are mapped to down-slopes, the up-slope case being similar. Suppose there is a down-slope centred at $x_n = b$, where $\mathbf{x}_{[n-1,n+1]} = (a,b,c)$. Then $(\Phi(\mathbf{x}))_n = b+\beta$. We either have a peak or a down-slope centred at a. Thus$(\Phi(\mathbf{x}))_{n-1} \geq a+\beta$. Similarly there is either a valley or a down-slope centred at c. Therefore $(\Phi(\mathbf{x}))_{n+1} \leq c+\beta$. Thus a down-slope centred at b is mapped to a down-slope centred at the image of b. □

Remark 3.9. Note that if Φ has spatial radius r and output radius s, we can generalise the definition of $\mathcal{G}$ to

$$\mathcal{G} := \{\mathbf{x} : |x_i - x_{i-1}| \geq s,\ i \in \mathbb{Z},\ 1 \leq k \leq r\}\,.$$

We can then define the same geographical features: for example a peak would have gradient $(L_r, \ldots, L_2, -s, -s, R_2, \ldots, R_r)$, with each L_i, R_i belonging to $\{-s, s\}$, and with similar definitions for a valley, an upslope and a downslope. Now let P, V, U, D represent the set of values returned by the SA Φ for positions coresponding to peaks, valleys, upslopes and downslopes respectively. So for example,

$$P := \{\phi(L_r, \ldots, L_2, -s, -s, R_2, \ldots, R_r) : L_i, R_i \in \{-s, s\}\}.$$

Then Φ is peak and valley preserving if and only if $\min P \geq \max\{U, D\} \geq \min\{U, D\} \geq \max V$. The proof of this is similar to the proof of Theorem 3.7.

However if $r > 1$, Proposition 3.8 does not generalise without considerable additional constraints on Φ's local rule, or modifying $\mathcal{G}$ so that points' gradients are only up or downslopes.

Remark 3.10. There are 105 choices of $(\alpha, \beta, \delta, \lambda)$ satisfying $\alpha \geq \beta, \delta \geq \lambda$. Thus the set of (spatial radius 1, output radius 2) sand automata that map $\mathcal{G}$ into itself make up $105/625 = 0.168\%$ of the space of all (spatial radius one, output radius 2) sand automata.

4 Equicontinuity and points of equicontinuity

In this section we investigate the equicontinuity of spatial radius one sand automata. In [4] and [10], the authors classify one dimensional sand automata as: either sensitive, or nonsensitive without an equicontinuity point, or non-equicontinuous with an equicontinuity point, or finally equicontinuous.

4.1 Vertical inducing points and equicontinuity

Given that a SA is topologically conjugate to a 2-dimensional CA, we use the following result:

Theorem 4.1 (Proposition 3.14, [10])**.** *Φ is equicontinuous if and only if $(\Phi^n)_n$ is ultimately periodic, if and only if $\forall \mathbf{x} \in X$, $(\Phi^n(\mathbf{x}))_n$ is eventually periodic.*

In order to classify Γ we introduce the following definitions. Let $n \in \mathbb{N}$. A configuration $\mathbf{x}$ is a *vertical inducing point* of order n for a SA Φ if $\Phi(\mathbf{x}) = \rho^n(\mathbf{x})$. For example, a fixed point for Φ is also a vertical inducing point of order 0. An example of a vertical inducing point for Γ is $\mathbf{x} = \overline{0,\ 1,\ 3,\ 1,\ 0} \cdot \overline{0,\ 1,\ 3,\ 1,\ 0}$. Then $I(\Pi(\mathbf{x})) = \overline{\binom{0}{1}, \binom{-1}{2}, \binom{-2}{-2}, \binom{2}{-1}, \binom{1}{0}} \cdot \overline{\binom{0}{1}, \binom{-1}{2}, \binom{-2}{-2}, \binom{2}{-1}, \binom{1}{0}}$, so that $\Gamma(\mathbf{x}) = \mathbf{x} + \overline{1} \cdot \overline{1} \neq \mathbf{x}$. In fact the SA Γ has an infinite number of vertical inducing points for each $-2 \leq n \leq 2$; we discuss this in the next section. The following lemma is straightforward:

