Exploring learners’ understanding of integration using structured interviews and construction tasks

Shafia Abdul-Rahman, MA (London)

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Abstract

This study explores learners’ understanding of integration. Understanding is a complex process, extending beyond learners’ ability to reproduce mathematical techniques in familiar, slightly modified situations. I see understanding as not only the ability to use knowledge to solve routine problems correctly but as the ability to act creatively in unfamiliar situations. Previous research into understanding of integration has focused on learners’ displayed behaviour in problem situations, without probing beneath the surface of their understanding. As such, the root of the difficulties in understanding integration remains obscure and learners continue to face problems. I would like my research to inform my future teaching.

In this study, I explore how the ‘Structure of a Topic’ framework, which focuses on awareness of mathematical ideas to inform behaviour, can be used as a research tool to reveal how understanding is structured. By prompting learners to construct several mathematical examples meeting specified constraints, what they choose to change reveals dimensions, depth and quality of their awareness.

Data was collected in the form of semi-structured interviews and construction tasks which were designed based on the constituents of the framework. Learners ranging from A-level students to graduates and, a spectrum of first year undergraduates in between, were interviewed and invited to construct relevant mathematical objects. Responses in the interview were compared with responses in the construction tasks to say something about the nature of their awareness and thus, understanding. The thesis provides a description of different dynamics and depths of awareness which contribute to the different forms of understanding displayed by learners with different backgrounds. The findings contribute to enhancing our knowledge of students’ understanding of integration and indeed any topic through the development of a theoretical framework that can assist exploration of students’ understanding.
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Chapter 1

Thesis Overview

1.0 Introduction

This study is concerned with the way in which learners come to understand mathematical concepts and how their understanding is structured. The calculus topic of integration is the focus of this research, although the applicability of the theoretical framework developed can be extended to the teaching and learning of a much wider range of mathematical topics. In this chapter, I outline what my interest is, how I became interested in it and why the study is important. In the section that follows, I consider things that I noticed in the course of teaching integration and my personal perspective on the difficulty of understanding integration as a topic that has led me to carry out the study.

1.1 The aims of my study

Seeing blank faces on my secondary students after spending a considerable amount of time on a topic, I often asked them this question out of concern and curiosity: ‘Which part of the topic do you not understand?’ and to my surprise the answer would often be: ‘All of it’. Though it provoked immediate anger and despair, many times I wondered what the students were missing out and why they did not see the links between ideas that were very obvious, at least to me, to make sense of the topic being taught.

My interest in the topic of integration began when I realised that my students appeared overwhelmed by the sight of the integral sign $\int$ and seemed confused by the number of rules to remember when solving integration problems. I also noticed that they were often
unsure as to which rule to use and were often caught out by simple rearrangement of facts in the problem, especially in the exams. My experience of teaching suggested to me that students often engaged in a ‘ritualised’ procedure of looking for the right technique to use to find the solution by inspecting the form of the integral to try and match it with the one they had access to in notes or texts. Many times they had echoed their concerns that there were too many rules to remember and that they often get confused between them. They had trouble remembering which rule/technique to use, and even if they did remember, they struggled to choose the appropriate one. The struggle became even more critical when the problem situation was slightly modified or was generalised beyond the familiar.

This situation may have resulted from the mathematics in school, which, as learners get older, increasingly places emphasis on skill acquisition for the purpose of passing exams and downplays the importance of understanding underlying principles. Perhaps due to the proximity of teaching integration and differentiation, many students did not even seem to know whether a problem or problem situation involved integration or differentiation.

These students often struggled not only to understand the problem but also to apply the correct technique. This struggle might result in students integrating (or differentiating) without knowing what they were doing or why, with little knowledge of what to expect and how to make sense of the answer. It is probable that this situation occurs when learners engage in rote learning of mathematics, although some researchers (Hiebert and Carpenter, 1992) disagree with the claim that rote learning does not eventually contribute to understanding. Indeed, in Confucian Culture educational systems, rote learning precedes but leads to understanding. Frustrations and failure in relating learned techniques and procedures into slightly generalised situations are likely to set even good students adrift. I was deeply concerned with the state of affairs faced by my students, realising that memorising the rules would only make them prone to forgetting them in
stressful situations such as examinations. I was grateful when an opportunity arose to study this in detail and this thesis is the result.

1.2 Background to the study

Having taught additional mathematics (equivalent to GCSE mathematics in the UK) to 16 – 17 year-olds in Malaysian secondary schools for many years, I struggled to understand the difficulties faced by students with integration. With hindsight, I noticed a number of principal difficulties associated with the learning of integration:

1. differentiation is a forward process, and the difficulties faced by students in this topic are not as complicated as those in the reverse or backward process of integration. This is because forward calculations are normally algorithmic and have single answers, thus they are easier to do than backward calculations which typically have multiple solutions and involve some element of creativity and uncertainty.

2. the sign \( \int \) seems to trigger learners to think in a certain direction by focusing on certain aspects of the topic, often anti-differentiation alone and not area, which may not be what the teacher had intended.

3. the dual nature of integration: it is both the inverse process of differentiation, where one can move from the knowledge of rate of change of something to knowledge of its state at some time in the future, and a tool for calculation of area, volume, length, centre of gravity, moments and probabilities.

My observations of students’ difficulties with integration led me to look at the curricular context of the topic and the variations in students’ approach to learning mathematical ideas.
1.2.1 Curricular context of integration in Malaysian schools

In Malaysian schools, integration is taught right after differentiation in Form 4 (equivalent to Year 10 in the UK), which seems logical since integration is the opposite of differentiation. However, in 2002 the topic was moved to the syllabus of Form 5 (UK Year 11). The topic is introduced both as the limiting process of summation and as anti-differentiation. The traditional approach to teaching integration had been to pose a question to students on how to find the area underneath a curve as opposed to area under a line graph. This inquiry-based approach was intended to encourage students to think creatively by drawing from their past, everyday experience and existing knowledge. Students were then presented with the idea of the limiting process of summation and eventually led to what appeared to be obvious to the teacher: the relationship between the process of integration, which is just a way of performing ‘infinite’ sums, and the process of differentiation. At this point, the history of calculus integration (and differentiation), in terms of people who were involved in the work and how it was conceived, was discussed. The ideal placing of the topic in the curriculum was intended to provide a sense of connection in relation to the place of previously learned topics and/or other topics.

The technique of integration by first principle proved to be very unpopular due to the introduction of too many parameters. It is briefly discussed before other techniques were taught, mainly the reverse process of differentiation, integration of polynomials, integration by substitution and definite integration. The techniques were introduced by ‘confronting’ students with fresh, textbook situations in which they could not use previously learned techniques. Emphasis was placed on identifying the kinds of problems that would require a specific technique. Uses in real-life situations were taught in applications of integration at the end of the topic and were usually confined to textbook problems.
One of the objectives of the curriculum for the Malaysian Certificate of Education examination states that students are expected to develop conceptual understanding of mathematical topics, with a special focus on problem solving, to develop the ability to think critically and creatively and to reason logically. The curriculum also highlights the need to acquire skills necessary for effective communication of mathematical ideas. For the topic of integration, the syllabus states that by the end of the lessons on integration, students are expected to:

- understand integration as the reverse process to differentiation,
- integrate functions in the form of $ax^n$ (n integer but $n \neq -1$); integrate polynomials,
- integrate using substitution method,
- integrate definite integrals,
- understand integration as a summation process; application to area and volume.

(Curriculum Development Centre, Ministry of Education, Malaysia, 2006, p. 5)

Much emphasis is placed on mastering the different techniques of integration and applying knowledge of integration to solve problems involving calculation of area and volume. Although the emphasis is not mentioned, learners’ attention is drawn (albeit deliberately or otherwise) to techniques. As such, they learn to associate mathematical topics with techniques and, consequently, become equipped to apply the techniques in well-formulated textbook problems. Their understanding is assessed by how well the students ‘reproduce’ the knowledge in a very slightly modified condition. However, when faced with questions that are more significantly modified, they could become incapacitated.
1.2.2 Possible variations in approach to learning mathematical ideas

One of the implicit aims of the teacher is to provoke learners to use their powers to make sense of, understand and more importantly appreciate the full sense of mathematical ideas. Understanding is necessary in order to retain knowledge and apply it to new situations. However, in many cases, the learning of mathematical ideas seems to be highly cognitively situated, which according to Lave (1988), normally occurs as a function of the activity, context and culture in which it occurs. Such conditions may explain why learners do not seem to be able to cope with situations beyond the familiar. Learners' behaviour in conditions beyond these may be a good indication of how flexible is the 'situatedness' of their understanding.

Understanding of mathematical ideas must not be confined to application of learned mathematical skills and techniques to familiar problem situations. I realise now that my undergraduate Chemistry lecturer's reminder that he would not test us on what we knew but on what we did not know was a call for generalising. Following Mason (2002), I see understanding as not only the ability to use situated knowledge to solve routine problems correctly but more importantly, as the ability to extend that situatedness appropriately and efficiently into unfamiliar situations. Dealing effectively with novel situations is likely to depend on, among other things, which aspect of the concept/idea becomes the focus of learners' attention, namely what they regard as important. Extending knowledge into fresh situations not only involves acting on previously trained behaviour in the use of techniques and special mathematical terms/symbols, but also requires awareness of the idea at a deeper level on the part of the learner.

Having taught secondary school students and student teachers for many years, I noticed that students arrive at understanding of mathematical concepts/ideas through different kinds of engagement with them. The kinds of understanding that result depend on what is
of import to the learners. Having undergone engineering training myself, I realised that
the mathematics conceived by engineers and scientists is not as rigorous and detailed as
that considered by mathematicians. Those who study mathematics as an applied science,
for example physics and engineering, appear to be more concerned with applications of
mathematics in real-life situations in which mathematics is regarded only as a tool.
Engineers seem to think in terms of numbers or at least ‘answers’ from the referential
aspect of their experience, rather than in terms of abstract concepts. Therefore, students
who have difficulties with analytic calculations can nevertheless become adept with
practical applications. Engineering students appear to use numbers (i.e. in examples) as
the principal means in moving from a surface approach to a holistic approach to
understanding.

In many institutions, engineering students take courses in mathematics from the
Mathematics faculty and learn applications of the concepts in the Engineering faculty.
Although mathematics is indispensable for the engineering and physical science
communities because the science that explains the laws of nature is expressed
mathematically, the engineering communities still believe that they are expected to
acquire as much abstract mathematics as that expected of professional mathematicians.
Practical engineers cannot avoid the in-depth study of mathematics before applying it to
engineering problems. Finding an appropriate balance between practical applications of
mathematics and in-depth understanding is difficult. Thus the depth of mathematics to be
learned by engineers and physical scientists is limited to understanding mathematics as a
language for describing physical and chemical laws. As a result, the learning of
mathematical concepts for physical science and engineering students is mainly focused
on applications and on making sense of the outcome rather than being dominated either
by mastering techniques or underlying principles. Many argue that these communities
rarely use any mathematics learnt in college in the workplace. They want solutions to practical problems. They are brought up through school to use techniques, i.e. what is testable, but lose contact as their education stresses applications and contexts. With the advent of number-crunching computer software, the need to understand the principles underlying mathematical concepts appears to have become less necessary for getting answers, however it is important to understand what the machine is doing when solving problems in a novel setting.

The students in school, on the other hand, may have different ways of understanding when compared to undergraduates studying Mathematics or Engineering at university. At school, emphasis is likely to be placed on students’ correct use of techniques to solve mathematics problems, probably due to pressures to pass examinations. This is because questions on the examinations are likely to be only a slight modification of what the students have been exposed to in the classroom. In fact, students learn ‘coping skills’ and things they have invented for themselves that will get them through the exams (Smith and Moore, 1991). Indeed, a ‘good’ teacher or text is likely to be seen as a source of exam-like problems and solutions. Therefore attention is likely to be focused on techniques when solving problems, which can test students’ understanding when faced with situations in which no readily available techniques are appropriate.

I imagined that for students who study mathematics as a subject at university, holistic approaches are likely to be related to their concentration on formulae and superficial approaches at the stage of working with numbers (i.e. examples). These students are exposed to different representations of mathematical topics and are assumed to know the meaning of each parameter in mathematics formulae. They are expected to understand the meaning and contextual use of the ideas. This suggests that mathematics students are taught to be aware that numbers are illustrative of formulae and therefore are variable.
This variability of dimensions constitutes the “dimensions-of-[possible]-variation” (Marton and Booth, 1997), a construct developed in Chapter 3.

It seems therefore, that although all of these students study integration, they may have different intentions and different routes to understanding. In this study, the psychological aspect of learning integration is considered rather than focusing on the sociological ones. In particular, methods for revealing learners’ awareness of properties of mathematical examples as the basis of understanding and appreciation of mathematical ideas are developed throughout the thesis. This idea will be discussed in further detail in Chapter 2.

The objective of this thesis is to develop a theoretical framework based on a traditional triad developed by Mason (2002) using Eastern ideas, and expounded in Griffin et al., (1989) and in Mason and Johnston-Wilder (2004). The framework was initially referred to as ‘Preparing to Teach a Topic’ and more recently known as the ‘Structure of a Topic’. This thesis investigates the possibility of using the framework effectively to study the nature of learners’ understanding of integration and to suggest ways of thinking that would reveal awareness and thus, enhance learning and understanding.

Previous research (see Orton, 1983; Hong and Thomas, 1997; Rasslan and Tall, 2002) into learners’ conceptions of integration has pointed out that learners have significant difficulties. However, I noticed that these pieces of research do not seem to probe beneath the surface to reveal the nature of their understanding but only report on their behaviour. As such, the root of the difficulties in understanding cannot be identified and learners will continue to face problems. In this study, I explore how the framework can be used to reveal how the understanding of mathematical phenomena of different groups of students is structured. In particular, I seek to reveal the nature of students’ understanding of integration in terms of what they are aware of, how their behaviour is directed by this awareness, and how this is influenced by their motivation to learn. Because examples
form an essential part of instructions in integration, and indeed in mathematics, I intended to use example-construction as a way of revealing aspects of learners' awareness. The extent to which learners appreciate generic features in mathematical examples can be revealed through getting them to construct examples. My conjecture is that, if learners' attention is shifted to structural and variable aspects of the examples rather than focusing on details of the particular example, it is likely that they (learners) will appreciate mathematical examples more fully and thus have richer understanding of mathematical ideas.

Since this study is not a teaching experiment, no deliberate attempt was made to intervene with the way learners learn. My goal was to probe and reveal learners' awareness of integration through example-construction. I conjectured that the example-construction task in itself could provide insight into learners' understanding and render richer appreciation of the topic. The nature of their engagement in fresh, novel situations may provide insight into their sense of generality and the constituents which comprise their understanding of integration.

I have interviewed A-level students who are just getting going with integration as a subject and PGCE students who have graduated, and, to look at a spectrum in between, I have interviewed first-year undergraduates studying engineering and mathematics in order to explore their conceptions of integration. I was interested in how their awareness, as revealed through the construction tasks, differed from that exposed in the interview. I interviewed them first to find out about their responses to questions based on the ‘Structure of a Topic’ framework and then gave them a number of example-construction tasks. Early investigations suggested that there is a tendency for learners to focus their attention on certain limited aspects and to overlook others. Particularly, techniques of integration dominated their attention while images, associations and connections were
less accessible, even though mathematicians and teachers consider these important. Some learners who were sensitised to change in their perception articulated that change and expressed their appreciation while others focused on the details of the tasks themselves.

In this research, a phenomenographic approach was adopted to study the range and variety of students’ understanding integration. I decided to employ this approach mainly because I believe learners arrive at mathematical understanding in a way uniquely characterised by the context of and purpose for learning mathematics, and the way they experience it, which influences how they conceive learning.

My aims in this study are to:

• expose the range of responses to be expected from learners from different backgrounds in relation to their understanding of integration using the ‘Structure of a Topic’ framework,

• reveal learners’ sense of generality through example-construction,

• use learners’ awareness of “dimensions-of-(possible)-variation” (Marton and Booth, 1997; Watson and Mason, 2005) to say something about their awareness and understanding of integration,

• suggest ways of teaching integration more effectively by explicitly directing learners’ attention to “dimensions-of-possible-variation” in the examples used and by encouraging them to articulate them clearly.

The specific research questions that I wanted to address are:

• Can the ‘Structure of a Topic’ framework be used effectively as the basis for probing students’ understanding of a topic (in this case, integration)? If so, what particular aspects does it reveal?
• What is the same and what is different about using construction tasks and probes based on the ‘Structure of a Topic’ framework for revealing students’ understanding of integration?

• What similarities and differences are revealed about engineering students, mathematics students and PGCE Mathematics students by using the two forms of probes?

It is my desire that what I find in this research will influence my practice as an educator in the future and perhaps inform the practices of others.

1.3 The structure of the thesis

In the next chapter, I review the relevant literature related to research in learners’ understanding of integration, including pedagogical and psychological difficulties. As the major concern of the thesis is enhancing learners’ understanding and appreciation, the literature review focused on the nature of learners’ understanding and how revealing and educating learners’ awareness can inform understanding and appreciation.

Chapter 3 considers the methodological choices that were available and justifies the methods employed in the study. A brief summary of the preliminary investigations which were conducted before the main study are reported in Chapter 4, which provided useful data that informed the development of analysis of the pilot study. Chapter 4 also provides accounts of the pilot study, which includes a description of the interview tasks, a description of the selection of participants and a brief account of the process of analysis. The details of questions to be used in the main study, which were tested in the pilot study, and the outcome of the pilot study is also presented. The chapter also presents an account of the major themes that emerged from the pilot study illustrated by extracts from the interview transcripts. Chapter 5 provides a more detailed account of the idea of revealing
aspects of awareness through example-construction as it emerged from the pilot study interviews. The chapter also considers issues for presentation of data analysis for the main study as informed by the preliminary and pilot studies.

Revealing aspects of awareness of students from different backgrounds is detailed in Chapters 6, 7a, 7b and 8, which consist of data from PGCE mathematics students, average mathematics students (Applied Math), strong mathematics students (Pure Maths) and engineering students, respectively. The chapters comprise data which gives qualitative evidence addressing my conjecture that by becoming aware of features of a concept not previously at the focus of their attention, some learners in fact revealed to themselves aspects of the concept that were not previously salient to them and that getting them to talk about these aspects reveals the dynamics of their awareness.

Chapter 9 brings together all the findings in the main study and present a summary of the results. Chapter 10 concludes the study with a summary of the findings and a discussion of recommendations for further research.
Chapter 2

Literature Review

2.0 Introduction

In this chapter, I present the relevant literature underpinning the theoretical background of the thesis. I begin with general theories relating knowing to understanding and then focus on research findings in learning and understanding of integration. Components which constitute understanding will play a vital role in informing teachers to provide opportunities for learners to gain rich understanding. A framework which organises and integrates these components will be developed, enabling probing of the aspects of understanding in order to reveal the structure of learners’ understanding.

2.1 Knowing and understanding in mathematics

In working with students for many years, I found that their understanding was often characterised by how much practice they had had with each particular technique. It seems that the more practice they had, the better they were able to reproduce the techniques in the exams by remembering without having to reconstruct the techniques for themselves. This situation appeared to put many students at risk when problems were slightly modified beyond the familiar. The students seemed to know but not to understand: the latter is often characterised by the ability to function in novel settings. I found it necessary to consider the questions “What does it mean to understand a mathematical topic?” and “How does one know one has understood?” Or again, “Having understood, what is the nature of the understanding and how rich is it?” These issues need to be addressed in order to appreciate the process of understanding better and to identify
sources of misunderstandings and to provide learners with a richer experience of learning mathematics. In order to gain insight into these questions, it has proved useful to consider the subtle distinctions between knowing and understanding. In what follows, I describe the literature and models of understanding that I have considered.

2.1.1 States of knowing

There are many ways of knowing. Ryle (1949) distinguishes between knowing-that, knowing-how and knowing-why. He refers to knowing-that as factual knowledge of things, knowing-how as knowledge of ways to perform, while knowing-why refers to having stories to account for phenomena and actions. Mason and Spence (1999) suggest that a more effective state of knowing is for learners to know-to use techniques that they have met, and powers they have developed, in new situations. While learners can often solve routine problems for which they have been trained, they do not appear to know-to use what they have learned when a slightly modified or generalised task is given. Mason and Spence (1999) further argue that knowing-to act in the moment involves “active, practical knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behaviour” (p. 136). Knowing is not a static property of a person but rather a dynamic and emergent relation between person and situation (Mason and Johnston-Wilder, 2004).

Relating knowing to awareness, Gattegno (1987) suggests that what matters is having an awareness become accessible rather than ‘knowing’ as some constant state. He suggests that “knowing is the awareness that one is aware of something, and according to whether we stress the something or the awareness, we progress in the subject, or in the education of our awareness” (p. 42). Very often evidence of awareness is provided by the focus of attention and so I use the two terms ‘awareness’ and ‘attention’ almost interchangeably.
depending on context and sense. Gattegno proposes focusing on the awareness rather than on what one is aware of and uses awareness to refer to learners' sensitivities to perceive change and to respond to that change with actions, whether rehearsed or informed. This idea resonates with Greeno et al.'s (1993) analogy of motion being a relation between a frame of reference and an object and not an exclusive property of the object.

Awareness plays a vital role in Vygotsky's (van der veer and Valsiner, 1991, p. 334) notion of development, which lies in transformation of a person's ability to act 'in himself', into an ability to act that way 'for himself'. Significant development, in Vygotsky's sense, does not refer to reacting to cues but rather responding to stimuli by making a conscious, explicit choice. Vygotsky's view concurs with Gattegno's (1987) suggestion that awareness is what can be educated and becoming aware of these awarenesses is key to successful engagement in mathematics.

Many researchers have referred to implicit and explicit forms of knowledge. For example, Davis (1996), based on Taylor's (1991) distinctions, describes two kinds of knowing. Formulated knowing consists of “those thoughts, behaviours and bits of knowledge that we have written into the text of our experience – those we are aware of, speak of and tend to link in narrative and causal chain” (p. 44). This form of knowing, Taylor argues, “represents only a small portion of our total action, even though it dominates our conscious awareness”. He suggests that another form of knowing, unformulated knowing, comes in the form of “negotiated movement through an interactive world during which our knowledge of that world and our way of being in that world are continuously enacted” (p. 45). This type of knowing has been referred to by many different researchers in different ways (habits below the level of consciousness, James (1899); theorems-in-action, Vergnaud (1981); tacit knowing, Polanyi (1958);
poetic knowing, Vico (1744) and Bachelard (1958)). This form of knowing helps in being awake to opportunities and to being able to respond rather than react. In other words, formulated knowing is that which is displayed and unformulated knowing is that which lies below the surface of awareness and needs to be triggered to be revealed. Davis (1996) asserts that learning takes place in the interplay between the formulated and the unformulated actions.

In the interplay between these forms of knowing, conception of mathematical ideas takes place, and seems to be highly situated. Learners’ cognition seems to be limited by the situation in which learning occurs. ‘Situated cognition’ (Nunes et al., 1993; Lave, 1988; Brown, Collins and Duguid, 1989) offers one possible explanation to account for learners’ inability to apply knowledge conceived in a specific context, to new situations. ‘Situated cognition’ is used to account for absence of transfer, a term used by behavioural psychologists to speak about the development in learning by applying knowledge into new, unfamiliar situations (see for example, Detterman and Sternberg, 1993; Lobatto, 2006).

Knowing to act in a fresh situation requires the need to know how and when to apply something learned in one context to another. This requires something in the new situation that resonates with past experience, which triggers, for example, techniques to come to the surface. For this to happen, the situatedness of the context in which the techniques were learned needs to broaden and extend. In my own research, the extent to which learners have access to their techniques and make use of them is part of what I explore. Actions which come to mind in a novel situation provide insight into learners’ focus of attention and hence suggest aspects of understanding they stress and others they may overlook.
2.1.2 What constitutes understanding?

Understanding in mathematics is a dynamic process. It can be thought of as an active process of constructing connections between knowledge by extracting relationships among mathematical facts, procedures, concepts and principles and forging links between new knowledge and what one already knows. To understand means to know what to do and to appreciate why that is what one does and why it works.

As I was intending to use example-construction as evidence of understanding in different groups of learners, I decided to look at the different theories of understanding proposed by various theorists.

2.1.2.1 Instrumental and relational understanding

For Skemp (1976), “to understand something means to assimilate it into an appropriate schema” (p. 46). He describes two types of understanding; instrumental and relational. The former is as a result of the training of behaviour in applying techniques and learning through rote memorisation of rules (“rules without reasons”) and the latter results from understanding at a deeper level (“knowing both what to do and why”). Some researchers (e.g. Hiebert and Carpenter, 1992) argue that rote learning may eventually result in understanding because “designing school learning environments that successfully promote understanding has been difficult” (p. 65). My own reservation about rote learning leads me to believe that relying on rote-learned techniques alone often results in short-lived, premature engagement in mathematical activity. Many instances with my students have shown that they were often disoriented when a slightly modified situation was presented. Instrumental understanding may give rise to difficulties in the form of cognitive dissonances (Festinger, 1957) when students are confronted with new problem
situations in which the techniques employed or memorised do not seem to be applicable, leading to frustration and loss of motivation.

Relational understanding, on the other hand, describes a flexibility of thought to make sense of new situations and the ability to make necessary connections and then apply it in the new situation. Of course, these two types of understanding are not two separate constructs. Skemp argues that learners and teachers may have different purposes in the learning of a topic. The learners’ purpose is to understand sufficiently to get answers using some kind of rule they know, while the teacher’s is for the learner to understand relationally so that knowledge can be applied in other settings. Skemp warns that “if pupils can get the right answers by the kind of thinking they are used to, they will not take kindly to suggestions that they should try for something beyond this” (p. 22). This endemic tension was elaborated by Brousseau (1984) as a tension arising from the ‘contrat didactique’.

Apart from instrumental and relational aspects of understanding, Watson (2002) suggests that knowledge of the underlying structure of the concept, knowledge about the usefulness of the concept in the context and different contexts and the overcoming of its inherent obstacles constitute understanding. Michener (1978) identifies types of mathematical objects which learners need to have access to in order to understand a topic—“results (theorems, facts); examples (illustrative); and concepts (including formal and informal definitions and heuristic advice)” (p. 362). She suggests that examples are related to construction methods, results to logical deductive reasoning and concepts to pedagogical ordering. This is very relevant to my study since I intend to use examples in order to reveal something about the nature of learners’ understanding.
2.1.2.2 The Pirie-Kieren Model of Understanding

Pirie and Kieren (1989) developed an onion-layer description of understanding to emphasise that understanding is a continuously evolving process. They noticed that people sometimes appear to retreat to previous ways of thinking and then suddenly emerge with more sophisticated and deeper understanding, a process they called ‘folding back’. They suggest that the process of coming to know starts at a level called ‘primitive knowing’ (involving actions on physical objects, figures, graphics or symbols) and proceeds to learners making images out of this ‘knowing’. Then, “these action-tied images are replaced by a form for the images”. This is identified as the “first level of abstraction” by the learner who does this by “recursively building on images based in action” (p. 8).

They continue that “the images can now be examined for specific or relevant properties” (p. 8). This may involve noticing distinctions, combinations, or connections between images. This level of ‘property noticing’ is the outermost level of unselfconscious knowing. The next level (Formalising) involves “thinking consciously about the noticed properties, abstracting common qualities and discarding the origins of one’s mental action” (p. 8). They suggest that “it is at this level that full mathematical definitions can occur as one becomes aware of classes of objects that one has constructed from the formation of images and the abstraction of their properties” (p. 8). This leads to the next level (Observing), in which “one is now in a position to observe one’s own thought structures and to organise them consistently. One is aware of being aware, and can see the consequences of one’s thoughts” (p. 8). For the purpose of validating the consequences of thoughts, “awareness of associations and of sequence among one’s previous thoughts, of their interdependence” (p. 8) is called for, hence the next stage (Structuring). At the highest level of recursion (Inventing) knowers act as free agents.
Chapter 2 Literature review

This is relevant to my research because in my studies, learners' fluency in constructing new examples was taken as evidence of understanding dominated by image having, with forays into image-making and property-noticing.

![Figure 2.1: The Pirie-Kieren Model of Understanding](image)

2.1.2.3 van Hiele levels of understanding

Parallel to Pirie and Kieren's ideas are the van Hiele levels of thinking (Burger and Shaughnessy, 1986), which describe levels at which a learner is said to be operating at any one time.

In Level 1 (Visualization), learners engage in activities that involve reasoning about basic geometric concepts without specific regard to properties of its components. The next level is Analysis, which entails reasoning by informal analysis of component parts and attributes with specific regards to necessary parts of the concept, using language developed at the previous level. Abstraction (Level 3) is described as the stage where concepts are regarded as a collection of properties of the concepts. Awareness of properties that are discerned at Level 3 is related to details of perception of a particular geometrical object and not necessarily independent of it. The fourth level, Informal
deduction, involves selection of sufficient conditions from previous levels of thinking about logically ordered properties. At this level, awareness that some properties are sufficient to give an object a certain quality is available. Logical organisation of properties in recognising similarities and looking for equalities of those similarities constitute awareness at level 5 (Rigour, Formal deduction).

However, at any moment a learner may be dealing with one aspect holistically without discerning features and being aware of other properties at the same time while engaging in formal reasoning in yet others. The structure of attention model (Mason and Johnston-Wilder, 2004) accounts for the different ways in which learners’ attention may be focused at any time and it is discussed next.

2.1.2.4 Structure of attention

A slightly different way of thinking about phases, which is in alignment with the van Hiele levels of geometric thinking and with the onion model of understanding (Pirie and Kieren, 1994), focuses on what learners are attending to at any moment and the nature of that attention. The principal difference among these models lies in what learners are attending to, and how they are attending and how these are structured: in van Hiele’s model, learners can be assigned to a specific level of functioning; in the onion model, there is a cyclic process of advancing and folding back. Instead of partitioning learners’ actions into ‘levels’ which implies restrictions in movements between levels, Mason and Johnston-Wilder (2004) offer a different model of operating which considers attention as the core of mathematical engagement. Influenced by Eastern ideas (see Bennett, 1966), Mason and Johnston-Wilder identify ways in which attention is focused on mathematical tasks through what is called the structure of attention, defined as: holding wholes, discerning details, recognizing relationships between aspects discerned, identifying
relationships as properties that objects like the one being considered can have, and reasoning based on agreed properties, formalised and stated independently of any particular objects, to form axioms. The critical feature is that learners’ and teacher’s attention can dart back and forth between these different ways of attending, or they can stabilise for a short period of time on one or other structure.

Both what one attends to and how one attends structures a person’s experience. In relation to this, Marton and Booth (1997) posit that the way a person experiences something is related to how their awareness is structured and involves the phenomenon as much as it does the person.

If we consider an individual at any instant, he or she is aware of certain things or certain aspects of reality focally while other things have receded to the background. But we are aware of everything all the time, even if not in the same way all the time. The structure of an individual’s awareness keeps changing all the time, and that totality of experience is what we call the individual’s awareness. ...A person’s awareness is the world as experienced by the person.

(Marton and Booth, 1997, p. 108)

They suggest that a person’s experience consists of both a ‘what’ aspect, which corresponds to the object of experience and a ‘how’ aspect (the act of experiencing). Therefore, the way of experiencing a phenomenon entails a dynamic relationship between the structural aspect (‘what’) and the referential aspect (‘how’) of the experience. This approach is the basis for a research method called phenomenography (described in Chapter 3) and accurately describes the way I sought a range of responses from learners in order to characterise their understanding.

One of the sources on which I draw is my own experience of teaching and learning mathematics. My analysis of observing students for many years is that the apparent lack of ability to extend knowledge conceived in a specific context causes problems when
learners have to apply their knowledge to other settings. Particularly, the examples used to illustrate a technique seem to be taken as the only ones that are available. Attention focused on details of the examples seems to prevent the learners from understanding what the example is exemplifying. Because learners 'know' how to do a specific problem only in the context in which it was encountered, it shows that they are aware of relationships but these are not necessarily available as properties to use in other situations. It is only when a new situation triggers the kind of awareness that will result in successful application of the methods learned that 'transfer' occurs.

For these reasons, I see conceptual understanding not only to be the ability to use situated knowledge to solve routine, textbook problems correctly but more importantly as the ability to recognise underlying structure through general properties. To solve problems correctly and confidently requires learners having a repertoire of skills, techniques and familiarity that comes with practice. An ancient Chinese idiom says “practice makes perfect” (Li, 1999, p. 33), a practise used by many teachers in China and East Asia. I remember my lecturer in my undergraduate Arts lesson used to remind us that ‘practice doesn’t make perfect but it sure helps’. As this was the core belief and practice of my teachers during my school years, I too carried on and in turn imposed this ideology on my students.

However, it took some time before I noticed that the more I tried to fill their minds with examples of a concept, the more they appeared to see each example as a class of its own and not as belonging to a general class. I realised that something was lacking in the way my students were approaching the topic. Extending knowledge appropriately and efficiently into unfamiliar situations requires flexibility of thought process to manoeuvre contexts and representations and intuitively to choose the most efficient action. I wondered whether, if a complex and rich awareness was triggered rather than a simple
memory exercised, learners would not need to rely on memory and could take on the problems more creatively and successfully. Hewitt (1999) suggests directing learners’ attention to what is arbitrary and prompting them to work out for themselves things which are necessary in mathematical objects, in order to reduce the necessity to memorise and in order to foster economic engagement in mathematics. It turns out that when learners are asked to construct examples, the examples they construct can reveal something of what they consider to be arbitrary and what they can reconstruct deductively as aspects of their understanding (see Chapter 3). The learner who depends on memory is fragile; the learner who can re-construct and adapt is robust.

Being motivated to approach and engage in a mathematical task creatively and confidently in mathematics undoubtedly contributes to learners’ understanding. The more learners feel that they ‘failed’ due to apparent lack of memory, the more they distance themselves from mathematics. The role of affect in learning mathematics is a crucial one and is discussed next.

2.1.3 Affect in learning mathematics

The centrality of the role of affect in mathematics learning is undeniable. Apart from the complexity of mathematical tasks, and attempts by learners to get a sense of them, learners’ internal drive to motivate themselves to do the tasks is crucially important. Many of my students appeared demotivated to continue working on a mathematics problem because they “could not do it” and because they “did not remember” the methods. Hewitt (1999) goes further and claims that if one has to remember, then one is not working on mathematics.

In relating lack of motivation among learners to teachers’ rigid adherence to the syllabus, Hogben (1938) suggests that “the primary task of an educationalist is to establish a
personal relationship by enlisting the personal interest of individual pupils in the exercise of their reasoning powers" and that "the recipe for good mathematics teaching is to put into the teaching of mathematics something which does not belong to the subject matter of mathematics as such" (pp. 111 – 113). This fits with Dewey’s (1933) *psychologizing the subject matter* principle, by which he suggests transforming the subject matter to suit learners. According to Dewey, instructing or imparting knowledge directly will not be as effective as inviting learners to confront problems naturally and resolve them, be it learning important ideas or techniques. Skemp (1979) attributes learners’ self motivation to the level of novelty in a situation and suggests changing learners’ environment in order to get the desired action. In doing so, the authority to command the desired actions must not come from external forces (e.g. teachers, parents) but from within the learner.

Disturbances to cognitive structure can often result in learners feeling demotivated. Learners who thought they knew often lose confidence when confronted with new, unfamiliar situations. The ability to make sense of the new situation, to relate it to previous experience and to integrate it into their existing web of connections ('schema') requires a conscious effort. Piaget (1971) introduced the notions of *accommodation* and *assimilation* to describe how a learner acts when faced with a novel situation. The principle of equilibration involves assimilating some things and adjusting to accommodate others in order to resolve disturbances. Influenced by Piaget’s work, Skemp (1979) adopts these ideas in learning. He writes,

> The existing structure of knowledge (schema) is thus an essential tool for further learning. Fitting new ideas into an existing schema is called *assimilation*. Sometimes a new idea cannot be thus assimilated without a modification of the schema, and this is called *accommodation*. Both of these complementary processes are necessary for understanding, and failure of accommodation is a common cause of difficulty.

(Skemp, 1966, p. 76)
Skemp suggests that failure to accommodate may result in lack of motivation. A positive form of disturbance, however, can result in the kind of reaction that can motivate learners. The ability to accommodate depends on the rigidity of the schema. One way for learners to gain confidence is to have access to a rich collection of examples and to techniques for constructing new examples. In my research, I investigate the nature of learners' example spaces by asking them to construct examples.

Proponents of what was, for a time, called ‘discovery learning’ such as Dewey, Bruner and Papert argue that learners learn better through inquiry and that the ‘Aha!’ moment is achieved through discovery. Papert (webref) wrote,

You can't teach people everything they need to know. The best you can do is position them where they can find what they need to know when they need to know it.

Papert (webref)

However, how are learners to know that they have discovered something if they have not any clue of what to discover? For some learners, especially in a novel situation, discovering something considered ‘trivial’ can be overwhelming. I believe the kind of tasks that I designed and used in my study afford a positive form of disturbance.

The ‘Structure of a Topic’ framework (Griffin et al., 1989; Mason and Johnston-Wilder, 2004) encompasses three aspects of understanding, namely enactive (observable behaviour), cognitive (awareness) and affective (motivation), in informing what one needs to pay attention to in order to teach a topic. The framework and how it is used as a research tool to investigate learners' understanding in my research is discussed in greater detail in Section 2.4.

To be able to engage creatively and confidently in mathematical tasks and to generalise ideas and concepts appropriately requires a special way of seeing and experiencing. In

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particular, how do learners know which aspects to discern and generalise and which aspects are peculiar to the context? In order to obtain a deeper insight into the nature of human understanding, I decided to look more closely at the ways in which a concept develops and how indications of understanding emerge and this is the focus of the next section.

2.2 Concept development

In school, integration is usually introduced first as the reverse process of differentiation and later as a tool to calculate area under a graph accurately using various techniques. Learners who can ‘see’ the generic form of the integrals seem to cope well with different forms of integrals. However, many learners appear to be caught up in details in an integral and, without the larger picture, they struggle to find the correct method of integration. Understanding may also differ depending upon learners' backgrounds and purposes for learning. According to my preliminary study (to be discussed in detail in Chapter 4), learners with different backgrounds approached integration differently. This emerged in the different ways in which they generalised properties of an example by constructing new examples, and suggests differences in the nature of their understanding. I conjectured in Chapter 1 that the examples constructed by learners’ reveal something about their awareness and, thus, about their understanding.

Such a conjecture requires a consideration of literature that addresses how learners’ experience of learning a concept leads to generalisation of mathematical ideas. Mason’s work contributes significantly to the development of these ideas.

In addition, I looked at the importance of learners expressing generality to reveal their awareness of structure in mathematical objects and how this awareness is revealed through example-construction. Fundamental to these ideas is the notion of dimensions of
variation formulated by Marton and Booth (1997) and the theory behind learner-generated examples (Watson and Mason, 2005), which will be elaborated further in Sections 2.5.3 and 2.5.6, respectively. Before that, I consider aspects involved in the stages of cognitive development in learners.

2.2.1 Stages of cognitive development

Piaget (1985) outlines four stages of cognitive development: sensory-motor, pre-operational, concrete-operational and formal-operational. Following Piaget’s ideas of conceptual growth, Bruner (1966) introduces three modes of representations which can be thought of as three worlds of experience – enactive, iconic and symbolic (material, imagined, symbolic). He writes:

Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic representation); and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (symbolic representation).

(Bruner, 1966, p. 44)

He theorises that an activity starts with an action, proceeds through imagery (diagrams) and ends with symbolic representation of relationships as properties but acknowledges that how learners move about within these three states is not known. He also acknowledges the importance of the use of language to distinguish the speaker from the action.

How transitions are effected – from enactive presentation to iconic, and from both of these to symbolic – is a moot and troubled question. To put the matter very briefly, it would seem as if some sort of image formation or schema formation – whatever we should call the device that renders a sequence of actions simultaneously, renders it into an immediate representation – comes
rather automatically as an accompaniment of response to stabilization. But how the nervous system converts a sequence of responses into an image is simply not understood.

(Bruner, 1966, p. 14)

Transition between or moving back and forth within these forms of representation were recast as movements up and down a spiral of *Manipulating–Getting-a-sense-of–Articulating* (Floyd *et al.* (1981); see also Mason *et al.*, 1982 and Mason and Johnston-Wilder, 2006). This spiral has a lot in common with ‘folding back’ movements studied by Pirie and Kieren (1989) which provides a fine grain of analysis of students coming to understand.

Transition between Bruner’s three modes of representations can be eased by identifying the purpose for engaging in the enactive mode of participation, for manipulating entities in order to get a sense of the generic features of the thing being manipulated and for articulating this. Bruner’s three modes of representation (EIS) together with language to express aspects of understanding are embedded in the MGA spiral to describe the process of approaching mathematical tasks. They resonate with two other frameworks (*Do-Talk-Record* (DTR) and *See-Experience-Master* (SEM), based on Floyd *et al.*, 1981) which echo the same ideas.

Picture mathematical thinking on a helix which loops round and round. Each loop represents an opportunity to extend understanding by encountering an idea, an object, a diagram, or symbol with enough surprise or curiosity to impel exploration of it by manipulating. ... Tension provoked by the gap which opens between what is expected and what actually happens provides a force to keep the process going and some sense of pattern or connectedness releases the tension into achievement, wonder, pleasure, further surprise or curiosity which drives the process on. While the sense of what is happening remains vague, more specialisation is required until the force of the
sense is expressed in the articulation of a generalisation. ... And achieved articulation immediately becomes available for new manipulating, and the wrap-around of the helix.

(Mason, Burton and Stacey, 1982, p. 155)

Manipulating affords learners opportunities for seeing new ideas and experiencing previously met ideas, and themes, and for mastering techniques. Recognising patterns, multiple relationships and attributes offers learners opportunity to ‘get a sense of’ the concept. Articulating these attributes and relationships leads to formalisation through discerning properties independent of the objects being manipulated and, hence, to appreciation of generality. Since the time of Plato, if not before, discussion and articulation of thought processes has been recognised as an important aspect of mathematical communication.

Tall (2004) also describes three worlds of mathematics, with slightly different emphasis. He suggests that development of concepts passes first through a stage of perceiving the world and making sense of it, both the physical world and the mental world. Reflecting on aspects of sensory experience affords opportunities for embodiment, hence the world of ‘conceptual-embodied’ or ‘embodied world’. Symbols are seen as facilitating actions that encapsulate concepts, both to do and to think about the concept, which Tall called the ‘proceptual-symbolic world’ or ‘proceptual world’. The third world refers to the ‘formal-axiomatic world’ or ‘formal world’ in which noticing structural properties and using axioms to define them is dominant.

Language plays a significant role in mediating and elucidating experience in these worlds. Experience is recast into language, which renders language an important tool to communicate the experience, but the act of expressing immediately distances the speaker from being immersed in the action being expressed. In my study, one of the components of the ‘Structure of a Topic’ framework is language patterns with which learners express
ideas and I intended to use language fluency when speaking about integration as an indication of richness of that component.

### 2.2.2 Process-object encapsulation

The process of turning actions into objects is fundamental to learning mathematics. Mathematical actions seen as process are as important as the result of carrying out those actions. Abstraction involves recognising properties of objects, which arise from actions and which are invariant across examples. It involves “drawing out” something from things that have that property (Piaget, 1970). Therefore, having seen a collection of things having the same property, a child learns the meaning of that property. Piaget (1972) suggested that abstraction results from action carried out on objects and coined the term *reflective abstraction* (Dubinsky, 1991) to describe performance, interiorization and coordination of actions to form generalisations. Skemp (1971) distinguishes classification, similarity, abstraction and concept as the following:

*Abstraction* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the same similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a concept.

(Skemp, 1971, p. 21)

In the case of mathematical examples, however, looking at and working out a wide range of examples does not guarantee that students will abstract properties structural to the examples unless they perceive them in a certain way, do something with them and have their attention drawn to those properties. This aspect is part of what I intended to explore in this study.
Gray and Tall (2007) see ‘abstraction’ as “dually a process of ‘drawing from’ a situation and also the concept (the abstraction) output by that process” (p. 1). They suggest focusing on “the thinkable concept (the abstraction) output by that process” so that “this thinkable concept is then available to be used as in more sophisticated levels of thinking” (p. 4). They point out that “abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking” (p. 14).

2.2.3 Reification

Sfard (1991) uses the term reification to describe the process of actions becoming objects and stresses that “process and object are … different facets of the same thing”. She uses operational and structural definitions and concepts to distinguish respectively between actions on and language of objects. The construction of a concept, according to Sfard, takes place in three stages: interiorization, condensation and reification. She suggests that interiorization takes place when “a learner gets acquainted with the processes which will eventually give rise to a new concept” (p. 18), in which case the actual activity does not need to be performed (this is what ‘Getting-a-sense-of’ refers to in the MGA spiral discussed in section 2.2.1). During the phase of condensation, lengthy sequences of operations are compressed into more manageable units. She writes, “at this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details” (p. 19). She defines reification as “an ontological shift – a sudden ability to see something familiar in a totally new light … an instantaneous quantum leap: a process solidifies into object, into a static structure” (pp. 19 – 20).
My research findings point to the fact that learners often focus on the process of integration and not so much on integrals as objects. In other words, processes and objects are seen as separate entities and not as integral aspects of the topic.