Lemma 4.2. *If $\mathbf{x}$ is a vertical inducing point of order n, then $\Phi^m(\mathbf{x}) = \rho^{mn}(\mathbf{x})$, for each $m \in \mathbb{N}$. Also $e(\mathbf{x}) \in S_{\binom{1}{0}}$ satisfies $\Phi^m_{\binom{1}{0}}(\mathbf{x}) = \sigma_V^{mn}(\mathbf{x})$, for each $m \in \mathbb{N}$.*

Corollary 4.3. *If a sand automaton Φ admits a vertical inducing point of nonzero order, then $\Phi_{\binom{1}{0}}$ is not equicontinuous.*

Proof. By Lemma 4.2, if $\mathbf{x}$ is a vertical inducing point of order $n \neq 0$, then $\Phi^m(\mathbf{x}) = \rho^{mn}(\mathbf{x})$ for each m. Thus the only way that $(\Phi^n(\mathbf{x}))$ is eventually periodic is if $\mathbf{x}$ consists entirely of sinks or sources; however in this case $\mathbf{x}$ would be vertical inducing of order 0. Because equicontinuity is a topological property, the result follows. $\square$

In the next theorem we identify a class of sand automata that have vertical inducing points.

Theorem 4.4. *Let Φ be a spatial radius one sand automaton. If there exists a nonzero m such that m appears in each row, or each column, of Φ's local rule table, then Φ admits a vertical inducing point of nonzero order.*

Proof. Suppose that the nonzero value m occurs in every row of the local rule table for the SA Φ. Note that $I \circ \Pi(\mathbf{x}) = (\binom{L(i)}{R(i)})_{i\in\mathbb{Z}}$. Thus $L(i+1) = -R(i)$ for each i. Conversely, if $(\binom{L(i)}{R(i)})_{i\in\mathbb{Z}} \in I(\Pi(X))$ is such that $L(i+1) = -R(i)$ for each $i \in \mathbb{Z}$, then there is some $\mathbf{x} \in X$ such that $I(\Pi(\mathbf{x})) = (\binom{L(i)}{R(i)})_{i\in\mathbb{Z}} \in I(\Pi(X))$. To find a vertical inducing point then, it is sufficient to find a cycle $\binom{L(1)}{R(1)}, \ldots, \binom{L(k)}{R(k)}_k$ where $L(i+1) = -R(i),$ for $1 \le i < k$, $L(1) = -R(k)$ and such that $\phi\binom{L(i)}{R(i)} = m$ for each $1 \le i \le k$. First select any $\binom{L(1)}{R(1)}$ such that $\phi(L(1), R(1)) = m$. If $R(1) = -L(1)$ then we are done. If not, select $\binom{L(2)}{R(2)}$ such that $L(2) = -R(1)$ and $\phi(L(2), R(2)) = m$. We know this this can be done because the local rule table has the value m in every row. If $R(2) = -L(1)$ or $R(2) = -L(2)$ then we are done; if not continue this process until we find $\binom{L(1)}{R(1)} \ldots \binom{L(k)}{R(k)}$ such that for some $1 \le j \le k$, $L(j) = -R(k)$; we know this has to occur since the gradient pair entries come from a finite alphabet. The desired cycle is $\binom{L(j)}{R(j)} \ldots \binom{L(k)}{R(k)}$. The proof when there is a nonzero m in every column is similar except that newly selected gradient pairs will come before the current gradient pairs. $\square$

Recall that an $n \times n$ *latin square* is a table filled using an alphabet $\mathcal{A}$ of size n, where each row and each column has exactly one occurrence of each member of $\mathcal{A}$. Theorem 4.4 and Corollary 4.3 then imply the following:

Corollary 4.5. *If a spatial radius one sand automaton Φ has a Latin square local rule table, then Φ is not equicontinuous.*

Remark 4.6. If we consider sand automata of higher spatial radius r, the equivalent of Theorem 4.4 is not so elegant to formulate. A first obstacle is that describing the constraints that a sequence of gradient tuples have to satisfy in order to have spatial periodicity k depends on k. Second, translating these constraints to a constraint on the rule table is not clear.

Let Φ be a spatial radius 1 linear SA. Then ϕ, the local rule for Φ, is a homomorphism on a cyclic group with group operation addition $\oplus$. In particular this implies that $\phi(L, R) = \alpha L \oplus \beta R$, where α and $\beta \in \mathbb{Z}_5$. This implies that Φ's local rule table is either a latin square, or has a nonzero element on the anti-diagonal.

Proposition 4.7. *Let Φ be a spatial radius 1 linear sand automaton. Then Φ admits a vertical inducing point, and so is not equicontinuous.*

Proof. It is sufficient to show that there is a word $\binom{L(0)}{R(0)} \dots \binom{L(n)}{R(n)}$ in $(\mathbb{Z}_5^2)^+$ such that $L(i+1) = -R(i)$ for $i = 1, \dots k-1$, $R(n) = -L(0)$ and there exists a nonzero m such that for each i, $\gamma(L(i), R(i)) = m$. Suppose first that there is a nonzero value on the anti-diagonal of the local rule table for Φ. Then there is a gradient pair $\binom{a}{-a}$ that returns a nonzero value m under ϕ. In this case, the word $\binom{a}{-a}$ is the desired one. If all anti-diagonal rule entries are 0, then for each $L \in \mathbb{Z}_5$, $\alpha L \oplus \beta(-L) = 0$. This implies that $\alpha L = \beta L$, so that $\alpha = \beta$. Thus $\phi = \alpha L \oplus \alpha R = \alpha(L \oplus R)$. But the local rule $L \oplus R$ corresponds to the linear SA Γ which has a Latin square as a local rule table. Therefore $\alpha(L \oplus R)$'s rule table is a permutation of this local rule table and so is also a Latin square. Corollary 4.5 now yields the result. $\square$

Note that linearity is not used in the case where there is a nonzero value on the anti-diagonal of the local rule table.

There are 5^{25} spatial radius one sand automata, and since there are 5^{20} local rule tables with only zero entries on the anti-diagonal, at least 99.968% of the total number of spatial radius 1 sand automata have a vertical inducing point, and so are not equicontinuous.

Next we show that while Γ is not equicontinuous, it does have equicontinuity points, putting it in Class (3) of the classification in [10]. We say that a word $\mathbf{w}$ is *blocking* for a SA Φ if there $\exists\, k, s \in \mathbb{N}$ such that $0 \le k + s \le |\mathbf{w}|$ and such that whenever $\mathbf{x}$ and $\mathbf{y} \in [\mathbf{w}]_i$, then $(\Phi^n(\mathbf{x}))_{[i+k, i+|\mathbf{w}|-s]} = (\Phi^n(\mathbf{y}))_{[i+k, i+|\mathbf{w}|-s]}$ for all natural n.

Proposition 4.8. *Let* $\mathbf{w} = 0\,3\,2\,3\,0$. *Then* $\mathbf{w}$ *is blocking for the sand automaton* Γ*, so that* Γ *has equicontinuity points.*

Proof. First note that the first statement will imply the second. Note also that for any integer i, if $[\mathbf{w}]_i$ is the cylinder set of points where we see the word $\mathbf{w}$ at position i, then $I \circ \Pi([\mathbf{w}]_i) = \binom{L_0}{2}, \binom{-2}{-1}, \binom{1}{1}, \binom{-1}{-2}, \binom{2}{R_4}$. This block satisfies $\gamma(\binom{-2}{-1}, \binom{1}{1}, \binom{-1}{-2}) = (2, 2, 2)$. Suppose that $\mathbf{x}, \mathbf{y} \in [\mathbf{w}]_{-2}$. Then $(\Gamma(\mathbf{x}))_{[-2,2]} = (x'_{-2}, 5, 4, 5, x'_2)$ and $(\Gamma(\mathbf{y}))_{[-2,2]} = (y'_{-2}, 5, 4, 5, y'_2)$, where $x'_i, y'_i \le 2$, for $i \in \{-2, 2\}$. Under the action of Γ the central three cells increase by 2 and the right most and left most cells can increase by at most 2. This implies that $(I \circ \Pi(\Gamma(\mathbf{x})))_{[-1,1]}$ and $(I \circ \Pi(\Gamma(\mathbf{y})))_{[-1,1]}$ equal $(I \circ \Pi(\mathbf{x}))_{[-1,1]}$, and Γ thus adds 2 to these three positions. An inductive argument completes the proof. $\square$