2.2.4 Procept

Gray and Tall (1994) show how a procedure or action carried out can become an object of study. They coined the term procept to indicate the dynamics of an action turning into an object. According to Gray and Tall, $2x + 3$ is both an addition operation and the answer to that operation being carried out. They write,

> We do not consider that the ambiguity of a symbolism expressing the flexible duality of process and concept can be fully utilized if the distinction between process and concept is maintained at all times. ... An elementary procept is the amalgam of three components: a process that produces a mathematical object, and a symbol that represents either the process or the object.

(Gray and Tall, 1994, p. 121)

In my research, the conceptual aspect of integration as area so far was often submerged beneath integration as a process to be carried out in order to achieve a numerical answer. The fact that many of those interviewed associated integration with area in the interview appeared to contradict the absence of use of area when solving integration problems. I was left with the desire for more insight into the nature of the displayed behaviour and how to reveal aspects of learners’ understanding other than behaviour.

2.2.5 Symbol sense

Gray and Tall (1994) suggest that symbols are a means of communicating mathematical ideas and do not represent, in themselves, the duality of process-object, at least not to the learners in the moment. Integration is particularly dependent on the use of symbols, which act as a mediating tool that may, but may not afford access to images and
associations of concepts. According to Skemp (1971), “making an idea conscious seems to be closely connected with associating it with a symbol” (p. 78). He further notes that symbols help to “reduce the cognitive strain of keeping the whole of the relevant information accessible” (p. 78). I found in my preliminary investigation that for many learners the symbol $\int$ did not trigger the necessary association to area as evidenced from their responses to the example-construction task, although they mentioned area as the meaning for the symbol in the interview. Therefore, I sensed that there was a gap between their procedural understanding and conceptual understanding.

In parallel with number sense as “non-algorithmic” feel for numbers (Sowder and Schappelle, 1989), Arcavi (1994) used *symbol sense* to denote “any of the special mathematical faculties by which meaning is aroused” (p. 24). He refers to it as the “algebraic component of ... sense-making in mathematics” (p. 32). Berger (2004) uses the term “functional use” of mathematical signs to characterise learners’ usage of signs before reaching maturity. She suggests that through phases of manipulation, imitation, template-matching and association, learners appropriate meaning to a new sign and gradually align themselves with the meaning held by the mathematical community.

Symbols carry meaning, both surface and deep. As such, learners need to see them in their totality. Hiebert and Carpenter (1992) emphasize the importance of connections and links between symbols and meaning. They say that meaning is derived by connecting symbols with other forms of representations or by establishing connections within the symbol systems. However, learners often make inappropriate connections or “represent information as isolated pieces” (p. 76) which result in difficulties. Askew *et. al.* (1997) found that teachers with a ‘connectionist’ approach to teaching and who are aware of how mathematical topics fit together turned out to be more effective in teaching. Therefore, if a learner is aware of the dual nature of a concept and has appropriated the meaning
denoted by the sign, the learner necessarily has a general sense of the concept. But what
does it mean to have a sense of the concept? How is this sense revealed or
communicated? In order to obtain deeper insight into the nature of the sense of ‘having
learned’ a concept, I decided to look at constructs such as concept image proposed by
Tall and Vinner (1981) and at the process of generalising.

2.2.6 Concept image

To have a concept means having available various images and propensities that come to
mind when a concept is used and having access to ways of thinking and to specific
techniques associated with the concept. Tall and Vinner (1981) coined the term concept
image to capture what it means to have a sense of a concept and to distinguish it from
concept definition as a formal statement. They use concept image to describe “the total
cognitive structure that is associated with the concept, which includes all the mental
pictures and associated properties and processes” (p. 2). By image, they do not mean only
mental pictures of the symbol or name but “the total cognitive structure which colours the
meaning of the concept” (p. 1). They extend the notion to evoked concept image to refer
to “the portion of the concept which is activated at a particular time” (p. 2). They termed
concept definition as “a form of words used to specify that concept” (p. 2). Given the
different mental manipulations and other associated processes of the individual, what
comes to mind when a technical term is used is often not what they were taught and
indeed may not be what the teacher has in mind.

Embedded in the notion of concept image are examples that are used to illustrate the
concept and ultimately having access to a rich example space. Examples are tied up with
a general class of things exemplified to obtain a general sense of understanding.
Constructivists argue that learners must actively construct knowledge on their own rather
than be passive recipients of knowledge (Piaget, 1972; von Glasersfeld, 1996). In working with examples, evidence of understanding can be seen in learners building on their understanding by constructing fresh examples rather than passively working out examples given by the teacher. Watson and Mason (2005) showed how example-construction by learners can be both a useful pedagogic tool to enrich learning and also a research tool to reveal learners’ appreciation and understanding of a topic. Zazkis and Leikin (2007) developed the research use of example-construction in a manner similar to mine. This idea will be developed in the topic of integration in Section 2.5.

2.2.7 Generalisation and expressions of generality

Generalising is the essence of human life. So too in mathematics, generalisation is key to mathematical understanding and children are naturally born with the power to distinguish and generalise. However, in mathematics, ideas are presented in specific contexts in order to facilitate understanding. Learners are then expected to discern, from the contexts, which relationships and properties of the ideas are general and to disregard those that specify details of the context. However, seeing generality through particular instances is not something that occurs naturally for many learners. The fact that generalisations are expressed in particular notations, which can be manipulated in order to abstract relationships independent of the context, presents further difficulty. However, it is often the case that contexts characterise mathematical examples that are used rather than aspects that are exemplified in the contexts.

Whitehead (1911) writes, “To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought” (p. 4). He continues to say that
... what the mathematician is seeking is Generality. Any limitation whatsoever on the
generality of theorems, or of proofs, or of interpretation is abhorrent to the mathematical instinct.

(Whitehead, 1911, p. 57)

Similar views have been expressed by Mason (1996) who comments on pedagogic strategies that overlook the importance of generalisation. He says that “a lesson without the opportunity for learners to generalise is not a mathematics lesson” (p. 65). According to Mason, learners come to lessons with natural powers to make sense and therefore, it is the teachers’ job to direct learners’ attention appropriately and effectively. He goes on to stress that,

If teachers are unaware of its [generalisation] presence, and are not in the habit of getting students
to work at expressing their own generalisations, then mathematical thinking is not taking place.

(Mason, 1996, p. 65)

In my research, I found that many learners have difficulty noticing or indeed expressing
generality that could be seen as exemplified in the examples given, and often they had to be helped to see it.

Krutetskii (1976) identifies types of mathematical generalisation that lead to “on the spot” generalisation. He suggests that apart from gradual generalisation from a diversity of particular cases,

... there is another in which able pupils, without comparing the ‘similar’, without special exercises or hints from the teacher, independently generalise mathematical objects, relations, and operations ‘on the spot’, on the basis of an analysis of just one phenomenon, into a number of similar phenomena. They recognize every specific problem at once as the representative of a class of problems of a single type and solve it in a general form – that is, they work out a general method (an algorithm) for solving problems of the given type.

(Krutetskii, 1976, p. 261)
He continues,

Capable pupils generalise mathematical material not only rapidly but broadly. They very easily find the essential and the general in the particular, the hidden generality in what seemed to be different mathematical expressions and problems.

(ibid., p. 262)

How these ‘capable’ students have a sense of generality in every aspect of mathematics and how other students do not seem to develop that sense is of great interest to me. I conjecture that those we call ‘more able learners’ are given that label because they operate at a more general level and perceive generality in every particular thing that they deal with, whereas those described as ‘less capable learners’ are often caught up with particulars which obscure their sense of generality.

Perceiving generality is one thing. Expressing it is quite another, and requires language fluency with technical terms and grammatical construction. Dienes (1963) stresses the use of symbols as language to express generality. He writes,

When a child becomes familiar with a mathematical structure he needs a language in which to talk about it, think about it, and eventually transform it. ... There is also the question of whether symbolism can be used as a tool for cutting through relevant noise during the abstraction process, or whether it can only be used to formulate what has already been abstracted.

(Dienes, 1963, p. 161)

Malara and Navarra (2002) suggest that there is a tendency in traditional didactics to separate meaning from rules of algebra by assuming that studying the rules and manipulating them will yield comprehension of meanings. They suggest that mental models of algebraic thinking are likely to be built up through initial forms of algebraic babbling (p. 230) in which the form is more or less conventional even though the content may be scrambled. Ainley (1999) referred to this phenomenon as emergent algebra. By
this, Malara and Navarra suggest that initial attempts by learners at generalisation must not be ruled out as incorrect or wrong. An analogy would be a young child making attempts to walk; judging it as not knowing how to walk is rather premature. It is necessary to avoid conclusions reached by categorising learners based on these forms of ‘babbling’. In my research, I intended to probe learners’ understanding further than their ‘babbling’ stage by building on their initial responses to example-construction in order to reveal their awareness and gain insight into their understanding.

Although the aim of teaching using the particular is to convey generality, this aim is not always clearly expressed in teacher’s objectives of a lesson, though it can be present in the teacher’s mind subconsciously. Generalisation involves stressing structural features of mathematical ideas and letting go of details that are irrelevant to these features. But how are learners to know which features are necessary and therefore structural, and which are irrelevant optional details, when the teacher appears to be stressing everything or does not draw learners’ attention to structural details?

Before initiating a discussion on learner-generated examples as evidence of understanding, I consider aspects related to the teaching and learning of integration. In what follows, I present issues concerning teaching and learning of integration including relevant research into students’ understanding of integration and misconceptions that contribute to the lack of flexibility in understanding.

2.3 Teaching and learning of integration

Calculus is a central branch of mathematics which is built on two major complementary ideas: differentiation and integration. Differential calculus studies rates of change, which are usually illustrated by the slope of a line or of a curve at a point. Integral calculus, on the other hand, studies the accumulation of quantities, such as areas under a curve, linear
distance travelled, or volume displaced. The two concepts, differentiation and integration, define inverse operations whose relationship is made precise by The Fundamental Theorem of Calculus.

2.3.1 The school scenario

In Malaysian schools, students are introduced to calculus concepts through differentiation at the age of 16. These students would have opted to study advanced mathematics (Additional Mathematics), depending on their grades in mathematics in a national examination in the previous year. At this stage, they would be taking all other pure science subjects as well such as Chemistry, Physics and Biology and Mathematics C (non-calculus). Their counterparts who did not do well in the exam, especially in Mathematics, would opt for Arts subjects such as Literature, Geography, and Arts although they do have to take the Mathematics C as well.

Integration is part of the syllabus for those students who opted for Advanced Mathematics the following year when they turn 17. Most schools use a textbook by Chew et al. (2006), recommended by the Ministry of Education. The teaching of integration begins with the introduction of integration as the reverse process of differentiation. However, the aspect of reverse that is emphasised is the technique alone (plus turns to minus, multiply becomes divide, etc.) and not its implications for the concept itself. Different techniques of integration are stressed, beginning with integration of $ax^n$ and moving on to multiple terms and algebraic expressions. The different techniques of integrating indefinite integrals such as by-parts, substitution and any special cases (e.g. $x^{-1}$, $e^x$) are stressed. When students are familiar with these techniques, integration of definite integrals is introduced, followed by its application in calculating bounded area and volume.
From my experience, learners do not appear to have a comprehensive view of the topic, although the introduction of techniques and applications of integration follows a sequential pattern of instruction. This is evidenced by learners’ lack of flexibility in extending the techniques learned into slightly generalised settings. Similar observations have been reported by Abdul Rahman et al. (2005) and Mohammad Yusof et al. (2005).

2.3.2 Research in teaching and learning of integration

Much of the research that has been done to look into learners’ difficulties in understanding concepts in calculus focuses on learners’ inability to think flexibly in terms of technique usage and to reflect on what they have done. Concern has been raised about the rote, manipulative learning that takes place in calculus courses (Cipra, 1988; Steen, 1988; White, 1990). Student difficulties in many areas of calculus such as rate of change (Orton, 1983), limit (Cornu, 1992; Tall and Vinner, 1981), tangent (Vinner, 1981; Tall, 1987) and function (Dreyfus and Eisenberg, 1982; Vinner and Dreyfus, 1989; Tall, 1993) have been well documented.

Tall (1993) identifies a number of fundamental difficulties associated with calculus learning including language confusions associated with the limit concept, restricted mental images of functions and student preference for procedural methods rather than conceptual understanding. To overcome the conflict, he suggests that students must “reconcile the old and the new [knowledge] by re-constructing a new coherent knowledge structure” rather than “keep[ing] the conflicting elements in separate compartments and never let[ting] them be brought simultaneously to the conscious mind” (p. 3). Dreyfus and Eisenberg (1991) voice similar concern in relation to images and representations when they say learners tend to “reduce the mathematics of calculus to a collection of algebraic algorithms, while avoiding graphics as well as geometrical images” (p. 25).
Both integration and differentiation have wide applications in physics, chemistry, engineering and technology. Although they are related, many researchers have focused on differentiation so that relatively little study has been done on integration. A small number of researchers have considered the topic of integration including Orton (1983), Hong and Thomas (1997), Rasslan and Tall (2002), Sealey (2006) and Tsamir (2007), whose work and pertinent details are elaborated below. They found that, in general, learners do not have a comprehensive view of integration and that learners could not cope well with slightly modified situations.

One of the early research studies into students' understanding of integration was that done by Orton (1983). Orton investigated students' understanding of elementary calculus integration and interviewed 110 students majoring in mathematics. He found that students' difficulties included understanding integration as a limit of sums and he described how students have problems with the reasoning behind integration methods, particularly when calculating areas bounded by curves. He identified errors that he categorized as structural and calculational/executive made by students when finding the area under a curve. He showed that although students were able to apply the basic techniques of integration, they lacked a fundamental understanding of the concept. Students' preference to use procedural skills and their apparent reluctance to use geometric interpretations may explain their strong inclination to move to an algebraic context (Dreyfus and Eisenberg, 1991) where they can use algebraic manipulations.

Students' understanding of calculus was also the focus of Artigue (1991). Discussing Orton's findings, Artigue found that performing routine procedural methods of finding area under a curve but explaining their procedures proved difficult for students.

Ferrini-Mundy and Graham (1994) reported inconsistencies between performance on procedural items and conceptual understanding, in that conflicting conceptions are held
comfortably and routinely in the development of calculus concepts, with separate understandings for geometrical and algebraic contexts. Students who are predisposed to familiar and frequently-used patterns formulate their own theories and make their own connections and construct meaning for problem situations, which are influenced by previous experience and knowledge. Norman and Prichard (1994) suggest that geometric intuitions about integration could become cognitive obstacles to the understanding of the concept. According to Selden, Selden and Mason (1994), even good calculus students often cannot solve non-routine problems. Their study showed that students exhibited a tendency not to use calculus, preferring arithmetic and algebraic techniques for solving calculus problems, even where the use of elementary methods would be inappropriate.

Recent research into students’ conception of integration includes Hong and Thomas (1997) who studied learners’ misconceptions and versatility in thinking about integration. They investigated the understanding of integration of 161 first year university calculus students. To do this, they used two sets of questions: Section I was based on standard textbook questions, which highlighted a process-oriented approach, and Section II was aimed at displaying understanding rather than applying procedure. Questions in Sections I and II were linked by common question forms to see whether the students had the techniques but lacked understanding. For example, one of the questions was:

Section I  Find \( \int_{9}^{16} x^2 \, dx \)

Section II  Given that \( \int_{9}^{16} x^2 \, dx = \frac{74}{3} \), what is \( \int_{9}^{16} t \, dt ? \)

They found that learners have a tendency to engage in rote application of procedures rather than calling upon a relational, concept-oriented understanding. This finding confirms my initial concern that led me to undertake this study. They then compared the
students’ performances using MS Excel and Maple™ computer modules as supplements and concluded that such experience may contribute to a more versatile conception of processes and objects of integration.

Rasslan and Tall (2002) studied the conception of the definite integral concept in 41 high school students who had undergone the SMP-A level teaching programme. They investigated the students’ understanding in terms of definitions, images and misconceptions related to the concept. A questionnaire consisting of six definite integral questions was designed and responses were categorized in terms of correctness and depth. Typical of the questions they asked are:

Find, if you can, a) \[ \int_{0}^{\frac{1}{2}} \frac{1}{(x-4)^{3}} \, dx \] and b) \[ \int_{-1}^{2} \frac{1}{x^{4}} \, dx \]. If you can, please explain the sign of the answer.

The function \( f(x) = x - \lfloor x \rfloor \) is given. Find the area directly below the graph and above the \( x \)-axis between \( x = 0 \) and \( x = 3 \).

The following function is given: \( f(x) = 1 - |x - 1| \). Find \[ \int_{0}^{2} f(x) \, dx \].

In your opinion what is \[ \int_{a}^{b} f(x) \, dx \] (the definite integral of the function \( f \) in the interval \([a, b]\)).

(Rasslan and Tall, 2002, p. 91)

They concluded that the majority of above average students do not define definite integrals meaningfully and that their interpretation of the topic to calculate area and beyond does not appear to be fully developed.

This finding is pertinent to my study because the aspects of understanding of integration that I intended to look at in my study included images and definitions. However, I elicited responses from students in two ways: through their responses in an interview via direct
questioning methods, and awareness of images and definitions as revealed through the examples they construct.

More recently, Sealey (2006) used teaching experiments to examine understanding of Riemann sums and definite integrals in students attending a calculus workshop. She set out to explore students’ ability to use area under a curve as a tool to solve problems, either using definite integrals or Riemann sums to approximate total accumulation. She gave two problems to a group of students, one of which was the following:

A uniform pressure $P$ applied across a surface area $A$ creates a total force of $F = PA$. The density of water is 62 lb per cubic foot, so that under water the pressure varies according to depth, $d$, as $P = 62d$.

a) Draw and label a large picture of a dam 100 feet wide and extending 50 feet under water.

b) Approximate the total force of the water exerted on this dam.

c) Find an approximation accurate to within 1000 pounds.

d) Write a formula indicating how to find an approximation with any predetermined accuracy, $\epsilon$.

(Sealey, 2006, p. 49)

A similar problem was given to another group of students. In both cases, the phrases “definite integral” and “Riemann sum” were not used by the instructor until after the activities were completed or until the students introduced the terms themselves. The study concluded that the students in the first group seemed to have a better understanding of the definite integral concept than the second group because they (the former) rephrased the problem in terms of area under a curve and in terms of Riemann sums. They were said to have pseudo-structural understanding. I am not convinced that the claims in this study are valid. The anthropological adage *absence of evidence is not evidence of absence*
applies here. Just because they don't say, it does not follow that it is not there. It merely reflects the fact that there has been no reason to express it. I suggest that all we can say was that it did not come to the surface sufficiently strongly to be manifested in behaviour. The fact that use of Riemann sums is not displayed cannot form the basis for justifying the claim that the students do not have the knowledge. Sealey goes on to suggest that area under a curve is not sufficient for understanding the definite integral and that knowledge of the underlying structure of the concept is needed for it to be a useful tool.

Tsamir (2007) examined students’ solutions to problems dealing with definite integrals, areas and volumes. She described five lessons segments taught by experienced mathematics teachers in classes of 25 to 28 12th graders who discussed problems dealing with integrals. The lessons included parameters, areas and volumes, adjacent areas, areas and volumes for \( f(x) = 2x \), composite trigonometric function and areas, volumes and intersection points. One such problem was:

1. Find the area enclosed between \( y = (1 - x)(x - 5) \) and the x-axis, between \( x = 0 \) and \( x = 3 \).

2. Find the volume created by rotating this area around the x-axis.

(Tsamir, 2007, p. 30)

Students’ errors were highlighted by drawing upon Stavy and Tirosh (2000) and Fischbein (1987) in identifying students’ repeated expression of erroneous \( \text{same area} - \text{same volume} \) ideas. She also noted that two students used incorrect formulas and one used a ‘pseudo formula’ (Vinner, 1997). She suggests that one theoretical approach does not describe the complex sources of student errors and that more than one theoretical framework is needed in analysing student errors.

What is salutary in these reports are the authors’ claims about students’ lack of flexibility and understanding of underlying principles. They point to the fact that students may have
knowledge of integration in terms of techniques and procedures, without adequate conceptual understanding of the underlying principles. With the exception of Orton’s study, none of the other studies involved interviews or other methods to get a full view of students’ understanding. The studies sought to explore understanding by observing behaviour alone. Little was done to investigate root causes of such problems.

Learners may mis-remember techniques and may not be able to check or reconstruct a technique when needed. This may explain what is happening when learners do not cope with unfamiliar situations or tasks. Learners’ apparent inability to think flexibly and the misconceptions that result can be attributed to rigidity in the way in which they set about learning techniques in mathematics. Trying to memorise procedures leaves learner vulnerable to mis-remembering or forgetting. Particularly in the case of integration, the integral sign links to many different methods of integration depending on the nature of the function being integrated and is involved in many different applications. This may explain why learners appear confused and resort to exercising their memory rather than dealing with the situation creatively and constructively.

One of the ways in which creativity can be fostered is by shifting learners’ attention from exercising memory to appreciating the underlying structure of mathematical problems. Rather than working on exercises which depends on memory, constructing examples that meet specified constraints may act as an internal force to be creative while at the same time enriching their associations and providing access to conceptual underpinnings.

2.3.3 Focus of study

As integration is central to calculus, it is studied by students from different backgrounds to varying depths. This includes students who study A-level mathematics to prepare themselves for various undergraduate courses such as Physics, Chemistry, Mathematics,
Economics, Engineering and other applied mathematics fields. These groups of students learn integration for different purposes and with different intensities.

In many countries, advanced level calculus is introduced in the secondary school since it is thought that it is good to push students through the rudiments of calculus as preparation for later study. However, I conjecture what is likely to happen is that students grasp a sufficient number of techniques to pass the exams without deep conceptual underpinnings. In fact, what they learn are special ‘coping skills’ and they invent certain ways of working to get them through the exams (Smith and Moore, 1991). When these students enter college and major in mathematics, formal analysis such as Riemann integration is introduced. These students may focus on principles underlying the topic and, thus, may develop a richer and more flexible understanding. Engineering students, on the other hand, may learn the topic for use as a tool to solve practical problems. Hence, they may rely on number-crunching devices such as calculators and computers to help them. Answers rather than concepts are important to them. To this end, concerns have been raised about the quality of mathematical understanding among engineers (Gill, 1999; Kent and Noss, 2002). During the course of my own engineering training, I found that I was relying too much on the tools and not appreciating the mathematics underlying the process of solving problems. Hence, any discrepancies in the solutions were hard to locate, whether they originated from procedural errors or from conceptual shortcomings.

In my view, what is missing in these students’ sense of understanding of integration is a comprehensive view of what comprises integration. They appear to have a fragmented view rather than significant conceptual understanding. How many and which aspects of integration they remember from early exposure of integration at A-level until they start their undergraduate studies is part of what I intended to investigate in this study. This
explains my choice of subjects for this study which comprise a spectrum of learners who encounter integration at various stages.

The fact that students learn integration for different purposes suggests that aspects which they stress may be different. Part of the focus of my study is the variation in richness in understanding of integration among learners, specifically A-level students, mathematics students, engineering students and PGCE mathematics students. A core aspect of the focus of my research is learners’ awareness of dimensions-of-possible-variation in mathematical examples as a window into learners’ accessible example spaces and their grasp of integration.

The ‘Structure of a Topic’ framework incorporates and extends Tall and Vinner’s (1981) notion of concept image by drawing on traditional views of the structure of human psyche. It also outlines the components of understanding needed for learners to gain a comprehensive view of a topic, namely awareness, behaviour and emotion. In the following section, I describe the individual components that make up the framework and develop and link the strands of the framework as an encompassing tool to investigate understanding.

2.4 The ‘Structure of a Topic’ framework

The ‘Structure of a Topic’ framework (Griffin et al., 1989; Mason and Johnston-Wilder, 2004) was originally designed to inform teaching by offering a structured framework for preparing to teach a topic (its original title) and was used primarily to structure teaching materials (see Figure 2.2). The constituents of the framework were intended to remind teachers to pay attention to three of the important components of the psyche in relation to what it means to understand a topic. The three components of learning a topic that
constitute the framework are *awareness, behaviour, and motivation*. As such, it occurred to me that it can be used for probing learners’ understanding as well.

Since awareness is based on actions which trigger further actions, the corresponding behaviour develops as a result of being aware of that awareness. As one becomes more aware, one is likely to refine behaviour and similarly, as behaviour is refined, sophistication in awareness develops. Actions cue a form of behaviour. Becoming aware of those *actions* (rather than focusing attention on the *objects* being acted upon) is how reification (Sfard, 1994a, 1994b) comes about.

Juxtaposing the notion of *concept image* (Tall and Vinner, 1981) with the three interwoven dimensions corresponding to aspects of the psyche (*cognition, enaction and emotion*) and informed by Gattegno’s (1987) assertion that *only awareness is educable*, the ‘Structure of a Topic’ framework was built around the components of *awareness, behaviour and emotion* and was devised in order to help teachers to gain an overview of a topic or a concept. In my research, I use the framework as a way to structure probing of learners’ understanding. Consequently, the framework is used both as a generative tool to design questions and tasks in my study and as an analytic tool with which to analyse the data that emerges. One of my research questions involves what is revealed when using this framework to devise probes and what is revealed when learners are asked to construct examples.
The behaviour strand relates to visible behaviour such as informal or natural use of language which the topic uses in a technical sense, and ways of expressing ideas in the topic, as well as particular techniques and methods. In Tall’s (2004) concept development model, this is the embodied form of representation seen as appropriate use of procedures and language. The top left-hand corner marks language fluency with which learners express ideas in mathematical tasks. The behaviour strand also marks competence which includes specific manipulative techniques and ‘inner incantations’ (inner talk) which a relative expert may employ when carrying out those techniques, positioned at the bottom-right end. This component of the framework has a lot in common with the proceptual/formal stage of concept development (Tall, 2004).

Mason and Johnston-Wilder (2004) write,

The behavioural thread stresses the terms which students may already know or use in a less formal manner, language patterns (terms, phrases, clauses), both those which the students already use but less precisely, and those which are the marks of competence and understanding. It includes also
specific manipulative techniques and any 'inner incantations' which a relative expert may employ when employing those techniques.

(Mason and Johnston-Wilder, 2004, p. 204)

The motivational-emotional strand entails different contexts, in which the topic originally arose, and root problems from which the topic came, in order to answer the question, "Why is this topic important"? These are the 'motivational examples' which are intended to interest learners and give them a sense of how the topic is used. Ideally these are phenomena which surprise or intrigue learners where analysis makes use of the topic. Sometimes 'root problems' and contexts have to be fabricated because the history is too complex to reproduce for learners. The ideas that students have in relation to uses of mathematical topics can influence learners' determination to understand and appreciate them. For example, learners' expectations of future use are likely to influence their engagement in solving integral calculus problems.

School topics are always built around a succession of techniques for solving classes of problems, which someone decides are teachable. The motivation strand also includes contexts in which the ideas have been known or can be expected to appear. For illustration and understanding purposes, a topic may be introduced in a simplified context, which may afford surprise or cognitive dissonance (Festinger, 1957) in learners, especially when the topic is used in contexts outside the classroom. If the surprise is a positive form of disturbance, then learners are more likely to engage in the process of accommodating and assimilating the new piece of information. However, in some cases, learners may fail to see the relationship and hence, fail to accommodate or assimilate anything.
The horizontal thread encompasses the motivational-emotional. This thread stresses both the contexts in which the idea or topic originally arose, (even if it is slightly fabricated or simplified story for students to understand), and in which it has been known to appear. It encompasses the virtue of surprising students, whether by challenging a preconception, or indicating an unexpected result. One of the roles of this overall framework is to support locating the surprise which helped to turn this topic from an ordinary result of perhaps passing interest into an actual topic.

(Mason and Johnston-Wilder, 2004, p. 204)

The motivational-emotional strand is a reminder of aspects of a topic, including origin and use, which could influence or motivate learners, a reminder of the affective affordances offered by or available in a topic. Despite these affordances, a learner may not be exposed to sources and other uses or may not have been particularly concerned about these aspects or they may have played a role at one time but are no longer salient. Far from drawing any conclusion about learners’ affective dimensions, the ‘Structure of a Topic’ framework includes origins of disposition and knowledge of applications and utility. I have altered the position of the elements in the emotion strand so that the future is to the right and the past is to the left. I recommend altering this position so that the flow of conception is reflected in left to right movement.

It is not my intention in this study to examine engagement of belief or what motivated learners when studying a topic. What I was looking for was an indication of aspects of the topic which one could anticipate might have influenced the learners. There is a delicate line between a learner’s actual affective state and reflection on the behaviour which one might anticipate to have influenced that learner’s state. Behaviour of individuals mirrors, however partially, what has influenced those individuals and I was looking to glimpse indications of what is available in the topic. Furthermore, what students are exposed to and what they are disposed to personally can be distinguished. All I was looking for is affective affordances that could be discerned through their
behaviour because what does come through is highly likely to have influenced them. What does come through requires a completely different study, including investigating reasons for why those things come through for those individuals in those situations. Thus no conclusions can be drawn from what is not discernable, but what is discernable indicates some sensitivity to that aspect of the emotional strand of the topic.

The awareness strand encompasses the concepts which the students already know and are familiar with. This may be in the form of previously met ideas that are related or relative positioning of the idea/concept in relation to other concepts learnt. The strand also includes historical analysis in terms of standard confusions or obstacles often encountered during the process of making precise the idea. Awareness of such misconceptions, either those formed by a learner in the past, or those that others have committed or are likely to commit, may prevent learners from doing the same. The other end of the strand encompasses the kinds of mental images, associations and connections which the teacher would like the students to develop as entailed by the idea/topic. The kinds of links, associations and connections and the images and sense-of the topic that the teacher would like students to have come to mind when the topic is called forth mark richness in understanding and are placed at the top-right end.

The awareness thread includes mental images, associations and connections which the teacher would like the students to develop, transitions from process to object which are entailed by or employed in the topic or idea, as well as standard confusions or misconceptions which students are likely to form because others have done so in the past. This may include historical analysis of obstacles encountered during the precising of the topic or idea.

(ibid., p. 204)

In relation to concept development, this framework bases individuals’ conception of mathematical ideas on awareness. The basis for behaviour is trained through practice but
training alone renders the individual inflexible. Flexibility comes from awareness which informs behaviour. For me then, learning involves educating awareness which in turn directs appropriate behaviour which draws upon and reinforces energy and motivation to learn.

The framework provides a finer structure to Tall and Vinner's (1981) notion of concept image if a broad interpretation is permitted. A narrow interpretation, however, would cover much of awareness strand of the framework (such as imagery, associations and connections but not obstacles) and behaviour strand (language patterns and techniques). The 'Structure of a Topic' framework extends the concept image notion by incorporating awareness of misconceptions related to a topic and by acknowledging the contribution of motivation in understanding and appreciation of a topic. My research looks at learners' structure of understanding of integration in terms of behaviour, awareness and emotion and the use of example generation as a way of both revealing and at the same time educating their awareness. I consider the 'Structure of a Topic' framework as providing a structure with which to reveal aspects of a concept image held by learners. With a structure, two things are facilitated: identification of aspects of a concept image that cause difficulty in learners and probing of those and other aspects.

Placing emphasis slightly differently, Kilpatrick et al., (2001) suggest five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. These strands can be explained in terms of awareness, behaviour and emotion. Conceptual understanding can be related to awareness as relevant actions are brought to attention when engaging in activities. Procedural fluency is attained by training of behaviour, although the training of behaviour alone may result in inflexibility of thought and a lack of adaptive reasoning. More appropriately training of behaviour must be guided by educating awareness so that
students not only practice recently met ideas but also gain experience with new concepts. Productive disposition refers to developing habits of identifying and tackling problems, taking initiatives to construct examples and trying to justify conjectures. Productive disposition and strategic competence relate to emotion (or affect), as they deal with developing experience of identifying problem solutions and justifying conjectures. The ‘Structure of a Topic’ framework spoke to my experience and so I used it both to structure my interviews and to analyse people’s responses to probes and tasks.

In terms of the ‘Structure of a Topic’ framework, rote learning merely trains behaviour in applying rules and techniques. For actions to be mobilised appropriately, to ‘know-to-act’, they must be guided by awareness. I conjectured therefore that when learners’ awareness is educated as well as their behaviour trained, they are more likely to attend to structural aspects of tasks and not simply details that are particular to the context.

In what follows, I look at the role of examples in mathematics instruction and the possibilities of using example-construction to enrich learners’ understanding.

2.5 Example-construction as evidence of understanding

The traditional way of working out examples, I believe, trains behaviour in applying methods and procedures. By contrast, I see constructing examples as dealing with learners’ awareness of structural properties of examples and thus mobilising their energy to engage in actions to produce results. By prompting learners to construct examples meeting specified constraints, what they choose to change can reveal dimensions, depth and quality of their awareness. The next section looks at the role of examples in mathematics instruction.
2.5.1 Examples in mathematics instruction

Examples form an integral part of the discipline of mathematics, in which methods, rules and algorithms and properties are illustrated by reference to particular, hopefully familiar, objects. Ranging from use in mathematical reasoning to proof, the function of examples extends well beyond their role as teaching aids in mathematical instruction. The use of examples to illustrate mathematical procedures and concepts has its roots in the earliest known records about mathematics learning which include Egyptian papyri, Babylonian tablets and ancient Chinese manuscripts. Its use in instructional explanations and mathematical discourse (Leinhardt, 2001) and as a device to communicate concepts between teacher and learner (e.g. Bruner et al., 1956; Tall and Vinner, 1981; Peled and Zaslavsky, 1997) have been widely discussed. Although particular instances of diagrams, symbols and reasoning are often used, the essence of their use to convey generality can be potentially perceived to illustrate different things by teachers and learners. In elucidating and making explicit use of the principle of generic examples, Hilbert (Courant 1981) suggests:

if you want to solve a problem first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as can be made, without losing any of its punch, and then you solve it. The generalisation is a triviality which you don’t have to pay much attention to. This principle of Hilbert’s proved extremely useful for him and also for others who learned it from him. Unfortunately, it has been forgotten.

(Courant, 1981, p. 161)

Hilbert sees generality in one ‘good example’ and only needs one such example to appreciate the generality present. While teachers use examples as a tool to explain mathematical concepts with implied generality, learners are likely to develop the thinking that only those particular examples or only very similar ones, are appropriate, though the
intention is for them to see the general through the particular. It may be the case that teachers and learners are operating on different levels; learners on the details and teachers on a more general viewpoint. Put another way, teachers and learners attend to different aspects, and attend differently.

The different roles and uses of examples may have contributed to this situation. Sowder (1980) distinguishes between 'examples' and 'illustrations' (examples of the application of a procedure). Michener (1991) lists four kinds of example used by teachers: start up examples, reference examples, model examples and counterexamples. Bills et al. (2006) distinguish further 'worked(-out) examples' and 'exercises'. This broad category 'examples', can be further distinguished into 'generic examples', 'counter-examples' and 'non-examples'. MacHale (1980) suggests that learners often disregard examples that they are not comfortable or impressed with. Mason and Pimm (1984) say that while a teacher may intend an example to represent a general case, students may only see the particulars of the example and fail to see what it represents. They propose the use of generic examples to explain examples that represent general cases which ignore the details of the example itself. Ineffective use of examples may lead to fragmentation of knowledge since students may not see the general through the particular as they are expected to. Wilson (1990) suggests that there are irrelevant characteristics in all examples and that students are sometimes inconsistent in their understanding of what is required to make up an example and its relation to a definition.

Other researchers (e.g. Burger and Shaughnessy, 1986; Lakoff, 1987) report on students' difficulties in distinguishing definitional properties and characteristic properties of examples, which Mason (1998, p. 22) explains by differences in the structure of their attention. Peled and Zaslavsky (1997) asked 38 in-service teachers to generate at least one counterexample for each of two unfamiliar, false geometry statements and found that
both teachers and student-teachers produced both particular and general counterexamples, but the former produced more semi-general and general counterexamples. Zaslavsky and Lavie (2005) made classroom observations and interviewed 12 secondary school mathematics teachers with regards to their use of instructional examples and concluded that activities such as professional development opportunities for teachers in classroom events do affect teacher’s choice of examples.

2.5.2 Generalising and specialising in mathematics

Mason (2002a) suggests generalising and specializing as two facets of the same activity. Confronted with a general assertion, one will necessarily check whether and when it is true by considering special cases. Attending to how particular cases work can point to its generality. However, the way in which generic objects such as examples are presented to students is often rather implicit in nature. For example, when a teacher shows an example to students, s/he is attending to the particular with an eye to how the techniques work in general. Although the students are expected to do the same, if this awareness is not communicated to the student, they may not know which numbers are structural and which are particular. Mason (2002) writes about two kinds of perceptions with regards to generalising and particularising.

- seeing the particular in the general (seeing not just a general assertion but the opportunity to try out specific particular cases and being aware of what constitutes particular instances of the general);
- seeing the general through the particular (seeing specific numbers or other aspects as placeholders for other possibilities).

(Mason, 2002a, p. 108)
It must be acknowledged that learners cannot be expected to make the shift in seeing the speciality and generality of objects automatically. Polya (1973) recommends that “specialization is passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set” (p. 190). Although the intention of such specialization is to use it to gain insight and guidance towards a general solution, students sometimes use it to explain a general case and do not return to the general case (Hazzan, 1999).

2.5.3 Awareness and dimensions of variation

According to Marton and Booth (1997), our awareness of things around us varies, not in its presence but in its intensity. In mathematics, we are aware of parts of mathematics in general. However, in a moment, our attention is focused on parts of the mathematics that we consider important and it is likely that teachers focus on different aspects of mathematics than do learners. For example, learners may focus on properties of mathematical objects that assist their memory for examination purposes while teachers may focus on underlying concepts which the object is supposed to illuminate for understanding purposes. Marton recommends exposing learners to variations of examples so that they become aware of those aspects as possible dimensions of variation.

Marton suggests that learning involves making distinctions, both discerning something from, and relating it to, a context. Learners’ sensitivity to notice opportunity to discern, to abstract structural properties and to construct new mathematical objects by extending awareness of dimensions of variation associated with tasks, techniques, concepts and contexts could give insight into their conceptual understanding. Tasks which promote such opportunities could also help some, if not all, students appreciate mathematical concepts more deeply. Therefore, if learners work on awareness of properties of
mathematical objects through constructing examples, they are likely to make an informed choice from the repertoire of techniques available to them and engage confidently with fresh, unfamiliar situations as well as with familiar ones. As I have mentioned in Section 2.1.1, transfer of knowledge to different contexts is then a question of how well the situated knowledge is explored with active awareness and extended to a new situation with accuracy, efficiency, appropriateness and confidence.

Awareness is a complex phenomenon. Display of awareness is rarely the whole of awareness. When someone says something, they display awareness of but that does not mean they are not aware of other things. Although what someone is explicitly aware of at any moment does not mean that they are not aware of other aspects of the object, the focus of attention needs to be on certain aspects of the concepts at some time in order for the necessary connections to be made and to emerge with more sophisticated understanding (‘folding back’ (Pirie and Kieren (1989)). Therefore, an explicit and conscious effort on the part of learners to shift attention to structural properties of mathematical objects is imperative in order to discern similarities and differences, which can help them see generality through the particular. In relation to this, Gattegno (1987) proposes that knowing means stressing awareness of something and uses ‘awareness of that awareness’ to describe the knowing of a relative expert.

When one is aware of properties of mathematical objects, one is more likely to be sensitive to the similarities and differences between them and to what is invariant when other properties of the object are changing. Conversely, the way to become aware of properties is to focus on similarities and differences so as to be sensitive to dimensions-of-possible-variation (described in Section 2.5.6). As meaning emerges from “what is unchanged across change” (Plato), awareness of what can vary when other properties
remain invariant in mathematical objects plays a vital role in the development of meaning.

As I have decided to study the nature of learners’ understanding in fresh situations, I felt the need to ask “What is it that comes to mind in the new situation that is different from the previous encounters”? Perhaps a more pertinent question to ask is “What is it about the tasks that afford the kind of engagement that entails a meaningful activity”? A consideration of affordance, constraints and the necessary attunements has proved useful to highlight this issue.

2.5.4 Affordance, constraints and attunements

Effective engagement in mathematical activities depends not only on how learners perceive the tasks/problems but also on what the tasks themselves afford. Gibson (1979) suggested that learning takes place through perception of, and interaction with, an environment and coined the term *affordance* to refer to the complex interrelationship between organism and environment.

The affordances of the environment are what it offers the animal, what it provides or furnishes, either for good or ill. ... I mean by it something that refers to both the environment and the animal in a way that no existing term does. It implies the complementarity of the animal and the environment …

(Gibson, 1979, p. 127)

Greeno (1994) developed the notion of affordance within mathematics education and pointed to the shifting of attention from individuals processing information to an understanding of what information is available to use.

... cognitive processes are analyzed as relations between agents and other systems. This theoretical shift does not imply a denial of individual cognition as a theoretically important process. It does, however, involve a shift of the level of primary focus of cognitive analyses from
processes that can be attributed to individual agents to interactive processes in which agents participate, cooperatively, with other agents and with the physical systems that they interact with.

(Greeno, 1994, p. 337)

Greeno (1998) views affordance as "qualities of systems that can support interactions and therefore present possible interactions for an individual to participate in" (p. 9). Because the systems lend themselves to norms, effects and relations which limit their possibilities, constraints are seen as "if-then relations between types of solutions ... including regularities of social practices and of interactions" (p. 9). Alongside affordance and constraints, the individuals acting in the system develop attunements, which are things they are disposed or attuned to recognise as possibilities. While Gibson (1979) used attunements to refer to the kinds of disposal that learners come up with, in my view, attunements also include things needed in order to access the affordances.

Features in a mathematical task that learners attend to can indicate what they regard as important in that task. Tasks that invite learners to recall different methods of doing the task may promote flexibility but may only contribute to further training of behaviour in applying rules of mathematics. This could simply impoverish learners' experience with regards to the task/concept. Probes used in such a task might only afford opportunity to display behaviour in carrying out applications of mathematical rules.

However, tasks that invite learners to discern features that can vary or that must be invariant in mathematical objects can reveal features of the task that are stressed by learners (thus regarded as important in the moment) and suggest features that may be being overlooked or ignored. Aspects or dimensions learners choose to change in the object can reveal something of the dynamics and depth of their awareness (Watson and Mason, 2005). As it turned out, through becoming aware of more features of a concept not previously at the focus of their attention, learners in fact revealed to themselves
aspects of the concept that were not previously salient to them. Getting learners to talk about these aspects acts to promote and enrich their understanding and appreciation of the concept.

2.5.5 Example-construction

Several pieces of research focus on the use of examples to explain general mathematical phenomena. Coming up with examples, on the other hand, requires different cognitive skills from working out given examples. Watson and Mason (2005) argue that learners who are asked to search in and bring their own experience to mathematical lessons are more likely to attain the kind of mental empowerment that good mathematics learners seem to have intuitively. Learners need to look at mathematical objects in terms of their structure to discern properties that are structural (invariant) and those that are optional (variant). Example-construction involves certain awareness, on the part of the learner, of what is allowed to change so that the example still maintains its fundamental characteristics.

Dahlberg and Housman (1997) were interested in how mathematics undergraduates construct concepts for themselves and studied changes in the development of the personal concept image as revealed by learners’ expressions of understandings in verbal and written forms. Since it was the first time example-construction tasks were used in the lesson, the students found that having to construct an example from the definition proved to be the most significant aspect of their learning for some learners, and for others, reinventing the given definition for themselves was important. They found that students who consistently generated examples and reflected on the process attained a more complete understanding of mathematical concepts by refining and expanding their
evoked concept image. They found that example generation significantly influenced the development of students’ concept images.

Hazzan and Zazkis (1999) studied learners’ mathematical and emotional difficulties in constructing examples. They conjectured that “while students are working on generating particular examples of a mathematical concept which satisfies certain properties, they construct a more general notion in their mind” (p. 4). They continued “constructing meaningful mathematical concepts requires the recognition of similarities of general ideas in different particular examples to discover their common structure and attributes” (p. 4). They showed how students had difficulty managing degrees of freedom of generated examples. They pointed out that students had a strong tendency to check the correctness of examples since they were often uncertain about how to proceed and were troubled by the novelty of having to make choices in mathematics. Further, the authors described how the tasks required students to construct links between concepts and their properties. They concluded that such tasks as “Give an example of …” not only promoted richer learning opportunities but also enabled learners to make links among concepts.

Weber and Alcock (2004, 2005) studied undergraduates’ use of examples in reasoning in proof production. They wanted to know how learners differentiate between specific examples and formal definitions and how learners switch between them. They found that example use by such learners is often illustrative and empirical rather than general and deductive. They identified semantic and syntactic proof production in which learners who produce proofs semantically attend to underlying principles rather than to details of individual examples. Alcock and Weber (2005) then distinguished between referential and syntactical approaches to proof. They found that learners who use a syntactical approach often do not regard examples as promoting structural understanding and often need help in seeing examples as a general case. The learner who constructs proof using
specific examples might benefit from them to make sense of her work. However, constructing examples is not something learners do naturally.

While pedagogical aspects of constructing example have been broadly discussed (e.g. Watson and Mason, 2005; Zhu and Simon, 1987; Leinhardt, 1993), its use as a research tool has not been fully explored. Zazkis and Leikin (2007) explore its use in researching prospective teachers’ grasp of number theory.