There are 24 nontrivial homomorphisms $\phi^* : \mathbb{Z}_5^2 \to \mathbb{Z}_5$. If $\phi^*(L, R) = \alpha L \oplus \beta R$, we use the notation (α, β) to represent ϕ^*. Eight of these maps have blocking words of the type described in Proposition 4.8. These are (a, a), where $a \neq 0$, and $(-1, 2)$, $(1, -2)$, $(2, -1)$, $(-2, 1)$.

4.2 Local-rule-constant configurations

The concept of a vertical inducing point is a special case of a more general type of configuration $\mathbf{x}$ where $(\Phi^n(\mathbf{x}))_n$ is easily computable. Let us say that the SA Φ is *local-rule-constant* at $\mathbf{x}$ (or $\mathbf{x}$ is a Φ local-rule constant point) if for some $\mathbf{y} \in X$, $\Phi^n(\mathbf{x}) = \mathbf{x} + n\mathbf{y}$ for each $n \in \mathbb{N}$. For example, if $\mathbf{x}$ is a vertical inducing point, then it is a Φ local-rule-constant configuration, with $\mathbf{y}$ constant. The existence of local-rule-constant points for a SA will imply the same non-equicontinuity results that vertical inducing points do, and as for vertical inducing points, the set of local-rule-constant points is a closed Φ-invariant set. As the set of such points will generally have empty interior, it will not lead directly to a proof that Φ is not transitive. However numerical experiments have suggested to us that this set generates some kind of an attractor for Γ, in a way that we make precise in Conjecture 4.10. In what follows we describe a family of local-rule-constant points for Γ.

Let $w = w_1 \dots w_k = \binom{L(1)}{R(1)} \dots \binom{L(k)}{R(k)} \in (\mathbb{Z}_5^2)^+$ satisfy $R(i) = -L(i+1)$ for $i = 1, \dots k-1$, and $\gamma_*(w_i)$ is constant $i = 1, \dots k$; then we say that w is a *cycle segment*. In Table 3, we list some cycle segments for Γ. Consider the directed graph $\mathcal{H}$ (see Figure 4) whose vertices are the cycle segments for Γ listed in Table 3, and such that there is an edge from vertex V to vertex V' if and only if the following are satisfied:

1. If the cycle segment corresponding to V ends with a gradient pair $\binom{*}{a}$ then the cycle segment corresponding to V' starts with a gradient pair $\binom{-a}{\star}$.

2. If the cycle segment corresponding to V ends with a gradient pair $\binom{*}{2}$ and has order j, then the order of the cycle segment corresponding to V' must have order at least j.

3. If the cycle segment corresponding to V ends with a gradient pair $\binom{*}{-2}$ and has order j, then the cycle segment corresponding to V' must have order at most j.

Note that vertices K, L and M in $\mathcal{H}$ are isolated. Define $\mathcal{G}^*$ to be the set of all configurations $\mathbf{x}$ in X such that $I(\Pi(\mathbf{x}))$ corresponds to an infinite path in $\mathcal{H}$.