In my research, the nature of examples constructed by learners from different academic backgrounds is studied. By studying aspects of learners’ awareness of dimensions-of-possible-variation as revealed in their examples, I seek to explore the different ways by which learners understand integration and the difficulties faced by learners in conceptualising integration in particular and indeed any mathematical topic generally.

2.5.6 Dimensions-of-possible-variation and example space

Although learners can have multiple awareness of many mathematical objects at any time, what comes to mind in the current situation is critical, because this is what informs and directs actions. Thus it is worthy of exploration. In constructing examples, learners are required to become aware of variant and invariant properties of the examples. Because teachers and learners may be attending to different features of an example, Watson and Mason (2005) extended Marton’s notion of “dimensions-of-variation” in examples to “dimensions-of-possible-variation” to highlight the nature of the potential difference in attention between and among teacher and students. It is crucial that learners appreciate the fact that examples are chosen from a whole class or space of examples. This led Watson and Mason to formulate the notion of an example space to refer to the collection of examples to which a learner has access. However, the examples that come to
mind in the moment depend on the situation that triggers them and thus, *accessible example space* is a more appropriate construct.

Encouraging students to generate examples of mathematical objects can expand their *example space* and shift their attention away from the examples to generalisations (Watson and Mason, 2005). Construction tasks actually have a dual purpose from a research point of view – to reveal learners’ awareness while at the same time affording the opportunity for educating that awareness. Generating examples can not only enrich students’ *example space* in terms of its content, but also provide an opportunity for students to explore its structure in terms of relationships among elements in the space, which in turn can reveal and alter students’ sense-of generality and enable them to develop construction techniques for extending their example spaces for themselves.

Zaslavsky and Lavie (2005) suggest that “specific elements and representation of an example or set of examples, and the respective focus of attention facilitated by the teacher, have bearing on what students notice, and consequently, on their mathematical understanding” (p. 2).

Using example-construction tasks as research tool, Zazkis and Leikin (2007) highlighted the need to establish criteria to evaluate examples generated by participants. They asked participants in a research project to give examples of odd numbers, arithmetic sequences, irrational numbers and multiple solution problems meeting various constraints. Based on the responses, they developed a framework through which to evaluate examples generated by the participants. They suggested that examples generated by participants can be used to analyse their knowledge and understanding by examining the quality and structure of the *example spaces* displayed and their relationships to the conventional ones. They suggested that example spaces can be described in terms of correctness and accessibility, richness and generality.
One of the features of example-construction that I noticed is displays of something of the aesthetic when the constructor (in this case, learners) rejects something as being unacceptable or acknowledges something as being good. This notion is discussed next.

2.5.7 Aesthetic component of example-construction

A sense of mathematical aesthetic enables learners to use value judgments to dismiss or appreciate something or evaluate other's contributions. I became sensitised to the aesthetic component when I analysed the undergraduate data. Although the notion of aesthetic component is not featured in the description of the 'Structure of a Topic' framework (Mason and Johnston-Wilder, 2004), it was very evident in the undergraduate data. That led me to go back through the transcripts from other students to look for evidence of mathematical aesthetic. A mixture of aesthetic judgments, competitiveness, and being caught up in a line of thinking influences a learner's motivation to engage in mathematical tasks. A sense of mathematical aesthetic contributes to motivation and a sense of personal identity as a competent learner. Language which can be interpreted as making value judgments about quality or appeal of an example or mathematical statement is a component of the emotional strand and a complex relationship with motivation. It was important in my study to record evidence of indications of an aesthetic component discernable in behaviour both in the interview and in the construction tasks. I was not looking for cause and effect but noting how being sensitised to the possibility of mathematical aesthetic can enrich the appreciation of what learners are doing and saying.

The role of aesthetic in the development and appreciation of mathematical knowledge is a fundamental one. Taking an objective viewpoint on aesthetic, Dreyfus and Eisenberg (1986) compared and contrasted responses which exhibited aesthetic dimensions. They questioned the extent to which learners can make aesthetic judgments and suggested that
aesthetic appreciation should be nurtured in learners. However, Sinclair (2004) and other mathematics educators including Papert (1978) suggest that aesthetic plays a more process-oriented and personal role in the development of mathematical knowledge. Sinclair writes,

"a student's aesthetic capacity is not simply equivalent to her ability to identify formal qualities such as economy, unexpectedness, or inevitability in mathematical entity. Rather, her aesthetic capacity relates to her sensibility in combining information and imagination when making purposeful decisions regarding meaning and pleasure."

(Sinclair, 2004, p. 262)

She identifies three distinct roles of aesthetic responses: evaluative, generative and motivational. The evaluative role of aesthetic "is involved in judgments about the beauty, elegance, and significance of entities such as proof and theorems" (p. 264). The generative role of aesthetic is "responsible for generating new ideas and insights that could not be derived by logical steps alone" (p. 264). The motivational role refers to "the aesthetic responses that attract mathematicians to certain problems" (p. 264). In my research, seeds of aesthetic judgments can be observed when learners evaluate their own and other's examples.

She also suggests that aesthetics are involved not only in the choice of structures (and, thus, the choice of mathematical objects, relationships and problems), but also in the communication of results about these structures.

In my research, I wanted to explore understanding of integration among learners from different backgrounds. As integration has many applications in science, students learn the topic for different purposes. I conjectured that for this reason, learners' understanding is structured differently. Therefore, I wanted to explore the nature of their understanding from the perspective of the structure of attention and awareness. Because learners are not
always articulate enough to express their understanding, I plan to contrast their responses in the interview with those in the construction tasks. A distinctive feature of my research is that, unlike other research which uses learners’ responses to inform the probes used and to categorise learners, I use probes in my research to inform my appreciation of the nature of these learners’ understanding and appreciation of integration.

2.6 Summary

In this chapter, aspects of learners’ understanding have been considered by reviewing different forms and models of understanding. Implied in these models of understanding is learners’ focus of attention on surface details and the ignoring of awareness of underlying structure in mathematical problems.

The cognitive development theories of Piaget and Bruner and the process-object encapsulation ideas of Sfard and Gray and Tall were reviewed in order to gain insight into the processes of conceptual development of mathematical topics. I used these ideas to make sense of the way in which learners come to understand.

Reviewing aspects of understanding highlighted the need to explore the components of understanding, which led to the consideration of the ‘Structure of a Topic’ framework as a tool to understand those processes and as a way to investigate the structure of learners’ understanding.

This brought me to the use of example-construction tasks as sources of evidence of understanding by providing opportunities for learners to reveal their awareness, based on Gattegno’s claim that ‘only awareness is educable’. By asking learners to construct examples that meet specified constraints, what they choose to change reveals the focus of their attention and aspects of their understanding.
Having reviewed the literature that was accessible to me, I highlighted a number of significant constructs that I find relevant to my study. The constructs that I find useful are those explaining different models of understanding (instrumental and relational, onion-layer, van Hiele levels), aspects of concept development (reification, abstraction, generalisation, procept, concept image) and tools to elicit evidence of understanding (example-construction, dimensions-of-possible-variation, example space). The data analysis in Chapters 7, 8a, 8b and 9 will either shed further light on the meaning of these constructs or use them to inform the interpretation of the data.
Chapter 3

Methodology and Methods

3.0 Introduction

I have stated, in the previous chapter, my view that understanding is an encompassing process, consideration of which involves not only what learners do (actions) but also what they are aware of. I contrasted this view with other studies which have considered learners’ understanding of integration that have been reported in the literature and deliberated on how my view relates to these studies. In this chapter, I outline the considerations that have shaped the development of the methods used in this study. In doing so, I draw upon the discussion in Chapter 2 to show how consideration of these critiques and my perspective on understanding of integration have shaped the development of the tasks used in this study.

3.1 Methodology

Underpinning my approach to the study is an approach known as phenomenography, identified by Marton (1981). As a methodological framework for identifying the variety of ways in which a person experiences a phenomenon, phenomenography fits closely with what I was seeking to achieve in my research, although not exclusively. Because I was interested to reveal learners’ awareness as one of the many components of understanding, I needed to probe further beneath the surface of superficial understanding to expose the nature of their awareness. The focus of my research is the exploration of the variations in experience that shape understanding, therefore I found my concerns related to phenomenography.
3.1.1 Phenomenography

Phenomenography, which originated in Gothenburg, Sweden in the 1970s, is a method of researching conceptions of learning and teaching (Marton, 1981; Saljö, 1982). Its applications in studying different ways of understanding the content of learning have been studied (Linder, 1989; Rensröm, Andersson and Marton, 1990).

In contrast to phenomenological research, in which the aim is to produce a ‘thick description’ of the phenomenon based on first-hand experience of the researcher (‘living in’ the phenomenon), phenomenography is the study of human understanding of specific phenomenon by using data from a range of subjects and producing a broad category to describe and highlight different ways of experiencing, perceiving, apprehending, understanding and conceptualising the phenomena in and aspects of the world around us (Marton, 1994). In my study, understanding of learners who study integration as a subject at school, who study it as a concept at university, and who study it as a tool for potential use in the workplace, is investigated. According to Marton (1988), since each phenomenon can be seen, experienced and understood in a limited number of idiosyncratically different ways, understanding is defined as the experiential relations between an individual and the phenomena, and changes in an individual’s understanding indicates a significant form of learning. Phenomenographic research aims to identify a limited number of qualitatively different ways of experiencing the phenomenon and to characterise these by focusing on relationships between them. This aim matches closely with what I hoped to achieve in relation to learners’ understanding of and thinking about integration.

Marton describes categories of description as consisting of a set of different understandings of the concept that indicate special characteristics. This, in turn, amounts to the formation of outcome space, the depiction of different ways in which the concept
has been understood. Reference to the outcome space makes it possible to compare the categories of description with one another to understand how appropriate is the understanding they represent. In my case, the way in which the students understood the content of learning (integration) is significantly related to the way in which they experienced their learning situation (the act of learning).

The different approaches learners adopt in a specific learning task result in a set of different understandings. These different ways of experiencing the specific situation call for a shift in focus of interest away from that which emerges from the specific situation towards learners' preconceived ideas about the phenomena dealt with in the situation (Marton, 1994). Consequently, learners' conception of what learning actually is affects the way in which they experience the act of learning, thus influencing the approach they adopt (Säljö, 1982; Marton, Dall’Alba and Beaty, 1992). In this study, learners' conception of understanding integration is studied in terms of their ability to discern dimensions-of-possible-variation and corresponding range-of-permissible-change. Following Marton and Booth (1997), it is assumed that revealing learners' awareness of possible dimensions in mathematical objects that can vary provides insight into what they consider as understanding. Further, a way of experiencing a phenomenon reveals something not only about the experienced phenomenon but also about the experiencing subject (Marton, 1988). This study of students' understanding of integration makes a contribution to the categorisation of integration by studying learners' different structures of understanding.

Phenomenographic data relates to many people's experience of the phenomenon under study. Data consists of many subjects' ways of experiencing the phenomenon, which is used to produce broad categories of description. The researcher then seeks to identify a limited number of qualitatively different ways of experiencing the phenomenon and to
characterise them by highlighting the relationships between them. In some criteria, hierarchies to show difference in depth are possible. Because my objective is to characterise understanding of learners from different backgrounds, the phenomenographic approach matches closely with what I hoped to achieve in my study.

Data in phenomenographic studies comes in the form of in-depth interviews, in which subjects are encouraged to bring to the surface aspects of the phenomenon which might not naturally come to surface for them. In other words, aspects that may not be the focus of attention and thus reside in the background are often brought to the foreground. In relation to this, Marton (1994) says, “the more we can make things which are unthematised and implicit into objects of reflection, and hence thematised and explicit, the more fully do we explore awareness” (p. 4427). Revealing implicit aspects of the subject’s experience involves encouraging them to express awareness of these aspects explicitly, which necessitates a genuine dialogue between the interviewer and the interviewee that addresses this issue.

Because data involves a range of possible responses from different subjects, analysis involves identifying “similarities and differences between the ways in which the phenomenon appears to the participants”. This requires the identification of “distinct ways of understanding (or experiencing) the phenomenon” by grouping the responses according to similar and different features. The way in which the group of responses is related and the nature of the relationship are investigated. What emerges is an ordered or semi-ordered set of categories of descriptions known as the outcome space, which becomes the outcome of the study.
3.1.2 Interviews

Some previous research into students’ understanding of integration has focused on their answers to integration problems and problem solving, using the answers to say something about the respondents (e.g. Rasslan, 2002; Sealey, 2006). Interviews have been used in conjunction with questionnaires in other research based on integration problems (e.g. Hong and Thomas, 1997).

My concern in the present study has been to explore the nature of understanding of integration. I wanted to probe learners’ understanding to a greater extent than only to study their behaviour in reproducing methods and techniques. I imagined using semi-structured interviews structured around a sequence of example-construction tasks. The tasks and the relevant probes were constructed to provide opportunity to reveal learners’ awareness of integration, which suggest where their attention is and which characterise the structure of their understanding.

I could have sat in lectures and made observations but these methods would not provide me with the kind of in-depth exposure to learners’ thinking processes. I chose to audio-tape and not to video-tape the interviews because not only would video involve ethical issues, my trials with audio-tape provided a sufficiently fine-grained level of details for the comparisons I wanted to make.

The clinical interview procedures developed by Piaget (and later developed by Ginsburg, 1981) seem very relevant to the kind of study that I intended to undertake. Ginsburg suggests that clinical interviews can be used in research to discover cognitive activities, to identify cognitive activities and to evaluate levels of competence. In my research, I sought to expose learners’ cognitive activities when doing integration. I did not intend to
identify cognitive activities recognised prior to the study nor did I seek to evaluate their competency.

My interactions with the subjects in this study come in two forms: semi-structured interviews and construction tasks. For the purpose of distinguishing the two kinds of interactions, I refer to the semi-structured interviews as interview and the construction tasks as construction tasks throughout this thesis.

3.1.3 Design and conduct of interviews

Kvale (1996) suggests that the aim of a qualitative research interview is to obtain a qualitative description of the interviewee’s lived experience and how meaning emerges for the interviewee from their experience. According to Kvale, the outcome of an interview is “knowledge … created inter the points of view of the interviewer and the interviewee” (p. 124). In a semi-structured interview, data emerging from it is an exchange between the interviewer, who tries to maintain the structure of the interview, and the interviewee, who is allowed to respond freely but be within the domain of the focus of the interview. No pre-formulated scripts are used and the interviewees are probed based on their responses as these emerge. A semi-structured interview, according to Kvale,

... has a sequence of themes to be covered, as well as suggested questions. Yet at the same time there is an openness to changes of sequence and forms of questions in order to follow up the answers given and the stories told by the subjects.

(Kvale 1996, p. 124)

Interview questions and probes need to be short, clear and straightforward so that the interviewee understands them and knows how to respond. Words that are likely to
mislead or confuse the interviewees must be avoided. Kvale recommends that short, easy-to-understand language, be used to keep the flow of the dialogue going.

Reading Kvale, I was particularly struck by the extent and the nature of probing beneath the surface of learners’ understanding, which proved to be more difficult than I first imagined. On analysing the preliminary and the pilot interviews, I realised that I was adhering too much to the questions I had prepared and did not follow up on the interviewee’s responses appropriately. My attention was focused on getting answers to my probes and not so much on what the interviewees had to say. My role was more of a teacher, coaching learners to guess what was in my mind, rather than a researcher trying to uncover their awareness and understanding. Arising from Kvale’s advice, I was alerted to watch out for the interviewees’ responses and to intervene only to guide the conversation in the desired direction or to encourage the interviewee to elaborate on the ideas expressed, as the aim was to get them talking about aspects of phenomena beneath the surface of consciousness they had not considered previously. I learned that excessive and unnecessary interventions may disrupt the flow of ideas and end up in the interviewer listening only to what s/he wants to hear. I also realised that an orchestrated and dishonest response from the interviewee might not reveal aspects of their understanding that I wanted to probe.

Since the primary aim of the interviewer is to probe beneath the surface of understanding, the interviewer must strive to keep the process as productive as possible. As the interview progresses, the interviewee may think that they have reached the boundaries of their understanding and may lose interest in the session, either because they did not understand the questions or because they could not withstand the interview process any longer. Or, as in my research where the interviewee is dealing with novel situations, it could also be the case that the interviewee feels that their understanding is being challenged and decides
not to continue with the interview. In either case, rephrasing the questions or reminding the interviewee about the purpose of the interview at this point can yield a more open exchange and enable deeper probing into the interviewee’s understanding.

Towards the end of the session, the interviewer ought to allow the interviewee to ask any further questions or comments. In order to emerge from the experience after a series of challenging questions, Kvale (1996) recommends that the interviewee be allowed time to let go and reflect on the experience.

Subsequent interviews will draw upon the experience and the kind of responses obtained from preceding interviews. Thus, it was important in the analysis to highlight issues that need to be explored further in later interviews.

### 3.2 Methods

My study has gone through several phases.

I conducted a preliminary study with three subjects in order to test out my interview skills, the kinds of interview questions, and the example-construction tasks: this is described in Chapter 4.

I conducted a pilot study with twelve students studying A-level to confirm and refine my techniques, to test out my conjectures and to practice analysis. This stage of the study is described in Chapter 5.

I conducted a main study with four PGCE mathematics students (described in Chapter 7), eleven mathematics undergraduates (described in Chapters 8a and 8b) and three engineering undergraduates (described in Chapter 9).

This inquiry started with my own search for explanations regarding the nature of students’ understanding of integration. It began with a preliminary study, to see how
different groups of people think about what integration meant and how they understood it. Early versions of example-construction tasks were used, which were later refined. This stage of the study also served as a testing ground to reacquaint myself with interviewing skills. The results from this stage helped in the development of the next phases of the study.

3.2.1 The pilot study

The pilot study involved a series of clinical interviews with twelve A-level Mathematics students. The interview sessions were organised around a number of questions generated based on the constituents of the ‘Structure of a Topic’ framework and a collection of example-construction tasks, which are described in Section 3.2.2.2 below. Interviews were tape-recorded and partly transcribed.

I decided to carry out the interviews, in pairs where possible, because I anticipated that students’ anxiety of being ‘under the spotlight’ might be reduced and they might feel more relaxed and free to respond as they would not be put to test. In some cases, there were not enough participants to work in pairs. In such cases, interviews were done with single subjects.

I also took field notes both during and after the interviews to remind me of the subjects, and the conduct and mood of the interviews. The field notes involved a description of the subjects’ body language, significant gestures and intensity in voice tones in order to describe the interview process better.

I chose not to transcribe all the interview sessions but to transcribe portions of the interviews that struck my attention as displays of interviewee’s aspects of understanding. I listened to the audio-taped interviews twice for each interview session in order to try to understand better what the students were saying and to try to establish reasons for what
they said. This was done by piecing together their articulations, gestures, body language and time spent on the task. These aspects contributed to enhanced transparency and trustworthiness of the interpretation of the interviews.

Analysis of the pilot study interviews included full accounts of the interview along with detailed accounts for them. Mason (2002) uses *account-of* to describe "as objectively as possible by minimizing emotive terms, evaluation, judgements and explanation" in order to "draw attention to or to resonate with experience of some phenomenon" (p. 40). He uses *accounting-for* to describe "explanation, theorising and perhaps judgements and evaluation" (p. 40) about a person's experience of the phenomenon. Although the distinction between these two forms of reporting an observed phenomenon is very subtle and one can easily slip into accounting-for when presenting accounts-of, the distinction is useful so as to try to present as truthful an account of the phenomenon as possible, removing the observer's value judgements. Perhaps more importantly, the accounts-of enable us to "learn about the sensitivities of the observer as well as about the incident" (Mason, 2002, p. 40).

Sometimes analysis coincides with the interview process itself. In such cases, it is important that the researcher is clear about the distinction between *accounts-of* the interview and *accounting-for* data in the analysis. Though I tried to be as objective as possible, I admit that it may not be possible to remain so at all times. *Accounts-of* the interviews I conducted included a full record of the examples constructed by the interviewee and field notes of the interviewee's voice tone, flow and fluency of language with which the interviewees expressed their ideas. The amount of time they took with a task, including pauses and hesitations, was also recorded.

When all the interviews had been completed, pertinent portions were transcribed and analysed, the interviews were compiled and further analysis was done to identify themes
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across the interviews. This part of the analysis was rather complicated because my aim was to compare and contrast responses given during the interview with those revealed through the construction tasks. Although contrasting responses in the aspects probed in terms of depth in what is expressed and revealed was rather apparent in the transcripts, a way of appreciating and displaying the contrast was needed.

3.2.2 The main study

The main study consisted of further clinical interviews with three groups of students: PGCE (Mathematics), Mathematics and Engineering undergraduates. The same set of questions and tasks that was used in the pilot study was used, with some modification to the probes in the construction task in order to gain clearer insight into the interviewee’s thinking. I decided to remove the application task (distance travelled), described in Section 5.1.2 from the tasks to be used in the main study. This decision was made because it occurred to me that the task was not revealing any significant findings. The finding from the pilot study was enough to convince me that many learners have a general predisposition to opt to use prescribed formula rather than thinking geometrically, consistent with Dreyfus (1994).

This time I paid close attention to the depth displayed in the aspects probed. Again, the interviews were tape-recorded, transcribed and analysed, and augmented with field notes.

In the analysis of the main study interviews, I identified and classified depths in responses into three categories: surface, immersed and deep sense of understanding. I realise that complete categorisation was impossible due to the complex nature of human understanding but the three categories that I identified seem to give a reasonable representation of that complexity. Based on these categories, I devised a method for
representing and appreciating visually the different depths in conceptions using what I called *density shapes*. I discuss this in detail in Chapter 6.

In order to strike a balance between qualitative and quantitative approach to data collection, this study draws upon both types of data. Supplementing qualitative data with quantitative data enables important features of both approaches to be utilised effectively. A set of questionnaires was administered to some 200 students in a calculus course (who were doing integration at that time) at a local university. The purpose of the questionnaire was not only to get a sense of variation in response from a wider population but also to substantiate the research findings by considering quantitative data. Data was collected through questionnaires which were administered electronically using Maple at the end of the lecture. Students for the interviews were selected opportunistically on the basis of voluntary participation. To avoid redundancy and to maintain the originality of the tasks, I wanted to conduct the interviews first before the whole-class questionnaires were to be administered. I wanted to capture the nature of the students’ engagement in fresh situations in order to shed light into their difficulties and thus, their understanding. However, none of the students was available for interview, although 21 students responded to the questionnaire. The responses are discussed in Chapter 5.

### 3.2.3 Permission

Permission was obtained from the participants for their responses to be used in the study. In order to protect the anonymity of the participants, names used in the study are pseudonyms.

### 3.2.4 Design of interview questions

The semi-structured interview came in addition to *construction tasks*. The development of the construction tasks is discussed in section 3.2.5 below. The interview centred
around a set of questions which was formulated to incorporate the constituents of the 'Structure of a Topic' framework. The six components that make up the framework are: language patterns and prior knowledge, techniques, common misconceptions, images and associations, scope of contexts and root problems. Because I was seeking to get a sense of learners’ current conceptions of integration, the questions were chosen to reveal not only their knowledge of techniques and misconceptions of integration, as some research does, but also to reveal their awareness.

3.2.5 Development of example-construction tasks

Previous pieces of research have attempted to investigate learners’ understanding of integration by exploring their flexibility in applying techniques of integration in slightly modified conditions than what learners are used to. In Chapter 2, I discussed the various tasks that have been used in these studies to investigate learners’ understanding of integration. I showed that what is often explored is learners’ behaviour in using procedures to carry out methods of integration. This, I argued, was only exposing their behaviour, which is trained to react to stimuli in a particular way. Responses to different tasks are taken as evidence of flexibility in understanding. I was not convinced that the conclusions reached in this way provided evidence of acts displaying creativity, as expected of learners who have rich understanding. Furthermore, aspects of understanding that are not displayed are not thereby proved to be absent from the mind of the learner. What is lacking in these pieces of research is a way of revealing what learners are aware of at any moment as indicative of the richness in their conceptions. I proposed the use of example-construction tasks as a way of exposing the complexity in learners understanding of integration.
In an attempt to expose the nature of learners’ awareness and to investigate their understanding given a fresh situation, I developed example-construction tasks meeting specified constraints for use in the interview. Although learners do not normally encounter this type of task in their normal learning environment, I conjectured that the tasks themselves could reveal learners’ knowledge of technique, dimensions-of-possible-variation of which they were aware and so reveal something of their awareness. Asking learners to engage in unfamiliar questions may indeed reveal whether or not they are flexible in making use of what has been conceived in contextualised setting in order to address the unfamiliar situation, by drawing on their productive disposition (Kilpatrick, 2001) and whether they make the necessary links/connections required to make sense of the new situation. In my research, I asked subjects to name the different techniques of integration in the *interview* and in the *construction task*, I gave them different examples of methods of integration and asked them to construct similar examples that are simpler and more complex. Constructing examples with certain characteristics reveals not only learners’ discernment of dimensions-of-possible-variation, but also their sense of understanding, through the expression of generality in the examples they construct.

To investigate learners’ conception of integration as both the reverse process of integration and a tool to calculate area, I gave the interviewees an integral that gives the answer zero and invited them to give another example of integral that also gives the answer zero.

\[ \int_0^2 (1 - x) \, dx = 0. \] Can you find another example like this for which the answer is 0?

With this task, I hoped to explore whether learners have a sufficiently rich understanding to see an integral as an object and a process at the same time (Gray and Tall, 1994). The interviewee was then asked to give a second example after the first example. Then, the
The interviewee was asked to give a third example. The intention was to shift learners' attention from the process of integration to integrals as objects and to see whether learners could make this shift. By asking interviewees to construct another example and another, I hoped to provide opportunity for them to shift their attention from perceiving the object as a particular instance to a representative of a general case. Mason (1989) suggests that to ‘abstract’ is to ‘draw away’ and that abstracting is a “delicate shift of attention” when learners draw away from the particular to the general in an “extremely brief moment” (p. 2). This shift of attention is necessary to enable abstraction of aspects that are variant and invariant in the example and to generalise.

The second task was designed to explore learners’ conception of integration as the reverse process of differentiation and to see whether they have awareness of the process when doing integration problems. The task invites learners to construct simpler and more complicated examples based on a mathematical statement that relates both the integral and the differential. I was inspired to design this task based on the kind of tasks typically used to assess learners’ understanding of the relationship between addition and subtraction, such as:

\[
15 + 8 = \square - 7
\]

I figured that tasks such as this highlight learners’ propensity to ‘give an answer’ rather than to inspect the statement to look for properties and discern relationships. In the task that I designed, an example of a statement relating an integral, the answer to the integral and the differential of the answer was given.

Given that \[2\int (\ln x + \frac{3}{2})dx = 2x \ln x + x = \frac{d}{dx}x^2 \ln x\]. Construct another integral, with its two corresponding expressions, which is simpler, and one which is more complex.
The idea was to explore the nature of learners' conception of the relationship between integration and differentiation. I deliberately left the arbitrary constants out of the expressions because I wanted to focus on the interviewees' awareness of what could change in the example and because the constants might confuse them. By asking them to construct a simpler and a more complicated example, I hoped to reveal learners' awareness of variant and invariant aspects in the integral and to gain insight into their sense of generality. Also, I hoped that by considering what makes an example simpler and what makes it more complicated, the interviewees would attend to what can be varied in the example. I expected that this might lead them to realise that complicated problems are, in fact, only complicated when you do not know what to do. They may also appreciate that tinkering with certain dimensions makes the example appear simpler or more complicated.

Subsequent tasks invited learners to construct simpler and more complicated examples of integrals which involve different methods of integration.

I must mention that it was not possible to do all the tasks with all the interviewees. However, I tried to get the interviewees to attempt to do as many tasks as possible when time permitted. The intention of inviting learners to construct simpler and more complex examples was to investigate the extent of their awareness of dimensions-of-possible-variation. Also, the aim was to explore the way in which learners appreciate the shift in attention from the details of the example to what is invariant and thus, structural. Of course, details of examples not mentioned do not mean that they are insignificant, nor does it mean that the person is unaware of them, but only that they were not considered important enough or relevant enough to mention. I concur with both Marton (1997) and Gattegno (1987) in their observations that experiencing a phenomenon entails back-
grounding and foregrounding: stressing certain aspects of the phenomenon and ignoring others.

3.2.6 The role of the interviewer

In probing beneath the surface of learners' conceptions, I saw my role as a researcher as being aware of the nature of my intervention. My role was not to interrogate, which could result in interviewee feeling too uncomfortable to open up and to access conceptions beneath the level of consciousness. Rather, I imagined a more persistent role to help the interviewee access domains that form part of the interviewee's conceptions which might not have been accessed or attended to before.

The kinds of task that are used in my research are novel to the interviewees. They were invited to construct examples meeting specified constraints rather than work out familiar or slightly modified examples (problems). In so doing, there is a strong possibility that interviewees might have reserved their ideas due to the unfamiliar nature of situation they were working in. My hope was that working on such tasks would stimulate learners' creativity and that they would appreciate aspects of understanding that the tasks were trying to develop. I planned to let them work out the example given, if they wished, and to invite them to articulate their thinking and reasons behind the examples they construct. I assumed that at some point during the interview after the interviewee had responded, simply looking at the interviewee without posing any further questions might prompt them to articulate their thinking and to explain further. Due to the novelty of the tasks used in my research, I was mindful that the interviewees might not understand what to do and, thus, might miss the real meaning of the tasks and so, I must make sure they understood the tasks.
3.3 Analysis of interview data

Analysis of interview data requires detailed description of the interview process. This description includes both what is said by the interviewee and how it is said. In my interview, a careful inspection of the examples constructed by the interviewee and a detailed record of their articulation for their choice of examples and the nature of the probes used was essential. Apart from this, the intensity of their voice tones and the fluency and flow of their language also informed me about their attitude and disposition towards the topic. To this end, I needed to pay close attention to aspects of the interviewee’s actions and reactions to the probes used in the interview.

For the analysis, I needed to prepare detailed accounts of the interview and then to account for the aspects addressed by the interviewee as fully as possible. In my research, I wanted to test out the ‘Structure of a Topic’ framework both for generating research probes and for analyzing the outcomes of the interviews. Therefore, I planned to classify the interviewee’s responses according to the depth and richness of the six aspects highlighted by the framework. This method of categorising includes responses to both the interview questions and to the example-construction tasks. By categorizing the responses according to the depth of understanding, I hoped to gain clearer insight into the complexity of learners’ understanding both in terms of what was said in the interview and of what is revealed through the construction tasks.

It needs to be mentioned that aspects of understanding that are displayed in an interview may not necessarily reveal all that the interviewee knows. In other words, the fact that the interviewee did not make reference to certain aspects in the interview does not suggest that they were not aware of them (see p. 56). In Chapter 2, I argued that awareness forms the core of understanding and that revealing learners’ awareness of aspects that form their
understanding is the aim of the example-construction tasks. Learners may be aware of certain associations/connections to integration but may have chosen not to mention those associations either because they were not triggered and therefore, did not come to mind, or because they did not seem relevant. In cases when subjects did not respond to a particular probe, I assumed that either the subjects had limited experience in these areas and therefore, did not have a lot to say or found no reason to express it at the time or the interview did not provoke responses in such areas. One would expect that if these aspects were significant, the kinds of probes that I have used would have revealed something of the nature of their understanding. A method of visually contrasting the different depths in responses in the interview and depths in aspects revealed through the construction tasks was developed and is discussed in Chapter 6.

3.4 The next few chapters

The next chapters report on what happened at each stage of the research and the conclusions reached. In Chapter 4, I present a summary of the informal first round investigation, during which I had the chance to try out the design of my study and to get a sense of the range of variations in what it means to understand integration as a topic. This chapter also reports on the pilot study, including a detailed description of the tasks used and a comprehensive analysis of data that resulted. The themes that emerged from the analysis are then identified and discussed. A major theme that emerged was the extent to which learners' awareness was revealed through the example-construction tasks. A way of displaying the different depths in awareness was needed and this forms the basis for Chapter 5. In Chapters 6, 7a, 7b and 8, I discuss the data and analysis of interviews from different groups of learners, namely PGCE students, average mathematics students, strong mathematics students and engineering students, respectively. Chapter 9 brings
together the results from the main study. And finally, I draw upon the findings of Chapters 6, 7a, 7b and 8 to conclude the study and discuss future research.
Chapter 4

Pilot study

4.0 Introduction

The pilot study tested the design and choice of questions to be used in the main study and the methods of collecting and analysing data. It also tested the effects of the construction task in revealing something about learners’ awareness. I begin this chapter with a summary of lessons learnt from a preliminary investigation. This is followed by a description of the design of the questions and tasks in the pilot study, the participants and how they were selected, and the methods of collection, transcription and analysis of the data. The purpose of this chapter is not to describe the analysis in detail but to give an overview of the type of data and some of the considerations that eventually informed the analysis in the main study.

4.1 Summary of preliminary investigation

The preliminary investigation was intended to test out the questions and tasks that were prepared and to refine my ability to conduct educational research. Since this was a preliminary investigation, it was not relevant to include transcripts and detailed descriptions. The investigation involved three participants: an A-level student, a mathematics educator and an IT lecturer. The participants were chosen from a colleague’s acquaintances, mainly to get a selection of people with different backgrounds. It was thought that their different backgrounds might reveal exposure to, and possibly understanding of integration in different ways; the student learning integration at school, the mathematics educator involved with teacher training at IOI
university and the user of mathematical concepts as a tool in the workplace. Of course the student and the educator may have had different exposure to integration and as such, I was not trying to compare the two or focus on aspects deficient in their responses. What I was interested in is to glimpse indications of what is available in the topic and what they are exposed to. Also, their backgrounds were similar to the backgrounds of students whom I expect to participate in the main study. The interviews were carried out individually and tape-recorded.

The preliminary study revealed a range of responses to be expected from learners from different backgrounds with regards to their understanding of integration. The different ways in which learners experience and understand a mathematical topic such as integration could point to the nature and structure of differences in how learners experience and understand mathematical topics in general. It also emerged that some aspects of integration may be the focus of learners’ attention/awareness and therefore stressed, while other aspects may be overlooked and thus pushed to the background. The investigation highlighted the fact that learners tended to overlook the significance of one or more of the strands in the six-fold framework, particularly concept associations, common errors and scope of contexts in which the topic was used. To a large extent, their attention was dominated by techniques. I intended to look for evidence that learners with different backgrounds might highlight different aspects of integration with varying intensities.

What transpired from the study also resonated with the works of Hazzan & Zazkis (1997) and Dahlberg & Housman (1997) and the theories of Marton (1997) and Watson & Mason (2005) that revealing learners’ awareness could support richer experience of learning mathematical topics. There was evidence that learners placed emphasis on technique without always displaying rich connections to suggest deep understanding. It
was also apparent that the construction task, which was a rather unfamiliar setting for the three interviewees, seemed to reveal a richer sense of connections for some learners than had emerged from the interview. This preliminary investigation inspired me to use variations of the construction task to try to reveal something about the scope and nature of learners’ awareness.

The preliminary investigation also drew attention to the fact that learners appeared to be relating to integration superficially. What they said in the interview did not always reappear in their responses in unfamiliar, novel situations such as the construction task. It may be the case that learners have a richer sense of connections with the topic than is revealed or fostered in normal settings. I planned to probe the (mis)match between what was said and what was revealed through the construction tasks.

4.2 Design of the questions/tasks in the pilot study

The questions come in three parts: first is a set of questions used in the semi-structured interview, which is discussed further in the next section. Secondly, the application task, which is described in Section 5.1.2. Finally, the example-construction task, which is elaborated further in Section 5.1.3.

4.2.1 The interview questions

The design of interview questions was based on the constituents of the ‘Structure of a Topic’ framework. I used the framework to distinguish between behaviour with technique, and language used in association with techniques, and awarenessess or concept images that might inform the carrying out of procedures.

Questions that sought to reveal awareness included:
Chapter 4

1. *What does the word “integration” mean to you?*

2. *What does the sign ∫ bring to your mind?*

3. *What have you discovered that you need to watch out for when you are doing integration?*

4. *What differences are there between ∫ x^2 dx from 2 to 0 and ∫ x^2 dx?*

These questions were designed to reveal awareness in relation to the concept image, associations, connections and also classic errors that are likely to be made by learners in the course of learning the topic. The questions were meant to elicit learners’ awareness of things that they might have struggled over or misconstrued with regards to what they normally do wrongly and why, and also ways to prevent them. The questions also sought to reveal learners’ mental images, diagrams, idiosyncratic ways of thinking and connections with other topics and what they stressed and consequently what they suppressed. Learners’ awareness of these aspects could influence their behaviour in using appropriate language when speaking about integration and in using appropriate technique when working on integration problems.

Questions that sought to reveal behaviour included:

1. *What are some of the technical terms that you can think of that are associated with integration?*

2. *What are the words in a problem/context that tell you that integration is relevant?*

3. *What techniques do you use most often for finding integrals?*

These questions were intended to reveal learners’ behaviour in terms of knowledge of technical terms, and words that trigger integration and techniques associated with it. Learners’ knowledge of terms/phrases that they may already know/use but in a less
formal manner together with language patterns that mark competence and understanding were expected to come to the surface.

Questions that sought to reveal emotion included:

1. *How were you introduced to the topic?*

2. *What kinds of problems does integration help to solve?*

Attention was also paid to the enthusiasm and disposition towards the topic, as revealed by voice tones, gestures and body postures and recorded as field notes. Motivation to learn was considered by attempting to locate surprises in learners from regarding integration as an ordinary subject for passing examinations to an actual appreciation of the topic as having real-life applications.

I had hoped that responses to the interview questions might reveal some sense of learners’ understanding with regard to the aspects probed.

### 4.2.2 Application of integration

To investigate further learners’ sense of awareness and behaviour in a problem situation, an application of integration task (Task 2: Finding distance travelled) was given. It transpired from my reading that learners prefer to use procedural skills and are reluctant to use geometric interpretations of situations (Dreyfus & Eisenberg, 1991), so it seemed reasonable to use a task which could be approached from an area or geometric point of view.

*A biker starts from rest and accelerates uniformly to 200 m/min for 1 minute and then coasts for 5 minutes. Below is the graph of velocity (v) against time (t) for the journey.*
1. Calculate the total distance travelled in the 6 minutes.

2. Given areas A and B underneath the graph:

2.1 What is the significance of area A and area B?

2.2 What would a sketch of the distance travelled graph look like?

**Figure 4.1: Task 2 – Finding distance travelled**

It was hoped that this task would reveal something about learners’ use of techniques as informed by their awareness of integration. This question was given in order to test whether claims about graphs at the heart of integration carried through into practice. Learners’ flexibility in using techniques of integration, and the possible links they could make with images and diagrams, were also intended to be considered in this question.
4.2.3 Example-construction

Task 3 consisted of a construction task in which students were asked to construct examples meeting specified constraints. The conjecture was that prompting learners to construct mathematical objects would reveal dimensions, depth and nature of their awareness. This is just one of the components of the development and exploitation of the framework for what it means to understand and appreciate a mathematical topic. The construction tasks were intended to reveal learners’ ability and disposition to cope with novel and unfamiliar situations with regard to their understanding. In addition to the task used in the preliminary investigation, two more questions were added.

I wanted to use students’ responses to constructing examples in which certain constraints were imposed in order to look for alignment between what they do in practice and what they say they do (from the interview). I also wanted to explore the potential of example-construction in revealing something about the structure of learners’ understanding of integration as they become aware of the properties of the topic. Learner’s flexibility in extending what they know in a familiar situation to a novel is one of the factors. The extent to which they coped beyond what was familiar could indicate the richness of their understanding. The research literature in Chapter 2 suggests that learners’ ability to extend mathematical knowledge into unfamiliar situations and beyond by generalising marks competence and understanding. The three examples they generate one after another could provide insight into their awareness and understanding by forcing them to see beyond the particulars of one example. Learners may soon realise that they are expected to come up with different kinds of example that are generalised by extending their example space (Watson & Mason, 2005).

In order to reveal something about learners’ conception of integration as area, I asked them the following question:
Given $\int_0^2 (1-x)dx = 0$. Can you find another example like this for which the answer is 0?

The following task was added to the construction task:

$\int_0^2 (1-x)dx = 0$

$\int_0^2 2(1-x)dx = 0$

$\int_0^2 \frac{1-x}{2}dx = 0$

*What is the same and what is different in the three integrals?*

The reason for adding this extra task was that the construction task appeared to be too unfamiliar for participants in the preliminary study, especially the student. It was not very revealing as the subjects resorted to a ‘don’t know’ response. This extra task, which was used only if the subject did not construct any example in Task 3, invited them to discern features that are the same and those that are different in the examples. It was intended to reveal something about the learners’ awareness of dimensions-of-possible-variation. In cases when time permitted, this additional question was asked:

*What can you say about $\int_0(2-x)dx$?*

This question aimed at further revealing learners’ associations of the object and their fluency with language in describing it.

The preliminary investigation had suggested that learners’ awareness of dimensions-of-possible-variation of mathematical objects varied considerably and so could be used to indicate the depth or richness of their understanding and appreciation of integration. Thus, if the students did have some kind of awareness of the variant properties of integrals, they were asked to generate more examples in order to reveal their sense of
generality and range of permissible change. Where students displayed naïve attempts at the “Give another example of …” question, their awareness of dimension-of-possible-variations and range-of-permissible-change were probed by asking the question “What can you change …”. On the other hand, if they did not reveal any sense of connections in terms of variable properties of the objects, questions seeking to reveal awareness in terms of discerning similarities and differences in three mathematical objects were asked.

The pilot study tested the final design of questions and how they were going to be analyzed in the main study. The following is a discussion of how the data was collected and how it was analyzed.

4.3 Data collection

Two kinds of data were collected in this study: quantitative and qualitative. In this section, I discuss how these data were collected.

4.3.1 Quantitative data collection method

A pilot study was arranged with a colleague teaching mathematics education at a university in the Midlands, United Kingdom to arrange for students to participate in the study. Two sets of data were sought: interviews with three or four pairs of first-year Mathematics undergraduates studying calculus, and questionnaires for all students on the course to respond to the same set of questions. However, the intended interview sessions did not take place because no student was available for interviews at the time due to other commitments. However, 18 students did respond to the questionnaire. The analysis of responses to this questionnaire will be presented later in section 4.4.
4.3.2 Qualitative data collection method

Another pilot study was arranged with a teacher from a local high school in the South East to get some students studying A-level to take part in the study. The selection of participants was done by the teacher, who was teaching the class. Six pairs of Year 12 students studying A-level (Mechanics and Statistics) participated in the study and were interviewed regarding their understanding of integration. They had studied calculus (differentiation and integration) in previous lessons.

Each interview was recorded on audiotape and later transcribed. The interviews were done in pairs in order to capture any kind of collaborative activities that might take place and how learners’ understanding developed in view of this collaboration. Working in pairs might be expected to prompt dialogue and so externalise their thinking. Sufficient time was allotted for each task so that interviewees had enough time to answer all the questions. During the interviews, some field notes of issues that were particularly striking were made. The interviews were planned to last about 45 minutes, although in some cases, they lasted only 30 minutes while others lasted as long as 90 minutes. At the start of each interview, the interviewees were told that I was interested in their understanding of integration and that they would remain anonymous. Since my interest was in their thinking, I encouraged them to talk to me about what they were thinking in the activities that followed.

The same set of tasks that was used in the preliminary investigation was used with the A-level students in the pilot study, except for the example generation task, in which some modifications were added in order to better understand learners’ responses and to reveal their sense of connections and relationships. The objective was not only to study students’ understanding when confronted with fresh, novel situations but also at the same time to circumvent naïve attempts at construction tasks.
The analysis was done by making a detailed transcription of the interviews from the audiotape. I also considered my own notes recorded during the interview. The transcriptions were made within a few weeks of the interview so that memories of the details of the interview could be triggered. The transcription was a record of all that happened during the interview including any pauses of 5 seconds or more.

Next, I wrote a commentary around extracts from the transcript, beginning with a brief description of the interviewee, their gender and their disposition during the interview, which also included the intensity of their voice and fluency of their utterances as an overall impression. The interviewee’s response to each task was described. The next stage of the analysis took the form of identifying common themes that emerged. In seeking these themes, I reminded myself of the issues identified in the literature review and those which arose in the preliminary investigation.

My attention was drawn to a particular theme highlighted in the literature review, namely learners’ sense of integration in terms of their associations and connections. As I tried to make sense of this through the interview transcripts, I noticed that many interviewees appeared to be stressing a particular aspect (namely techniques) and ignoring others (associations and connections), particularly diagrams and area. Nonetheless, their engagement in the construction task revealed a good deal more about the nature of their awareness. This contrast between what was said in an interview and what was revealed through the construction task was quite apparent and became the focus of my study.

I also discovered that it was not enough to ask for examples because sometimes it is unclear what they think they are exemplifying. It seemed that the interviewees’ attention needed to be shifted and explicitly drawn to what is being exemplified in the examples in order to appreciate its connections.
4.4 Qualitative analysis of the result

4.4.1 The Interview

There were twelve students who participated in the interviews. They were interviewed in pairs, with either one or both students responding to any particular question. Analysis of the interview responses took the form of looking for a range of variations in response to questions seeking to reveal the variety of learners’ concept images and associations with integration in the first three questions. These probes were intended to reveal learners’ focus of attention and connections to integration. I have summarised where appropriate the responses from the participants in the interview and not provided detailed transcripts, choosing to use the available space for analysis of the construction tasks.