Theorem 4.9. *If* $\mathbf{x} \in \mathcal{G}^*$*, then* $\mathbf{x}$ *is* Γ *local-rule-constant.*

Proof. Choose an $\mathbf{x} \in X$ such that $I \circ \Pi(\mathbf{x})$ is represented by an infinite path $\mathbf{V} = \dots V_{-2}\, V_{-1} \cdot V_0\, V_1\, V_2 \dots \in \mathcal{H}$. Let $\mathbf{y}$ be the point in X obtained by applying γ_* to the representative in $I(\Pi(X))$ of $\mathbf{V}$. We claim that $\Gamma^n(\mathbf{x}) = \mathbf{x} + n\mathbf{y}$. Suppose that V_j corresponds to the cycle segment $(I(\Pi(\mathbf{x})))_{i_{j-1}+1}, \dots (I(\Pi(\mathbf{x})))_{i_j}$, and it ends with a gradient pair of the form $\binom{**}{2}$. Then in $\mathbf{x}$, $x_{i_j+1} - x_{i_j} \geq 2$. Geographically speaking there is a "steep hill" to the right of x_{i_j}. By condition (2), V_i can only be followed by an V_{i+1} whose corresponding cycle segment has order at least that of V_i's. Therefore in $\Gamma(\mathbf{x})$ the "steep hill", if it changes, can only get steeper. Thus $I(\Pi(\Gamma(\mathbf{x})))_{i_j} = I(\Pi(\mathbf{x}))_{i_j}$. Similarly if V_i ends with a gradient pair of the form $\binom{**}{-2}$, Condition (iii) guarantees that if $I(\Pi(\mathbf{x}))_{i_j} = -2$ then $I(\Pi(\Gamma(\mathbf{x})))_{i_j} = -2$. This fact is true for all j. Finally if $\mathbf{V}$ corresponds to the infinite loop at K, L or M, then $I(\Pi(\mathbf{x}))$ is constant and $\mathbf{y} = \overline{0} \cdot \overline{0}$, so that $\Gamma(\mathbf{x}) = \mathbf{x}$ in which case Γ is (trivially) local-rule-constant. Thus $I \circ \Pi(\Gamma(\mathbf{x}))$ is also represented by $\mathbf{V}$. By induction it follows that $\mathbf{x}$ is Γ-local-rule-constant. $\square$

For example, suppose that we want a configuration $\mathbf{x}$ that under the action of Γ we have $\Gamma^n(\mathbf{x}) = \mathbf{x} + n\mathbf{y}$ where $\mathbf{y} = \overline{2}, \cdot 1, 1, 1, 1, \overline{2}$. Then to build such a configuration using the cycles from the vertical inducing points we can let $I \circ \Pi(\mathbf{x}) = \overline{\binom{2}{0}, \binom{0}{2}, \binom{-2}{-1}, \binom{1}{1}, \binom{-1}{-2}} \cdot \binom{2}{-1}, \binom{1}{0}, \binom{0}{1}, \binom{-1}{2}, \overline{\binom{-2}{-1}, \binom{1}{1}, \binom{-1}{-2}, \binom{2}{0}, \binom{0}{2}}$. This point leads to an infinite number of configurations in the preimage set $(I \circ \Pi(\mathbf{x}))^{-1}$. One such point is $\mathbf{x} = \overline{2, 2, 4, 3, 4}, \cdot 2, 1, 1, 2, \overline{5, 4, 5, 3, 3}$.

Label	Cycle segment	Order
A	$\binom{2}{-1}\binom{1}{0}\binom{0}{1}\binom{-1}{2}$	1
B	$\binom{-2}{-2}$	1
C	$\binom{2}{0}\binom{0}{2}$	2
D	$\binom{-2}{-1}\binom{1}{1}\binom{-1}{-2}$	2
E	$\binom{2}{-2}$	0
F	$\binom{-2}{2}$	0
G	$\binom{-2}{0}\binom{0}{-2}$	-2
H	$\binom{2}{1}\binom{-1}{-1}\binom{1}{2}$	-2
I	$\binom{-2}{1}\binom{-1}{0}\binom{0}{-1}\binom{1}{-2}$	-1
J	$\binom{2}{2}$	-1
K	$\binom{-1}{1}$	0
L	$\binom{1}{-1}$	0
M	$\binom{0}{0}$	0

Table 3: All of the cycle segments that can be used to create local-rule-constant points.