Question 1: Meaning of integration

Responses from the students to the interview questions are shown in Table 4.1 in Appendix A. Of the nine students who answered, four of them mentioned area under a curve or volume of revolution (shown in bold), while two mentioned the reverse process of differentiation (shown in bold). One of these two students displayed confusion between integration and differentiation. Another two linked the topic with methods or techniques of integration such as ‘increasing the power and dividing by [that] new power’. One of them mentioned all three. It seems that learners’ responses ranged from rich associations between differentiation, area and volume to surface connections based only on a few techniques (increase power and divide by new power). I was mindful that a way of distinguishing these connections was necessary to relate to their awareness. Some learners also voiced concern for use of the topic, which reveals their attitude towards the topic and towards learning mathematics more generally.
Question 2: The integral sign

The integral sign $\int$ seemed to evoke aspects related to techniques of integration, with attention of many of the students drawn to limits. Surprisingly, only one student mentioned ‘finding area’. Two students commented on the physical look of the sign, saying that ‘it was a squiggly S and looks Greek’. One student boldly declared that ‘it’s like posters on the doors; you just go in and start doing’. What struck me in these responses is the fact the students appeared to be anticipating a request to ‘solve’ something. The sign seemed to trigger methods and techniques of integration for many of these students. They also displayed recollection of actions seen in class such as partitioning. Table 4.2 in Appendix A shows their responses.

Question 3: Things to watch out for

When asked about things to watch out for when doing integration, almost all of the students referred to aspects related to method or technique of integration. It seems that aspects that needed to be cautious about were strongly associated with using the right technique and getting correct answers. These responses suggest limited connections to the topic because no reference was made to behaviour of functions or definability of limits, which would suggest more sophisticated awarenesses. Many showed considerable degree of self-awareness and self-monitoring in terms of checking calculations and modifying what they had said (see Table 4.3).

Question 4: Technical terms

Language pattern, as revealed through knowledge of technical terms related to integration, suggested that the students had considerable ease with the language. *Arbitrary constants, limits, function* and $dx$ were some of the most frequent terms
mentioned. This fluency with language is one component of fluent behaviour (see Table 4.4).

**Question 5: Words that trigger integration**

Another aspect of language pattern that forms part of understanding is recognition of words that trigger integration. Students in this study used a wide range of words that mark relevance of integration, with references not only to words such as *integrate*, *differentiated*, *area under a curve*, techniques of *substitution*, *by-parts*, but also a range of words used in problems involving applications of integration (see Table 4.5).

**Question 6: Techniques**

Knowledge of techniques forms another feature of revealed behaviour in connection to the topic. The students in this study displayed knowledge of several techniques of integration, but *increasing the power and dividing by the new power* seemed to dominate the attention of many of them (see Table 4.6). This is not unreasonable as they are only meeting calculus for the first time.

**Question 7: Use of integration**

The question on the use of integration was intended to reveal their commitment to the topic through its use or relevance, which could have the potential to motivate them. Responses to this probe varied from solving maximum/minimum problems, area under a curve to applications in Architecture, Physics, Statistics, Engineering and Mechanics. They displayed a sense of connections in terms of contexts in which the topic appears in real life. Nevertheless, remarks from a considerable number of students that these applications were ‘very theoretical’, ‘it does not apply to real world’ and ‘not something that perhaps could work in real life’ suggest either a disconnection with applications and
little relevance outside of lessons, or a predisposition to pure mathematics (see Table 4.7).

**Question 8: Differences between** $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$?

What the students mark as similarities or differences between a definite and indefinite integral shows a good deal about their associations with these objects and the focus of their attention. Many students pointed out the limits as the difference and suggested that the definite integral gives ‘an answer’ and that the indefinite integral was rather ‘incomplete’. Faced with this situation, no student in this study made reference to any process of summation to obtain area under the graph, or to the indefinite integral as a function (see Table 4.8).

Responses from the students are shown in Appendix A.

Having noticed the limits, the students associated the definite integral with solving for area and suggested that they were uncertain with what they got with the indefinite integrals. As a result, they suggested that they were more confident in dealing with definite than with indefinite integrals.

Clara: It’s not too clear [what the indefinite integral is]. We don’t know. … whereas with this we’re more confident [because] you know you don’t have to do “open” type of thing like that. This [definite integral] is easier to integrate but you don’t know what you want there [indefinite integral]. You just get $x^3 / 3 + C$.

It’s an area, it’s not … It’s still an answer but you know area under the curve when you calculate that [definite integral].

It seems that learners were so accustomed to ‘finding answers’ that they appeared to be overlooking the significance of other associations of the objects they were attending to.
Furthermore, they displayed limited facility and fluency in language to describe objects that did not entail ‘answers’.

4.4.2 The distance travelled task

Responses to Task 2 suggest that students in this study preferred the formula for calculating distance travelled to finding area under graph, even though a geometrical solution would be more feasible. Almost all the students in this study associated distance travelled with area under the speed-time graph when the graph was shown, drawing from the relationship between distance, speed and time. However, when the graph was not available, many students suggested that they would use the formula because ‘it makes it so much easier to just use the equation’ and that ‘if you draw it, then you lose time and if you get it wrong, you dig yourself into a hole’. They appeared unconfident about procedures to produce correct diagrams for mathematical problems, which supported their inclination to use the formula. Further, they suggested that they would use integration only if the graph was a curve.

On the same note, it could be drawn from the exchange between John and Alan below that although learners knew the feasibility of using graphs, in this instance, that awareness did not come to mind.

John: If not given a graph, I wouldn’t think that way. I would use the formula, I don’t know why. If I was given just the text, I would think, that bit is doing that and 2 times, not equation, just the simple $s = \frac{d}{t}$. Yeah, I wouldn’t use graph. I wouldn’t picture the graph in my mind. I would just look at basically what it is asking and use $s = \frac{d}{t}$ and stuff like that and rearrange.

Alan: You can even draw your own graph, yeah but I don’t.

Interviewer: Do you prefer the graph?
John: If it is a bit more complicated. If I am reading something this simple, I would think acceleration is that bit times time and halve it. Maybe I would draw a graph simply to see. Before today, if a graph was not given, I wouldn’t have used a graph. It is easier to use a graph. It is easier to picture what’s happening but faced with that question, I wouldn’t necessarily have drawn a graph.

Interviewer: Do you have any preference?

John: It is much easier to use a graph because it’s more visual, like you definitely know what’s what and it’s very hard to get yourself lost for something that simple.

It appears as though learners’ reluctance to use diagrams is two-fold: their uncertainty in transferring information from texts to graphical representation correctly and their sense of ‘reasoning’ from numbers rather than making sense from visual interpretations. Dreyfus (1992) suggested that learners’ reluctance to use geometric images results in them focusing on algebraic manipulation of mathematical problems. One possible explanation for this reluctance is the focus of attention on getting answers through manipulating symbols using stock techniques.

4.4.3 The construction task

The pilot study included one construction task. This task invited learners to construct successive integrals that give the answer zero. It was hoped that this task would reveal not only learners’ focus of attention but also the richness of their understanding to see the integral as objects as well as process.
Jonathan and Alex

In constructing examples, Jonathan changed the limits from 0 to 1 and the function to $x - 1$ and constructed $\int_0^1 (x - 1)\,dx$. It seems that he ‘ignored’ the integral sign and only substituted the numbers into the expression. The extract below demonstrates not only both Jonathan and Alex’s focus of attention on ‘substitution’ aspect of integration, but also their limited connections to the topic.

Interviewer: Given that $\int_0^1 (1 - x)\,dx = 0$. Can you give another example like this for which the answer is zero?

Jonathan: $\int_0^1 (x - 1)\,dx$ because if you take $1 - x$ you get zero, whatever you do to zero, you are not going to get anything out of the zero out.

Interviewer: But if you plug in the lower limit....

Alex: [Long pause] So $\int_1^i (x - 1)\,dx$

Jonathan: You could have just $x$ limits -1 and 1 $\left[ \int_{-1}^1 x\,dx \right]$. What you do is you take the lower limit from the upper limit, so $1 - 1$ is zero. So $\int_{-1}^1 x\,dx$, no... that makes 2, doesn’t it? Unless it is $\int_{-1}^1 x\,dx$.

Interviewer: If it is 1 and 1, what is happening there? You can have any function there, can’t you? Because whatever you have, you are taking away.

Jonathan: Yeah.
Interviewer: What is same and what is different?

Jonathan: You have core function of $1 - x$, limits and answer's the same.

Interviewer: Can you work out why the answer's zero?

Jonathan: I see why because you've got 2 minus 2.

Interviewer: What is the most general example that you can think of for which the answer is zero?

Jonathan: Don't know.

Interviewer: What are the things that you can change in the integral?

Jonathan: Things out of the brackets, change numbers and change the numbers as well.

Put limits $a$ to $b$ ... of something. ... Can't think of one.

Both of them attempted to construct examples by simply substituting numbers in limits into the expression, ignoring the $\int$ sign. They were unaware of the fact that the expression they constructed was what a mathematician would consider a degenerate case giving zero area. Further probes revealed that they manipulated details in the example superficially. For example, although Jonathan constructed the correct example that featured area cancellation $\left[ \int_{-1}^{1} x \, dx \right]$, his remark that ‘it makes 2’ suggests that he substituted the limits into $x$ and found their difference. He seemed to have his attention focused on substituting limits and overlooked the process of integration and also what the object signified. The $\int$ sign was seen as a command to do something and this turned into a command to substitute numbers. Their awareness of connections to area seemed to be pushed to the background as they focused on techniques (behaviour) and displayed no
awareness of connections of the object given as example and the example they constructed.

In the construction task, I also noticed hesitance in their speech and lack of intensity in their voices, especially Alex. It would be reasonable to suggest that they displayed limited fluency with language associated with integration and little familiarity with constructing in their own example, which was consistent with their responses in the interview.

**Clara and Sam**

Given the task and invited to construct another example, Sam suggested that the function be changed to \(-x + 1\). Her partner, Clara, asked whether the limits had to be the same. She seemed to display a sense of possible dimension that could vary. She constructed the same example as Jonathan \(\int_0^1 (x-1)\,dx\). However, when Sam questioned whether the integral actually gave the answer zero, Clara realised that she had not integrated it and constructed a different example.

**Clara:** Oh ... I didn’t integrate that. So ... you want it to be a half ... \(\int_0^{-1} x\,dx\), because when you integrate you get \(\frac{x^2}{2} - \frac{1}{2} x\), so get \(\frac{1}{2} - \frac{1}{2}\).

**Sam:** It has to be \(2x - 1\) because normally you don’t put a half in that thing.

**Clara:** I was thinking when you integrate that you get \(\frac{x^2}{2} - \frac{1}{2} x\) when you put the 1 in, you get a half and a half which is zero.

**Interviewer:** Can you give me another example?
Clara seemed to be attending to the arithmetic details of the example. She retained the term $x$ in the expression and worked the second term and the limits accordingly. Although the example she constructed was correct, she appeared to be manipulating the details of the example and did not display any connections to area. Sam appeared to have a preconceived notion of *ideal* forms of integrals (no fractions) that she had to put in the right form.

When asked to construct another example, Clara displayed a rather interesting perception when she suggested that one of the trigonometric functions would work because of the nature of the reversibility of differentials of the functions. It would be reasonable to suggest that she thought the reversibility of the differentiation of the functions would cancel each function out when multiplied together. However, it was not clear whether her notion of ‘cancelling’ was referring to area or functions.

**Clara:** I think maybe one of the trig identity, $\cos x$ and $\sin x$... put them together. When you differentiate $\cos$ you get $\sin$, $\sin$ you get $\cos$. I was thinking having both $\sin x$ times $\cos x$ together ... This is very complicated. ... I was thinking about the graph and I want 90 on top of zero...

**Sam:** When is $-90$ ...

**Clara:** I don’t know.

**Sam:** She is trying to say one [limit] is minus 90 and the other one is 90

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x \, dx$$

[...]

[121]
Clara: I just think ... when you differentiate cos you get –sin, when you differentiate
sin you get cos... I'm not sure how it works....

Sam: Integration of sin is ... unless you use tan or something.

Clara: I think I was thinking about differentiation.

Although Clara revealed some possible dimensions that could vary (i.e. limits and
functions in this case), based on her reasoning, she did not display evidence of
particularly rich connections to the topic.

In order to understand their awareness of dimensions-of-possible-variation, I invited them
to consider the issue specifically by asking about things in the integral that could be
changed. To this prompt, Sam suggested multiplying the expression by 2.

Sam: You can times it by 2 to get [answer] zero ... that'll be $4x - 2 \int \log(4x - 2) dx$.

Clara extended the dimensions that could vary by suggesting that the 1 in the lower limit
could take a different form (log 1).

Clara: I was thinking cancel ... to get you to the log of ... emm ... is it log of 1 is

$$\log 1$$

[Discussions resumed about logs, natural logs and e]

To get an impression of their awareness of connections to area, I probed them further by
asking why the integral was coming to zero. Without any hesitance, Clara made reference
to area and related it to limits. However, her remarks did not give me the impression that
she had the connections to area in mind without my prompt.

Interviewer: Why is it coming to zero?
Clara: Is it because it has a negative area, you can’t get a negative area.

Interviewer: Where do you get the negative area?

Clara: If \( x \) gets bigger, this will just keep getting negative because it’s 1 minus ... As \( x \) gets bigger the whole area gets smaller and smaller and more negative.

Interviewer: What’s happening to the area?

Clara: The area is getting bigger but it’s negative. I’m not completely sure.

Seeing them struggling to understand, my teacherly instinct overpowered my role as a researcher in this setting and I attempted to show them why the integral is coming to zero by drawing out the graph. Having seen it visually, they realised that the areas cancel out, and this prompted Clara to suggest graphs such as \( x^3 \) and \( xy \) between limits of 1 and -1 for further examples and refuted Sam’s example of \( x^2 \). She made reference to the function being odd. Sam then engaged in rapid fire production of an example that had certain features of the given example (\( x \) to the power, part of the area on either side of the axis). Working in pairs seemed to benefit the students in conjecturing and reasoning, at least in this instance, and because of their interaction, I acknowledge that more of learners’ thinking were externalised than if they had worked alone, although there is a potential silencing of a less vocal member of a group.

Interviewer: Can you give me another example?

Clara: You can have an \( x^3 \) graph or \( xy \) graph and integrate that between 1 and -1 (works it out).

Sam: It can work with \( x^2 \) as well, you can have -1 to 1, -2 to 2.

Clara: Oh ... it might not happen with \( x^2 \) because that area there is not negative. You
need to start with an odd function.

Interviewer: How about the limits?

Clara: \(-n \text{ to } n\), as long as \(n\) is the same number.

Clara seemed to display a web of connections, having suggested odd functions and revealed a sense of range-of-permissible-change within those dimensions. The hesitancy in their voices and halting speech with pauses for more than 10 seconds suggest restricted fluency in language when speaking about integration, although Clara appeared to be more articulate than Sam, or else they were having to think in new ways.

**John and Alan**

John was one of the very vocal participants in this study. Given the task, John and Alan worked out the example first. John then suggested that it had to be 'something minus something', in which case both were equal, and constructed \(\int_{0}^{2} x - x\). He did not construct any other example. His partner, Alan, suggested that 'no matter what value you put in, you still get zero’. Both of them appeared to engage in algebraic manipulation of the integral and wrote \(0 = x^{3} - 4\). They did not display any evidence of connections to area.

After an explanation was offered as to why the integral was coming to zero, both John and Alan constructed examples such as \(\int \tan x \, dx\) and \(\int \sin x \, dx\) that revealed richer connections. An interesting observation emerged when Alan noted that the areas cancel each other out only in theory. This observation was supported by John, who suggested that they cancel because of the positive and negative areas, although in practice the areas exist physically (but in different parts of the axis).
John: If you take tan [limits] that will come out to zero because it’s equal area top and bottom.

Alan: If you take sin [limits], area above x-axis is equal to area below x-axis. All equal zero because there is actually area there even though it shouldn’t.

Interviewer: What shouldn’t?

John: That’s a good point because there’s area there but it cancels itself out. If you are using numbers, it doesn’t cancel itself out because there’s positive and a negative area. If you are using it practically, you’ll know.

In constructing subsequent examples, after having had a sense of connections to area, John decided on the function to be a single term with a negative and a positive number as lower and upper limits. However, John struggled for quite some time to ‘generalise’ the situation to an odd powered function, algebraically. There seemed to be some confusion between the function being integrated and the integrated function in which to substitute numbers, which seemed to me resulted from a focus on details of the example and not on the connections and imagery it signified.

Interviewer: Can you give another example?

John: tan $x$ would give zero because it cancels itself out because of the negative area. If you had only single term that you are integrating, then you have negative number and positive number on either side of the integration sign that should always come up to zero.... Oh no ... you have the $x^2$ one, it’s never ... it’s always positive. It’ll have to be a line (odd power).

Interviewer: Can you give an example?

John: If you take $5x$, no 5, it will come out $5x$, if you put 3 and -3, then that cancels out.
Interviewer: Basically, [you are saying] you want a positive number and a negative, doesn’t matter what goes in there?

John: Does matter if you had $5x \ldots$ turn into square when plus minus number, it will come out positive eventually. You’d have to start with 5 or $5x^3$ then it turns itself into … once integrated it has odd power it should, so you want 5, $5x^3$ after you’ve integrated.

Alan: Integrate even powers $\int$even powers$. I think that would work because wouldn’t work for positive negative numbers, works for $\pi$ to $-\pi$. Trig functions you want certain amount you need a difference of 2$\pi$. That will guarantee one complete cycle but don’t add 1 because it makes up by one.

[...]

John: $\int_{-2}^{2} 4x^2 dx \text{ becomes } \frac{4}{3} x^3 \bigg|_{-2}^{3}$, comes out as plus… oh dear, it’s the other way round. You want odd power.

Alan: When you’ve written it down with the integral sign, that then is your curve so you want [to integrate] odd power [turn to even power].

John: Generally, integral of any number $\int$any number$\text{, oh no ...}$

Alan: You want to end up as a number.

After a long discussion on whether the integral of a number or the answer to the integral must be a number, both John and Alan agreed on integrating a number from $-n$ to $n$ $\left[ \int_{-n}^{n} \text{number} = [\text{number}]_{-n}^{n} \right]$ and suggested it could be any number. John wrote down
John displayed an inclination to focus on algebraic manipulation of the example and not to visualise. The evidence for pattern spotting from the given example in order to generalise was apparent. I also noticed that John’s attention was focused on techniques of integration so that he ignored the $dx$ when working out the example.

Interviewer: Did you have an image?

John: No, I did it algebraically. I didn’t visualise. I didn’t see a graph in mind. $\int_{0}^{a} 2-x \cdot$

I think $\int_{0}^{a} f(n-x)dx$. 

In order to gain insight into their thinking about the integral, I asked why they thought the integral was coming to zero. John referred to the fact that there were two things taken away. However, the referent ‘things’ was unclear, whether he was referring to numbers or areas.

Interviewer: Why is the integral coming to zero?

John: There are two things taken away. You’ve got two exactly same things that you are taking away, which is going to be zero. So it’s like any 1 minus 1, infinity minus infinity is infinity, like nothing is actually happening to it.

Interviewer: What is the integral doing?

[Silence]

Interviewer: Is there anything that you can change?
John: You can change whatever you want; you can change limits, change numbers in it. Don’t know what you change to but you could change (limits). You are putting it further along the graph.

Interviewer: Can you change the function?

John: You can change that also but can’t think what they are.

John’s remarks that ‘you can change whatever you want’ suggest that he displayed awareness of dimensions that could vary. His suggestion that the limits could vary and its implications to the graph evidenced awareness of connections. However, his reference to limits alone and the fact that reference was not made to area highlight limitations to this awareness.

**Heidi and Geoff**

To construct another example, Heidi suggested zero multiplied by the function $\int_0^1 (1-x) \, dx$ because ‘zero times zero [integral] is zero’. Asked to construct another example, she suggested:

Heidi: Limits the same, $(1-x)$, equal zero the same ... multiply by 2, divide by 2, times 1. Since the answer equals zero, nothing is changed, just adding more numbers, putting in different numbers but also multiply with zero ($dx$). ...

either $(1-x)$ or $dx$ equal zero, $x = 1$, $1 - 1 = 0$

It seemed that Heidi was treating the integral as an arithmetic expression $\int (1-x) \, dx$ and therefore, another integral for which the answer was zero was produced by multiplying the integral by any number, suggesting $\int 2 \times 0$ and $\int 3 \times 0$. It appeared as though she was applying the multiplicative rule $a \times b = 0$, $\therefore a = 0$ or $b = 0$. Heidi then suggested that if one of the limits was zero, the integral would equal zero because ‘if you
substitute zero into the equation, it gives you zero’. She wrote \[ \int_{0}^{2} (equation) = (2 \times equation) - (0 \times equation) \] and proclaimed that it amounts to nothing. The same argument was used with the third example \[ \int_{0}^{3} (equation) = (3 \times equation) - (0 \times equation). \]

When invited to construct something different, Heidi constructed \[ \int_{0}^{4} (2 - 2x) \] and \[ \int_{0}^{8} (4 - 4x) \] and generalised to \[ \int_{0}^{2n} n(1 - x)dx. \] She pointed out that the difference was that the coefficients were different, either multiplied by 2 or divided by 2. There was no evidence of checking to discern the mismatch between limit and function.

Her partner Geoff appeared to be very reserved and only commented on explanations offered by Heidi. He expressed concern that both examples suggested by Heidi \[ \left[ \int_{0}^{2} (equation) \text{ and } \int_{0}^{3} (equation) \right] \] were not the same. He pointed out that the difference [in limits] had to be the same and suggested changing the limits to 2 and 4.

Geoff: No change in limits as long as the difference is the same ... Don’t have to have 0 ... if you change the limits from 2 to 4, the difference is the same, so ... same answer.

Heidi, who was more vocal than Geoff, seemed caught up with ‘getting the answer’ to be zero that she tried to find ways of getting zero by manipulating the expression. Having listened to this, Geoff appeared to focus his attention on the subtraction aspect of integration and suggested that the difference in the limits should be maintained.
**Abby and Jay**

Abby responded to the task by constructing \( \int_0^1 (2x - x^3) \) and suggested that it had to be differentiated to get another integral. She said that ‘something is needed to cancel each other out’. Her example was incorrect, which meant she was not checking her answers. This suggests a focus on powers but forgetting the multipliers. She too, like Heidi, seemed to be treating the integral as an arithmetic operation, in which two equal amounts taken away was equivalent to nothing. She then constructed \( \int_0^1 (3x - x^2) \) while suggesting that it might not work. Subsequently she constructed \( \int_0^1 (9x - x^3) \) and pointed out the ‘need to cancel because zero doesn’t matter’. She then suggested that the two terms need to be the same and constructed a general expression \( \int_0^1 (n^{-1})x - x^2 \). She showed no evidence of realising that she had lost the integral sign. Her partner, Jay, did not make any contribution in this part of the study. Lack of interaction meant less thinking was revealed than other pairs.

**Adam & Ian**

When asked to construct another example, Adam suggested that he could change the limits to \( \int_0^2 (1 - x^2) dx \). To construct another example, he suggested ‘you can put squared outside the brackets \( \int_0^2 (1 - x)^2 dx \), so you can get different curve (change shape)’. It seems that Adam was engaging in rapid triggered responses apparently without actually thinking about the examples.

In order to gain insight into his thinking about the examples, I asked him ‘Why do you think the integral is coming to zero’? Adam’s response was:
Adam: When $x = 1$, you get 1 minus 1 equals zero.

His partner, Ian then offered an explanation:

Ian: Is it because the fact that when you are integrating a curve like that, basically you are dividing it into an infinite amount of trapeziums and the area of the strips is actually zero because there are infinite amount of them, the area of the trapezium is zero … I don’t know. They’re infinitely close together.

Ian seemed to be attempting to relate the explanation he was offered when relating integral to area under a curve (summation of infinitesimally small strips of area) to the fact that the integral in the example was zero. There seemed to be a gap in understanding between what the summation process was doing and the integral in this example being zero. The limiting process of $\Delta x \to 0$ and the cancellation of area that amounted to the integral in this example being zero appeared to be confused.

In the next section, I discuss analysis of the quantitative data.

4.5 Quantitative analysis of data

The quantitative data was collected by administering the same set of questions in the form of a questionnaire. A mathematics lecturer who was willing to carry out the tasks with his first year mathematics undergraduates was approached and electronic versions of the interview questions and the construction task were administered. A total of 21 students volunteered to participate in the study. The questions which were presented in a Virtual Learning Environment (VLE) was later copied and transferred into a spreadsheet.

4.5.1 The interview questions

Appendix B presents responses for questions on the interview that attempt to address learners’ awareness of integration.
Ten out of 18 students articulated connections between integration and area while seven of them related it to the reverse process of differentiation and one just mentioned ‘integrate something’. Surprisingly, only one of the students mentioned the process of summation, denoted by the letter S. Others recalled techniques of integration and a symbol for integration. It seems that for almost all students, the sign initially triggered techniques that come with it and not concepts associated with it. With regards to things to watch out for, almost all of the students expressed the need to look for aspects related to techniques of integration such as arbitrary constant, definite/indefinite integral, integration by substitution, integration by parts, limits and with respect to, with little or no reference to function behaviour or definability of limits, which would have suggested richness of connections (see Table 4.9).

Almost all the students (fourteen) identified area, among others, as word(s) that triggered integration. Two of them wrote just ‘integrate’ while one referred to rate of change and differential equations. One student did not answer the question. It seems that most of the students in this study expressed connections with area.

The students displayed a wide range of technical terms and techniques that are associated with integration. Some of the technical terms mentioned were definite/indefinite integral, arbitrary constant, limits, with respect to, integration by substitution, by parts and derivatives. The most frequently mentioned techniques were integration by substitution and integration by parts, with some other techniques such as increasing power and dividing by new power, partial fractions and trigonometric functions were also mentioned. It would seem to me that these students appeared to be well-versed in language patterns and techniques associated with integration, judging by the wide range of responses provided (see Table 4.10).
To the question addressing the difference (or similarities) between $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$, most of the students referred to the integrals as only indefinite and definite integrals and therefore the former needs to have a constant and the latter doesn’t. What struck me about these responses was that the definite integral was perceived as being more exact and gave answers while the indefinite integral was perceived as giving just an expression with $x$’s. Only one student made a reference to area under the curve between the limits (see Table 4.11).

The probe ‘What kinds of problem does integration help to solve’ revealed a range of responses from the students. A significant number of them (twelve) referred mainly to area under the curve problems, volume of revolution, differential equations and calculus problems in general. A more sophisticated connection was revealed by five of the students who mentioned uses in contexts such as probability distribution, kinematics, mechanics and energy (see Table 4.12).

4.5.2 The construction task

Due to a technical error in the construction task (the question read $\int_0^1 (1 - x) \, dx = 0$ on the computer screen instead of $\int_0^2 (1 - x) \, dx = 0$), the outcome of the task was rather flawed. Nevertheless, some students did respond to the question ‘What are the things in the integral that you can change?’ while six of them attempted to construct examples. They suggested that dimensions such as the limits, the integrand sign or the integrand could vary. They seemed to display awareness of what could vary, although most of them suggested changing the upper limit to zero or the lower limit to 1, which was a degenerate case giving zero area. Three of them suggested multiplying the integral by
zero or a constant. This supports the basis for two conjectures: first, the focus of their
attention on limits and second, evidence for lack of imagery in associations. One of them
(S10) suggested modifying the integral numbers, 1 to any number \( n \) and the 0 to 2 – \( n \)
\[
\left[ \int_{\frac{-n}{2}}^{n} (1-x)dx \right],
\]
which was correct and revealed a sense of the range of change permissible
for the limits.

In the successive construction of examples, two of the students who did attempt it varied
the limits (S10, S14), one varied the function (S13) and two varied both the function and
the limits (S2, S14). Student S1 suggested reversing the order in the function for the
second example and multiplying by a constant for the third. Student S18 constructed
examples which both had the same values for upper and lower limits. Table 4.13 in
Appendix B presents the students’ attempts at constructing examples.

It seems to me that many students engaged in a ‘trial-and-error’ strategy to construct the
examples and displayed little or no evidence of richer connections since no reference to
or use of area was made. Their attention seemed to be focused on manipulating details of
the example and not on what it signified. Another feature that caught my attention was
their tendency to treat the integral as an ‘arithmetic’ operation, in which what seemed to
come to mind upon seeing the integral was the method of ‘upper limit minus lower
limit’. In these circumstances, the students tended to ‘ignore’ the function being
integrated.

Two features of the analysis highlighted my concern, and these features became themes
for the main study and are discussed in greater detail in the next chapter. The first feature
that I noticed was the range of variation in response to the interview questions which I
had based on the six-fold framework. The responses seemed to suggest that for each of
the strands, learners displayed different levels or degrees of densities of understanding of
the aspects considered. Some students displayed rich connections of the aspect being probed while others demonstrated superficial associations. It also emerged that learners tended to stress some aspects and to ignore others. In particular, aspects concerning technique and language pattern were emphasised while those related to imagery and connections were overlooked, hence resulting in impoverished experience on the part of the learner.

Having said that, I realised that for some learners, the construction task revealed greater sophistication than the interview. This formed the basis of the second theme that emerged from the study, which was that the construction task revealed more and in some cases different aspects of learners’ understanding than the interview alone.

4.6 Levels of understanding

Under this heading, I present the different levels of associations and understanding that emerged from learners’ responses to the questions, both in the interview and the construction tasks. Some learners demonstrated deep understanding of the aspect being probed while others displayed only superficial connections and yet others revealed emergent state of sophistication in which understanding has not fully developed. I have divided these responses into three levels, rich connections, surface connections and connections that I categorised as something in between. The next chapter looks at the categorisation of these levels in detail.

4.7 Contrasting awareness

In this section, I present an analysis of Clara and Jonathan who displayed contrasting degrees of awareness of integration in the interview and the construction task. Clara displayed richer connections in the construction task than she did in the interview.
Jonathan, on the other hand, appeared to be in contrast to what he mentioned in the interview: his association of integration to area was not revealed in the construction task.

4.7.1 Clara

In the interview, Clara related the meaning of integration to volume of revolution and mid-ordinate. She did not specifically make reference to area herself, although she appeared to build on her partner Sam’s reference to area under a curve earlier. The sign seemed to remind her to ‘integrate something with respect to $x\ldots \, dx$ at the end’. In terms of things to watch out for, Clara appeared to attend to a specific technique of integration that used partial fractions and suggested that one also needed to be careful about taking away values when negative numbers were involved (she referred to negative areas). Her responses in the interview about awareness seemed to highlight a focus of attention on technique but with limited connections.

In the construction task, she started out by varying the limits and the function simultaneously and constructed $\int_{0}^{1}(x-\frac{1}{2})dx$ because ‘half minus half is zero’. She constructed $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin x\cos x) \, dx$ and suggested that ‘when you differentiate cos you get $-\sin$, when you differentiate sin you get cos… I’m not sure how it works...’ When asked why the integral was coming to zero, she made reference to negative area and pointed out that ‘you can’t get a negative area.’ After I pointed out the fact that the areas cancelled each other, she suggested functions such as $x^3$ and $xy$ integrated between -1 and 1. She also ruled out the function $x^2$ because ‘area there (left of y-axis) is not negative’ and generalised to odd functions between $-n$ to $n$. 
It seems that a greater awareness of connections and links to integration than was captured by the probes in the interview was revealed through the construction task. She revealed more sophisticated connections and articulated deeper awareness of integration, although at a very early level. Perhaps this awareness was triggered by the interviewer’s unintended introduction of ideas about areas. When her attention was drawn to a sketch of the function, she appeared more articulate and displayed richer connections.

4.7.2 Jonathan

In the interview, Jonathan referred to meaning of integration as area under a curve and the opposite of differentiation. He was reminded of ‘certain lines of thinking’ and that ‘you get the will to take numbers in front of an \( x \) and put them up as a power’. He also said that ‘you probably know that is something you are calculating the area’. Aspects related to techniques of integration such as not to ‘confuse numbers, adding and not multiplying, doing differentiation by mistake and negatives when you are trying to find areas between upper and lower limits’ dominated his thinking of things to watch out for.

The things he said in the interview did not appear in the construction task. Although he mentioned area and the need to watch out for negative areas, he constructed \( \int_0^1 (x - 1) \, dx \) as an example and reasoned that ‘if you take \( 1 - x \) you get zero, whatever you do to zero, you are not going to get anything out of the zero’. He appeared to focus mostly on the substitution aspect of integration. In the second example, he constructed \( \int_{-1}^{1} x \, dx \), but his remark that ‘what you do is you take the lower limit from the upper limit, so \( 1 - 1 \) is zero … no, … that makes 2’ reaffirmed an ‘arithmetic’ focus. He seemed to ignore the function being integrated and simply plugged in limits. He then suggested \( \int_{-1}^{1} x \, dx \).
(degenerate case). To the question ‘Why is the integral coming to zero?’ his response was ‘I see why ... because you’ve got 2 minus 2’. The probe did not appear to shift his attention because he seemed to attend to the arithmetic of the operation.

### 4.8 Summary

In this chapter, I have introduced the different levels in responses to questions based on the six-fold framework indicating depth of understanding in each aspect that emerged were briefly discussed. These levels, which signify learners’ associations, ways of speaking and motivational drive to learn the topic, can be divided into three: low, medium and high. The levels not only indicate the nature of understanding in that aspect of the framework, they also reveal aspects that are stressed and aspects that are ignored in relation to elements encompassing understanding and appreciation of integration. It is very likely that the more depth they showed in each aspect, the more that aspect was emphasised in relation to other aspects that make up understanding. These levels of understanding will be discussed in detail in Chapter 5.

A distinct contrast between learners’ responses in the interview and what was revealed in the construction task had also emerged. In the main study, the distinctiveness of this contrast was explored with learners from three different backgrounds. It appeared from the pilot study that learners who think about connections (associatively) were more sensitised to notice opportunities to generalise and more articulate in marking the aspects in question. On the other hand, learners who referred only to superficial aspects may indeed have only a superficial appreciation of integration. In some cases, learners whose attention was re-focused with the help of the probes displayed greater sophistication in their sense of integration. More importantly, they acknowledged their own ways of thinking prior to constructing examples and appeared more sensitised to notice
connections and to think associatively once engaged in example-construction. Others, however, attended to arithmetic details of the examples and did not display evidence of changed perception.

The next chapter places the themes I have identified in focus and discusses them in greater detail prior to the main study. In the main study, four additional construction tasks were given to the participants, in order to explore learners’ awareness of dimensions-of-possible-variation in greater detail by constructing examples that met various specified constraints.
Chapter 5

Exploring the data …

5.0 Introduction

In this chapter, I present the method I developed for categorising and displaying learners’ responses in the interview and the construction task. Since different probes elicited different responses, a way of differentiating the depths in responses triggered by the interview probes and those revealed as learners engaged in construction tasks is necessary in order to capture effectively any forms of contrast in awareness, language and disposition towards the topic. As I analysed their responses to each of the probes, I realised that responses from the students in the interview ranged from displays of superficial connections to evidence of considerable sophistication. The aim of this chapter is to provide a systematic and transparent presentation of the data analysis process in the chapters that follow.

5.1 Interview

The interview responses mainly focused on what the interviewees said, with particular attention to their use of language, gestures and intensity of their voices. As responses in the interview could only be analysed based on how they related to the aspect being probed, what they said seemed to be a good indication of the kinds of associations and connections triggered by those aspects and, thus, what they regarded as important. In the construction task, the examples the interviewees constructed themselves revealed a great deal about their awareness, together with their reasoning and the language and the tone of their voices. Therefore, what they noticed in the examples and what they made remarks
about could be taken as indicating something about their sensitisation to these features and more importantly as revealing the dimensions, depth and nature of their awareness. Great care was taken not to assume that absence of evidence of a feature meant absence of that feature, but rather only absence of evidence.

Due to the complexity of placing responses in appropriate stages of depth in the aspect being probed and in order to differentiate between these responses, I looked through the interview transcripts and paid close attention to the kinds of responses, looking for evidence of the kinds of connections I referred to earlier. As hard as it would seem to categorise the responses exclusively in any category, I went through each transcript as many times as possible until it became clear to me the sorts of evidence that could be detected. Types of response were accumulated in each category and the frequency of occurrences of certain kinds of response was recorded. I found that it was possible to distinguish three intensities or depths in understanding: superficial understanding, deep understanding, and an intermediate, emergent sophistication in connections, which I have termed immersed state. Therefore, the responses were categorised according to these three categories which indicated different depths in association and understanding (see Table 5.1).

5.2 Categories of response

The categories were charted under the headings of fluency in language, facility with technique, awareness/connections, misconceptions, scope of context and scope of root problems that constituted the ‘Structure of a Topic’ framework discussed in Chapter 2. Due to the fact that two kinds of comparison were envisioned, namely contrasting the same learner’s responses in the interview and the construction task and also contrasting responses between the pairs, a way of indicating responses from the two learners was
needed. At first, I used big and small dark circles to represent different depths in understanding of one of the pair and light circles to indicate that of the other. However, two concerns made me change representations. First, I was concerned that the dark circles tended to dominate vision and undermine the significance of the light circles. Second, I realised that the impact of the smallest of the circles was compromised, although they indicated some form of understanding. Therefore, I developed a way of presenting the data using, what I called density shapes.

5.2.1 Density shapes

The density shapes enable:

- the distinguishing of subjects in a paired interview
- the contrast in depths in subjects’ understanding in terms of the aspects probed
- a visual appreciation of the contrast

I used a circle shaded in different intensities of grey to indicate the first of the pair’s responses and a square shaded in different intensities of grey to indicate responses from the second of the pair. Three intensities of grey were used: light, medium and dark, to indicate the three depths in understanding discussed in Section 5.1, namely ‘surface’, ‘immersed’ and ‘deep’, respectively. In order to facilitate the comparison between responses in the interview and the construction tasks, I presented the density shapes in a two-row table indicating interview and construction tasks, with columns indicating the six components of the ‘Structure of a Topic’ framework. There were times when the second of the pair built on what was said by the first student. In cases such as this, the grey circle or square is marked with a † to indicate the student’s remarks. A commentary was made about this in the analysis.
5.2.2 Categories for the interview

The categories for responses in the interview are presented in Table 5.1 below. The categories for response are divided into two: interview responses and construction tasks responses.

<table>
<thead>
<tr>
<th>Awareness Associations/ Connections</th>
<th>Misconceptions</th>
<th>Behaviour Fluency in Language</th>
<th>Facility with Technique</th>
<th>Motivation Scope of Context</th>
<th>Scope of Root problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface Does not mention or display awareness of connection/ association of core ideas such as area or volume.</td>
<td>Mentions limits, arbitrary constant.</td>
<td>Mentions integrate, area under a curve or volume of revolution or surface features of an integral.</td>
<td>Recalls specific techniques and/or difficulties with it (technical aspects).</td>
<td>States limited scope of use of concept.</td>
<td>Indicates awareness of what problems integration solves beyond exercises. Limited to one of anti-differential, area or differential equations.</td>
</tr>
<tr>
<td>Displays awareness of association with area/volume but does not display links with images, graphs or other connections.</td>
<td>Mentions common errors related to specific techniques.</td>
<td>Apart from the above, also mentions specific techniques of integration.</td>
<td>Recalls specific techniques and displays understanding of some techniques.</td>
<td>Mentions a substantial number of uses of concept dominated by textbook concepts. Mentions limits, reverse process of differentiation and rules, use in a few related contexts.</td>
<td>Indicates awareness of sources which lead to integration as a topic.</td>
</tr>
<tr>
<td>Mentions association with area/imagery/other associations and/or demonstrates awareness of this connection.</td>
<td>Mentions common mistakes related to sense-of-the concept including continuous functions, area (positive/negative), images.</td>
<td>Mentions process, integrate, limits, definite/indefinite, arbitrary constant, function, graphs, areas, solve differential equation.</td>
<td>Mentions several techniques and/or displays broader sense of understanding of the techniques.</td>
<td>Mentions limits, reverse process of differentiation, limit of sums, connections with rate of change, use in range of other contexts.</td>
<td>Rich account of origins of integration in quadrature and differential equations and anti-differentiation.</td>
</tr>
</tbody>
</table>

Table 5.1: Categories of responses in the interview

5.2.1.1 Interview responses: Awareness/Misconceptions

Under this heading, I categorised responses in the interview that indicated depths in awareness. Questions that probed this were about, for example, meaning of integration, associations with the integration sign and misconceptions related to the topic. Surface connections, represented with a light-shaded grey circle or square, were the coding for responses that mentioned only superficial aspects of integration such as technique of increasing power and dividing by new power or being the reverse of differentiation.
These responses do not reveal the concept image of integration in relation to area under a curve or summation of the limiting process to find area. Although subjects might have been aware of these associations, awareness of these was not expressed. A dark-shaded circle or square, representing richer connections, was the code for responses that not only displayed attention to details but also a marked sophistication in connections such as area, techniques and process of integrating. Responses that indicated links to area/volume but did not demonstrate rigour in explanation to display this connection were coded a medium shade of grey to signify immersed state.

Awareness of common mistakes related to integration also marks understanding and appreciation. If the learner mentioned ‘things to watch out for’, this signified the students’ awareness of them, which reminded them of things to avoid doing. Responses that indicated aspects related to surface features, such as adding the arbitrary constant and limits, were coded as surface connections to indicate restricted awareness. Attending to technical aspects of common errors suggests focus of attention on technique and does not provide evidence of richer connections to other aspects of integration. Awareness of connection was displayed by some students who mentioned common mistakes related to some techniques. Responses such as these were coded as an immersed stage, which demonstrated a modest sophistication in awareness compared to responses that were coded as surface. Deep sense of awareness that marks rich associations was the code used for responses that included errors related to having a sense-of integration such as definability and continuity of the function, positive/negative areas and images of the function.

5.2.1.2 Interview responses: Fluency in Language/Facility in Technique

In aspects grouped under the category of language fluency and facility in technique in the interview, I looked for evidence of restricted facility with language/technique and for
responses that indicated fluency. I used the code ‘deep understanding’ for responses that indicated fluency and ‘surface ease’ with language for those which showed limited fluency. Aspects of language fluency emerged in the use of technical terms and words that triggered integration. Responses that were coded as ‘surface facility’ with language were those which contained only aspects such as integrate, opposite of differentiation, area and/or volume of revolution, indicating restricted facility with language. A dark shading indicating fluency was the code used for remarks going beyond these surface features to include evidence of appreciation of language patterns to do with function, graphs, process of summation and applications in solving problems. Responses that built on further from surface references but which were not used robustly or frequently enough to suggest fluency were coded with medium shading (immersed state).

The second category through which aspects of behaviour were probed was facility with technique. A good indication of trained behaviour through which understanding can be revealed is facility and flexibility with techniques associated with integration. Responses to this probe were obtained not only from a direct probe such as ‘What are some of the techniques associated with integration?’ but also from responses to other probes in which techniques of integration were mentioned. If the student recalled a specific technique and/or difficulties with it, referring only to technical aspects of it such as arbitrary constant and limits, ‘surface understanding’ was the code used, indicating limited facility. Exclusive reference to technical aspects and notation were marked with a light-shaded circle/square in technique. In this category, depth in knowledge of several techniques associated with integration was not displayed, including nature of the function being integrated, things to watch out for in using those techniques and reasons for using the techniques. When a student mentioned these aspects of techniques, this was coded as deep understanding to mark facility with technique. Responses coded as immersed state
of understanding included mentions of some of the techniques and some aspects related to them.

5.2.1.3 Interview responses: Emotion/Motivation

Probes that addressed affective aspects of understanding of integration included reflections on access to origins and possible uses. Aspects related to uses not only support a strong foundation to understand uses and origins, it also provides a reason for learning the topic. Limited access to these contexts means learners may not appreciate a topic fully. Responses that touched on a limited scope of problems such as area, volume and reverse process of differentiation were taken as indicating restricted access to uses. These contexts, often including or referring to textbook problems, suggested a narrow scope of applications. These types of response were coded as ‘surface understanding’. I must stress that I was not intending to draw any conclusion about what motivated learners but what I was looking for was an indication of aspects related to origins and uses of the topic which one would anticipate might have influenced them. I was looking to glimpse indications of what is available in the topic in this aspect.

A fairly rich awareness was indicated by responses that included uses of integration in a number of contexts; this was coded as an ‘immersed state of understanding’. Responses in this category appeared closely related to learners’ A-level subject choices, ranging from uses in mechanics to statistics. A ‘deep understanding’ was the code used for responses that marked an encompassing sense of use in a range of contexts, such as physics and architecture. Responses in this category suggested a sense of direction and commitment and which one would expect to influence learners.

Access to origins of integration could further support emotional commitment to learn in a dynamic way. Limited access to this aspect may render an impoverished learning
experience. Responses in this category which reveal a uni-dimensional knowledge of the contexts were coded as ‘surface understanding’. Most of the responses from the students in this study were categorised in this category since reference was made only to the reverse process of differentiation. A richer sense of connection revealed through in-depth knowledge of root problem such as the limiting process, and summation to obtain the area under a curve, would be coded as ‘deep understanding’.