Note that the set $\mathcal{G}$ defined in Section 3.1 is contained in $\mathcal{G}^*$, and that $\mathcal{G}^*$ is closed and Γ-invariant. We have conducted simulations where the spacetime diagrams of several initial configurations are generated, and empirically what seems to be happening is that the iterates of the initial configuration converge "almost everywhere" to a configuration in $\mathcal{G}^*$. In particular, define the words $O = \binom{-2}{-2}\binom{2}{0}\binom{0}{2}\binom{-2}{-2}$, $P = \binom{-2}{-1}\binom{1}{0}\binom{0}{1}\binom{-1}{-2}$ and $Q = \binom{2}{2}\binom{-2}{0}\binom{0}{-2}\binom{2}{2}$, $R = \binom{2}{1}\binom{-1}{0}\binom{0}{-1}\binom{1}{2}$.

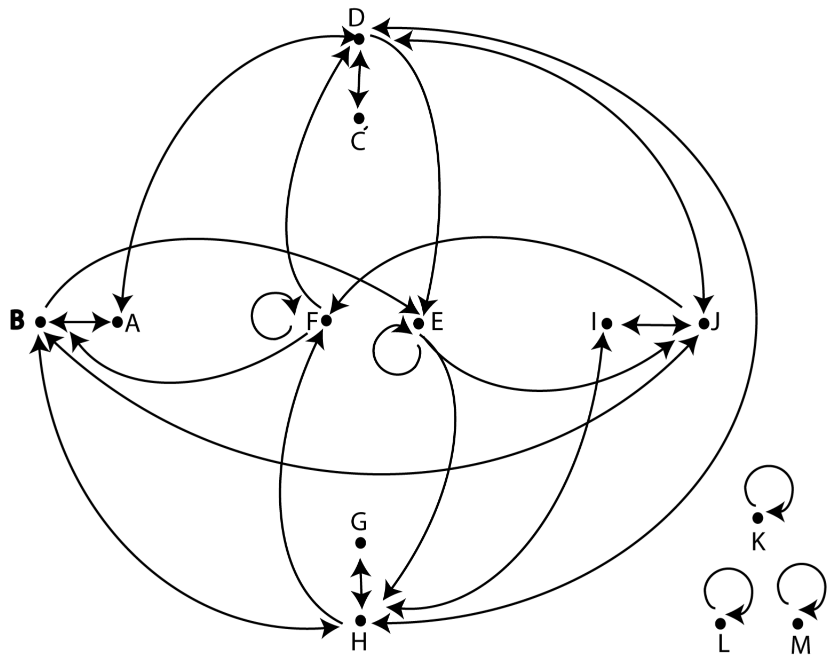

Figure 4: The graph $\mathcal{H}$. See Table 3 for definitions of the vertices.

These sets of words are *2-periodic* in the sense that if $I \circ \Pi(\Phi^n(\mathbf{x}))_{[i,i+3]} = O$ then $I \circ \Pi(\Phi^{n+1}(\mathbf{x}))_{[i,i+3]} = P$ and if $I \circ \Pi(\Phi^n(\mathbf{x}))_{[i,i+3]} = Q$ then $I \circ \Pi(\Phi^{n+1}(\mathbf{y}))_{[i,i+3]} = R$. If we include these in a new graph $\mathcal{H}'$ (see Figure 5), then this seems to describes the asymptotic behaviour of Φ more accurately. This leads to the following conjecture. Similar to our definition of $\mathcal{G}^*$, let

$$\mathcal{G}' := \{\mathbf{x} : I(\Pi(\mathbf{x})) \text{ corresponds to an infinite path in } \mathcal{H}'\}.$$

Conjecture 4.10. *The set $\mathcal{G}'$ is an attractor for Γ, in that if $\mathbf{x} \in X$, then* $\lim_{n\to\infty} d(\Gamma^n(\mathbf{x}), \mathcal{G}') = 0$.

5 Conclusion

In this article we have studied a specific family of one-dimensional sand automata Φ, and investigated the topological dynamical properties of the corresponding family of two-dimensional cellular automata $\Phi_{\binom{1}{0}}$. The local rules for these sand automata have a partly algebraic structure, which imply that some (generally undecidable) results, such as surjectivity and equicontinuity, are computable. Despite the chaoticity of many linear cellular automata our investigations show that some of these sand automata do not have chaotic dynamics, a question raised in [4]. We have defined the concepts of vertical inducing points, and more generally, local-rule-constant points and have shown that a large class of spatial radius one sand automata have these configurations, which imply that they are not equicontinuous. We work with a specific SA and show that it does have equicontinuity points. We conjecture the existence of an attractor set. To what larger family of sand automata will these results extend?