5.2.2 Categories for the construction task

Categories for the construction task were established for the awareness and behaviour aspects of understanding. Since knowledge of contexts was not applicable in this instance, emotional commitment was identified through displays interpreted as surprise and despair and acknowledgements of such feelings. The intensity of the students’ voices and the flow of their speech were taken as indicative of their excited or casual approach to the task. The categories for the construction task are presented in Table 5.2 below. I have removed the columns under emotion/motivation as they are not applicable to the construction tasks.
Language patterns

Uses language that shows lack of fluency in displaying awareness of structure, often stumbling, incomplete thoughts.

Technique

Use

Shows limited facility with technique. Uses technique inappropriately or incompletely.

Recall

Refers to techniques inappropriately or inarticulately.

Awareness

Images/Associations

Misconceptions

Demonstrates limited awareness of dimensions of possible variation, wrong examples constructed.

Table 5.2: Categories of responses in the construction tasks

5.2.2.1 Construction task: Awareness/Misconceptions

In the construction task, the students’ sense of connections was analysed in terms of the discernment of dimensions-of-possible-variation in the example through which awareness was displayed. If the student constructed a wrong or incorrect example, the code used was a ‘surface connection’. By constructing a wrong or incorrect example, the student not only displays his/her focus of attention, s/he also reveals an absence of disposition to check and a limitation in his/her awareness of possible dimensions that could vary in the example. As discussed in the literature review, what the students choose to vary can reveal the dimensions, depth and nature of their awareness. The more dimensions they vary simultaneously, the more likely it is that they have attended to what the example exemplifies and generalised from the particulars of the example.

In some cases, the students varied some dimensions in the example without a clear indication of awareness of structure. The student may have varied dimensions successively without having discerned the varying dimensions. In this case, the student
Chapter 5 Exploring the data ...

seemed to pay attention to details in the example potentially with an awareness of structure but it was not articulated explicitly. Such cases were coded as ‘immersed understanding’. When several of the possible dimensions were varied and awareness of connections and relationships between the dimensions was clearly articulated, then the code used was ‘deep understanding’. In many such instances, the students expressed the relationship with a clear awareness of the structure which was perceived in the example. In other words, they appeared to attend to the form rather than the details that make up the example.

Another aspect of awareness that was looked for was knowledge of classic errors related to the topic. If the student made classic errors in the course of constructing the examples and did not display awareness of them, it is suggestive that restricted awareness was displayed in this aspect, hence the coding ‘surface understanding’. Any limitation in awareness of common mistakes that are possible may affect learners’ understanding and result in impoverished experience. Constant lack of understanding and failure to make sense could discourage them from continuing to be enthusiastic about the topic they are learning. On the other hand, having an awareness of their own slips and mistakes from past experience could prevent them from repeating those errors and so foster competence if not understanding. Responses which display such awareness were coded as an ‘immersed state of understanding’. A more sophisticated form of awareness of misconceptions related to the topic would be awareness of other’s slips or possibilities of them. If the students were able to identify mistakes which most learners are prone to make, then it is highly likely that the student him/herself would not make the same mistakes. ‘Deep understanding’ was the code used for responses that indicated such awareness.
5.2.2.2 Construction task: Fluency in Language/Facility in technique

The construction task also provided a platform to capture students' language patterns when talking about integration. Fluency and flow of language were discernable in the way the students described the examples they constructed. Articulating thoughts using language that is accurate and flowing suggests a considerable degree of understanding. Where students were often stumbling, with halting speech and utterances of incomplete thoughts in describing their choices in constructing the examples and in explaining their thoughts, then this was coded as 'surface facility with language', to indicate limited fluency.

In many cases, subjects used language fluently but not always in a way which was relevant to the topic. Although they demonstrated fluency, more often they were not articulate enough in communicating their thoughts to suggest language fluency. Such responses were coded as an 'immersed state'. Utterances may relate to particular aspects but with inexplicit references. Displays of fluent and articulate expression of awareness of structure in the examples they constructed and usually in relevant manner were coded as a 'deep understanding'.

The second aspect of behaviour manifested through the construction task that was examined was knowledge of and facility with techniques of integration. Students' facility in using techniques of integration appropriately and flexibly in an unfamiliar setting, such as the construction task, may suggest whether they have the skills necessary to engage in mathematical tasks effectively. I have divided this category into 'recall' and 'use' of techniques to differentiate the two forms of connections. If the student chose techniques inappropriately, or referred to techniques inarticulately, then the responses were coded as demonstrating 'surface connections' to suggest limited facility. The ability to use techniques appropriately at this stage of their learning is an important indication of
competence and understanding. Responses that suggested exclusive technique-orientation were marked with a light-shaded circle/square under technique.

Responses coded as ‘immersed connections’ were those which displayed restricted facility and flexibility with techniques. These responses include ones which showed appropriate use of techniques but not always in an efficient way. The student may get diverted at times but make an attempt to correct his/her own slips. These kinds of response suggested emergent understanding but lacked rigour.

Some of the students displayed a rich facility and flexibility with techniques. They demonstrated appropriate use of techniques and dealt with unexpected answers efficiently. References to several techniques were made appropriately and matter-of-factly. Such responses were coded as ‘deep understanding’ in this aspect.

5.2.2.3 Construction task: Emotion/Motivation

The construction tasks also provided opportunity for students to display emotional commitment. If the student displayed a laid-back disposition and engaged unenthusiastically with a flat voice tones and periods of waning, the code used was ‘surface connection’. One would assume that if a learner has motivation, the learner would engage in the task somehow.

In some cases, the students showed high energy and engaged in the tasks confidently. A highly motivated learner would demonstrate a disposition to engage and engage confidently. Such cases were coded as ‘deep understanding’. Instances when a learner displayed a limited degree of willingness to engage, were coded as ‘immersed understanding’.

One feature of example-construction that emerged from the data with mathematics undergraduates is judgments related to a mathematical aesthetic. Aesthetic judgments result from and contribute to emotional commitment. Aesthetic sensitivity provides one
set of criteria for a student to use to evaluate their own examples or those of colleagues. Evidences of mathematical aesthetic were commented as side notes.

5.3 Comparing the responses

Initially, I attempted to present the density shapes for the purpose of comparison on the six-fold framework itself in order to appreciate the stressing and ignoring of one or more of the aspects in the framework. However, the clusters around one or more of the strands did not make visual appreciation and contrasts as clear as I had hoped for. Therefore, I chose to present the density shapes in a table with the six aspects probed as column headings for the interview and the construction tasks.

In order to ensure that my analysis of the responses is systematic and transparent and to illustrate the coding scheme I developed, I present a ‘unit analysis’ of two of the interviews next. I chose these two interviews because they contain much of the information that I wanted to represented. The tables present four-column entries which are: interviewee (first column), interview responses (second column), their analysis (third column) and the coding (fourth column). I have selected parts of the semi-structured interview and the construction tasks to show how the responses are analysed. Responses I have chosen to illustrate are from Matt and Beth (‘average’ mathematics students - Table 5.3.1) and Harris and Danny (‘strong’ mathematics students – Table 5.3.2) because they contain a variety of responses that are illustrative of the coding.
### 5.3.1 Responses from Matt and Beth (‘average’ mathematics students)

<table>
<thead>
<tr>
<th>Interview probes</th>
<th>Analysis</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Int:</strong></td>
<td>What does the word “integration” mean to you?</td>
<td>Matt confused integration with differentiation and first suggested that integration was area underneath a graph before changing his mind to say that it was differentiation (area). Beth declared that she didn’t know what integration was but admitted knowing how to do it. She then attempted to correct Matt at which point Matt realised that integration was area under a curve between two points. Finally, they both agreed that integration was area under a graph.</td>
</tr>
</tbody>
</table>
| **Beth:** | Integration ... | The integral sign \( \int \) reminded Beth of ‘\( dx \)’ and Matt of ‘integrate’. They both agreed that the sign meant ‘integrate’.
<p>| <strong>Matt:</strong> | Isn’t that finding the area underneath a graph? No, that’s differentiation. | |
| <strong>Beth:</strong> | Don’t know, I haven’t done it yet [laughs]. Emm ... I know how to do it ... | |
| <strong>Matt:</strong> | Differentiation is area. | |
| <strong>Beth:</strong> | I thought differentiation was where you find out the ... | |
| <strong>Matt:</strong> | Integration is the area between two points under a curve, isn’t it? | |
| <strong>Both:</strong> | It is, yeah. | |
| <strong>Int:</strong> | What comes to mind when you see the sign ( \int )? | |
| <strong>Beth:</strong> | ( dx ). Emm ... don’t know, you just know you’ve got to integrate when you see the sign. | |
| <strong>Int:</strong> | Anything else? | |
| <strong>Matt:</strong> | Just integrate, yeah. | |
| <strong>Int:</strong> | What sorts of things have you discovered you need to watch out for when you are doing integration? | Beth’s awareness of things to watch out for centres around confusion between rules of differentiation and integration and techniques. Matt worries about the same thing. |
| <strong>Beth:</strong> | Sometimes I get a bit confused trying to differentiate when I know I’m suppose to integrate. | |
| <strong>Matt:</strong> | Sins and cosines as well when you are integrate them, they both turn negative or positive, you’ve got to remember all forms. | |
| <strong>Beth:</strong> | Sometimes I forget the order of how it works whether goes from sin then cos then negative sin then negative cos. | |
| <strong>Matt:</strong> | Differentiation is the other way round so it confuses the brain. I think differentiation is much more in the mind, integration ... you’ve got to really think about it because differentiation I find it easier. | |
| <strong>Beth:</strong> | I think it’s because we were taught differentiation first so we have that set in our heads. | Aspects related to technique and confusion with differentiation dominate Matt and Beth’s articulated attention with regard to things to watch out for. There was no evidence of awareness of connections such as area. This part of the interview is summed up with light-shaded circle and square (OD) under ‘awareness (misconception)’. |</p>
<table>
<thead>
<tr>
<th>Int:</th>
<th>What are some of the special words/language that you use when you are talking about integration?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt:</td>
<td>It always has ( dv ) at the end of an integral ... when you integrate you've the limits, you have square brackets around it, you put numbers up and below.</td>
</tr>
<tr>
<td>Int:</td>
<td>Anything else?</td>
</tr>
<tr>
<td>Beth:</td>
<td>You just use integrate an integral function and ... In ( x ). That's about it I can think of.</td>
</tr>
<tr>
<td>Int:</td>
<td>What words in a problem/context that tell you that integration is relevant?</td>
</tr>
<tr>
<td>Matt:</td>
<td>Find the area underneath a graph,</td>
</tr>
<tr>
<td>Beth:</td>
<td>You just look for either the specific sign or that. If it's not on there then you wouldn't be sure.</td>
</tr>
<tr>
<td>Int:</td>
<td>How were you introduced to integration?</td>
</tr>
<tr>
<td>Beth:</td>
<td>During A-levels, just randomly ran into electrical (inaudible) and we started a new chapter called integration.</td>
</tr>
<tr>
<td>Matt:</td>
<td>[We] started with basic examples of how to integrate, right from the beginning.</td>
</tr>
<tr>
<td>Beth:</td>
<td>They did differentiation first and then we were told that integration was similar but in fact the opposite form that when you differentiation you did something and when you integrate, it'll be what you started off.</td>
</tr>
<tr>
<td>Int:</td>
<td>What kinds of problem do integration help to solve?</td>
</tr>
<tr>
<td>Beth:</td>
<td>It depends on what the graph is about.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Construction tasks</th>
<th>Analysis</th>
<th>Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Int:</td>
<td>Given ( \int_0^1 (1-x)dx = 0 ). Can you find another example like this where the answer is 0? [Long pause]</td>
<td></td>
</tr>
<tr>
<td>Beth:</td>
<td>If you change the 1 into another figure and the higher limit will still hold that figure, would that work? ... Like for example, if it was ... [writes ( \int_0^4 (2-x)dx )], would that be true? [checks] Wait, you have to integrate that before you do the values. ... actually that's wrong now.</td>
<td></td>
</tr>
<tr>
<td>Beth suggested changing the function ( (1-x) ) to ( (2-x) ) and changing the upper limit to 4.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>She constructed ( \int_0^4 (2-x)dx ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>She appears to have spotted a pattern between the constant in the function and the upper limit.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beth’s response is coded with a light-shaded circle under ‘awareness’ ( ) because of the limited dimension varied and a medium-shaded circle (○) under ‘technique’ because the example she constructed was wrong but she was quick to realise it.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Int:</td>
<td>[To Matt] Can you find another example?</td>
<td></td>
</tr>
<tr>
<td>Matt:</td>
<td>No, I’m just trying to think.</td>
<td></td>
</tr>
<tr>
<td>Matt first substituted the limits directly into the function before Beth</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Int:  | What is going on in your mind? | Matt: | suggested to him that he had to integrate it first. | Beth: | Yeah but you’ve got to integrate it before you do the ... | Matt: | Oh yeah, that’ll be ... so that’ll always be naught when you put in the naught. | Beth: | Yeah. | Beth: | So if you integrate that, it’ll be $2x$ minus half $x$ squared. (put 0 ... put 4) that would work. Woohuh ... I’m not blonde, well ... I am blonde but I’m not having the blonde fame then, that’s wicked! | Int: | How did you decide? | Beth: | Just popped up in my head. Just felt like it was 2. | Int: | Did you use information from here (given example)? | Beth: | I doubled that and doubled that. ... I don’t know why I thought that would work but it appears to have worked. | Matt: | So if that was 8 and 4 it would work again? | Beth: | It might possibly, [checks] so that must also work. | Int: | Can you find another example? | Matt: | It’s double the 8, 4. | Beth: | I reckon if you did 6 and then you did zero and you did 3, because as long as that’s half that, that seems to be okay. [checks] | Int: | What is the most general example you can think of for which the answer is zero? | Beth: | So if you had $a - x$ with a limit of $2a$ and zero, that should always work, fingers crossed. I don’t know how it will apply in negative numbers but ... | \hline
| Int:  | Why is the integral coming zero? | Beth: | Now that ... I do not know. Emm ... | Matt: | Both of those sides (referring to $\int \frac{x - x^2}{2}$) clearly cancel each other out, don’t know why. That side will always be double that, and you divide by 2. | Int: | That’s because you’re using ... | Beth: | That’s just this specific one there, not for every single ... I think it’s just potluck there. | Int: | What can you change? | Beth: | The upper limit and the constant | \hline

**Chapter 5**

Exploring the data ...
(number). .. But I think it’d only be for specific circumstances because if you change those to the same thing (constant and upper limit) it wouldn’t work. If you change them to any random numbers it wouldn’t work, only for specific values.

relationship between the two dimensions (constant and upper limit) than Matt.

| Chapter 5 | Exploring the data ...

Table 5.3.1 - Analysis of response from ‘average’ mathematics students (Matt and Beth)

Below is the illustrative analysis of parts of the interview and the construction task for Harris and Danny (‘strong’ mathematics students).

5.3.2 Responses from Harris and Danny (‘strong’ mathematics students)

<table>
<thead>
<tr>
<th>Int:</th>
<th>Danny:</th>
<th>Harris:</th>
</tr>
</thead>
<tbody>
<tr>
<td>What does the word “integration” mean to you?</td>
<td>Everything comes together, reverse of differentiation.</td>
<td>Finding area under a curve, volume, solving differential equations.</td>
</tr>
<tr>
<td>What sorts of things have you discovered you need to watch out for when you are doing integration?</td>
<td>Whether the integral is defined, continuous functions.</td>
<td>Whether there is a limit.</td>
</tr>
<tr>
<td>What are some of the special words/language that you use when you are talking about integration?</td>
<td>By-parts, substitution, double integral, surface integral, integrate.</td>
<td>Solve differential equations, Fourier series, statistical functions, continuous functions, integral tests, series, continuous random variables, rate of change, area, volumes, blocks, work done.</td>
</tr>
</tbody>
</table>

Integration triggered! everything comes together and reverse process of differentiation in Danny. Harris mentioned area, volume and solving differential equations.

Both Danny and Harris displayed fluent language patterns that included a range of techniques and applications.

This part of the interview is summarised with a dark-shaded circle and square under ‘awareness’ ( ) because of reference to area/volume and connections to differentiation and technique.

This part of the interview is summed up with a dark-shaded circle and square under ‘awareness (misconception)’ ( ) because of reference to common mistakes beyond techniques.

This part of the interview is summarised with a dark-shaded circle and square under ‘language’ and ‘technique’ ( ) for fluency of language and access to technique.
<table>
<thead>
<tr>
<th>Int:</th>
<th>Given ( \int_0^1 (1-x)dx = 0 ). Can you find another example like this where the answer is 0?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harris:</td>
<td>You can get odd function.</td>
</tr>
<tr>
<td>Danny:</td>
<td>It’s gonna be like that [sketches]. [Quickly]</td>
</tr>
<tr>
<td>Harris:</td>
<td>You are gonna have half of the area above x-axis. Something like this [sketches ( y = 1 + x ) graph].</td>
</tr>
<tr>
<td>Danny:</td>
<td>It’s not gonna be something like that, you got that [shows sketch].</td>
</tr>
<tr>
<td>Harris:</td>
<td>Yeah.</td>
</tr>
<tr>
<td>Danny:</td>
<td>Then you could double it. You could put 0 to 4 ( \int_0^4 (2 - 2x)dx ). You are multiplying [later changes limit to 0 to 2].</td>
</tr>
<tr>
<td>Int:</td>
<td>Can you tell me what you are doing? This here is that kind of graph so you got positive and negative area, so you could double the amplitude so that would still be 0 to 2. Or you could double the limit and change the function, so it would still be twice the height and then change the function.</td>
</tr>
<tr>
<td>Danny:</td>
<td>You stretch it along the x-axis to an angle [but] you still retain the shape.</td>
</tr>
<tr>
<td>Harris:</td>
<td>You stretch it along the x-axis to an angle [but] you still retain the shape.</td>
</tr>
<tr>
<td>Int:</td>
<td>Can you give another example? Integral from 0 to 4 … We can have sin function that looks similar to this, the limit here is ( \pi ) and this one is ( \pi ) from 0.</td>
</tr>
<tr>
<td>Harris:</td>
<td>He suggested changing the limits (although his limits are incorrect) and the suggested sin ( x ) from 0 to ( \pi ).</td>
</tr>
<tr>
<td>Danny:</td>
<td>As you change the limit, the function will change.</td>
</tr>
<tr>
<td>Harris:</td>
<td>How about if we say from 1 to -1?</td>
</tr>
<tr>
<td>Danny:</td>
<td>Will it work for integral of ((1 - x)^n), ( n ) will have to be odd.</td>
</tr>
<tr>
<td>Harris:</td>
<td>Yeah, good generalisation.</td>
</tr>
<tr>
<td>Int:</td>
<td>What is it about your example that is like this example? They have equal amount of area as the main example, part above and part below.</td>
</tr>
<tr>
<td>Danny:</td>
<td>You’ve got a general [example] ((1 - x)^n). I think this is gonna be odd [and] this is gonna be even if ( n ) is odd.</td>
</tr>
<tr>
<td>Harris:</td>
<td>Danny generalised to ((1 - x)^n) and related to the integral having equal amount of area above and below the x-axis. Harris generalised on the oddness of the function.</td>
</tr>
<tr>
<td>Int:</td>
<td>Can you give another example? So if we raise the power so we can get a polynomial that looks like this. We can take this degree 1 and we can make one with degree 2 so that it looks something like this.</td>
</tr>
<tr>
<td>Harris:</td>
<td>Danny expressed awareness of relationship between what could vary and change allowed. Harris expressed appreciation for Danny’s generalisation.</td>
</tr>
<tr>
<td>Danny:</td>
<td>As you change the limit, the function will change.</td>
</tr>
<tr>
<td>Harris:</td>
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<tr>
<td>Danny:</td>
<td>Will it work for integral of ((1 - x)^n), ( n ) will have to be odd.</td>
</tr>
<tr>
<td>Harris:</td>
<td>Yeah, good generalisation.</td>
</tr>
<tr>
<td>Int:</td>
<td>Harris generalised the example to polynomials</td>
</tr>
<tr>
<td>Harris:</td>
<td>Danny expressed awareness of relationship between what could vary and change allowed. Harris expressed appreciation for Danny’s generalisation.</td>
</tr>
<tr>
<td>Danny:</td>
<td>As you change the limit, the function will change.</td>
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<td>Harris:</td>
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<td>Danny:</td>
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<td>Harris:</td>
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<tr>
<td>Int:</td>
<td>Danny generalised to ((1 - x)^n) and related to the integral having equal amount of area above and below the x-axis. Harris generalised on the oddness of the function.</td>
</tr>
</tbody>
</table>

Both Harris and Danny displayed rich connections between function property, imagery and association to area. They also displayed language fluency that reflected this awareness. This part of the interview is summed up with dark-shaded square and circle (●) under ‘awareness’ and ‘language’.
Chapter 5

Exploring the data ...

<table>
<thead>
<tr>
<th>Int:</th>
<th>What can you change in the example?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Danny:</td>
<td>You can change whatever you like.</td>
</tr>
<tr>
<td>Int:</td>
<td>Does it matter what you change?</td>
</tr>
<tr>
<td>Danny:</td>
<td>Yes.</td>
</tr>
<tr>
<td>Harris:</td>
<td>Scalar multiple, you can multiply by 2 and still have zero or you can change the variable.</td>
</tr>
<tr>
<td>Danny:</td>
<td>You can change the function but then you’d have to change the limit.</td>
</tr>
<tr>
<td>Int:</td>
<td>Why do you have to change the limit together with the function?</td>
</tr>
<tr>
<td>Danny:</td>
<td>Because the thing you are considering includes different areas, so [the limit has to change together with the function].</td>
</tr>
<tr>
<td>Harris:</td>
<td>You can change the x, doesn’t have to be x. It can be y, any variable can be used instead of x.</td>
</tr>
<tr>
<td>Danny:</td>
<td>The fact that you have zero gives you lots of relationships between limits and the function.</td>
</tr>
</tbody>
</table>

Harris displays awareness of another dimension (variable x) that can be varied. This part of the interview is summed up with a dark-shaded circle (●) under ‘awareness’.

| Danny: | The fact that you have zero gives you lots of relationships between limits and the function. |

Table 5.3.2 - Analysis of response from ‘strong’ mathematics students (Harris and Danny)

5.4 Summary

It seems that learners may have access to more forms of knowledge than those that are demonstrated in interviews alone. Because different kinds of probes revealed different depths in different aspects of their understanding, I set out to compare learners’ responses in the interview and the construction task in observable aspects related to behaviour and aspects that reveal awareness. It became necessary to devise a way of visually appreciating the contrast because of the range of responses that were analysed. Each of the aspects probed was categorised into three major categories that emerged from the responses, namely ‘surface or elementary understanding’, ‘immersed state of understanding or emergent thinking’, and ‘deep understanding’.

In this chapter, I have outlined the kinds of response that were coded with each of the three categories, in order to make comparison possible between responses that addressed different aspects of integration to be made. I have illustrated with examples from the interview transcript how the coding works. In the next chapter, I present analysis from the
main study, in which I use the form of analysis and presentation of data that I have developed and discussed in this chapter.
Chapter 6

Main Study: Data Analysis

Interviews with PGCE students

6.0 Introduction

This chapter is the first of four chapters describing results of the main study. Because of the different backgrounds of the participants, each chapter provides the information available about these participants. In this chapter, findings from PGCE students are presented. As described in Chapter 3, the interviews were intended to reveal:

- learners’ responses to aspects of integration that were not previously expressed or focused upon in other research;
- aspects of integration that learners’ are aware of/ focus their attention and the constituents of that awareness;
- the possible (mis)match between learners’ responses in the interview and the construction tasks vis-à-vis their awareness, behaviour and emotion;
- the effectiveness of the construction tasks as probes to reveal awareness of aspects of integration in behaviour rather than as thought about.

In order to obtain a visual summary of the responses made during interview and the construction task, I developed a system of shading of circles and squares as described in Chapter 5. The aim was to develop a visual appreciation of the contrast between what was said in the interview and what was revealed when learners engaged in construction tasks.
Students in this group were on a one-year PGCE course at a local university. The course focuses on learning to teach mathematics in secondary school. During the one-year course, they studied Education and Mathematics Pedagogy (12 weeks) and taught at school (20 weeks). They did not have much time to work on their own mathematics subject knowledge. Because they were doing their PGCE some time after their degree, their degree grades were not available. There were four students who volunteered to be interviewed in this group. They were interviewed in pairs. The first to be interviewed in this group were Jon and Denise.

6.1 Pair P1 (Jon and Denise)

Jon was a mature male student who was very vocal and expressive about his ideas. Denise was a young female student who was also very vocal and vibrant. Jon held a degree in Economics and Statistics, while Denise had a degree in Mathematics and an MBA (IT). Where appropriate, some parts of the interview have been clarified with transcripts and comments while other parts have been summarised with comments.

6.1.1 The interview

In the interview, Denise (marked with circles) displayed a rather limited connection with integration. She recalled the meaning of integration only as the ‘reverse of differentiation’. Jon (marked with squares) showed a relatively deeper connection when he mentioned ‘summation, working out areas, volumes and to some extent, inertia’. The sign $\int$ reminded him of indefinite integration. Denise said that the sign brought to mind indefinite [integration] and she added that there were ‘no limits, you just need to find the value’ (□ under ‘awareness (connectedness)’).
Denise: I mean ... if you got any number under the sign, then you just find the value and then you know the limits ... it lies between this and that particular value.

Thus her recall of what is involved was not as rich as it might be.

Denise displayed connection with techniques or methods involved to obtain ‘the value’. It is evident that her attention was dominated by technique to find ‘the answer’. For Jon, however, the topic seemed to trigger deeper connections. The following probe provided exemplification of these observations.

Interviewer: Do you have any images/conversations when you see the sign? What’s the first thing that comes to your mind?
Jon: An integration exercise in the textbook.
Denise: Yeah ... whenever you see the sign, it’s an indication that this is an integration question and you have to apply particular rule or something to find value. ... We were never told exactly what integration is. Like the reverse of differentiation and whatever you apply to differentiation (rules) just reverse of that. That’s how we find integration, we were never told exactly... in real life where you can use integration.
Jon: I suppose the summation of the [normal distribution] diagram to accumulate the finite stripes and working out from the first principle the value of integration. I have not applied it. I suppose I have in Statistics so normal [distribution] the proof that it is equal to 1. That’s the only example in stat apart from pure math. I used integration in Mechanics in school for centre of gravity.
Denise: The sign $\int$ has certain significance. When you see it you apply certain rules.
Jon: Apart from use in area under a curve and the first principle, but then you quickly get on to the rules and different techniques.
Jon displayed considerably deeper awareness of things to watch out for in the topic than Denise as he mentioned common errors related to specific techniques while Denise was concerned with applying rules (\[\text{under 'awareness (misconceptions)'}\])

Interviewer: What sorts of things have you discovered you need to watch out for when you are doing integration?

Jon: You've got to work out what rule/algorithm to apply to a particular question. For example integration by substitution, the formula for substitution, the kind of complication you don't get in differentiation.

Denise: Need to look for limits because your answer differs when you got limits.

Jon: And that the expression is continuous, there aren't any discontinuities within the limits. So you got to be careful in applying the limits and then plus C. For solving problems (application) have you got the right technique, some things you've got to integrate, some things you've got to differentiate.

Denise displayed propensity to relate integration to techniques in which words that were used to relate to the topic concentrated on technique. She mentioned limits, substitution, trigonometry, integration by parts, integral by first principle and product rule as some of the special words that she used when talking about integration (\[\text{under 'language' and 'technique'}\]). Jon did not mention any.

Denise recalled being introduced to the topic as the reverse process of differentiation. Possible origins of integration did not come to the fore and so, there was no evidence of exposure to or awareness of them. Jon recalled being introduced from differentiation from first principle and then integration (\[\text{under 'origins'}\]). As I noted in Chapter 2, I am not making any assumption that exposure to origins of integration will necessarily influence engagement but exploring what aspects of a topic are inherent to be called upon in learning. What is important is the residue you find of that emotional component. Of
course even when a student find a mathematical topic useful and applicable they may still disengage from it and similarly, even when students find a mathematical topic not particularly useful or applicable they may still engage with it. There is no causal effect as Boaler (2000), Brown (1999) and Nardi & Steward’s (2000) studies show. Rather, I was interested to look at the affective affordances offered by the topic that reflected in learners’ behaviour.

6.1.2 The construction task

The construction tasks revealed, to a greater extent, distinct features in the language, awareness and emotional aspects of the students’ understanding. In Task 1, Jon constructed $\int_{0}^{3} (3-x^2) \, dx$ by changing both the upper limit and the function. He displayed awareness of the fact that some dimensions in the example could be changed simultaneously, but in relation to each other (unlike some other subjects), namely limits and function. His attention seemed to be directed to the upper limit, increasing it by 1 and at the same time changing both terms in the function. Denise simply reversed the order in the function and constructed $\int_{0}^{2} (x-1) \, dx$. In this instance she did not display awareness of other possible dimensions that could be varied. She then decided to work on Jon’s example (under ‘awareness (connectedness)’).

Jon constructed a second example by increasing the upper limit by 1 and simultaneously changed the terms in the function $\int_{0}^{4} (4-x^3) \, dx$. He appeared to be guessing at a pattern between the upper limit and the first and second terms in the function, admitting that he had not worked out the general rule for the pattern. Having been corrected by his partner,
he then adjusted the first term to accommodate the new upper limit and went on to
generalise his example (under ‘technique’).

Jon: It would have to be ... 16. Make that any number you like. So $16x$ ...
[works out]

Denise: No, it should be $x - 4x$. Or you can make it $\int_0^4 (x - 4x^3)\,dx$ ... It won’t
work.

Jon: $1 - 4x^3 ... 16 - x^3$, that’s alright. [Checks] $16x - \frac{x^4}{4}$.

Denise: Yeah ... it will work if you do multiples of 4.

Jon: So then we’ll have $25 - x^2$ ...

Denise: If you do $\int_0^5 (25 - x^4)\,dx$ it works. [Checks] $25 - \frac{5^4}{4}$. No, it has to be
another multiple of 5 ... 125.

Interviewer: Can you give another example?

Jon: $\int_0^6 (6^4 - x^3)\,dx$.

Denise: [Checks] Yeah.

Based on his first two examples, I conjecture that Jon looked for a pattern and generalised
(partially correctly) for the third, and subsequently spotted a pattern and generalised. He
appeared to be caught up in the act of trying to find the relationship between the limits
and the answer, and it was not apparent whether he was aware of the properties of the
integral. Indeed, he did not display any awareness of the relationship between the
symbols and the object being generalised.

In producing a general example for the task, Denise constructed $\int_{-1}^1 x\,dx$ and $\int_{-1}^1 x \,dx$, and
after checking whether they worked, suggested that making one [limit] negative and one
[limit] positive might work (☐ under ‘awareness (connectedness)’). Responding to the probe ‘What can you change in the example?’ Jon suggested that the limits and the expression could change. To this, Denise added:

Denise: When you change [upper] limits, take multiples of limits. For example 5 [you take] 125.

Jon followed up with this remark and constructed $\int_0^n (n^{x-2} - x^{n-1})dx$ as the general example and suggested that it could be proved by induction (☐ under ‘awareness’). Jon and Denise’s responses are also coded with ☐☐ under ‘technique’ for their efficient but inappropriate use of technique.

It seems that both Jon and Denise were so caught up in the act of finding relationships and generalising that they overlooked other useful connections to the integral. They did not display any awareness of associations of the integral with area, although they displayed language fluency to some extent (☐☐ under ‘language’).

The prompt “Why is the integral coming to zero?” seemed to trigger awareness of this association and might have shifted Jon’s attention to make explicit reference to this connection.

Interviewer: Why is the integral coming to zero?

Jon: Because if it is the area under the curve…

[Sketches the graph]

[Long pause] We’ve actually integrated across … We’ve probably got it wrong, have we? Well, $y$ equals $1 - x$ is that straight line [sketches] So in fact you’ve got two areas that have been taken away from each other. So in fact the area under the curve, depends how you define area under a curve … so should we not have integrated from 0 to 1 and then form 1 to 2?

[Pause]
Interviewer: Now can you think of another example?
Jon: *Any value between there [points to the limits]. So where you got the area is opposite sides of the *x*-axis. In this case you should integrate it from there to there [points to the limits].*

Having become aware of the connection, Jon displayed awareness that limits could vary. He also implied a range-of-permissible-change within that dimension. The fact that he did not mention that other dimensions could change (for example, function) suggests that in this situation his sense of generality appeared to be limited to and dominated, at least for the moment, by his attention to limits. I conjecture that both Jon and Denise attended to the *process* aspect and not to the *concept* of integration. Their attention needed to be explicitly directed to this aspect for them to associate the integral with area, although they expressed association with area in the interview.

In Task 2, Denise constructed a simpler example by replacing the fraction $\frac{3}{2}$ in the example with 1 and maintained the format of the example [$\ln x + \text{constant}$]. She seemed to be attending to details of the example, focusing on reducing complexity by removing the coefficient and replacing the fraction with 1. There is no evidence to suggest that she was aware of the *form* of the integral expression because she did not attend to it and did not produce the differential form of the expression (\( \circ \) under ‘awareness (connectedness)’).

Denise:  
\[
\int (\ln x + 1) \, dx = x \ln x - x + x \quad \text{because you minus } x \text{ and then you've got plus } x \text{ then you get } x \ln x , \text{ which is very simple ... that's the best. And then [for the complex one]} \quad \text{if you do } \ln x + 1 \ldots , \text{ if you do times } \ldots , \ln x + x ,
\]
what do you get... \( x \ln x - x + \frac{x^2}{2} \).
She appeared to be looking for a *pattern* to generalise the situation and did not check her conjectures by differentiating. She displayed evident involvement and disposition to try, to adjust, to conjecture and to generalise. Having generalised, she seemed content (evidenced by the speed at which she launched into generalising) that she could manipulate the generalisation to produce simpler and more complex examples.

[Long pause]

Denise: I think every time you add each $x$, you involve higher term ... $x$ becomes $x^2$, $x^2$ becomes $x^3$.

Interviewer: Maybe you would like to tell me what are the things you considered when you are making it simpler or more complex?

Denise: Could I generalise the problem. ...

\[
\int (2 \ln x + x) \, dx = 2(x \ln x - 2x) + \frac{x^2}{2}
\]

\[
\int (3 \ln x + x^2) \, dx = 3(x \ln x - 3x) + x^3
\]

\[
\int (n \ln x + x^{n-1}) \, dx = n(x \ln x - nx) + x^n
\]

is the generalised form. You can do it with $4x$, $5x$, ...

Interviewer: How do you make a complex one?

Denise: I think the simpler one is easy, you just write \( \int \ln x \) plus any constant.

Interviewer: What number do you want to put in for $n$ [for the more complex example]?

Denise: You can have \( \int (8 \ln x + x^7) \, dx = 8(x \ln x - 8x) + x^8 \). ... You can involve positive [term], you just generalise [some of] these and if you involve negative term like \( \frac{1}{x^3} \) might be a difficult one because you have to change it to $x^3$. If you involve any negative number instead of $n$ ...

[...]

Interviewer: How about the second part of the expression?

Denise: It is just the differential of this.
Having generalised, she simplified and complexified the generalisation within a very narrow range, suggesting domination of technique and suppression of forms and previous connections (under ‘technique’). However, I conjectured that the fact that she mentioned awareness of the form of the example and did not attend to it and did not give an answer suggests that her attention was on the act of generalising itself and not on the manipulability of the generalisation as an object. She showed no interest in or concern to check her conjectures, although she was quite articulate in expressing her thinking (under ‘language’).

Jon, on the other hand, displayed some awareness of dimensions that could vary when he constructed a simpler integral which was not in the format of \( \ln x + \text{constant} \) (under ‘awareness (connectedness)’). He seemed to have discerned and maintained two terms in the expression and the addition operation. \( e^x \) was used in the simpler example presumably because of its recurrent property.

Jon: I’m thinking of having a different expression [altogether] ...

\[
\int e^x + x^2 = e^x + \frac{x^3}{3} = \frac{d}{dx} \left( e^x + \frac{1}{12} x^4 \right).
\]

Similarly, Jon too seemed to look for patterns and generalised based on the properties he discerned. Although he claimed that he had the ‘generalised form’, it is not evident whether he was aware of the form when constructing examples. He suggested that a more complex integral could have inverses, square root 3’s and ‘something horrendous’. For him, complex example was synonymous with ‘hard-to-integrate’. Although he came across as very fluent, his articulations are not always accurate (under ‘language’). It appears that Jon was also caught up in getting an answer and he did not display awareness of relationship or form of the example. The fact that Jon did not attend to the form of the example may explain his act of not varying other dimensions in the example.
(limited *awareness-in-action*, Mason 1998). The remark about the second part of the expression by Denise was in fact, incorrect.

Interviewer: What is your complex one?
Jon: I haven’t found one; I’ve just [the] generalised form.

Interviewer: Give me an example.
Jon: \[\int \frac{1}{\sqrt{x^3 + x^4 + x^5}} \, dx\]

Interviewer: Why do you think that will make it harder?
Jon: It’s a complicated substitution.
Denise: And then you have to look at ... it’s 1 over something, there’s root and you apply a different formula.

Interviewer: How about the second expression?
Denise: It is just the differential of this (points to the answer in the middle).

When asked to describe what came to mind in the example \(\int_{\pi}^{2\pi} x \sin(x^2) \, dx = 1\) in Task 3, both Jon and Denise’s attention was directed to technical features of the example (under ‘language’). Jon appeared to be drawn to the \(\sqrt{\pi}\) in the upper limit and Denise pointed to ‘calculating the value \(x \sin^2\)’. Both agreed that the graph of the function did not come to mind, although they associated the concept with working out area under a graph (in the interview). Attention seemed to be focused on limits and how to solve the problem, which brought into mind application of rules and which method to use. The way Denise ‘read’ the integral as ‘easier to apply rules’ illustrated her facility (but not always accurate) with technique but likewise, it also pointed to her fragmentary connections since no reference was made to functions or associations such as area.

Interviewer: Do you have any images?
Jon: No, the graph doesn’t come to mind.
Denise: Yeah, when you look at the expression, it’s just the limits and then how are you going to solve this [integral] expression.

Interviewer: How would you describe the integral in your own language, as you understand it?

Denise: I’m reading it as ‘$x \sin u$’, just give any value to $x^2$. It makes it easier to apply rule.

Based on this example, Denise constructed a simpler example $\int \frac{1}{x - x^2}$ by recalling the substitution method of integration from past experience, which used direct trigonometric substitution method. For a more complicated example, she suggested:

Denise: You can put $x$ equal to sine, and then $1 - \sin^2 \theta$, is it right?

$\sin^2 \theta + \cos^2 \theta = 1$. If you put $x = \sin \theta$, it will be $\sqrt{1 - \sin^2 \theta}$,

$\frac{1}{\cos \theta} = \sec \theta$ ... Before you start you must know about the formula

$\sin^2 \theta + \cos^2 \theta = 1$, what relation you can get because you are using the positive one, it’s $\tan^2 + 1$, then you have to remember you can’t substitute $\sin \theta$ in this one ... If it’s $1 - x^2$, you know you can put sine. That (refers to Jon’s example) should be $\tan^2 \theta$.

Her remarks about this method of integration suggest Denise had considerable facility with technique (\(\bigcirc\) under ‘technique’). However, the dimensions she varied did not suggest a very rich connection because she used only a particular, inappropriate substitution (trigonometric) and did not make explicit reference as to how this method of integration worked (\(\bigcirc\) under ‘awareness’).

Jon constructed a simpler example $\int \frac{1}{\sqrt{x^2 + 1}}$. In his question to Denise, he appeared distracted by Denise’s remarks about this method of integration.

Jon: Is that simpler than that or more complex?

Denise: They are definitely simpler.

Interviewer: Why do you think they are simpler?
Denise: When you start with trigonometry, you know there’s only three terms: sin, cos and tan. So you can straight away put sin $\theta$, if it doesn’t workout then you put cos and then you try with tan.

Jon did not display awareness of this method of integration in his construction of a more complicated example (under ‘awareness’ and ‘technique’). This was apparent when he constructed $\int x \sin x^3 \, dx$, which was quite a complicated integral to solve using the substitution method. He might have constructed the example without actually thinking about the basis by which he constructed it. He had lots of ideas, but most did not actually work. Denise constructed $\int \frac{1}{x^4 - x^2} \, dx$ as a more complicated example, suggesting that that could be reduced to $x^2 - 1$ (under ‘awareness’ and ‘technique’).

Asked explicitly about the dimensions that could be varied in the example to make it simpler or more complicated, Denise confidently remarked:

Denise: Whenever you do substitution the first thing is right substitution, second changing the whole expression in terms of $t$, third changing limits and working out answers.

In Task 4, Denise displayed awareness of connections between terms in this method of integration. She made implicit reference to this relationship by providing more examples. However, her remarks about this relationship were not very precise (under ‘awareness’ and under ‘language’). Jon did not display awareness of this relationship, nor did he display any facility with this method of integration because he talked about substituting $x$ with tan $\theta$ (under ‘awareness’ and under ‘technique’). He might be hanging on to the previous task (integration by substitution).

Denise: $(x + 1) \sin (x + 1)$ but it’s the same you can let $t = x + 1$ so you can derive that expressions. You can do, if don’t involve any trigonometric term,
(x+1)(x+1+2x) or any term you can make it like this is t, this is t+2 if this is $x^3+x$ and this is $x^3+x+2$.

Jon: $\sqrt{x(1+x^2)}$, you can let $x = \tan \theta = x + c$. That’s by substitution. It is easier than applying the product rule.

In Task 5, both Denise and Jon seemed to be attending to the complexification of the example and did not display awareness of the form of the example that required this specific method of integration. It is likely that, because attention was focused on complex terms to be put together to make the example more complicated, they had overlooked critical features which made the example work. Denise, however, mentioned changing $x$ to $z$, which suggests awareness of one dimension that could vary (denoted under ‘awareness’).

Interviewer: How did you make it more complex?

Denise: You involve any division function, like 1 over something and then the multiplication.

Jon: And some square roots.

Denise: Or you can make it 1 over square root.

Interviewer: So to make it harder what you have done is …

Denise: Increased the powers, add more terms or constant numbers, 1 over something … then multiplication, you can make it negative or 1 over square root of something and then $x^3 + x$ or something. And then to confuse them you can write $z + 3$ or $\frac{z^2 + 1}{\sqrt{z^3 + d}}$ because they are familiar with $x$.

6.1.3 Making comparison

Denise displayed a strong tendency to attend to techniques. Her language and the intensity of her voice tone characterised confidence in her facility with and dominance of this aspect. In the interview, she revealed limited connections and her associations appeared to be overlaid by her attention to technique. Her awareness of common
obstacles linked with the topic also highlighted this tendency. Access to uses seemed rather limited.

These marked features are also revealed in the construction tasks. The examples she constructed and the dimensions she varied in them revealed a great deal more about her attention to details and her restricted awareness of connections. Her responses showed significant emphasis in techniques which structured her concept image of integration. Elements in her example space appeared to be characterised by this emphasis on technique and appeared isolated. It is likely that her attention to details displaced her ability to be aware of relationships and connections. However, she had a sense of overall form, but not always correct. She demonstrated a high level of disposition to try.

The interview responses revealed Jon’s richer awareness in terms of connections and common mistakes compared to Denise. By contrast, his responses also highlighted limited language fluency marked by his hesitant speech and moderate voice tone. He also displayed a tendency to stress techniques of integration, but to a lesser extent than Denise. Access to uses was displayed to a reasonable degree, unlike Denise. The interview probes shed light on the structure of Jon’s concept image, in which technique aspects were emphasised but not always correctly. The examples he constructed evidenced the fragmentary nature of his example space.

Although the construction tasks did not reveal any advancement in language fluency, they highlighted richer connections for Jon in some instances. However, on a number of occasions Jon displayed restricted facility in techniques of integration. Responses in the construction tasks showed considerable depths in associations and imagery aspects of their concept image of integration but not the technique aspect. He showed a willingness to try novel tasks.
Table 6.1 below shows a summary of Jon and Denise’s responses in the interview and the construction tasks.

<table>
<thead>
<tr>
<th>Pair</th>
<th>Interview</th>
<th>Construction tasks</th>
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<tr>
<td>Jon &amp; Denise</td>
<td>◯</td>
<td>◯</td>
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</table>

Table 6.1: Jon and Denise’s responses in the interview and construction tasks

6.2 Pair P2 (Rob and Hanna)

Rob and Hanna were the second pair in this group. Rob was a male student who was very vocal. Hanna was a female student who was equally vocal. Rob had a degree in Mathematics and Its Applications while Hanna held a degree in Chemical Engineering.

6.2.1 The interview

When asked, Rob (marked with squares) referred to integration as finding area under a line and the reverse process of differentiation (◯ under ‘awareness (connectedness)’).

He displayed feelings of discontent from the beginning when he claimed that he had used integration a lot in his undergraduate studies but ‘never really knew what it was for,’ although he knew ‘it has got a lot of uses’. Hanna (marked with circles) said that she first used integration in statistics, to find mean, including distribution functions. Although she claimed being introduced just to the method of integration, the explanations were done later in pure [mathematics] (◯ under ‘uses’).

For Hanna, the sign $\int$ brought to mind connection with ‘area straight away’. She related to the fact that it ‘looks like a curve’ and assumed integration was to find out the area
under a curve bounded by the axes (\( \bigcirc \) under ‘awareness’). Rob related the sign with technique (\( \bigcirc \bigcirc \) under ‘awareness’ with an emphasis on technique (\( \bigcirc \)) (see Table 6.2).

Rob: Just to know that it is an integration question and just look for the \( d \) whatever at the end and see what I’m going to integrate in terms of, for example \( dx \) in terms of \( x \). I don’t have any idea what it is for necessarily. Just go in and integrate it. […] I’m just going to use my \( u^m = \frac{u^{m+1}}{m+1} \). Literally I’ll be thinking about which one [technique] I’m going to use rather than what the question is about.

Probed further to find out whether other associations came to mind, both Rob and Hanna displayed negative reaction towards the topic (\( \bigcirc \) under ‘uses’).

Interviewer: Do you have any images/conversations when you see the sign?

Rob: Yeah, can I get away by using this [laughs] or will I need to integrate by-parts or by substitution and if it is substitution, that’s what I fear. And … just knowing what substitution to use that will help.

Hanna: It’s hard. You have so much with the … And with the by-parts, there’s lots of rules to practice. When you see that sign, you are thinking ‘Please let it disappear’.

Rob: It’s not the nicest sign to see…this curly with a bit.

Their awareness of common errors also revealed an inclination towards techniques. Rob declared the integration constant as the thing to watch out for when doing integration. Hanna revealed relatively richer connections in terms of awareness of misconception when she talked about technical aspects such as getting the formulas right for [integration] by substitution and [integration] by parts and things she used to do incorrectly (\( \bigcirc \) under ‘awareness (misconceptions)’) (see Table 6.2).

Words such as substitution, partial, constants, powers, expressions, logarithms, trigonometry and finding out regions formed part of Hanna’s language pattern. Rob added to the list words such as [integration] by parts and boundaries. It appears that Rob
displayed language fluency restricted to techniques in comparison to Hanna who displayed a richer source of language. For Rob, words in a problem/context that triggered integration seemed to be ‘Find the area under this curve’ and he claimed that he could not think of anything more. Hanna related to working backwards to find the arbitrary constant given \( f' \) and \( f' \) and she illustrated it with an example \( f'' = x^3 + 2x^2 + x \) (under ‘language’ and ‘technique’).

The distinction between \( \int x^2 \, dx \) and \( \int_0^a x^2 \, dx \) was marked by Rob and Hanna as getting ‘an answer’ for the first integral and as pointless action for the second. The extract below reveals their limited awareness and their tendency to stress technique aspects (under ‘awareness’).

Hanna: You don’t get an answer for that [first]. It almost seems no purpose because you get no actual number. With that one [first] you are just getting an expression.

Rob: I don’t really know what you’re actually getting when you do that. Yeah … \( \frac{x^3}{3} \) but what does that really represent, I don’t know. I know that is the region between there and there [draws] but without the boundaries, what have you really got? You got \( \frac{x^3}{3} \) so you got a new graph come up but what is the point?

Hanna: You could have a 9 there and say the line \( 2x + 5 \) you are integrating it and all you know is that the area underneath is \( \frac{2x^2 + 5x}{2} \) but it seems weird to represent an area in terms of \( x \) because you only think of \( x \) as points in the bottom.

Rob: Plus a \( C \).

Hanna: Yeah plus a \( C \).

Rob: \( \frac{x^3}{3} + C \) is the general area whenever you put your values in you will have …

Hanna: Yeah but it just seems meaningless.
Rob: No point, yeah.

Possible uses of integration did not surface as Rob suggested area as the only problem integration helped to solve. Hanna was drawn to contemplating which method to use when looking at the problem and whether she could simplify it. She also suggested volumes and areas as contexts where the topic was used in her background studies (engineering). There was no evidence of exposure to or concern for other uses (☐ under ‘uses’ and ‘origins’) (see Table 6.2).

6.2.2 The construction task

The construction tasks revealed a good deal more about aspects that were stressed or overlooked by Rob and Hanna in the interview. In Task 1, Rob displayed richer connections when he related to oddness and evenness of functions and associative link to area (□ under ‘awareness’).

Rob: It’s going to be one of those even functions, isn’t it, $x^2$? No, not $x^2$. Oh, I haven’t done this for so long... That would, wouldn’t it [refers to the example]? That line going from there to there and that and that cancel.

Hanna displayed limited connections when she engaged in algebraic manipulation of the details in the example (☐ under ‘awareness’).

Hanna: [checks by writing $x - \frac{x^2}{2}$ ...] I can’t think of any. I suppose could you work back and say, ‘Well, I know $25 - 25$ is zero, how can I get the $25$ like $5x$ and I can have $x^2$. That’s how I might do it so I must have integrated that and that should be $x - 5$ and do that between $5$ and ... oh no, $x^3 + 5$, wouldn’t it? Oh, that’s not zero, coz then I would have $5^3 - (5 x 5)$, which is $25$ minus ... That’s the same kind of thing, I suppose ... You knew it from a scientific version, didn’t you. I just worked backwards.
Rob: Yeah, it is just symmetric, I don’t even finish it so whenever you’ve got the same
thing on both sides, actually opposite things on both sides, like where you stick the
same value and plus and minus and that bit cancels that bit but I can’t think of any
example but that does it.

 Asked to construct another example, Rob constructed $\int x^3 - 4$ and suggested ‘because
that raises the power and that will become positive.’ The evidence suggests Rob revealed
stronger connection in this task, although he appeared rather predisposed only to
techniques in the interview. Although reference to graph was not voluntary, he claimed
to have visualised the function and constructed another odd function (trigonometric) as a
third example. A general example was constructed based on the oddness property of the
function (☐ under ‘awareness’). However, Rob was not very articulate in expressing his
views (☐ under ‘language’). Hanna did not construct any example in this task.

 Interviewer: When coming up with these examples, are you envisioning a graph or are you
substituting limits?

Rob: Yeah, thinking about the graph as well and more about the power of x there. I
thought of $x^2$ but $x^3$ wouldn’t work. When you integrate that it is going to be a
cubic power and the positives and the negatives are going to stay rather positive,
negative and negative is going to be positive.

 Interviewer: Can you give another example?

Rob: Along the same line or I could try and think of another odd function, is it sine or
cos? Yeah … sin or cos, one of them because if it goes under the graph evenly you
can just take it from there to there [draws $y = \sin x$ and $y = \cos x$ graph].

 Interviewer: Can you think of a general definite integral for which the answer is 0?

Rob: $-a(f) = af$? I can’t remember the formula but it’s whether a function is odd or
even, isn’t it? Aaah! … is it $f(x) = f(-x)$ [checks] … I think that’s it.
The next task revealed Rob and Hanna’s connections with integrals, which highlighted the kind of associations that came to mind. Rob related the integral \( \int (2-x)dx \) to area while Hanna appeared to be thinking algebraically about ‘the answer.’ Although Rob’s reference to area was not entirely accurate, it highlighted his principal associations (☐□ under ‘awareness’).

Interviewer: What can you say about \( \int (2-x)dx \)?

Hanna: Is it always negative because when you integrate it you get \( 2x - \frac{x^2}{2} \), that’s always going to be positive and that’s always going to be negative, or more likely to be negative, when bounded... I just see the answer, the simple one.

Rob: It’s got to be that bit of area, isn’t it?

Hanna: I’d do it if I’m doing a curve but what is it bound by? It could be integrated between 2 and 5 or something like that.

Interviewer: So you’re thinking of the answer?

Hanna: Yeah.

Rob: I was thinking about the answer as well. When it is not bounded, what is the point?

Interviewer: So what would you say now about \( \int (2-x)dx \)?

Hanna: The answer, \( 2x - \frac{x^2}{2} \).

Rob: It defines the region under \( 2-x \).

Rob’s remarks in Task 1c highlighted richer connections to relationships to geometric representations. While Hanna’s remark reflected a link to superficial features of the examples, Rob displayed deeper awareness in terms of relationships between limits, functions and connection to area (☐☐ under ‘awareness’).
Hanna: All with respect to x.

Rob: All linear. That crosses between positive and negative x, it goes either side of the axis. It becomes zero because that cancels out because the boundaries you have used because this is going from 0 to 2, this is going from -2 to 2. When you start getting bigger, you start to get purely negative.

Interviewer: Any other similarities or differences?

Rob: As these numbers grow smaller, it starts off positive and then grow negative and because we’ve chosen these boundaries, that’s why it’s zero. If it is any more, it’ll be very negative. Same difference, I think except bigger boundaries.

Hanna struggled to understand Task 2. She noted that she did not know that logarithmic functions could be integrated and continued to ask more questions about the task. Having clarified about integrating natural logs, Rob asserted that he did not like logs and suggested taking the ln x out would make the example simpler. He constructed \(2 \int (x^2 + 2x)dx = \frac{d}{dx}...\) as a more complicated example, suggesting that putting an x on the constant would make it more complicated and putting a natural log x would make it simpler. He provided just the integral part of the expression and did not produce the differential part of it. It is not clear whether he was attending to the form of the example or to the complexification of the integral. The dimensions he varied and the fact that he did not produce the differential side of the expression suggest the latter. Hanna did not construct any example for this task, which provided no evidence to suggest the nature of her awareness, but it is tempting to conclude that her difficulty with ln x blocked her from seeing the form (□ under ‘awareness’ and ‘technique’).

In Task 3, Hanna was reminded of the graph \(y = \sin x\) because of the expectation of ‘something to be simplified’. Rob displayed signs of negativity when he said that he disliked working with radians and suggested he did not know anything about substitution method. Hanna mistakenly related the task to the method of integration by-parts,
assigning $x$ as $u$ and $\sin(x^2)$ as $dv$. They both displayed limited language fluency (under ‘language’).

Hanna: It’s just remembering the $\sin x^2$, that comes to $2 \sin x$ squared or something like that or half $\cos x^2$. So you have $u = x$, $dv = \sin(x^2)$, $v$ equal to ... is it half $x$ squared...

When suggested that the integral was usually done using $u$-substitution, Hanna constructed $\int \frac{x}{\sqrt{x^2 + 3}}$ as the simpler example, stating $u$ is $x^2 + 3$. She observed:

Hanna: It’s really hard when you say, “Can you think of a simpler one” because you just think of an integral as an integral, don’t you ... you don’t think of it as having that many forms.

Hanna revealed fragmentary and disjointed awareness of integration. Rob also revealed a limited range of connections, which seemed to be a deliberate decision (under ‘awareness’).

Rob: You know enough about how the mental look to make ones that you want.

To make the example simpler, he suggested having $x$ instead of $x^2$. Hanna erroneously carried on with the ‘by-parts’ method and suggested selecting the $u$ carefully to make the example simpler or more complicated.

Hanna: You’d have square root of 1 over $x$ $\left[ \frac{1}{\sqrt{x}} \right]$. You pick the $u$ really carefully so that you don’t end up ... $\sqrt{u}$, $du$ over $2x$ times $u$ over $\sin x$ $\left[ \frac{du}{2x} = \frac{u}{\sin x} \right]$, and then you have your $du$ is tan or something like that.

Both of them also thought that the substitution method was far more demanding than partial fractions. Rob noted that, in the partial fractions method of integration, there was an expectation to ‘sort of follow rules’. Hanna observed that choices were limited in the substitution method.
After being reminded that she was dealing with the method of integration by substitution, Hanna suggested that trigonometric functions and divisions of functions would make an example of this type more complicated. She also suggested that by making it more difficult to find the substitution \( u \), she would produce a more complicated example and the more complicated term was chosen to be \( u \) (under 'technique').

In Task 4, Hanna constructed \( \int (\cos x + \sin x) \), adding a \( \sin x \) to the \( \cos x \) in the example. She noted that, with this method of integration, one needed to add equations together to get the answer. It seems that she had conveniently chosen to add another trigonometric function so that this method of integration would still apply. Although her example was correct, she did not display enough evidence to suggest richer connection or facility with technique (under 'awareness' and under 'technique').

For the same task, Rob constructed \( \int x^2 \ln x \, dx \) and observed:

Rob: Just choosing something that can't be broken down by normal, like what I was saying earlier, these little rules and just attaching something onto it and not really mattering what it is as long as you got a \( uv \) and that is still if you've got one is \( u' \).

Rob's remarks suggest deeper awareness of form and facility with technique compared to Hanna because not only did he reveal more connections, he also expressed this connection explicitly (under 'awareness', 'technique' and 'language').

Task 5 provided further evidence for Hanna's limited facility with technique and fragmentary connections. She chose to change the second function to a linear one and constructed \( \int (x + 1)(x + 1)^3 \). Apparently, Hanna thought that the integral was solved using the by-parts method. She suggested that part of the integral was done by substitution. Rob constructed \( \int (x)(2x + 4)^3 \, dx \) for the simpler example and \( \int (x + 1)^3(2x^3 + 4)^3 \, dx \) for the
more complicated example and suggested that his example was very similar to the given example. Hanna, after struggling for some time with how she would find the integral, suggested that one ‘could write it out and multiply all out’. Rob agreed and suggested that he would do the same, expand it and integrate it manually (under ‘technique’).

One crucial observation worth pointing out is that Hanna left out the $\,dx$ term in all her examples, which is not the case with Rob. This apparent lack of sensitivity to conventional notation gives me the impression that she was more concerned with application of rules than what the object signified. After learning that the example was usually done using substitution, Hanna was annoyed by her apparent lack of understanding and exclaimed that she was ‘so dumb’ and ‘would not have thought of doing that (substitution) for some reason’, suggesting that because of two terms multiplied together, she thought of integration by-parts method. She then constructed $\int x^2(2x^3 + 5)$, with $2x^3 + 5$ as $u$ and related it to the one given example by suggesting that ‘one is a simpler version … differential of the other’. Rob noted that the integral would be $u^2 - 9$, however, he later suggested that it was not possible because some terms still remained. He added that one must ‘make sure that something cancels’. It appears to me that Hanna was quite quick at picking up possibilities, or refreshing her memory and awareness (under ‘awareness’ and ‘technique’).

6.2.3 Making comparison

Rob appeared to be negative as he expressed his limited views of integration in terms of the techniques involved, its associations and applications. However, despite this seemingly negative predisposition, he displayed relatively rich connections to the topic. Task 1 revealed his geometric thinking and his association with area. Examples constructed in other tasks provided enough evidence to suggest rich connections and
facility with technique. Responses in the construction tasks revealed significant depths in his concept image of integration, highlighting aspects related to connections of the topic. His rich awareness was characterised by the interconnected nature of elements in his example space. However, he still remained unenthusiastic about integration at the end of the session.

From the onset, Hanna displayed restricted perception, stressing techniques of integration over other associations. Although in the interview she said the integral sign reminded her of area, she did not make use of this connection in Task 1. Not only did she display limited facility with technique, she also revealed restricted connections to the topic. She seemed to be attending to superficial aspects of the examples. She engaged in the tasks instrumentally and her responses revealed restricted concept image of integration and isolated example space. She did react perceptively to suggestions which reminded her of possibilities that did not come readily to mind.

The table below summarises Rob and Hanna’s responses in the interview and the construction tasks.

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
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<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
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<tr>
<td>Interview</td>
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<tr>
<td>Construction tasks</td>
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Table 6.2: Rob and Hanna’s responses in the interview and construction tasks
6.3 Summary

Responses in the interview suggested that the students displayed limited connections to integration while stressing techniques and displaying limited access to uses. Although the construction tasks provided affordance to reveal connections, for some students (Denise, Hanna and to an extent, Jon), the connections appeared rather fragmentary. The students tended to follow the format of the example given closely and to restrict the dimensions they varied. In most of the tasks, Rob demonstrated richer connections, but his remarks indicated a basically negative disposition. One feature common to all of them is that they were not very articulate in expressing connections.

In the course of analysing the interview of this group of students, I realised that I did not probe the learners’ examples as intensely as I could have done. I failed to find out more about their sensitivity to which features were the same and which were different in the examples they constructed and the example that was given. The reason for my concern about this was that with hindsight some students appeared to be constructing examples without necessarily attending to the salient features of the example. I wasn’t always clear
what it was that was being varied. I wanted the interviewees to have a sense of freedom in constructing the examples but I hoped that they would state explicitly what they varied. As a result, the construction task questions produced a set of responses in which I could not see the reason behind the decision to produce those examples.

In subsequent interviews, I added the question “In what ways are the examples you have constructed like my example?”, which forced the interviewees to explicitly address the similarities and differences in the examples, and the dimensions they chose to vary.
Chapter 7a

Main Study: Data Analysis

Interviews with ‘average’ mathematics students

7a.0 Introduction

Chapters 7a and 7b both discuss issues concerning interviews with mathematics undergraduates. However, as I had five ‘strong’ students from a leading university and six ‘average’ students from an average university, I decided to present analysis of their interviews in separate chapters. This chapter discusses qualitative issues concerning interviews with the average mathematics students.

Students in this group were studying Applied Mathematics at an average university and they were in their first year of study. By average, I mean most of the students got a B or a C in their A-level Mathematics. There were six students in this group and they were interviewed in pairs. All except Craig obtained a B in their A-level Mathematics. Craig obtained a C.

7a.1 Pair M1 (Marlene and Craig)

Marlene was a mature student and was very vocal and candid in her responses. Craig was rather reticent and was not very articulate. Both Marlene and Craig were Mathematics majors. Marlene had taken Mathematical Methods, Statistics, Mathematical Modeling, Discrete Mathematics, Structural Mechanics and Internet and Multimedia. Craig took Financial Mathematics, Discrete Mathematics, Statistics and Operational Research, Numerical Mathematics, Mathematical Modeling and French.
7a.1.1 The Interview

Extracts from pair M1’s responses in the interview are shown below:

Interviewer: What does the word “integration” mean to you?
Marlene: Especially finding area under a curve.
Craig: You can find volume.
Interviewer: Anything else?
Marlene: Well ... you use the area you find to find other things ... (inaudible).
Interviewer: What other things?
Marlene: We have been using them, haven’t we ... for statistics because we know under a normal curve, the whole area underneath is one, so you can find the mean, you can find median, you can divide it in half...
Interviewer: What comes to mind when you see the sign \( \int \)?
Marlene: Integration straight away. I also look for boundaries.
Interviewer: Do you visualise the function?
Craig: Yes.
Marlene: I can’t visualise the function.
Craig: I visualise it unless it’s really bizarre.

Marlene (marked with circles) expressed awareness of integration as area under a curve and a sense of purpose and application of the concept. The integral sign invoked for her the procedure or technique of integration with attention to ‘boundaries’. Craig (marked with squares) also mentioned applications of the concept as triggered by Marlene’s idea (○ under ‘awareness (connectedness)’ and ‘uses’ because of Marlene’s reference to area and to application of the topic in statistics (see Table 7a.1). I summed up Craig’s reference to volume with □† under ‘uses’ because it was unclear whether he was
building on what he just heard from Marlene or whether it was a fundamental connection for him.

Interviewer: How were you introduced to the concept of integration?

Marlene: From the limits, the differential side of it then, undo the differential ... and also the trapezium rule.

Marlene’s response to how the topic was introduced suggests that for her, the scope of root problems are manipulation (under ‘origins’). Despite the direct probes, little came to the fore in relation to possible sources of integration and so, there was no evidence of exposure to or concern about this aspect.

As described in Chapter 2 Section 2.4, the ‘Structure of a Topic’ framework includes a reminder that knowledge of misconceptions or common mistakes that can be made during the learning of the topic adds an important dimension to learners’ awareness. Responses to the question that addressed this issue prompted the following:

Interviewer: What sorts of things have you discovered you need to watch out for when you are doing integration?

Craig: With by-parts, if you get the $u\,dv$ the wrong way round, it can cause opposite effect, the result gets more and more complex as you go on.

Marlene: I think with integration by-parts you can go in random circle and you think, ‘Arghh.’

Interviewer: Why is that?

Marlene: Probably using the wrong technique for the wrong question.

Both Marlene and Craig appeared to focus their attention on aspects concerning use of a specific technique (under ‘awareness (misconceptions)’, * indicates orientation towards technique). Craig appeared to be meta-commenting (Pimm, 1994) on use of the technique of integration by-parts, thus demonstrating awareness of process and not just
technique. No reference was made to other techniques. Neither was there any reference to explicit mistakes related to understanding or sense-of the concept itself.

In relation to the behavioural aspects of integration, fluency with language when talking about the topic and facility with technique were considered.

Interviewer: What words in a problem/context tell you that integration is relevant?
Marlene: Density function, definitely density or area perhaps.

Interviewer: Any other words that trigger integration as relevant?
Marlene: Integrate this or integrate to find...

Interviewer: Okay... What are some of the special words/language that you use when you are talking about integration?
Marlene: Integral of limits, functions I suppose.
Craig: I can’t think.

Marlene displayed a limited pattern of language when asked directly to recall special words that triggered the use of the topic. She mentioned a number of words related to integration (under ‘language’ and ‘technique’) (see Table 7a.1). Craig did not contribute to this part of the interview.

The next prompts were intended to reveal Marlene and Craig’s connections to the given objects to other parts of mathematics.

Interviewer: What differences are there between \( \int x^2 \, dx \) and \( \int_0^2 x^2 \, dx \) ?

Marlene: Definite and indefinite integral.

Craig: Have plus C on that one. You can get a function with that one and you can get a complete result with that one because of the boundaries unless this (indefinite integral) is equal to something. Then you can rearrange to find the C.

Marlene: [They are] very similar, the \( x^2 \), the \( dx \), the integral sign, …
The integrals triggered associations such as arbitrary constant, function and ‘complete results’ in Craig (□ under ‘awareness’). Marlene’s reference to definite and indefinite integrals is summarised with ○ under ‘awareness’.

Awareness of the use of the topic was addressed by prompting with this probe:

**Interviewer:** What kinds of problem does integration help to solve?

**Marlene:** Probability.

**Craig:** If you want to find volume of something, that you can’t physically measure, it’s large.

**Marlene:** Distance-velocity.

**Craig:** Integrating acceleration you get velocity, integrating velocity you get displacement.

Marlene’s reference to use of integration in two familiar settings (probability and distance-velocity) is summarised with ○ under ‘uses’ and so is Craig’s reference to finding volume (□) because both of them reflected on access to rather limited aspects of possible uses.

Table 7a.1 below summarises Marlene and Craig’s responses in the interview.

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connections</td>
</tr>
<tr>
<td>Marlene</td>
<td>Interview</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Craig</td>
<td>Interview</td>
<td>□</td>
<td>□</td>
</tr>
</tbody>
</table>

Table 7a.1: Summary of Marlene and Craig’s responses in the interview

In the next section, I analyse these learners’ responses to the construction task.
7a.1.2 The Construction Task

As Marlene engaged in Task 1, the interchange between Marlene and the interviewer is shown below:

Marlene: \[ \int_0^1 (1-x)dx \]

Interviewer: Can you find another example?

Marlene: I’m suspecting you can change it to whatever you want.

Interviewer: Give me one example.

Marlene: \[ \int_0^4 (1-x)dx \]

Interviewer: And another?

Marlene: I think 5 will work \[ \int_0^5 (1-x) \]

Interviewer: What is the most general definite integral you can think of for which the answer is 0?

Marlene: \[ \int_0^a (1-x)dx \]

Marlene chose to change the upper limit in the example given, which was consistent with her remark in the interview when she said she looked for boundaries (limits). She gave no indication of checking her examples to make sure they were correct. It could be the case that Marlene’s attention was focused on the operation (minus) and the answer (zero). Her attention appeared to be shifting between applying rules of integration and plugging in values. With the way Marlene’s awareness was structured, it was only natural for her to generalise the integral to \( \int_0^a (1-x)dx \). This can be accounted for by suggesting that her attention was focused on limits and techniques of integration and she was ignoring the connection to area. This limited appreciation of what can change and what must remain constant in the example has been summarised with \( \bigcirc \) under ‘awareness’ in Table 7a.2.
Craig:  I think that works \[ \frac{1}{3} \int (1 - 2x) \, dx \,.

Craig chose to change both the function and the limit. He constructed his example by changing the upper limit and the second term in the function and did not check whether the example was correct. It is likely that he got the example wrong because his attention was focused on the result of integrating first \( x \) then \( 2x \). In this case, \( x - x^2 \) did give an answer zero for the limits he had chosen. Although he could be aware that he could change the lower limit as well, he did not choose to do so, possibly because he wanted to reduce the number of changing properties (the number of subtractions he had to deal with). Since zero would give zero when substituted so therefore, he only had to deal with the upper limit. If so, he seemed to display awareness of several properties of the integral (☐ under ‘awareness’).

When asked what she considered when constructing the examples, Marlene replied, ‘Integrate it and find out what they are’. She displayed awareness that she could integrate to find out what the answers were, if she so chose. Having been probed further to articulate her thinking, she submitted that her examples were not going to work because the second term had squared terms over a number, which did not equal the first term (☐ under ‘technique’). Marlene’s conjecture was that the denominator in the second term would be the same as the new upper limit. Her attention seemed to be shifting between the second term and its integral. The probe to construct another example seemed to have helped her to become aware of this attention shift.

Interviewer:  So what have you got as the integral?

Marlene:  \( x - x^2 \) over \( 2 \left( x - \frac{x^2}{2} \right) \). \( \ldots \) I’m thinking 3 is not going to work. It should be 2 on the bottom there not 3 and 4 is not going to work either, that
should be 2 not 4 on the bottom. Because I was thinking $x$ cubed [$x^3$] and then $x$ to the four [$x^4$] but it doesn’t work that way.

She expressed her previous thinking of working out $\int_{0}^{n}(1-x)\,dx$.

Craig: I think this works as well $\int_{-1}^{0}(1-x)\,dx$ because of the square, 1 minus another 1 ... oh ... no ...

Craig now displayed awareness of another dimension that could be varied. He chose to change the limits (but keeping one as zero) and holding the function constant. However, he then realised that it was not going to work (under ‘technique’). His attention seemed to shift between the integral of the second term $\left[\frac{x^4}{2}\right]$ and the lower limit (-1) because (-1)$^2$ is 1. Craig displayed uncertainty in recognising the degrees of freedom to change several properties at once in the integral. The act of mentally integrating together with the minus sign may have led to mathematical slips.

Marlene: Changing the gap, changing the gap between limits should work, shouldn’t it? If the function is the same, if it is there we might get it up to this one. If we bring it to 4, the gap is still 2. So $\int_{\frac{1}{2}}^{4}(1-x)\,dx$. That’s not going to work, is it?

[Long pause]

Interviewer: What are you trying to do?

Marlene: Try and improve it.

Marlene’s attention then shifted to the limits (previous reference to boundaries), namely the difference between the upper and lower limits. However, seeing difference in limits as a critical ‘dimension’ could be a temporary misconception. She changed the limits but maintained the same function and the same difference in the limits. This awareness of
variability of some of the dimensions, while keeping constant some other dimensions, suggests the extent to which she was aware of the degree of freedom in mathematical objects. Self-checking then suggested to her that the example was not going to work. Further prompts suggested that her attention remained focussed on changing the limits and checking the answer. I conjectured that her remarks about ‘changing the gap’ suggested that Marlene may have some awareness of areas. She also displayed considerable language fluency (\(\bigcirc\) under ‘awareness’ and \(\bigcirc\) under ‘language’) (see Table 7a.2).

With the intention of revealing whether Marlene was still aware of and can make use of what she said in relation to her associations with the sign \(\int\), I prompted her with the following probe:

<table>
<thead>
<tr>
<th>Interviewer:</th>
<th>You might want to ask yourselves why it is coming to zero.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marlene:</td>
<td>Because there’s no area underneath it touching the graph, touching the (x)-axis.</td>
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</table>

[Long pause]

| Marlene:     | [After sketching] Aaahh .... they’re cancelling each other out. Look at that! Nifty! Because part of the area is underneath and it’s negative and it cancels out. ... So we can do [change] both limits, couldn’t we? So if we take \(x\) [equal to] 3 we get -2 ... plus 2, in which case you want 1 minus (-1) which is equal to 2, because when you integrate and it crosses the \(x\)-axis, you have to integrate the parts separately because otherwise they cancel each other out because one is negative and one is positive, then area can’t be negative as such. So we’ve simply got the situation here, we’ve got these little areas, when \(x\) is naught we got 1, naught and -1 and then naught to 2, if we sum both of them, we are going to get little areas that are going to cancel each other out. |
Marlene’s attention now appeared to be shifting from focusing on the task as process to object. This delicate shift of attention (Mason, 1984) did not come naturally for Marlene because she seemed to be focusing on one aspect at a time. Although Marlene rehearsed the fact that integration was area under a graph, what she said and what she did when faced with construction task did not match. It could be that in the interview she was only repeating a stock phrase without actually being aware of its import. The fact that she did not think in terms of area with what she was doing suggests that this association did not come to mind on this occasion. She could have associated the object with area but did not think about it in this instance. Having been reminded of this awareness, Marlene immediately made good use of it (\( \bigcirc \) under ‘awareness’).

Interviewer: So you are saying the integral from ...

Marlene: For example \(-\int_{-3}^{1}(1-x)dx\).

Interviewer: Can you find another example?

Marlene: \(\int_{3}^{4}(1-x)dx\). I guess -2 would work with ... 4. \(\int_{\frac{1}{2}}^{4}(1-x)\).

Interviewer: Can you give me a general example for which the answer is zero?

Marlene: \(\int_{n}^{n+2}\).

Interviewer: Is there anything else that you can change?

Marlene: You can change the function; involve any straight line function that cuts through the origin .... It doesn’t necessarily have to go through the origin, does it? You have to set the limit from either side of the point where it did go through. Any straight graph would work.

Interviewer: Can you give a general form of integral for which the answer would be zero?

Marlene: I’m trying to think. The gaps are getting bigger, so how do we describe that?
Marlene now displayed awareness of area by changing both limits (but maintaining the 
function). She constructed two more examples by changing both limits simultaneously. 
When prompted to reveal her awareness of other dimensions that were variable, she 
displayed awareness of the function as a variable dimension, although she limited the 
function to any straight line graph. Although she expressed appreciation of the symmetry 
in words, she did not reflect that appreciation in the symbolic expression of her general 
example. For a general example, Marlene did not express symbolically the dimension that 
was changing (gap between limits). No comment was made about the use of different 
letters in place of \( x \) which was a syntactic rather than semantic dimension-of-possible-
variation. What seemed salient for her at this point was the changing of the limits, which 
was reflected in the general example she constructed in which she was still acting in 
terms of a difference of 2 rather than the symmetrical orientation of the area under the 
graph.

The next probe was intended to display a much greater fit with the learners’ response to 
questions about the understanding of the topic and what they did when faced with 
construction task. It helped to deal with an issue that arose.

**Interviewer:** Task 1b: What can you say about \( \int (2-x) \, dx \)?

**Marlene:** Straight line, indefinite integral.

**Interviewer:** What comes to mind?

**Marlene:** Area under that line \( y = 2 - x \).

**Craig:** Just so that we have another way of finding the boundaries.

**Interviewer:** So do you need the boundaries to make sense of this object?

**Craig:** Just so that I could look at the boundaries and the curve.

Marlene associated the object with the type of function and the type of integral. The 
object as area under the graph now came to mind. Craig’s attention was, however,
focused on the limits, possibly with the intention to determine an answer. His comments suggest that he did not make real sense of the integral without the limits (under 'awareness').

Interviewer: Given these integrals. What is the same and what is different in the three integrals?

\[
\int_{-2}^{0} (1 + x) \, dx = 0
\]

\[
\int_{0}^{2} (2 - 2x) \, dx = 0
\]

\[
\int_{\frac{-1}{2}}^{3} \frac{(1 - x)}{2} \, dx = 0
\]

Marlene: They are all equal to zero but the limits are different. They’re all straight lines, aren’t they? The powers ...

Interviewer: Any differences?

Marlene: Well, the limits are different. On the last one, you could take [the] ½ outside the integral sign....

Craig: If it makes sense, if you integrate the first bit, you get the second bit, so you got x there, you got 2x there.

Marlene: It’s a feature common to all.

Craig: The difference between the boundaries is all the same, all 2, no, it’s not, it’s 3, -1, ... 4. That’s the difference.

Marlene could be attending to the fact that the integral equalled zero as an answer (process). It is unclear whether the referent *they* was referring to the answer (process) or area (object). It is highly likely that Marlene was referring to the answer because she connected it with the limits being different. The functions were compared but only at the level of being straight lines. The difference in limits was noticed but Marlene did not make any explicit connection between the change in limits and the function and how that
gave the answer zero. Craig attended to the limits (differences in relationship between the first term and the second term). He also attended to the difference in limits without reference to their relationships to the functions. It reinforced the suggestion that his focus was on limits. This suggests that Craig was attending to the process aspect of the object, and awareness of the object as area was not displayed (−10 under ‘awareness’).

The following task was offered to probe the learners’ awareness of the relationship between integration and differentiation and the extent to which they were aware of the structure in the expression given.

Interviewer: Given that \( 2 \int (\ln x + \frac{3}{2}) \, dx = 2x \ln x + x \) and \( 2 \int (\ln x + \frac{3}{2}) \, dx = \frac{d}{dx} x^3 \ln x \).

Construct another integral with its two corresponding expressions, which is simpler, and one which is more complex.

[Long pause]

Craig: You want different answers?

Interviewer: I don’t think different answers.

Craig: Different forms...

[Long pause]

Marlene: I think I’ll try \( 2 \int x^2 \, dx = 2 \frac{x^3}{3} = \frac{d}{dx} \frac{x^4}{4} \). [Marlene explains to Craig] That’s just the differential of that, two versions of the same answer.

Interviewer: How did you make sense of that?

Craig: I’m not making sense of it at all. I can understand the first bit. I’m having trouble with the \( d/dx \) form.

Marlene: It’s like the differential of the integral.

Marlene: More complicated, you have to have sin and cos, \( \ln \). If you have sin and cos and log that will make it really complicated.

Craig: \( \ln x \sin x \left[ \ln x \sin x \, dx \right] \)
Chapter 7a Main Study: ‘Average’ mathematics students

Marlene and Craig took a long time (more than 2 minutes) to make sense of the example given. Having discerned the form of the example, she constructed a simpler example by integrating $x^2$ but chose to maintain the coefficient 2 in front of the expression. It could be that she did not see the coefficient as variable. However, she comfortably left out the coefficient $\frac{2}{3}$ in the final expression, presumably because her attention was momentarily focused on integrating $x^3$. She revealed a sense of form (function with derivative first and integral third) (☐ under ‘awareness’ and ‘technique’) (see Table 7a.2). When asked for a more complicated example, Marlene did not construct an example but suggested that having trigonometric and logarithmic functions would make the example more complicated.

Craig made sense of the usual form of forward integration but faced with the unfamiliar form of the answer, he did not make sense of the second expression. This suggests that his attention was focussed on the integral sign as an instruction to ‘get an answer’ and so he did not cope with the unfamiliar form. Even when the form was explained to him, he did not display awareness of the relationship between integration and differentiation. In fact, he constructed an example which did not resemble the form. It is likely that he jumped around following whatever came to mind. His mathematical thinking might be at the mercy of triggered surface associations to which he reacted rather than responding thoughtfully (☐ under ‘awareness’ and ‘technique’).

The task below was given to probe the learners’ focus of attention and associations with integrals by asking them to say what they saw and what came to mind.

Interviewer: Task 3: Given that $\int_{a}^{b} x \sin(x^2) dx = 1$. Describe what comes to mind when you first see the integral.
Craig: [As soon as seeing the integral] That’s quite nice because you can cut it into
different parts, take it out and then $\sqrt{\pi}$ squared is $\pi$, so you really
integrating a ... oh ...

Marlene: $\sqrt{\pi}$ is a definite integral anyway.

Craig: Yeah, so actually we don’t have a ...

Marlene: What was the question?

As soon as he saw the integral, Craig started to talk about the method of integrating
the integral and quickly associated the $\sqrt{\pi}$ with $x^2$. Craig’s attention seemed to be
focused on the limits and plugging in the limits to somehow simplify the integral,
although he then expressed a connection between the limits and their substitution for
$x$ after integration. Again, the symbols acted as triggers that brought out many
disconnected surface associations.

Interviewer: What comes to mind when you first see the integral?

Craig: Integration by parts, you separate the $x$ and the $\sin x^2$. You integrate $x$ you
get $x^2$ so that’s the integral part, emm... $\sin^2 x$ ...

Marlene: $\sin$ of $x^2$ ...

Craig: Oh yeah ... it’s in brackets.

Marlene: $\sin$ squared, you are just ‘$\sin$’ ning it twice, aren’t you?

Craig: Yeah.

Interviewer: How do you try and understand it?

Marlene: When I see $\sin$, I think of $\sin$ curve. Anyway, that’s squared, I’m not sure
the shape of it and $x$ times $\sin x^2$.

Craig: It could make it steeper, couldn’t it?

Marlene: It could make it taller.

Interviewer: What conversations do you have in your head when you see it?

Marlene: I’m seeing the $\sin$ wave and then I’m thinking how am I going to do this?
The area under the curve is obviously 1. The $\sqrt{\pi}$ seems rather strange.
The method of integration seemed to be the association triggered by the task. However, he did not spot the fact that the integral was solved using substitution, unlike \( \int x \sin x \, dx \) which was done using the method of integration by-parts. The fact that area under the graph was not mentioned suggests that his attention in this instance was dominated by algebraic structure (relation between \( \sqrt{\pi} \) and \( x^2 \)) but he appeared to have abandoned its geometrical representation (area) (\( \bigcirc \) under ‘language’).

Marlene displayed awareness of the image of the sine function and area. It is likely that she was augmenting her earlier ‘non-visual’ stance. Her attention seemed to be shifting between the process of integrating the integral and the integral as an object (\( \bigcirc \) under ‘language’). She displayed some confusion in reading the notation – what exactly was being squared.

In order to shift their attention from working on the example (properties) to generalising the form of the example (relationships), the following prompt was used:

Interviewer: The integral is usually solved using substitution. Can you construct another example which is simpler and one which is more complex?

Marlene: The simpler one would be to replace the first \( x \) with a constant like \( 2 \sin x^2 \, dx \). You could still substitute \( x^2 \) as \( u \) but you don’t have to worry about integrating the \( x \).

Interviewer: How do you understand \( u \)-subs?

Craig: I would take \( u \) to be \( x^2 \) because it makes it a lot nicer.

Interviewer: How do you decide what to substitute?

Craig: Because I’m not fully confident in using an angle like that... an \( x^2 \) angle. \( \sin x \) I’ll be happy but \( \sin x^2 \), it may confuse. So \( \sin u \) instead of \( \sin x^2 \).

Marlene: And it’s particularly valuable if you have more than just \( x^2 \) in the bracket... a long thing in the bracket... oh, I can get rid of all that, put a \( u \) and now that I’ve done that bit, I can tackle the other bit. Breaking it down and substituting is only further assistance it might give us. Yeah... it’s just...
breaking it down into manageable pieces... so I can do that, and I can do
that, and I can do that, in fact if I put them correctly, I can do the lot.

Interviewer: Can you construct a more complicated example?

Craig: If you make the angle $2x - 4$, so $\int x \sin(2x - 4)dx$. If you have $\log \sin$...

Marlene: You could make it more complicated by making either the angle more
complicated but you could also add on the $\cos x \sin x^2$.

Marlene: Add another long function... $\int \ln x^2$, that would be nasty.

Craig: I think the thing that makes it feel more complicated is that you are having
lots of different things like you got $\cos$ mixed with $\sin$ mixed with $\log$ and
tan...

Interviewer: But then we are talking about $u$-sub. Can that be solved using $u$-
substitution?

Marlene: Sometimes you can when you put $\cos$ of $\sin$. Then you make the $\sin$ [as] $u$.

$\int \cos(\sin(x^3))dx$.

Interviewer: What are the things you need to watch out for when using $u$-substitution?

Marlene: I think converting it back at the end what you substituted exactly not a
version of it. You put in the integral of differential back in as $u$ rather than
what you originally started with.

Craig: Try and recognise when it gets nasty then you do it the wrong way round,
[You] realise that you are not getting anywhere. You start in the wrong
direction.

Interviewer: Have you developed any technique of your own?

Marlene: Not really.

Craig: Try and break it down if I can. Like if I get $\tan$, I break it into $\frac{\sin}{\cos}$, that
makes it a lot nicer. And I prefer integration by-parts. Try and see if I can
recognise it. If I see something out of the blue that I haven’t seen before, I
change it into something I have seen before.
Having been helped (using the prompts) to shift her attention from working on the example to working on the relationship, Marlene constructed a simpler example by maintaining the method of integration but reducing the abstractness of $x$ by replacing it with a constant ($\bigcirc$ under ‘awareness’). However, she did maintain the trigonometric function $\sin x^2$ and so lost the integration by-parts and created an integral that could not be expressed using elementary functions. She displayed verbal fluency but went astray from integration by-parts. Her sense of substitution was incomplete, because she did not mention the derivative part ($\bigcirc$ under ‘language’ but with $\bigcirc$ under ‘technique’).

Craig expressed the possibility of varying the substitution as an indication of his awareness of structural properties of this example. However, he kept the trigonometric function constant, so any awareness of variability of this aspect was not displayed. He demonstrated an awareness of replacing complexity ($\square$ under ‘awareness’).

Marlene suggested that what was being substituted could vary and could be more complex. Later, she suggested that both terms could change and be made more complicated. Her attention to structural relationships triggered recall of how substitution was used, but an incomplete sense of how it worked. She suggested that a relationship was discerned and receded to the background and awareness of this meant that it could be called upon when needed. Variability of properties remained in the foreground, awareness of which distinguished between the simpler and the more complicated example. Her remarks on things to watch out for when using this method showed her recognition of a potential obstacle and her articulation of technique ($\bigcirc$ under ‘awareness (misconception)’ and ‘technique’).

Craig constructed his example by varying the angle $[\sin x^2]$ and making it more complicated $[\sin(2x-4)]$ but maintaining the form of the example $[x \sin u]$. His attention to the variable $u$ in integration by-substitution revealed his awareness of the
relationship between the substituted and the remaining term in the integral and the
relationship between examples, but not between the outside \( x \) and the inside \( x^2 \), which is
the essence of integration by substitution (☐ under ‘technique’) (see Table 7a.2). His
meta-comment about technique and recognition of encountering problems when using the
technique incorrectly are summarised with ☐ under ‘awareness (misconception)’ (see
Table 7a.2).

Interviewer: Task 4: The integral \( \int x \cos(x) dx \) requires integration-by-parts. Construct
your own similar example that also requires integration-by-parts.

Craig: Integral of \( \ln x \).

Interviewer: How about a more complicated example?

Craig: Integral of \( 2x \cos x \), no that’s not that complicated … integral of
\( \cos x \tan x \). I can integrate \( \cos x \) easily than \( \tan x \). So that’s what I
choose my \( u \) to be. I probably break \( \tan x \) into \( \frac{\sin}{\cos} \).

Craig recalled (\( \ln x \)) when asked to construct a simpler example. It is likely that \( \ln x \) was
chosen because of its use as an example in class, and its unique nature of having one term
(as contrasted with two terms, which is more typical for this method of integration). A
more complicated example was constructed by combining two trigonometric functions.
However, Craig again displayed a tendency to utter the first idea that came to mind
without much thought and no checking (☐ under ‘awareness’). Marlene did not
contribute to this part of the interview.

The following task was intended to say something about the understanding of \( \int f'f \) method
of integration.

Interviewer: Given \( \int (x + 3)(x^2 + 6x)^3 dx \). Construct an integral which uses the same
idea, which is simpler, and one which is more complex.

[Long pause]
Craig: \((x + 1) \sin^2 x\). [In the brackets] power times one linear times \(\sin^2\) isn’t too bad.

Interviewer: How is it worked out? What method [do you use]?

Craig: [Integration] by-parts.

Interviewer: Can you construct a more complex one [example]?

Marlene: Longer polynomials with higher power.

Craig: [Integral of] tan to a power or tanh. That’ll be evil [laughs].

Craig discerned the form of the example as linear function multiplied by quadratic function without any particular reference to the relationship between the two (i.e. the derivative of one is a constant multiple of the other). This suggests that Craig’s attention seemed to be focused only on superficial aspects of form and not on the relationships between a function and its derivative (under ‘awareness’ and ‘technique’). There was no evidence of rich awareness of this method of integration from Marlene’s responses (under ‘awareness’).

Table 7a.2 below summarises both Marlene and Craig’s responses in the construction task.

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
</tr>
<tr>
<td>Marlene</td>
<td>Construction task</td>
<td></td>
</tr>
<tr>
<td>Craig</td>
<td>Construction task</td>
<td></td>
</tr>
</tbody>
</table>
7a.1.3 Making Comparison

The purpose of the summary is to enable me to make comparison between different interviewees and between what was said in the interview and what they did when constructing examples. Table 7a.3 below presents a summary of both Marlene and Craig’s responses.

<table>
<thead>
<tr>
<th>Average ranks</th>
<th>Pair M1</th>
<th>Interview</th>
<th>Facility with technique</th>
<th>Connectedness</th>
<th>Misconceptions</th>
<th>Uses</th>
<th>Origins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marlene &amp; Craig</td>
<td>Construction tasks</td>
<td>QOO</td>
<td>O</td>
<td>Q</td>
<td>Q</td>
<td>QOO</td>
<td></td>
</tr>
</tbody>
</table>

Table 7a.3: Marlene and Craig’s responses in the interview and the construction tasks

The summary highlights a number of important aspects of the learning of integration.