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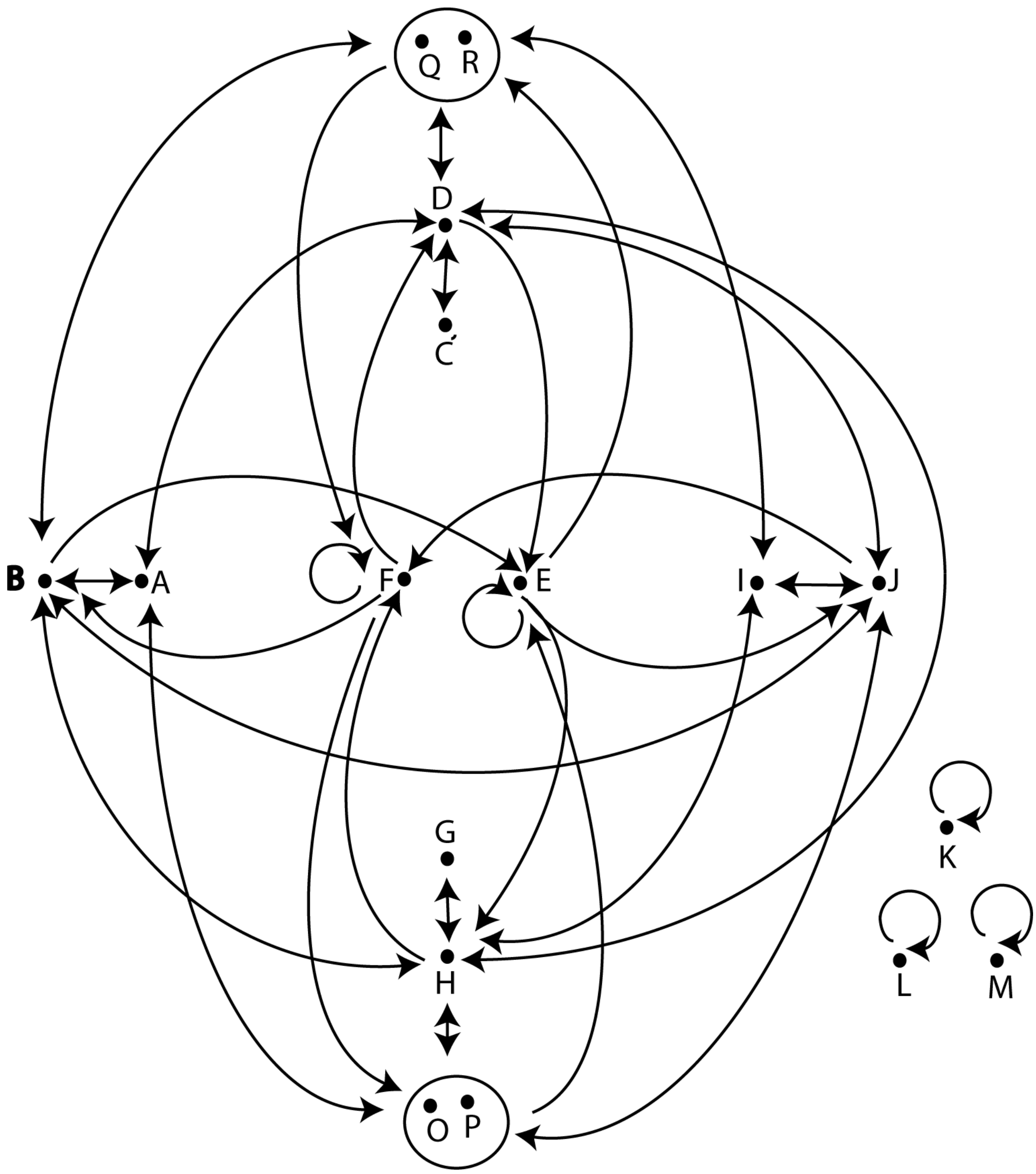

Figure 5: A potential attractor set for Γ.

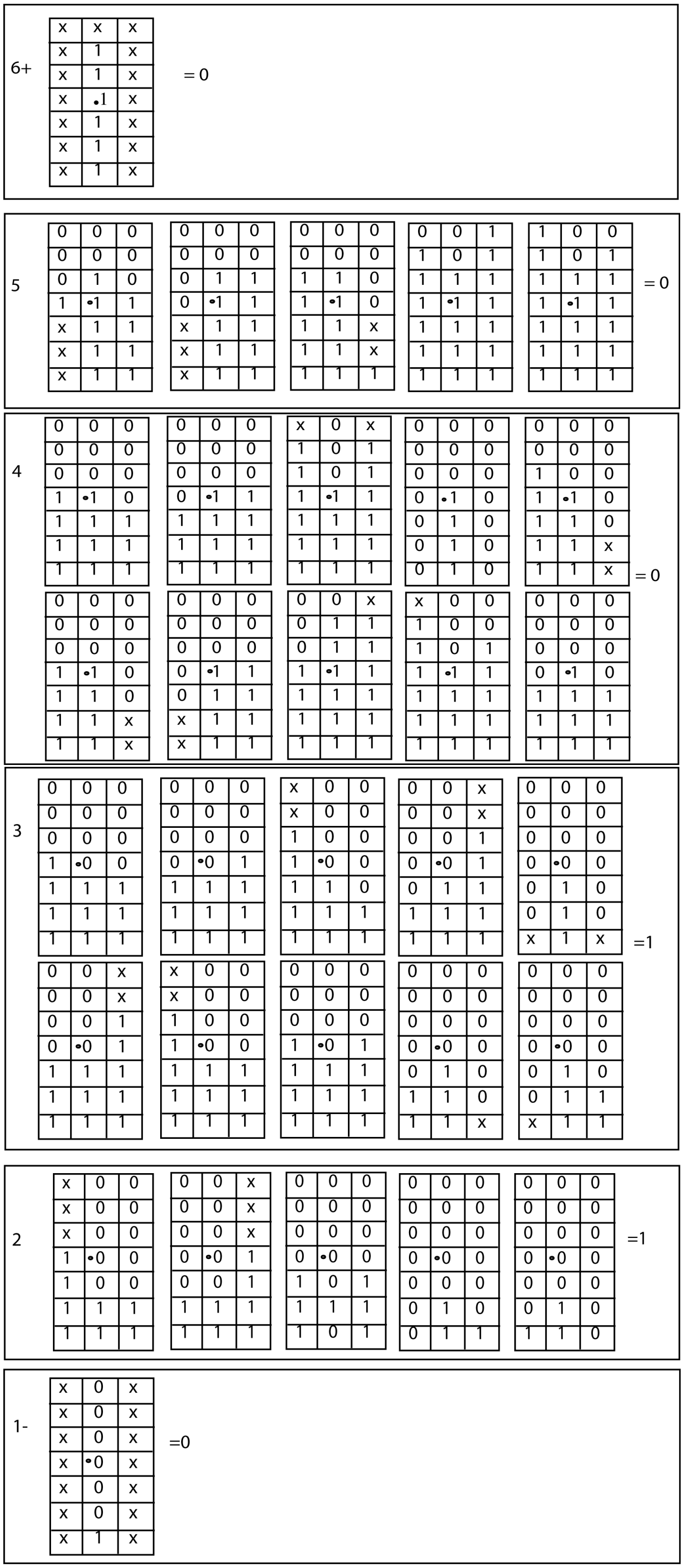

Figure 6: The local rule table for $\Gamma_{\binom{1}{0}}$. The number on the left of each rectangle represents the number of ones in the reference column. An 'x' represents a cell that can have any value. The $\cdot$ represents the central cell. Only configurations where the central cell changes are listed.