In the interview, Marlene did not display a strong sense-of or rich connections to integration. Her fluency in language and knowledge of common mistakes seemed restricted to superficial features. There is not enough evidence to suggest that her reflections on access to possible uses of integration were robust, nor did she display substantial reflections on access to origins of integration.

The construction task, however, acted to provide opportunity for Marlene to display more aspects of awareness, behaviour and emotion. She revealed more fluency with language patterns when talking about the tasks, although she often focused on technical aspects of the topic. Her awareness of connections and relationships was revealed more through the construction task, although she did not display enough evidence to suggest rich connections or sense-of the topic. Her awareness in terms of common mistakes appeared to be substantiated in her act of talking about the examples she constructed. She
Chapter 7a  Main Study: 'Average' mathematics students

displayed a willingness to try and was quick to respond to changes in her perception. Her responses suggest limited depth in her concept image and the fragmentary nature of her example space.

Craig, however, did not show enough evidence to suggest real fluency in language. His remarks suggested a focus on technique, which appeared to trigger him into making hasty remarks. The connections he made were not evident enough to suggest his awareness of them was all encompassing. The extent to which he reflected on access to possible uses of integration appeared rather limited. He often built on an association with what Marlene had just said.

In the construction task, Craig did not show any evidence that his awareness had been reinforced. There was no evidence that his fluency in language was enhanced through his engagement in the tasks. What was revealed was his focus on technique; in the substitution and by-parts tasks, even that focus was incomplete, facility to seep in mind the necessary relationships. It is also revealed in his remark about misconceptions related to the topic. This focusing of attention appeared to be limiting his awareness of connections and his use of language.

The construction tasks produced more animated voice tones and longer sequences of dialogue between them, pointing to some disposition to try to make sense, though their failure to check any of their constructions suggests they had not developed a disposition to explore mathematically without some outside authority checking their work.

Comparing Marlene and Craig’s responses, I am able to say that the use of the six-fold framework revealed aspects of the topic on which both of them focused their attention as well as aspects that, despite prompting, did not appear to figure in their sense of integration. Aspects related to connections and misconstruals appeared to be at least in

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the background, including area, while aspects related to technique appeared to be stressed. These students displayed at best, low key enthusiasm for the topic. They did engage with the construction task, albeit without checking their constructions. However, what strikes me is Marlene’s expressive disposition and ease with language that helped reveal more aspects about the topic and about herself. Craig’s hesitancy with language and his ‘expectation’ of being tested appeared to focus his attention on technique so that he overlooked other aspects.

The same process was applied to two other pairs and below is the discussion with some extra notes that raised pertinent questions.

7a.2 Pair M2 (Simon and Sarah)

Simon and Sarah were studying Mathematics and Statistics in their first year. Simon was a very friendly and outspoken male student while Sarah was a rather reticent female student. Both Simon and Sarah were majoring in Mathematics and Statistics. Both of them had taken modules including Statistics and Operational Research, Mathematical Methods, Problem Solving and Modeling, Introduction to Discrete Mathematics and Information Systems Fundamentals. Sarah also took Computer Programming while Simon did Introduction to Mathematical Economics in addition.

7a.2.1 The interview

Extracts from their interview exchange are presented below.

<table>
<thead>
<tr>
<th>Interviewer:</th>
<th>What does the word “integration” mean to you?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sarah:</td>
<td>Mathematical process to … reverse of differentiation.</td>
</tr>
<tr>
<td>Simon:</td>
<td>The word integration for me means, if you don’t think about Math, it’s mixing of two things.</td>
</tr>
<tr>
<td>Interviewer:</td>
<td>Anything else?</td>
</tr>
<tr>
<td>Sarah:</td>
<td>I just think of substitution … part of Mathematics.</td>
</tr>
</tbody>
</table>
Simon: Now what comes to mind is graph of a shaded region.

Interviewer: What comes to mind when you see the sign $\int$?

Simon: Obviously the process of integration.

Sarah: Being asked the question that needs to be solved.

Simon: Looking at the sign, I immediately look at limits.

Interviewer: Why?

Simon: I guess that's the way you do mathematical process, you see the sign, it says integration. I immediately think … okay, between what points?

Interviewer: Anything else?

Sarah: I just start thinking how I would solve the question, what method I could use.

Interviewer: Any images?

Simon: No, not really, just that … looks like a snake [laughs]. No, not really.

The word integration, for Sarah (marked with circles), meant a mathematical process related to the reverse process of differentiation. Simon (marked with squares) was reminded of a more general reference to ‘mixing of two things’. Probed further, he mentioned a shaded region while Sarah made a remark about a technique of integration. The integral sign $\int$ reminded Simon straight away of the process of integration so that he looked for limits to perform calculations. Methods and techniques of integration dominated Sarah’s attention upon seeing the sign. Simon also amusingly referred to the sign as looking like a snake. From what they said, I could gather that both of them displayed limited awareness of integration in relation to what came to mind (under ‘awareness’).

In relation to things to watch out for, Simon suggested that images of a function could help in identifying regions underneath the graph more than equation could. He also suggested checking if the function was integrable. For Sarah, remembering to put the
arbitrary constant and making sure if the integral was definite or indefinite appeared to be her main concern.

Simon: Especially if you have a graph in front of you, the curve dips underneath below $y = 0$. If you have an equation in front of you, it’s not that easy to tell.

Interviewer: Why?

Simon: I guess it’s much more difficult to picture a graph of an equation.

Sarah: I always think whether it’s a definite or indefinite integral where there’s limits involved and where you need to think about constants, that could be ... I always used to ... tend to forget when I integrate something to put a plus C at the end. I always try and bring it to my mind straight away so I don’t forget it.

Simon: Obviously you also got to check and make sure that the function is integratable.

Interviewer: How do you do that?

Simon: You’ve got to look at the one variable like if it’s $x$ when there’s $x$ and together, whether you are gonna integrate it ... you’ve got to keep an eye up for that.

Simon displayed a richer sense of awareness of things to watch out for when dealing with integration than Sarah, who was only concerned with arbitrary constants (under ‘awareness (misconception)’).

Asked about technical terms related to integration, after a long pause, Simon mentioned ‘integrate’. Examples of words in a problem that triggered integration to the students included area and probability density in Statistics. Simon suggested that words such as velocity and acceleration in differentiation reminded him of the reverse process of integration.

Below are their responses for the probe concerning words that trigger integration:

Interviewer: What are some of the words in a problem context that tell you integration is relevant?

Simon: Area.
Sarah: Yeah, finding area. Probably it follows in Statistics, you have to do probability density functions, you have to do some integration.

Sarah: Because probably when we were taught how to devise a problem, we were just kind of shown, you have to integrate this particular problem and still ... that's what goes on with me.

Interviewer: So you think of a similar problem that you've done before?

Sarah: Yeah.

Simon: I also think, it's a ridiculous thing but there was an incident with differentiation like ultimately you translate it back like when you got for example velocity, speed and acceleration of something because we used to do differentiate speed you get acceleration, so instead of acceleration you ultimately think speed and so ... I don't know, probably key ones like acceleration, speed, ...

Both of them displayed somewhat limited fluency with technical terms and words related to integration, however, mention of area was taken as displays of 'immersed state of understanding' (see Table 7a.4).

According to Simon and Sarah, integration was introduced as the reverse process of differentiation and that was how they remembered it.

Interviewer: How were you introduced to the concept of integration?

Sarah: As ... in second year.

Simon: I was taught differentiation and then we kind of did a table and then we were shown how we could go backwards from that.

Sarah: Just like we could have a function and then we'd have what differentiates it and we could show how to get back. That's how we were first introduced.

In terms of applications and utility of integration, Simon mentioned speed and acceleration while Sarah referred to probability.

Interviewer: What kinds of problem do integration help to solve?

Simon: As I said, the speed and acceleration stuff, .. other than the constant at the end, when you diff you lose the C, things like volumes of glasses, volume q.
Sarah: I’d say you use it for probabilities.
Simon: I never think about it that much.

Reflections on access to aspects related to origins and uses of integration appeared limited. This restriction could influence the way in which they approached the topic and the way they learned. From what she said, Sarah displayed limited access to uses (☐ under ‘origins’ and ‘uses’). Simon, however, displayed a somewhat richer access than Sarah (☐ under ‘origins’ and ‘uses’).

For differences between definite and indefinite integrals, Simon mentioned limits, which was picked up by Sarah, who commented on other surface aspects such as functions and integration with respect to $x$ and definite/indefinite integrals. Probed about the difference, Simon referred to definite and indefinite integrals and related them to area under a graph.

Interviewer: What differences are there between $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$?
Simon: Second one’s got limits, ...
Sarah: Yeah, they’re both the same functions. We’re integrating them both with respect to $x$ .... One is a definite integral (second one).
[Long pause]
Interviewer: Any difference?
Simon: Not to an extent that I mean, both of these are the same equation except one is restricted to finding the area between 0 and 2 and the other one is endless, you couldn’t find the area under first one as it’s ... it’ll never cross $y = 0$, so it’s not a logical way of thinking. If you graph $y = x^2$, I wouldn’t think I wanna get the point from here to here because the line keeps rising forever so unless I have the second one saying find the area between 0 and 2, then I wouldn’t really integrate that. If there was another graph that did go up and come back down again, logically I might think what is the area between where it crosses the ...
In distinguishing the two integrals, Sarah displayed limited sense of what the two integrals represented. The fact that she commented on the surface aspects suggests that she focused on these aspects. There was no evidence that she was aware of connections to area. The fact that Simon commented on both the surface aspects and what the integrals signified suggests that he was aware of connection to area and of the surface similarities and differences (☐ ☐ under ‘awareness’).

Table 7a.4 below summarises Sarah and Simon’s responses in the interview.

<table>
<thead>
<tr>
<th></th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Connectedness</th>
<th>Misconceptions</th>
<th>Uses</th>
<th>Origins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sarah Interview</td>
<td>☐</td>
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<tr>
<td>Simon Interview</td>
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<td>☐</td>
<td>☐</td>
<td>☐</td>
<td>☐</td>
</tr>
</tbody>
</table>

Table 7a.4: Summary of Sarah and Simon’s responses in the interview

The next section discusses how Simon and Sarah performed in the construction tasks.

7a.2.2 The construction tasks

Their responses in the construction tasks are discussed below. The extract that follows is Simon’s attempts at Task 1.

[Long pause]

Simon: Obviously the line never leaves the x-axis, we get ... no area. The line never moves up or down, y never increases or decreases as x increases. Basically that line is never going to move up, ...obviously you can always do ... that's gonna be like that graph [sketches the graph], I guess .. you need to move them up, you cancel them out, this area there and this area here. I guess you can always do the same, crossing that over. That’d be ...

Interviewer: Same limits?
Simon: Yeah, same limits.

Asked to construct another example, Simon first commented on the given example and suggested that area underneath the graph cancelled out to give the answer zero. He then suggested crossing the line over \( y = -(1 - x) \), which would give the same answer. Asked to give another example, he suggested that functions such as \( x^2 \) became more difficult. He then suggested that the function and the limits could change and constructed \( \int_{0}^{4} (2 - x) \, dx \).

Interviewer: Can you find another example?

Simon: It starts getting difficult when you’ve got things like \( x^2 \) and ...

Interviewer: What can you change?

Simon: You could change the 1 to 2, and then change the limit to ... \( 4 \left[ \int_{0}^{4} (2 - x) \, dx \right] \)?

Sarah did not construct any example at this point and declared that she could not think of any. She suggested ‘scrap[ping] 2 and still get naught’. Simon then suggested functions such as \( x^3 \), \( \sin x \) between 0 and \( 2\pi \), referring to them as having the same area.

Simon: Things like ... \( x \) cube .... obviously it’s still similar like symmetry but there you’ve got ... I don’t know [laughs] [draws] if it goes like that, like the ... is it the sine graph between 0 and \( \pi \) ... \( 2\pi \), it’s got the same area.

Interviewer: Why?

Simon: Because the area between 0 and \( \pi \) are the same underneath between pi and \( 2\pi \) and you take away the area underneath.

Simon revealed richer awareness in terms of connections of integration to area and in terms of awareness of dimensions-of-possible-variation. Although he correctly constructed the examples and varied many possible dimensions, Simon was not very
articulate in expressing the generality in the examples. He said that he was still confused (□ under ‘awareness’ and ‘technique’ and □ under ‘language’). There was no evidence as such for Sarah because she did not construct any example.

In the next task, I asked them to comment on the integral \( \int (2-x) \, dx \). My intention was to explore further their sense of understanding of integration. Sarah immediately declared that it was an indefinite integral while Simon suggested on the behaviour of the function and what happened to the area underneath. Probed further, he mentioned that there were no limits.

Simon: Obviously as \( x \) increases you’re not getting as much area as you did before.

Interviewer: Anything else?

Simon: No limits.

It goes on to prove that Sarah’s attention was focused on the nature of the integral (and possibly technique of integration). The only reference that came to mind was that it was an indefinite integral. Simon displayed a deeper appreciation and saw the integral as a function under which certain amount of area was covered (□□ under ‘awareness’).

The next task invited Simon and Sarah to say something about the similarities and differences between three definite integrals.

Interviewer: Given the integrals \( \int_{0}^{1} (1-x) \, dx = 0 \), \( \int_{0}^{2} (2-2x) \, dx = 0 \) and \( \int_{-1}^{1} \frac{(1-x)}{2} \, dx = 0 \). What is the same and what is different in the three integrals?

Sarah: The first thing I noticed is that they’re all definite and they’re all with respect to \( x \).

Simon: I’m just looking at the limits …

Sarah: Just linear functions to integrate.

Simon: This a nice trick question, they’re all exactly the same [laughs]. I bet they all add up to the same …
Interviewer: In what ways are they the same?

Simon: The first and the second one are both graph and then it's different by 2 but apart from that, ...(limits) by 4 ... I'm not doing any calculation but I'm saying then divided by 2 ... I'm picturing the graph.

Sarah commented on the surface aspects of the integrals such as definite integrals with respect to $x$ and linear functions. Simon inspected the limits and tried to relate them to the answer and suggested that 'they all add up to the same ... ', although he did not do any calculations. He admitted picturing the graph for the functions.

Sarah displayed surface connections when she commented on the surface aspects of the function (under 'awareness'). Aspects mentioned by Simon revealed richer awareness as he tried to look for similarities and differences among the functions (under 'awareness').

Task 2 invited them to construct simpler and more complicated examples to an expression relating integration and differentiation. Both Sarah and Simon took a long time to understand the task. Asked how she understood it, Sarah expressed the relationship in the expression clearly. She then constructed $f(l-x)$ for the first part of the expression but struggled with the differential part although she knew how it was related (under 'technique').

Interviewer: How do you understand it?

Sarah: It's saying twice the integral of that expression is equal to the differential of that expression.

Interviewer: So for simpler one you've decided to use $\int(1-x)$ equals ... something and ... what are you thinking of doing now?

Sarah: The differential of something but I can't ...

[Long pause]
Sarah: I've tried to put it in that form again, so ...

Interviewer: What have you considered?

Sarah: Integrating it again ... so that that is now the differential of that one.

After a while, I asked her to construct a more complicated example. She was not sure what ‘complicated’ meant, whether the example had to be non-linear or had some other characteristics.

Although Sarah expressed the relationship in the expression clearly, she managed to construct only the first part of the simpler example. Her responses suggest that she had awareness of form but that awareness was not translated into action (☐ under ‘awareness’ and ☐ under ‘technique’).

Simon constructed \(\int 2x^4 \, dx = 0.4x^5 = \frac{d}{dx}0.4\frac{x^6}{6}\) for a simpler example and suggested that he randomly thought of something, integrated it and thought about the derivative for the second one. For a more complicated example, Simon suggested having natural logarithm (In) or \(e^x\) (☐ under ‘awareness’ and ☐ under ‘technique’).

Sarah, having declared that she was ‘just not very good at making up a question’, suggested making the example a product so that product rule could be used for a harder example. Simon then suggested using two variables, \(x\) and \(y\).

Simon constructed \(x^3\cos x\) and suggested that he just put the functions together randomly to make it difficult.

[...]

Sarah: [constructs \(\int \cos x \sec x \, dx = ... = \frac{d}{dx}...\)]

Interviewer: What have you considered?

Sarah: I just thought that it’d more complex if it had a product then you have to kind of, it’s not basic integration it’s actually a rule that you have to integrate it and do all
the things like that. ... I put a cosh in there coz I just thought that cosh is
integrated to sinh, although it’s kind of a more complicated function, it’s not that
complicated an integration that I have to do.

Simon: \[\int \ln(x - 2e^x) \, dx\]. I know \(e^x\) is not that complicated.

Task 3 involved an example involving integration by substitution \[\int_0^\pi x \sin(x^2) \, dx = 1\].

When asked to describe what came to mind upon seeing the integral, Simon declared
‘product rule’. He explained that aspect of the integral that caught his attention was the
upper limit (\(\sqrt{\pi}\)). Sarah commented on surface aspects of the integral (\(\bigcirc\) under
‘language’).

Simon: I’m beginning to look at the root pi, why root pi? .. I’ve never come across root pi.

Sarah: It’s definitely defined limits, it’s product ... that’s what comes to mind.

Simon: I’m looking at the limits ... the limits ‘d be the easy way of integrating something
like that.

I then suggested that the integral was usually solved using the method of substitution
and invited them to construct a simpler and a more complicated examples. Looking at
the integral, Sarah remarked (\(\bigcirc\) under ‘technique’):

Sarah: Loads of things going through my head now. I just remember like substituting part
of it for \(u\) or something like that and then prove it in terms of the \(u\) or something
... \(dx\) in terms of \(du\) or \(du\) in terms of \(dx\) ...

Simon did not contribute to this part of the interview.

In Task 4, an example of integration by-parts was given \[\int x \cos(x) \, dx\] and subjects were
invited to construct simpler and more complicated examples. Simon first remarked that
he did not want to construct anything similar to the given example. After some time, he
noted that he was going on a circle possibly because the integral he constructed had
another integral when he integrated it. Sarah also encountered the same problem and suggested having a linear function for the first part of the integral and something which she could integrate for the second (under ‘technique’).

Sarah: I started to make the first term linear so that when you get to do the differential of the first term which it goes to a constant, which makes the second part a lot simpler. The second term, something which I could integrate (cos x).

Table 7a.5 presents a summary of Simon and Sarah’s responses in the interview and the construction tasks.

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
</tr>
<tr>
<td>Sarah</td>
<td>Construction task</td>
</tr>
<tr>
<td>Simon</td>
<td>Construction task</td>
</tr>
</tbody>
</table>

Table 7a.5: Summary of Sarah and Simon’s responses in the construction task

7a.2.3 Making comparison

The categories probed based on the six-fold framework revealed differences in aspects dominating attention. For some learners, the contrast between what was said and what was done appeared to be more marked than for others.

Table 7a.6: Sarah and Simon’s responses in the interview and the construction tasks
Simon (M2, marked with squares) displayed considerable evidence to suggest fluency in language but highly focused on technique, awareness of connections and common mistakes, and also access to origins and uses. However, his engagement in the construction tasks revealed even richer awareness of connections and relationships. Opportunity to display awareness also seemed to have afforded him with a chance to articulate his thinking more clearly. He also displayed facility with technique. Simon appeared emotionally enthusiastic about the topic and engaging with the construction tasks. The construction tasks also revealed considerable depth in his concept image and richness of his example space.

It is evident from her response in the interview that Sarah (M2, marked with circles) displayed language fluency. However, she too was technique-focused and displayed little indication of awareness of connections and associations. Her remarks on misconceptions and his reflections of aspects related to origins and use of integration were at best superficial and technical in nature.

The construction tasks led Sarah to display more awareness of connections and a richer sense of the topic. Her facility with technique was shown, although her fluency with technical terms remained limited. On the whole, Sarah demonstrated a low level of motivation and disposition to working on integration. Her responses suggest considerable limitations in her concept image and example space.

7a.3 Pair M3 (Matt and Beth)

Matt was a male student and was rather reserved about Mathematics. Beth was a female student and was very outspoken and friendly. Matt was majoring in Mathematical Sciences and Beth was majoring in Mathematical Sciences and Finance. Courses they have taken include Statistics, Mathematical Methods and Information Systems.
7a.3.1 The interview

Below are extracts from their interview.

Interviewer: What does the word “integration” mean to you?

Beth: Integration ...

Matt: Isn’t that finding the area underneath a graph? No, that’s differentiation.

Beth: Don’t know, I haven’t done it yet [laughs]. Emm … I know how to do it …

Matt: Differentiation is area.

Beth: I thought differentiation was where you find out the …

Matt: Integration is the area between two points under a curve, isn’t it?

Both: It is, yeah.

Interviewer: What comes to mind when you see the sign \( \int \)?

Beth: \( dx \). Emm … don’t know, you just know you’ve got to integrate when you see the sign.

Interviewer: Anything else?

Matt: Just integrate, yeah.

Matt (marked with squares) confused integration with differentiation and first suggested that integration was area underneath a graph before changing his mind to say that it was differentiation (area). Beth (marked with circles) declared that she didn’t know what integration was but admitted knowing how to do it. She then attempted to correct Matt at which point Matt realised that integration was area under a curve between two points. Finally, they both agreed that integration was area under a graph. The integral sign \( \int \) reminded Beth of \( dx \) and they both agreed that the sign meant ‘integrate’.

Both Matt and Beth displayed some awareness of integration but it was restricted to ‘area’, ‘\( dx \)’ and ‘integrate’, with further reference to what they meant. Links to functions, graphs and other associations were not mentioned. The integral sign triggered what to do
and not what it is associated with. There was no reference to limits, function etc. They were not very articulate in expressing their thinking (☐☐ under ‘awareness’).

In relation to the sorts of thing to watch out for, both Matt and Beth commented on aspects related to techniques of integration and confusion with differentiation. There was no evidence of awareness of connections such as area (☐☐ under ‘awareness (misconception)’).

Beth: Sometimes I get a bit confused trying to diff when I know I’m suppose to integrate.

Matt: Sins and cosines as well when you are integrate them, they both turn negative or positive, you’ve got to remember all forms.

Beth: Sometimes I forget the order of how it works whether goes from sin then cos then negative sin then negative cos.

Matt: Differentiation is the other way round so it confuses the brain.

Matt: I think differentiation is much more in the mind, integration ... you’ve got to really think about it because differentiation I find it easier.

Beth: I think it’s because we were taught diff first so we have that set in our heads.

According to Matt, technical words related to integration were $dx$, limits and square brackets. Beth added integral functions and $\ln x$ to the list. Asked about words in a problem that triggered integration, Matt suggested area underneath a graph, while Beth noted looking for the integral sign.

Interviewer: What are some of the special words/language that you use when you are talking about integration?

Matt: It always has $dx$ at the end of an integral ... when you integrate you’ve the limits, you have square brackets around it, you put numbers up and below.

Interviewer: Anything else?

Beth: You just use integrate an integral function and ... $\ln x$. That’s about it I can think of.
Interviewer: What words in a problem/context that tell you that integration is relevant?

Matt: Find the area underneath a graph,

Beth: You just look for either the specific sign or that. If it’s not on there then you wouldn’t be sure.

Matt displayed limited fluency with language when he commented on surface features of an integral, although he mentioned area as words that trigger integration. Beth also displayed limited fluency when she referred to techniques (under ‘language’ with an emphasis on technique). Also, they both displayed propensity to stress techniques of integration (see Table 7a.7).

Asked how integration was introduced to them, they suggested that they were introduced to integration during A-levels, starting with basic examples of how to integrate. They did differentiation first and were told that integration was similar but the opposite of differentiation (under ‘origins’). Beth suggested that the kinds of problem that integration helped to solve ‘depends on what the graph is about’, indicating somewhat deeper connection (under ‘uses’). Their responses showed limitations in access to affective affordances of integration that might have influenced them.

Asked about the similarities and differences between $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$, both Matt and Beth commented on limits and arbitrary constants.

Matt: That one (first) doesn’t have limits so you plus a constant.

Beth: You can physically work out a value for that, I don’t think you can with this one.

Matt: Plus C.

Beth: Don’t you do on both though? The two Cs cancel out, it doesn’t matter.

Interviewer: Any similarities?

Matt: Apart from the two limits, it’s all the same.

Beth: It’s the same function.
Beth and Matt attended to surface aspects of the integrals, commenting on limits, functions, arbitrary constant and the answer. What the integrals represented in terms of area was not mentioned, although Beth suggested that a value could be worked out for the definite integral (○□ under ‘awareness’).

Table 7a.7 below summarises Matt and Beth’s responses in the interview.

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td>Beth Interview</td>
<td>○</td>
<td>○</td>
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<tr>
<td>Matt Interview</td>
<td>□</td>
<td>□</td>
</tr>
</tbody>
</table>

Table 7a.7: Summary of Matt and Beth’s responses in the interview

The next section discusses Matt and Beth’s responses in the construction tasks.

7a.3.2 The construction tasks

Their responses in the construction tasks are discussed below.

Interviewer: Given that \[ \int_{0}^{2}(1-x)dx = 0. \] Can you find another example like this where the answer is 0?

[Long pause]

Beth: If you change the 1 into another figure and the higher limit will still hold that figure, would that work? ... Like for example, if it was ... [writes \[ \int_{0}^{4}(2-x)dx \].] would that be true? [checks] Wait, you have to integrate that before you do the values. ... actually that’s wrong now.
Beth suggested changing the function \((1-x)\) to \((2-x)\) and changing the upper limit to 4.

I conjectured that she spotted a pattern between the constant in the function and the upper limit. She constructed a wrong example but was quick to spot it (\(\bigcirc\) under ‘awareness’ and \(\bigcirc\) under ‘technique’). Matt first substituted the limits directly into the function before Beth suggested to him that he had to integrate it first.

Interviewer: [To Matt]: Can you find another example?

Matt: No, I’m just trying to think.

Interviewer: What is going on in your mind?

Matt: When you put the naught into there, it’s always going to equal naught that bit. So when you put in the naught, it’s gonna be just 1. 1 minus naught.

Beth: Yeah but you’ve got to integrate it before you do the …

Matt: Oh yeah, that’ll be … so that’ll always be naught when you put in the naught.

Beth: Yeah.

Beth worked out her example and was thrilled that it worked. She suggested that she doubled the constant in the function and the upper limit and generalised the integral to

\[
\int_{0}^{2a} (a-x)dx.
\]

She admitted that she was not sure how the general integral worked with negative numbers.

Beth: So if you integrate that, it’ll be 2x minus half x squared. (put 0 … put 4) that would work. Woohuh … I’m not blonde, well I am blonde but I’m not having the blonde fame then, that’s wicked.

Interviewer: How did you decide?

Beth: Just popped up in my head. Just felt like it was 2.

Interviewer: Did you use information from here (given example)?

Beth: I doubled that and doubled that. … I don’t know why I thought that would work but it appears to have worked.

Matt: So if that was 8 and 4 it would work again?

Beth: It might possibly, [checks] so that must also work.
Interviewer: Can you find another example?

Matt: It's double the 8, 4.

Beth: I reckon if you did 6 and then you did zero and you did 3, because as long as that's half that, that seems to be okay [checks].

Interviewer: What is the most general example you can think of for which the answer is zero?

Beth: So if you had a \(-x\) with a limit of \(2a\) and zero, that should always work, fingers crossed. I don’t know how it will apply in negative numbers but ...

Beth generalised based on the two examples that she came up with, varying only two dimensions (the constant and the upper limit) (\(\bigcirc\) under ‘awareness’). I conjecture that the examples she constructed were purely by chance. In order to probe her understanding further, I asked her why she thought the integral gave the answer zero. She admitted that she did not know. Following Matt’s observation that the two sides of the answer cancelled out, she suggested that it was only for specific cases.

Interviewer: Why is the integral coming zero?

Beth: Now that ..., I do not know. Emm ...

Matt: Both of those sides (referring to \(\int x - \frac{x^2}{2}\)) clearly cancel each other out, don’t know why. That side will always be double that, and you divide by 2.

Interviewer: That’s because you’re using ...

Beth: That’s just this specific one there, not for every single ... I think it’s just potluck there.

Interviewer: What can you change?

Beth: The upper limit and the constant (number). .. But I think it’d only be for specific circumstances because if you change those to the same thing (constant and upper limit) it wouldn’t work. If you change them to any random numbers it wouldn’t work, only for specific values.
I conjectured that both Beth and Matt focused on technique of integration so that they did not see the integral as an object, attending only to details of the example (process). No reference was made to area. Beth, however, revealed somewhat richer awareness, at least, of the relationship between the two dimensions (constant and upper limit) and more fluent in language use than Matt (under ‘awareness’, ‘technique’ and ‘language’).

In the next task, I asked Matt and Beth to express the similarities and differences between three definite integrals.

Matt: The areas equal to naught.
Beth: You’re either adding or subtracting something from it or $x$ from something. They’re all $x$ from something but ...

[Pause]

Interviewer: Anything else?
Beth: Brackets ... which means you’ve got to do that first, which is helpful when you get more complicated examples, ... they’ve also got girl, two o’clock thing indicated ($dx$).
Matt: One has been divided by 2, don’t know why that is though.
Beth: They’re all different graphs.
Matt: Two of them are negative $x$ and one of them is a positive $x$.
Beth: I think it all depends on which way it’s going. So it depends on which value underneath if you’re working out values underneath.

Matt announced that areas equalled naught as an example of a similarity among the three integrals, suggesting his awareness of area. Beth first commented on surface aspects of the integrals such as adding or subtracting $x$, brackets and $dx$. She then mentioned different graphs. Matt also commented on aspects such as negative and positive $x$.

Both Matt and Beth displayed limited awareness of connections to area as could be observed from their remarks. Matt displayed somewhat richer awareness than Beth as he
related the three integrals as having no area. However, he did not follow up his remark with further explanations. There was some evidence of connection when Beth hinted that ‘it depends on which value underneath if you’re working out values underneath’ (☐ ☐ under ‘awareness’ and ☐ ☐ under ‘language’).

Task 2 invited them to construct simpler and more complicated examples to a given expression.

Interviewer: Given that $2 \int (\ln x + \frac{3}{2}) \, dx = 2x \ln x + x = \frac{d}{dx} x^2 \ln x$. Construct another integral with its two corresponding expressions, which is simpler, and one which is more complex.

Beth: Emm … [long pause] I’m not entirely sure how they are the same.

[Interviewer explains]

Beth: So they’ve integrated it in two different ways? … I don’t think I could do that, don’t think I have enough knowledge or knowledge of integration to be able to do that.

[Interviewer explains again, with pauses]

Matt: Still doesn’t help me at all.

Beth: I think the $\ln$ is just throwing me off a bit. I’m bad with log. Sort of made my head go ‘Arrghhh’.

Interviewer: Can you construct another example which is simpler?

Beth: It doesn’t have to have $\ln x$?

Interviewer: What do you think?

Beth: I hope not.

Both Matt and Beth struggled to understand the example at first. Beth thought that the integral was solved in two different ways and declared that she could not do it. Also she suggested that the $\ln$ made her uncomfortable and she was unsure whether to keep it in
her example. She was also perplexed at the existence of $\frac{d}{dx}$ in the expression and at the fact that the integral equalled two different things.

Interviewer: Can you construct a simpler and a more complicated integral with its two corresponding expressions?

Matt: I don’t know where to start.

Beth: I’m just thinking, how can I integrate in two different ways of the same thing.

[Long pause]

Beth: That bit there ($\frac{d}{dx}$) is plaguing me now because I’m just looking at it and thinking, I’ve never seen that in integration, I’ve only ever seen it in differentiation that I can remember. ... I’m confused, does that mean that or does that mean that? And you’re saying it doesn’t mean either. It means something completely different.

Beth revealed limited awareness of the relationship between integration and differentiation and did not display flexibility in relating the two processes. Matt ‘somehow’ worked out the relationship and constructed a simpler example (under ‘awareness’ and ‘technique’). Observing what Matt was doing, Beth tried to express explicitly what he was doing in order to understand what was going on.

Matt: What about that [$\int xdx = \frac{1}{2}x^2 = \frac{d}{dx} \frac{1}{6}x^3$]?

Interviewer: Can you explain?

Matt: If you started with $x$, you integrate that you get half $x$ squared and I thought ... $d/dx$ of something ... don’t know how I’d done that. When you differentiate that you get that. I somehow worked it out. ... took ages though.

Interviewer: A more complicated example?

[Long pause]

[ Matt constructs $\int 2x + \sin x = x^2 - \cos x = \frac{d}{dx} \frac{x^3}{3} - \sin x$ ].

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Beth: So you’ve integrated it then you integrated it, is that what’s going on?
Matt: I don’t know, I didn’t do that.
Interviewer: How did you come up with this [example]?
Matt: First you integrated it and then you sort of … working backwards. First I thought that’ll be an $x^3$, involves a lot of minus 1, minus 1 … minus sin get back minus cos.
Interviewer: What are the things you have considered in coming up with the more complicated example?
Matt: What functions can you call complicated and what can’t you?
Beth: They’re not really any more complicated, just working out how to do it.
Interviewer: But you have added $2x$?
Matt: I don’t know why I’d done that, just randomly wrote $2x$.

Although he varied a number of dimensions in his complicated example and hence, revealed rich awareness, Matt was not very articulate in expressing his actions (ठ under ‘awareness’ and ठ under ‘language’). Beth did not construct any example in this task (see Table 7a.8).

Task 3 involved constructing examples of integrals using substitution method. Asked what came to mind seeing the integral, Matt suggested that it was impossible to do but later suggested using method of substitution. Beth also suggested starting with the brackets but did not specifically mention substitution (ठ under ‘language’).

Interviewer: Given that $\int \sqrt{x} \sin(x^2) \, dx = 1$. Describe what comes to mind when you first see the integral.
Matt: First thing I might do is it’s impossible to do. Is just looks so hard to do because bit there and bit there, got lots of different parts to it and I don’t know where to start.
Beth: I’d start with the brackets and think have I seen that before.
Matt: What I’ll probably do is integrate that first and probably stick in like $x \sin u$ then having the $u$ later on.
Interviewer: How would you describe the integral?
Matt: Integrating function of a function.

I then suggested to them that the integral was usually solved using a substitution method. Matt constructed \( \int \cos 2x \) as a simpler example and suggested substituting \( 2x \) with \( u \). For a more complicated example, he suggested adding another function such as a square root. Beth constructed \( \int \frac{\cos x}{\sin 6x} \) for a more complicated example, suggesting that integrating \( \cot \) would be difficult.

Interviewer: The integral is usually solved using substitution. Construct an integral which uses the same idea, which is simpler, and one which is more complex.
Matt: Something like that \([\cos 2x]\), substituting \( 2x \), so \( \cos u \) with \( x \) in front?

Interviewer: How about a more complicated example?
Beth: Emm ... you sort of mess with it.
Matt: Can you add another function like square root of all that?
Beth: That wouldn’t be fun doing.
Interviewer: Why?
Matt: Three functions instead of two.
Beth: All I know is \( \sin \) over \( \cos \) is \( \tan \) and \( \cos \) over \( \sin \) is \( \cot \) so I figured that would be pretty difficult for me to do. I’m not sure where I’d going with that. ...because how do you integrate \( \cot \)?

[...]
Beth: Don’t know much \( u \)-substitution.

Interviewer: When do you use substitution?
Matt: When there is more than one function.

Neither Matt nor Beth displayed familiarity with this method of integration (no entry for this part of the interview in Table 7a.8).

The next task invited them to construct examples using by-parts method.
Interviewer: The integral \( \int x \cos(x) \, dx \) requires integration-by-parts. Construct your own similar example that also requires integration-by-parts.

Matt: Isn't that substitution? Isn't there formula for by-parts?

Beth: Maybe. [constructs xsinx] Just because I'm not entirely sure whether you do (with) use it. Trig function ... would you use it with like exponentials something like that say if you had \( xe^x \).

Interviewer: When do you use by-parts?

Beth: Isn't that just to separate two parts of the equation so that you can work it out easier?

Matt: Doesn't \( u \) come out or something comes out? Don't know, haven't done it for ages.

Because she was not quite sure about integration by-parts, Beth constructed \( \int x \sin(x) \, dx \) as a simpler example. Both Beth and Matt were not sure about integration by-parts (○ under ‘awareness’ and ‘technique’) (see Table 7a.8).

Task 5 involved integration by-parts in the form of \( \int (f'g) \, dx \).

Interviewer: Given \( \int (x + 3)(x^2 + 6x)^3 \, dx \). Construct an integral which uses the same idea, which is simpler, and one which is more complex.

Matt: Is that using by-parts?

Interviewer: What do you think?

Matt: Probably both (substitution and by-parts). So you've got to put those two up and then use \( u \)-subs there.

Beth: Would that be anywhere ... if we did that, \( x^2 + 6x \), all squared.

Interviewer: How did you decide on this?

Beth: Because that one's got square outside the brackets, I'm inside the brackets. I thought, Oh, let's play with squares or to the power of something.

Interviewer: How about a more complicated example?
Beth: Emm ...

Matt: You can stick in like a third function of ... something like that.

Beth randomly constructed \( \int (x^2 + \sec x)^2 \, dx \), copying the given example and changing the cube to squared. She did not pay any attention to the \( f^g \) relationship in the example.

Matt suggested inserting a third function for a more complicated example.

Neither Matt nor Beth displayed any sense of awareness of structure in this method of integration. Hence, the dimensions varied in the example did not reflect deep sense of understanding (under 'awareness' and 'technique').

Table 7a.8 shows a summary of Matt and Beth’s responses in the construction tasks.

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
</tr>
</tbody>
</table>

Beth Construction task: 

Matt Construction task: 

Table 7a.8: Summary of Matt and Beth’s responses in the construction task

**7a.3.3 Making comparison**

Matt’s (M3, marked with squares) responses in the interview suggest a limited fluency with language, a focus on technique, a low level of associative thinking and little evidence of richness in reflections of aspects related to origins and use of integration. The
construction task did, however, reveal aspects of his restricted connectedness and facility with technique.

Beth (M3, marked with circles) displayed considerable language fluency with a propensity to focus on technique. Her awareness in terms of associations, connections and common mistakes seemed limited, as was her reflections on origins and use. The construction tasks revealed some evidence of connectedness and facility with technique, although they also revealed, at the same time, a focus on technique.

Both Matt and Beth demonstrated a limited disposition to work on integration. Their responses suggest considerable limitations in their concept image of integration and fragmentary nature of their example space.

### 7a.4 Summary

**Table 7a.3: Marlene and Craig’s responses in the interview and the construction tasks**

<table>
<thead>
<tr>
<th>Average tasks</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Connectedness</th>
<th>Misconceptions</th>
<th>Uses</th>
<th>Origins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marlene &amp; Craig</td>
<td>Interview</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Construction tasks</td>
<td>○○○○</td>
<td>□</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>□</td>
</tr>
</tbody>
</table>

**Table 7a.6: Sarah and Simon’s responses in the interview and the construction tasks**

<table>
<thead>
<tr>
<th>Average tasks</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Connectedness</th>
<th>Misconceptions</th>
<th>Uses</th>
<th>Origins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sarah &amp; Simon</td>
<td>Interview</td>
<td>○</td>
<td>□*</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Construction tasks</td>
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<td>○□□□□</td>
<td>○□□□□</td>
<td>○□□□□</td>
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</tr>
</tbody>
</table>

**Table 7a.9: Matt and Beth’s responses in the interview and the construction tasks**

<table>
<thead>
<tr>
<th>Average tasks</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Connectedness</th>
<th>Misconceptions</th>
<th>Uses</th>
<th>Origins</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matt &amp; Beth</td>
<td>Interview</td>
<td>○</td>
<td>□*</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>Construction tasks</td>
<td>○□□□□</td>
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<td>○□□□□</td>
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</tbody>
</table>
The use of the six-fold framework as a background structure to prompt learners to talk about encompassing aspects of the topic revealed a good deal about the subjects. Subjects in this group displayed a range of behaviour, awareness and emotion in relation to integration. Analysis of the data suggests that subjects in this group displayed a richer view of integration in construction tasks than when probed directly in interview. The group’s focus of attention on technique was evident and consistent with A-level teaching which was directed to passing exams (modular). Engagement in the construction task revealed more dimensions of their awareness. However, their awareness was still rather limited as evidenced by the small number of dimensions they varied in the examples. Subjects in this group showed disposition to engage which focused on techniques, which was not very strong. As such, their concept image of integration and their example space appeared considerably limited.
Chapter 7b

Main Study: Data Analysis

Interviews with ‘strong’ mathematics students

7b.0 Introduction

In this chapter, I discuss findings from the ‘strong’ mathematics students. Participants in this group were first-year Pure Mathematics students at a leading university in the Midlands. There were five students altogether, so four of them were paired up and one student was interviewed alone. All of them obtained A’s in their A-level subjects including Mathematics. Interviews with the students were conducted in one afternoon when they had some free time.

7b.1 Pair M4 (Kim and Darren)

Kim, a female student, was a very friendly and cheerful person. Darren, a male student, was very outspoken and witty. Kim and Darren were the first pair to be interviewed in this group. Kim was majoring in Mathematics and Philosophy while Darren was majoring in Mathematics. Kim had taken courses in Real Analysis, Algebra and Logic and Philosophy while Darren took Real Analysis, Algebra, Calculus and Statistics.

7b.1.1 The interview

Extracts from pair M4’s responses in the interview are given below.

Interviewer: What does the word “integration” mean to you?

Kim: If you are given a function such as \( f(x) \), imagine you plot a graph if you would integrate the function it is calculating the area underneath the graph of that function between certain values, between that function and
Chapter 7b  Main Study: ‘Strong’ mathematics students

that x-axis. Integration generally indefinite would be a general rule to work out the area underneath the graph and then integrating definite within certain boundaries would be a way of finding out within certain x-values.

Interviewer:  What comes to mind when you see the sign ∫?

Darren:  Integration...basically finding the integral of the function.

Kim:  In terms of differentiation, like if you differentiate something you think of, this is calculating the rates of change is the line of this graph so it is sort of the reverse of that, so it’s like finding out you have the rate of change, you are finding out what the value is in the first place.

Kim (marked with circles) articulately expressed connection to area under a graph, drawing on knowledge of functions and on integration as a ‘tool’ to calculate area underneath the graph. She pointed out the distinction between a definite and indefinite integral in relation to area. She also identified the connection between integration and differentiation as a process when triggered by Darren (● under ‘awareness’ and ‘language’). For Darren (marked with squares), the integral sign brought to mind the process of integrating (□ under ‘awareness’ and ‘language’). It is likely that in his articulations, he was thinking of the integral as a function, but this is only surmise.

The following probe revealed their awareness of origins of the topic.

Interviewer:  How were you introduced to the concept of integration?

Darren:  By splitting the graph, basically looking at the area under the curve, so given the curve splitting it into finitely small ones and then going down to infinitesimally small ones. As the size tends to zero it tends to do this thing.

Kim:  I only learnt it in terms of the reverse of differentiation.
Darren described the process of finding area using the idea of limit of sum (□ under 'origins'). Kim's remark about being introduced as the reverse of differentiation is marked with ○ under the same category. Possible uses of integration in other contexts did not come to the surface. Thus, there was no evidence of exposure to or relevance of this aspect in this instance.

Awareness of common obstacles associated with a topic was addressed using the following probe:

**Interviewer:** What sorts of things have you discovered you need to watch out for when you are doing integration?

**Kim:** When I first started doing integration, just remembering to add on constant for indefinite integration. It is quite an important thing to remember because the way the sum progresses you need to have the constant in the right places or it might affect your answer to the whole problem when you combine functions.

**Interviewer:** Anything else?

**Darren:** Stupid mistakes like when you have expression of two things and I do expression of one thing plus expression of the other instead of multiplying. It's nothing to do with integration; it's mainly remembering how to do integration by-parts and substitution and especially remembering … I change the variable and then forgot to do the proper addition in the proper function and the actual integral.

**Kim:** Substituting the variable in the right context or something.

**Darren:** Like √(x^2 + 1), u = x^2 + 1, du = 2x dx and I put du without bothering about the x's. So that you've got to be careful about that.
The remark about technical aspects such as adding the integration constant to watch out for in the initial stages of learning the topic suggests that Kim focused on technique (☐ under ‘awareness (misconceptions)’). Darren’s remark about specific aspects of integration suggests deeper awareness (□ under ‘awareness (misconceptions)’). Of course it may be that in dialogue they felt no need to re-express what the other has said.

Fluency in language and facility with technique are also indications of competent behaviour. To this means, the following probe was posed:

Interviewer: What words in a problem/context that tell you that integration is relevant?

Darren: Integrate!

Kim: Find the area under the graph.

Darren: Solve the differential equation, anything to do with calculus.

Kim: Separation of variables. I suppose you could have a question about the derivative of something is … Find the function or something like that.

Interviewer: What are some of the special words/language that you use when you are talking about integration?

Darren: Limits, definite/indefinite, C, function, graphs, areas.

The language patterns displayed by Darren suggest fluency associated with method and also applications of integration (□ under ‘language’). Kim also displayed fluency with language but to a lesser extent (☐ under ‘language’). Again, their focus was entirely within the topic itself, without any reference to other contexts in which integration is used.

In order to get a clearer insight into their associations, I prompted them with the following probe:
Interviewer: What is the same and what’s different between $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$?

Kim: It’s the integral of the same function.

Kim noted the sameness of the function; no remarks were made on the properties of the objects or associations of them, either because they were thought as irrelevant or they did not come to the fore (under ‘awareness’).

Table 7b.1 below summarises Kim and Darren’s responses in the interview.

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
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<tr>
<td>Kim</td>
<td></td>
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<td></td>
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<tr>
<td>Darren</td>
<td></td>
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</tbody>
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Table 7b.1: Summary of Kim and Darren’s response in the interview

The next section discusses how Kim and Darren performed in the construction tasks.

7b.1.2 The construction task

Kim and Darren’s responses in Task 1 are presented below:

[Long pause]

Darren: That ... equals zero, ... the integral of $1-x^2$ from 0 to 2 $\left[ \int_0^2 1-x^2 \, dx \right]$.

Interviewer: Can you find another example?

Kim: The same $\left[ \int_0^2 (1-x^3) \, dx \right]$.

Interviewer: And another?
Kim: Integral from 0 to 4 of something \( \int_0^4 \frac{1-x}{16} \, dx \). A general rule could be ...

Darren: [tries something]. Arghh ... that doesn't work, does it?

Kim: \( \int_0^n \frac{1-x^n}{n^{n-1}} \, dx = \left[ x - \frac{x^n}{n^{n-1}} \right]_0^n = n - \frac{n^n}{n^{n-1}} = 0, n \geq 2 \)

Darren: That is a really complicated generalisation. I have a simple one,

\( \int_0^2 (1-x)^{2n+1} \, dx, n \leq N \cup \{0\} \).

Kim: It's not any more complicated.

Darren: It only has one variable, one thing that you change, integral from 0 to 2 of \((1-x)^{2n}\), any natural number, \(n = 0\), then what you get is \((1-x)\) to an odd power over something, which doesn't really matter, and then if you put in 2 you get minus and if you put in 0 you get plus. Is that the other way round, I'm not quite sure. No, ... it has to be to an even power which means that should be 3 that should be 5

\[ \int_0^2 (1-x)^3 \, dx = 0 \]

[ tweeple.net](http://www.tweeple.net)

Darren constructed an example by varying the function and keeping the limits constant. However, having worked it out, he realised that his first answer was incorrect. One possible explanation is that his attention was focused on the function, raising it to a power. Then, generalising the situation, he maintained the same limits but raised the function to an odd power. His attempts at generalisation seemed to be structured by his awareness of possible variability of this dimension. No reference was made to area (under ‘awareness’ and ‘technique’) (see Table 7b.2).
Chapter 7b   Main Study: ‘Strong’ mathematics students

Kim constructed examples by increasing the upper limit by 1 and changing the second term in the expression to accommodate the upper limit. The form of the function \((1 - x)\) was maintained because 1 will integrate to \(x\), so that the second term had to work out to the new upper limit. Kim displayed awareness of a relationship between the function and the limits. She varied both the limits and the function. She seemed to work on a pattern between the upper limit and the second term in the expression 

\[
2 \rightarrow x, 3 \rightarrow \frac{x^2}{3}, 4 \rightarrow \frac{x^3}{16}.
\]

Her generalisation seemed to confirm this. However, her particular examples were incorrect, leading her to an incorrect generalisation which contained seeds of a correct idea \([\int_{0}^{n} \left(1 - \frac{x^{n-1}}{n^{n-2}}\right)dx, n = 2, 3, ... \]) (○ under ‘awareness’ and ○ under ‘technique’) (see Table 7b.2).

In order to gain further insight into the components of their awareness, I probed them as follows:

**Interviewer:** In what ways are the examples you’ve constructed like my example?

**Kim:** In this sense, it is exactly the same in that all the examples I have been giving has just been increasing 1 to the limit and adjusting the integral to make sure that when you integrate it, \(1 - x\) whatever you’ve got there, the integral of that is always going to be zero. And the example that you have given is just a general \(specific\?) case when \(n\) is 2. I’m trying to think quite how it will work when \(n\) is 1, I think it has to be \(n\) greater than 1, ...

**Darren:** Mine is almost identical except I raised \((1 - x)\) to an odd power, as long as it is raised to an odd power, it’ll always equal zero.

**Interviewer:** What can you say about the general integral that you have constructed?

**Darren:** It’s a polynomial of an odd degree.
Kim: Where mine, I take a polynomial of degree \( n - 1 \), so the degree is 1 less than the upper limit. My general [integral is] integral of \((1 - x)^n - 1\).

Interviewer: How have you come to generalise this?

Kim: Having looked at the two solutions I’ve got here, what I’ve done each time to get that, because first of all I just started doing integral from 0 to 3 of \((1 - x^2)dx\) \[\int_0^3 (1 - x^2)dx\], okay that will work, but then I realized ... oh no, it won’t work because it’ll give 3 - \(\frac{27}{3}\), which is 3 - 9 that’ll be – 6.

So in order to get that to 0, I needed 3 minus 3 so to do that, I need to divide 3 by 3 and then in this example, I needed to divide 3 by 16 to get 4 minus 4 so that the upper limit is raised to the power 2 less than that limit each time in order to divide through to get that to equal zero.

Kim constructed examples by adjusting the limits to get the answer zero, which indicates the focus of her attention. Although Kim seemed to be aware of the general case by seeing through the particular, her awareness of what could change in the integral seemed to be limits of functions and relationship needed between them to preserve her perception of the form \(1 - \frac{x^n}{\beta}\) (\(\square\) under ‘language’).

Darren maintained the function and generalised the power of the function (raising it to an odd power) to get the answer zero, suggesting facility with technique but a limited range of functions to try. He kept it relatively simple (\(\bigcirc\) under ‘language’).

In order to reveal more about their awareness of dimensions-of-possible-variation, I probed them using the following prompt:
Interviewer: What are the things in the integral that you can change and still get the answer 0?

Darren: You can change the limit, you can change the function, you can even change the term \( \text{d}f \) if it makes any difference.

Interviewer: The variable?

Darren: Yes.

Kim: I think you could use another variable and say integrate with respect to \( u \) or something instead of \( x \).

Darren: That’s interesting. If you set \( u = 1 - x \) then \( du = -dx \) and then that integral becomes the integral from 1 to -1 of \( u \) \( du \), \( u \) is an odd function, anything like that is an odd function, so that automatically \( y = 0 \), so any odd function like that, so long as that’s an odd function, so long as it’s

\[
\int_{-1}^{1} u \, du, \text{ it'll always equal 0. } \left[ \int \frac{u^{2n+1}}{3} \, \text{d}x = 0 \right]
\]

Kim: I suppose so but this works! I think if you can’t find anything that works, great, do that. But if you can, this seems a better one.

Interviewer: How about the limits?

Darren: The limits were defined by when \( x = 2, u = -1 \) and when \( x = 0, u = 1 \) but you have \( -du \) so you substitute the limits and you get that odd integral.

Darren expressed several dimensions that could vary. Although he made an interesting observation about the oddness of the function, it only confirmed his conjecture about his generalisation. He appeared to be seeing the example as a specific instance of a generality and did not seem to recognise the possibility of changing the limits independently of the function. I conjectured that he implicitly thought in terms of area or else algebraically because he was familiar with integrals of odd powers (under ‘awareness’).
Kim was not comfortable with the idea of an odd function and maintained her view on the relationship between limit and the function as shown in her generalisation. She demonstrated an urgency to generalise algebraically (under ‘awareness’). Although this seemed to be a clear display of having picked up elements of a mathematical aesthetic to justify her approach, it is difficult to be sufficiently precise to get an agreement as to what constitutes a mathematical aesthetic (Dreyfus and Eisenberg, 1986; Sinclair, 2004). I was alerted to it in the interviews with the ‘strong’ mathematics students but it was too hard to operationalise and so it is not included in the overall analysis.

Given Task 2 to find out these learners’ awareness of the relationship between integration and differentiation and the extent to which they were aware of structure in mathematical objects, Kim and Darren struggled to understand the task at first.

Interviewer: [reads the question out] How do you see it now?

Darren: You have the same integral and you have two different forms of the answer.

[Interviewer reads out the question again, with pauses]

Kim: Sorry ... I’m understanding it now, it’s a different integral with two expression not the same one. It’s the same integral with two different expressions.

[Long pause]

[...]  

Kim: Oh I see ... one is written in terms of a derivative and one is written in terms of just the function without any ...

[...]
Darren: So you want one in terms of the thing and one in terms of the derivative and then something horrendously more complicated than that.

Interviewer: [To Kim] Are you making any sense of it?

Kim: No.

[Darren explains]

At first, both Kim and Darren struggled to understand the form in which the expression was presented. Their attention seemed to be focused on ‘getting an answer’ to the integral but not the reverse process. I conjectured that they rehearsed the meaning of integration as reverse of differentiation without necessarily having an awareness of its relevance here. Kim’s remark about her understanding of the task triggered Darren to observe the same thing (under ‘awareness’) (see Table 7b.2).

Upon recognising the form of the expression (after a discussion with Darren), Kim expressed generality that it ‘could be any function’ (under ‘awareness’). In the next extract, the examples she constructed are shown.

Kim: That could be any function.

Darren: You can just do [2∫dx = 2x = d/dx(x^2)] (simpler).

Interviewer: Do you have any criteria for choosing from ‘any’ function?

Darren: For being simpler just have a constant. Zero would be even nicer but if you are doing definite integral that just gives you zero which is the derivative of zero. It is kind of pointless, so you have to have a constant in there.

Kim: I think I understand it now.

Darren: ... -1 and so I decided to get d/dx tan x and then, which was a mistake because I can’t remember what the integral of x^2 is, ...the differential of
Darren: $x^3$, so leave that out. And then the differential of $\log x \ldots \frac{1}{x} \ldots$ minus 1, that can’t be right. Umm... let’s not use trigonometric functions ... those are evil things ... $u = \sin x \cos x \ldots$ I think that works ... but I’m not sure whether that’s actually right. It’s the integral from -2 …………

Interviewer: You said it could be any function. How did you decide to pick that for your simple one and that for your more complicated one?

Darren: I decided on this one to make it as simple as possible. Basically just use a constant and then you’ll get a simple answer which is very easy to evaluate as a differential. For the complex one,

$$\int (-2 \sin x \cos x - 2 \cos x \sin x + \frac{1}{x} - \frac{1}{x^2})dx = \cos^2 x - \sin^2 x + \log x - \frac{1}{x}$$

I worked backwards because if you start with a complex integral then you have to spend ages working it out. Then you have to spend ages working out what differentiates to get your answer. So with an expression, differentiate it to see what you get and then differentiate it again so then put that as your function for your integral.

Darren gave the simplest form of expression but did not choose to change the multiple 2 in front. He could be partially aware of the structure. He displayed awareness of form of the expression. He constructed a simpler example by forward integration, starting with the integral and using the simplest number possible so that the structure remained. His labelling of trigonometric functions as ‘evil’ offers insight into his emotional relationship with different functions. The more complicated example was constructed by working backward, starting with the differential form of the expression which included complex terms like trigonometric and logarithmic functions. He displayed an awareness of and facility with the inverse relation between integration and differentiation. There was a shift of attention in using differentiation as inverse of
integration. However, he made an error in differentiating $x \log x$ (taking $\log x$ to be base $e$) (☐ under ‘awareness’ and ☐ under ‘technique’).

Interviewer: What is it about your example which is like the one I gave you?

Darren: Umm... that one is similar to that one in a way because they both involve $\log$. $\log$ and trigonometric functions are nice when you are looking for complex integrals because they don’t differentiate away to zero. They sort of keep being ‘evil’. Whereas constants and polynomials differentiate away to zero which means you can work with them quite easily. So... and this is sort of a mixture of the two because you have a logarithmic function but you also have just a constant.

Interviewer: Are there any other ways that they are like this one?

Kim: Just the fact that they are both functions, integrals that can be written in terms of the derivative or just as the function.

Interviewer: In that way, the ones you’ve constructed ...

Kim: ... can be written like that.

Interviewer: How are they different?

Kim: The simpler one $\int x \, dx = \frac{x^2}{2} = \frac{d}{dx} \frac{x^3}{6}$ is just in terms of polynomials, whereas this one is in terms of $\log$ and polynomials; whereas the more complicated one

$$\frac{d}{dx} \sin^3 x \cos x = 3 \sin^2 x \cos^2 x - \sin^4 x = \int (6 \sin x \cos^3 x - 10 \sin^3 x \cos x) \, dx$$

is in terms of trigonometric functions, and that one is $\log$ and polynomials again, that’s just what I have decided to do.

Darren: Logs are slightly easier to work with than trigonometric functions.

Kim constructed a simpler example by integrating $x$ (polynomial of degree 1) and a complex example by differentiating trigonometric functions. She too started with the differential form (☐ under ‘awareness’ and ‘technique’).
Kim and Darren's responses to Task 3 are discussed next.

**Darren:** That's the derivative of that or part of the derivative thus you can easily integrate it, thus if you took \( \sin x^2 \) or \( \cos x^2 \), there is no need for any substitution. At first sight you think \( x \) is \( \sin^2 \), let's use substitution but then you look at it closely you think actually you don't need to use substitution because that's a direct derivative.

**Interviewer:** Anything else?

**Darren:** Umm... that's what I immediately see. I don't even try and predict that graph. It's some wiggly graph and yes, you like take area under it to be 1 but just work it out.

**Kim:** Like [what] Darren said, that being the derivative of that, so it can be carried out in that way, just by observation. I think if you are given something like that, you wouldn't think about the graph if you are given a complicated function, you think about just the general rule, like with trigonometric function to use particular method to integrate, just think about it in that way rather than particularly looking at trying to work out what the graph look like.

Darren recognised the form immediately and associated the integral with the method of solving it. He was articulate about the form of a simple integration by substitution. His decision not to picture the graph (although he seemed to be aware of it as a possibility) demonstrates his association of integration (\( \square \) under 'technique', 'language' and 'awareness' sections in Table 7b.2).

Kim too showed awareness of associative images but focused on a method of integration for the complicated algebra. She appeared to build on what Darren said, but her articulation in her own words suggests she too saw the form directly (\( \bigcirc \) under 'awareness').
Interviewer: The integral is usually solved using substitution. Can you construct an integral which uses the same idea, which is simpler, and one which is more complex?

[Immediately]

Darren: \( \frac{1}{x^2 + 1} \int \frac{1}{x^2 + 1} \, dx \) which is just \( \tan \) substitution. That’s very, very simple.

Kim: This one \( \int \frac{2x}{x^2 + 1} \, dx \) is just the logarithmic substitution because that is [the] derivative of that. It’s a logarithmic one but you just say that is \( u \) and then \( du = \frac{1}{2x} \), \( 2x \) cancel out you get \( \frac{1}{u} \) and it’s just the log of that.

Darren and Kim constructed simpler examples by using a substitution method that immediately came to mind. It is evident that Darren was aware of the relationship present in this method of integration. Kim, however, marked this relationship in words. Both appeared well-versed in this method of integration, judging by the speed at which the examples were produced.

Darren: For more complex, let’s do several stages so …

Kim: Well it doesn’t need to be terribly complicated.

Darren: Yeah, I would like it to be really, really complex.

[Pause]

Kim: \( \frac{2}{\sqrt{1 - 3x^2}} \), so that’s going to be sin or cos or hyperbolic. I think it would work if you do either sin or cos. Something like \( u = \frac{\sin x}{\sqrt{3}} \) because if you square \( u \), then you can get \( \frac{1}{3} \sin^2 x \), \( x = \frac{1}{\sqrt{3}} \) if you substitute \( u \) in there, the whole thing is \( -2 \sin x^2 \).
Interviewer: In what ways are the examples you’ve constructed like my example?

Kim: I suppose in the way it is trigonometric example in the way that you have to use substitution or you can use substitution.

Interviewer: In that $u$ is not a trigonometric function.

Kim: No, $u$ isn’t. That’s the difference that for this one the function itself is polynomial but then the substitution involves trigonometry whereas this one (given example) is a trigonometric function which involves substitution of polynomials.

Kim: No, I don’t think so.

Interviewer: What have you considered when coming up with these examples?

Kim: If you think that one, you can almost do it intuitively, so I’m thinking with this sort of example, I think I would look at that and think log $x^2 + 1$ because that is derivative of that, whereas with this example, I’m trying to think of something which I thought it’s a bit more complicated than that in that I would want to definitely do a substitution if I was calculating that but something that I would have some idea what I was looking, I might need to experiment a bit to something which might be more difficult to know what to do straight away.

Darren: This is … you just [do] substitution $x = \tan u$ so then you get $dx = x^2 du$ and so you substitute back.

Kim: I wouldn’t say that is simpler than that.

Darren felt ‘compelled’ to construct a complicated example that involved a few stages (■ under ‘awareness’ and ‘technique’). Kim constructed an example that used a different type of substitution. She retained the form (method of integration) but decided on obscuring the obvious so that the method was not readily perceivable (■ under
‘awareness’ and ‘technique’). Both Kim and Darren showed disposition to engage in mathematical thinking.

Interviewer: How is it like this [example]?

Darren: When you are using substitution, I think that it is simpler than that.

Kim: But that’s true with learning, not really matter. Like that might mean simply because you know that it is the universal trigonometric function but if you didn’t know that that was a universal trigonometric function, if wouldn’t just come to you like that. It’s a standard result.

Darren: Even then, this one to do by substitution, I think does one use \( \sqrt{x} \)? Does one use \( x^2 \)? I’m not exactly sure what to do. This one I see and I think ... aha... that has a form of a trigonometric substitution, even if I don’t know it’s \( \tan x \). And this one \( \int \frac{1}{\sqrt{x} (x + 1)} \) is more complicated only in that ... I look at that and I think what sort of substitution am I going to use there. Look at that and think... ah... there’s a \( \sqrt{x} \), lets use \( x = u^2 \).

Kim: That’s probably a matter of familiarity in knowing that if you have something that looks in a particular way, you think OK, I’ll use trigonometric substitution or log substitution or something, isn’t it because obviously if you were not familiar with the things, like if you have something like that to use a trigonometric substitution, you wouldn’t have a clue; whereas with that one, even though it isn’t like a particular standard or you would ultimately work out like \( u = x^2 \).

Interviewer: In what ways are the examples you’ve constructed like my example?

Darren: Umm... well I suppose that one uses different sort of substitution in that it’s trigonometric substitution that uses two substitutions that’s why I had it slightly more complex. If you substitute \( x = u^2 \), you get \( \frac{2u}{u^2 + 1} \) which is a log integration, which you need another substitution for so that would be why I’d say that was slightly more complicated because you need to do it twice.

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Kim was articulate about her awareness when she explicitly identified familiar situations in which she recognised what to do. She displayed meta-awareness of complexity in choosing which function to substitute in this method of integration (under ‘technique’ and ‘language’ in Table 7b.2). Darren constructed a complex example by making it difficult to decide what to use for substitution and to do the substitution twice. This marks facility in technique and fluency in language in describing the construction (under ‘technique’ and ‘language’ in Table 7b.2). Again, he made elementary errors in the details.

Interviewer: Task 4: The integral \( \int x \cos(x) \, dx \) requires integration-by-parts.

Construct your own similar example that also requires integration-by-parts.

Darren: \( \log x \int \log x \, dx \).

Interviewer: In what ways is this example like my example?

Darren: It's like that in that it uses integration by-parts to produce the integral to a simpler form that you can actually integrate because you can't actually integrate that as it stands but if you use integration by-parts you can make it into a nicer integral.

Interviewer: In what ways is it different [from my example]?

Darren: It is different in that a lot of integration by-parts and especially ones which involves just \( x \) terms, you differentiate the term that goes down to zero so that you can get a term which you can continually integrate on its own. With this one, it's very different because you have to imagine that it's actually \( 1 \cdot \log x \) and then differentiate the more complex part and integrate the thing that goes to zero. I'd say that's a more complicated one.
Darren produced a simpler example (but later decided it was more complicated) by recalling an example of integration by-parts. He chose the example because integration by-parts problems usually had two parts (one to integrate and one to differentiate). I conjectured that he ‘stored’ examples of integration in terms of methods of integration. He seemed to be aware that integration by-parts needed two terms but \( \int \log x \, dx \) was considered unique and more complicated because whoever was solving it had to transform it into two parts, which was not natural.

**Kim:**

\[
x \sin x \left[ \int x \sin x \, dx \right]
\]

the same sort of thing as that but just uses \( \sin \) instead of \( \cos \). It’s similar in that they are both a product of \( x \) and trigonometric function on \( x \) and different in that one is \( \sin \) and one is \( \cos \), they require the same treatment pretty much.

Kim varied only one of the terms in the function. Her attention and awareness seemed to be confined by the \( \sin \)-\( \cos \) relationship.

**Darren:**

That is simpler because integral of \( e^x \) is \( e^x \) so integral of \( xe^x \) is simpler because one of the terms when you differentiate or integrate it just remains the same, so all you need to do is look at that term. So with this one \( \int x \cos(x) \, dx \) and this one \( \int \log x \, dx \), you have a choice, do I integrate that or differentiate that, whereas this you can almost ignore the \( e^x \) and so you just think do I integrate or differentiate the \( x \) and in the end gets differentiated.

Darren constructed the simpler example \( xe^x \) in order to have two terms multiplied together using the fact that \( e^x \) behaved nicely. Eliminating the choice for which term gets integrated and which one gets differentiated was considered a simpler example of integration by-parts. For a more complicated example, Kim constructed \( \int e^x \, dx \). It
is unclear whether she constructed the example based on her awareness or she duplicated Darren’s example (under ‘awareness’ and ‘technique’).

To gain insight into their awareness of structure and what could be changed in \( f'g \) method of integration, I gave them the following task:

**Interviewer:** Given \( \int (x + 3)(x^2 + 6x)^3 \) \( dx \). Construct an integral which uses the same idea, which is simpler, and one which is more complex.

**Kim:** For this I would probably multiply out the polynomial using binomial coefficient and I suppose if it’s more complicated one the powers might be such that it’s not possible, the powers might be too large to conceivably do that.

**Kim:** \( \int (x + 1)dx \)

**Darren:** Err… that’s not technically a polynomial.

**Kim:** Yes it is because that has a greater order, so it won’t ultimately have a negative order because that will cancel that out.

Kim discerned and varied the polynomial nature of the function and constructed a simpler example which was the simplest polynomial. She also retained the addition operation and added 1. The more complicated example that she constructed did not use the same idea as the given example in terms of method of integration. Aspects Kim discerned in the integral (multiplication of polynomials) seem to bear the consequences of what she varied in her example. She overlooked the fact that one term was a constant multiple of the derivative of the other.

**Interviewer:** What is it about your example which is like the one I gave you?

**Kim:** It’s the integral of polynomial of degree 1 so it’s a simpler polynomial whereas more complicated one a polynomial with more complicated coefficient. \( \int (3x^3 + 4)^8(17x + 1)^3 \) \( dx \)

**Interviewer:** How is it different?
Kim: It has a greater power for that and negative power that and perhaps more awkward coefficient than there to multiply out with using same idea.

Kim: That one I will just multiply out.

Darren: Oh ... hang on, I see. Have a look. That is \((x+3)\) times by 2, \(2(x+3)^3\)
so there's an \((x+3)\) \(u\) there. So it's quite similar. But that might be completely irrelevant.

Kim: I don't see things like that, so I just multiply that out.

Darren: There's a pattern there, which there definitely isn't in my more complex example.

Kim: It depends on how you view that one [the given example]. I didn't really see that.

Darren: I only just noticed that, so I can construct new ones which do have connections between each term.

Kim: If I was faced with that in an exam or something, I don't think I would notice that because I don't tend to notice these things.

Kim was aware that she did not always notice relevant relationships. It reflected her propensity to engage in manipulative details. She displayed awareness of a disposition to jump in and calculate, which she was aware of but did not manage to block (\(O\) under ‘awareness’, ‘technique’ and ‘language’).

Darren became aware of the \(f\,g\) form of the integral having constructed simpler and more complicated examples. It is highly likely that this awareness was brought about by his tinkering with the integral to produce a simpler and a more complicated example.

Interviewer: In what ways are your examples like the one I gave you?

Darren: Basically this one (more complex)

\[
\int(42x^6 + 18x^2 + 96x^3)^7(7x^9 + 3x^5 + 16x^5)^{19}\,dx
\]

has extra threes buried in there in several different ways. And this one \(\int(2x + 1)^2(4x^2 + 2x)\,dx\)

I've decided to go \(2x + 1\) and beside, to make it even more obvious the way that it's buried in there in that in there it's just \(2x\) times that in there.
So I raised it to lower powers and make it sort of more obvious
connections between the two.

Interviewer: How are they different [than my example]?
Darren: It’s different in that the function itself is ... the embedded function is
about on a par but the way in which the two terms are related is much
more obvious and it’s raised to a lower power. Whereas this one is more
complex simply in the fact that it’s got stupid powers. You don’t usually
go around raising x to 39 and then [raise] all that [expression to] 21. That
makes it slightly more complex and the embedded function is infinitely
more complex as well, although I still made it slightly obvious in that and
the function is not quite as obvious as it is in this one because to get from
that to that I both multiply by something and divide by something. So
neither of those is the actual embedded function. The actual embedded
function would be $7x^36 + 3x^{12} + 16x^2$ because to get from that to that
you multiply by 6 and to get from that to that you multiply by $x^3$.

Interviewer: Are they connected in any other way to my example?
Darren: Apart from the fact that they both have embedded functions, they both
have two terms which were also raised to single power then multiplied
together. But in this the terms are more complicated and each term has
more terms and you really would not want to multiply that out.

There is a slip between what Darren stressed mentally and what he actually wrote
down. He constructed a complicated example by discerning a relationship between $f'$
and $g$ in the integral. He made the integral complicated by *burying* the $f'$ further in the
embedded function $g$ so that the relationship was *concealed*. He did not always manage
to retain the structure perceived when he leapt to a conjectured more complicated
version. The simpler example was constructed by making the relationship between $f'$
and $g$ *more obvious*. Darren varied the terms and relationships between them, possibly
having become aware of the method and the relationship between the terms (under ‘awareness’, ‘technique’ and ‘language’ in the table).

A summary of Kim and Darren’s responses in the construction task is presented below.

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<th>Behaviour</th>
<th>Awareness</th>
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</thead>
<tbody>
<tr>
<td>Language fluency</td>
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<td>Connectedness</td>
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<tr>
<td>Kim</td>
<td><img src="image" alt="Kim's responses" /></td>
<td><img src="image" alt="Kim's responses" /></td>
</tr>
<tr>
<td>Darren</td>
<td><img src="image" alt="Darren's responses" /></td>
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</tr>
</tbody>
</table>

Table 7b.2: Summary of Kim and Darren’s responses in the construction task

7b.1.3 Making comparison

The table below presents a summary of what Kim and Darren said in the interview and what they did in the construction task.

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
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<tbody>
<tr>
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<td>Facility with technique</td>
<td>Connectedness</td>
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<tr>
<td>Interview</td>
<td><img src="image" alt="Interview responses" /></td>
<td><img src="image" alt="Interview responses" /></td>
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<tr>
<td>Pair hist Kim &amp; Darren</td>
<td><img src="image" alt="Construction responses" /></td>
<td><img src="image" alt="Construction responses" /></td>
</tr>
</tbody>
</table>

Table 7b.3: Kim and Darren’s responses in the interview and the construction tasks

Responses from Kim in the interview suggest a good deal of facility with language. Her use of language was, however restricted to surface features and did not show much evidence of possessing a sense-of or understanding. She did not make any remark about techniques in relation to the topic. Her associations between terms within the topic appeared comprehensive. However, she did not display (chose to or was not prompted
to) evidence of robust awareness of common errors or access to sources and origins of integration.

In the construction task, she displayed considerable language fluency when talking about the examples she constructed. Her facility with technique seemed rather good, although she displayed a tendency to be technically oriented. However, she did not display enough evidence to suggest rich awareness of associations and relationships. In some tasks (Task 2 and Task 3) she did display this awareness but only after lengthy explanation by her partner and the interviewer. The restraint in her voice tone gave me the impression that her attention was dominated by technique and she appeared to have difficulty comprehending unfamiliar mathematical objects, perhaps due to a limited collection of associations within the topic. Kim showed disposition to make sense and to build on what she heard from Darren.

Darren was very fluent and displayed reasonable facility in technique, as indicated in his responses in the interview even though he overloaded details, which then led to errors. However, his responses did not suggest that his awareness of the scope and application extended beyond the pure mathematical structure and use of techniques, evidenced by what he expressed as his associations in terms of what the topic reminded him of and his awareness of common mistakes.

The construction task revealed even richer language fluency as he talked about the examples. He also demonstrated a marked facility with technique as revealed by the examples he constructed and the extent to which he displayed understanding of the method involved. Again he made simple errors, although his intuitions were often correct in principle even though not in detail. He also revealed heightened awareness of connections and associations, evidenced by the dimensions he chose to vary in the examples he constructed.
There were a few comparable features in Kim and Darren’s responses in the interview and the construction tasks through the use of the six-fold framework. In the interview, Kim not only revealed aspects of the topic she did not previously focus her attention on but also the extent to which she focused on certain aspects. The construction tasks seem to have afforded opportunity to reveal, to a greater extent, aspects she focused on. Her behaviour, as highlighted in her language pattern and technique facility, gave me the impression that this was the aspect which formed the basis of her appreciation of the topic and as a result, she overlooked useful and important associations linked to it. Her apparent reluctance to consider alternative methods and the moderation of her voice tones suggest that Kim was more cautious than her colleague Darren. She wanted to be able to ‘do’ her examples, whereas Darren was willing to make more complex conjectures.

In Darren’s case, his responses in the interview suggest that he had language fluency but did not display particular facility with technique or associations. Nor did he show any significant awareness of any misconceptions. He did not seem to take notice of contextual or root problems associated with the topic. The construction tasks, however, revealed a good deal of fluency with language and facility with a range of techniques. They also revealed an even richer awareness of the topic, judged by the awareness he displayed of things that could vary in the examples he constructed. His eagerness to push his limits in constructing examples with different constraints and his confident tone of voice suggest a willingness to engage, a disposition to generalise. Neither was stimulated to relate integration to specific applications beyond narrow confines of differentiation and integration.

The same process was applied to another pair (Harris and Danny) and an individual (Rick). The next two sections discuss the students’ responses, accompanied by extra
notes that are worth pointing out. I must point out that in the next two interviews and in
the interviews in Chapter 9, I have summarised where appropriate what the participants
said in the interview and not provided detailed transcript because I thought that the
space could be better used for analysis of the construction tasks.

7b.2 Pair M5 (Harris and Danny)

Both Harris and Danny were males. Harris was an exceptional 15-year old foreign
student while Danny was 19-year old local student. They were both studying
Mathematics. Both Harris and Danny had taken courses in Analysis (Continuity &
Differentiability (no integration)), Linear Algebra, Partial Differential Equations,
Probability and Statistics. Danny had taken Geometry in addition while Harris took
Fourier series and two-variable Calculus.

7b.2.1 The interview

Both Harris (marked with circles) and Danny (marked with squares) displayed notable
language fluency when they mentioned a wide range of words associated with or that
triggered integration such as techniques (by-parts, substitution, double integral, surface
integral, integrate, solve differential equations), functions (Fourier series, statistical
functions, continuous functions), integral tests, series, continuous random variables and
applications (rate of change, area, volumes, blocks, work done). The fact that
integration triggered a wide range of associated language and awareness in them
suggests a considerable degree of fluency with the language of the topic ('language' and 'technique').

They displayed some form of connectedness and awareness when they referred to
meaning of integration as everything comes together, reverse of differentiation, finding
area under a curve, volume, solving differential equations but it was not evident
whether they were aware of integrals as limits of sums (• under ‘awareness’). Displays of sound awareness of possible misconceptions were clear by their reference to more essential errors such as whether the integral is defined, continuous functions and existence of limits (• under ‘awareness (misconception)’. Reflections on access to uses of integration appeared robust when they recalled a number of applications of the topic. However, neither of them displayed much evidence to suggest exposure to origins, only making reference to area and textbook introductions (• under ‘uses’ and ○ under ‘origins’ in Table 7b.4).

Interviewer: How were you introduced to the topic?
Danny: Area, I would say. … Later there was a link between differentiation and integration.
Harris: So in our textbook … this is rate of change of area with respect to $x$ is the $y$. That’s what the textbook says.

Table 7b.4 below summarises what Harris and Danny said in the interview.

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<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
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<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td>Harris</td>
<td>Interview</td>
<td>•</td>
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<tr>
<td>Danny</td>
<td>Interview</td>
<td>○</td>
<td>●</td>
</tr>
</tbody>
</table>

Table 7b.4: Summary of Harris and Danny’s response in the interview

7b.2.2 The construction task

In the construction task, both Harris and Danny displayed profound language fluency and facility with technique. Here is their exchange in Task 1.
Interviewer: Given \( \int_{0}^{2} (1-x)dx = 0 \). Can you find another example like this where the answer is 0?

Harris: You can get odd function.

Danny: It's gonna be like that [sketches].

Harris: You are gonna have half of the area above x-axis. Something like this [sketches \( y = 1 + x \) graph].

Danny: It's not gonna be something like that, you got that [shows sketch].

Harris: Yeah.

Danny: Then you could double it. You could put 0 to 4 \( (2 - 2x) \left[ \int_{0}^{4} (2 - 2x)dx \right] \). You are multiplying [later changes limit to 0 to 2].

Harris immediately associated the example with an odd function and both of them made reference to area, saying half below and half above the x-axis (under ‘awareness’). When asked to construct another example, Danny suggested that doubling it could still get the answer zero. However, his referent it was unclear whether he was referring to algebraic manipulation of zero or area. He first doubled the function and the limits as well but later maintained the limits from 0 to 2 which suggest that he was attending to the object as area. He then talked about doubling the amplitude which accounted for his maintaining the limits while Harris talked about stretching it [the function] along the x-axis, as this next excerpt shows (under ‘language’):

Interviewer: Can you tell me what you are doing?

Danny: This here is that kind of graph so you got positive and negative area, so you could double the amplitude so that would still be 0 to 2. Or you could double the limit and change the function, so it would still be twice the height and then change the function.

Harris: You stretch it along the x-axis to an angle [but] you still retain the shape.
Harris’s remark about ‘stretching’ the graph along the x-axis suggests his awareness of what could vary (i.e. function, limits) and what must remain constant (area).

Having expressed his sense of generality, Harris constructed examples such as \( \sin x \) from 0 to \( \pi \) (although the limits were incorrect). Asked to construct another example, he quickly generalised to ‘raise the power so we can get a polynomial that looks like this’ (referring to the example given) (under ‘awareness’). Both of them then worked on symbolic generalisation.

Danny: Will it work for integral of \( (1 - x)^n \), \( n \) will have to be odd.

Harris: Yeah, good generalisation.

Interviewer: What is it about your example that is like this example?

Danny: They have equal amount of area as the main example, part above and part below.

Harris: You’ve got a general [example] \((1 - x)^n\). I think this [referring to \( n \)] is gonna be odd [and] this [referring to \( n + 1 \) after integration] is gonna be even if \( n \) is odd.

They generalised the example and constructed \( \int (1 - x)^n \), \( n = \text{odd} \) (under ‘awareness’).

They also explicitly expressed their awareness of what could change and what must remain constant and the corresponding range-of-permissible-change, thus revealing a heightened sense-of and appreciation of the underlying principle (under ‘language’). Harris displayed awareness of another dimension (variable \( x \)) that could be varied (under ‘awareness’).

Interviewer: What can you change in the example?

Danny: You can change whatever you like.

Interviewer: Does it matter what you change?

Danny: Yes.
Harris: Scalar multiple, you can multiply by 2 and still have zero, ... or you can change the variable.

Danny: You can change the function but then you'd have to change the limit.

Interviewer: Why do you have to change the limit together with the function?

Danny: Because the thing you are considering includes different areas, so [the limit has to change together with the function].

Harris: You can change the $x$, doesn’t have to be $x$. It can be $y$, any variable can be used instead of $x$.

Danny: The fact that you have zero gives you lots of relationships between limits and the function.

In Task 2, Danny immediately recognised the form in which the expression was given and constructed his examples. However, he chose to change a number of dimensions, in which case it was not evident whether he was aware of yet other dimensions that could vary (under ‘awareness’).

Danny: So you want another one where you have the integral and the integral of, so it’s like integrating it twice.

[ Danny constructs $\int xdx = \frac{1}{2} x^2 + C = \frac{d}{dx} \left( \frac{1}{6} x^3 \right) + C$ as simpler example ]

Interviewer: In what ways is your example like this example?

Danny: It expresses the integral as the derivative of another function.

Interviewer: How is it simpler?

Danny: It’s a simpler function to integrate, a standard integral, uses the same variable. I could have used $s ds$.

Interviewer: How about a more complicated one?

Danny: $2 \int \sin x e^{2 \sin x} - (2 \cos^2 x) e^{2 \sin x} \, dx = 2 (\cos x) e^{2 \sin x} = \frac{d}{dx} (2 e^{\sin x})$

[...]

Interviewer: Did you need to have the 2 {coefficient}?
Danny: [It] depends on what you mean by ‘the same’ but they are expressed in the same way as this.

Interviewer: So ‘likeness’ for you means expressed in the same way and have the 2 in front?

Danny: Yes.

Interviewer: Do you think likeness has to have the 2 in front?

Danny: No, but there is a sort of a degree of how similar they are. I’m saying that makes it more similar. If I replace that with $(\log x + \frac{3}{2})$ then that will make it exactly the same.

He demonstrated facility with technique by starting with the differential side of the expression, having varied a number of dimensions to make the integral more complicated (under ‘technique’).

Danny: $2\int \sin x e^{2\sin x} - (2\cos^2 x)e^{2\sin x} \, dx = 2(\cos x)e^{2\sin x} = \frac{d}{dx}(2e^{\sin x})$

Interviewer: In what ways is your example like this example?

Danny: I just put in a function that I know has a slightly messy derivative, I started with that function and differentiated. Well ... I started with this one [differential side], that [integral side] is blobby and I didn’t know how to integrate it.

Interviewer: So you didn’t start with the integral?

Danny: That is how I approach the problem.

Harris also constructed simpler and more complicated examples that reflected rich awareness. He went one step further and varied the method as well (under ‘awareness’ and ‘technique’)

Harris: This one $\int (1 + x)^3 \, dx = \frac{d}{dx} \frac{(1 + x)^4}{4} + C$ I can integrate straight away and this one $\int x e^x \, dx = xe^x - \int e^x \frac{d}{dx} xe^x - 2e^x \, dx$ is created by-parts, which is much more messier.
In Task 3, Harris constructed $\int x e^{-x^2} \, dx$ and $\int \cos^{999} x \sin^{100} x \, dx$ (simpler and more complicated, respectively). His examples revealed his awareness of form and his sense of generality associated with substitution method. He noted that ‘it is more or less having the same form, $e^{-x^2}$ and you’ve got $\sin x^2$ here so … could all be different. You’ve got exactly the same technique of integration’. Danny picked up on Harris’s remark and added:

Danny: I’d say those two are virtually the same. You are replacing the exponential series with this, aren’t you? Because it’s the same argument.

Harris: You can generalise it with any function, $x^2$ … whereas this one

$$\int \cos^{999} x \sin^{100} x \, dx$$

you’ve got a trigonometric function, you are gonna have fun integrating this one. I think eventually you can do it by-parts somehow… no, no, no by inspection, you’ve got a $d$ …

Danny: You’ve got an odd power because … so you could say that $\cos x$ and then express $\cos^{998}$ in terms of $\sin^2$ and then you just substitute you get $\sin$, you get polynomial.

Danny’s examples $\int \frac{1}{u \ln u} \, du$ and $\int \frac{1}{\sqrt{1-x^2}} \, dx$ (simpler and more complicated, respectively) also revealed his awareness of scope and range. He suggested that ‘it’s the same method’ and ‘a great deal of similarities in the function’, noting that it is more complicated ‘because instead of writing $u = \text{a function of } x$, you have to do $x$ as a function of $u$’. Apart from revealing their awareness of form, the above exchange also demonstrates Danny and Harris’s fluency with language and facility with technique (under ‘awareness’, ‘technique’ and ‘language’).
Similar instances of revealed depth of awareness could be gathered from the examples they constructed in Tasks 4 and 5. For the simpler example in Task 4, Danny and Harris constructed \( \int x^2 \sin x \, dx \), and \( \int x^{999} \sin x \, dx \), respectively. Danny remarked that it was ‘just two straight forward simple functions multiplied together, two functions that are simply easily integrated and differentiated’. Harris noted that one could ‘apply reduction formula [...] and apply by-parts a few times. He added that ‘there are variations you can have: \((x+1) \sin (x+1) \, dx\), \(2x \sin 2x \, dx\) ... they all [use] integration by-parts’. He further observed that ‘when you get composite of two functions, you can have \(e^x\) to power, \(e\) to power, ... \([e^x]\), more composite functions’.

Their comments suggest not only facility with the method involved but also awareness of what must remain unchanged and what could vary in the example given

(\(\square\) under ‘awareness’, ‘technique’ and ‘language’).

In Task 5, Danny constructed \( \int (2x + 6)(x^2 + 6x)^3 \, dx \) and \( \left( \sum_{n=0}^{\infty} n x^{n-1} \right) \left( \sum_{n=0}^{\infty} x^n \right) \, dx \) for his simpler and more complicated examples, respectively. His remarks about scalar multiples and ‘turning it into exact derivative of this and the function rather than half its derivative’ and having ‘undetermined number of terms which still [has] got the same idea that make use of differential’ suggest considerable depth of his awareness and facility in technique. He also showed considerable language fluency (\(\square\) under ‘awareness’, ‘technique’ and ‘language’).

Harris constructed similar example for the simpler one \( \int 2x(1 + 2x)^2 \, dx \) and decided to push his limits to construct an example ‘as complicated as possible’. He referred to the simpler example as having ‘two polynomials multiplied together, one that is raised to a power, one that is linear term’. For his complicated example
\[ \left( e^x + 3 \right) \left( e^{2x} + 6e^x \right) e^x \, dx \quad \text{so} \quad \left( y + 3 \right) \left( y^2 + 6y \right) \, dy \], he introduced a new term, suggesting that ‘the basic structure is very much similar but you’ve got \( e^x \) instead of \( x \) here but still using the substitution’. He justified the complicatedness of his example by commenting that, ‘on the first sight, it looks very different and you’ve got an extra term \( e \) as well’. He too demonstrated significant depth in awareness, facility in technique and language fluency (under ‘awareness’, ‘technique’ and ‘language’).

Table 7b.5 below presents a summary of Harris and Danny’s responses in the construction task.

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<tr>
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<th>Behaviour</th>
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<tbody>
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<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
</tr>
<tr>
<td>Harris</td>
<td>Construction task</td>
<td></td>
</tr>
<tr>
<td>Danny</td>
<td>Construction task</td>
<td></td>
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</tbody>
</table>

Table 7b.5: Summary of Harris and Danny’s responses in the construction task

7b.2.3 Making comparison

Table 7b.6 below presents a summary of Harris and Danny’s responses in the interview and the construction tasks. From the table we can make a comparison between aspects of the structure of a topic as delivered by the six-fold framework.

Table 7b.6: Harris and Danny’s responses in the interview and the construction tasks
The comparison of these students’ responses in the interview and the construction tasks strengthens my view that the use of the six-fold framework as an analytic tool affords opportunity not only to reveal aspects previously not talked about in the interview, but also highlights aspects they stress as important. It provides a structured analysis of their activity. The construction tasks seem to provide opportunity for them to reveal a good deal more about their language fluency and facility in technique. The language fluency and facility in technique displayed reveals a good deal of flexibility and connectedness.

Both Danny and Harris displayed rich awareness of integration, evidenced by their associations with the examples given and the range of dimensions they chose to vary. From the analysis above, it also emerges that Harris perhaps appeared slightly more sensitised to notice generality in his awareness than Danny. His facility with technique and fluency with language also suggest that they were informed and guided by his heightened awareness. Although Danny also displayed comparable language fluency and technical facility, the dimensions he chose to vary in the examples he constructed did not give me the impression of quite as robust an understanding and appreciation as Harris.

The clarity of Danny and Harris’s language, the intensity of their voice tones and the speed at which they recognise forms and construct examples gave me the impression that they were confident and motivated. From the fluency with language and facility with technique in explaining the examples they constructed, I am led to interpret a confident demeanour. This appears to be informed and guided by their awareness of associations within the topic, which appears to be very richly connected and deep. The depth of their awareness and fluency with language, coupled with facility with technique, give me the impression that Danny and Harris had a robust understanding
and appreciation of integration. Their responses suggest the dynamics and depth of their concept image and the richness of their example spaces.

**7b.3 Single M6 (Rick)**

Rick was a 19-year old who was very bright but was not very vocal in expressing his thoughts. He was majoring in Mathematics and Statistics and had taken courses in Analysis (Continuity and Differentiability), Linear Algebra, Partial Differential Equations, Probability and Fourier series and two-variable Calculus. Because Rick was interviewed individually, I will present the table that summarises his responses both in the interview and in the construction tasks at the end of the construction task analysis.

**7b.1.3.1 The interview**

Rick’s (marked with circles) responses suggest a rather fragmented awareness when he recalled the meaning of integration as ‘summing up of infinitesimally small elements’ and ‘grouping things together’. The integral sign seemed to invoke the question of which method to use to ‘integrate the function’ (○ under ‘awareness’) (see Table 7b.7). His awareness of common mistakes associated with the topic also pointed to his tendency to look out for limits and to remember to change them accordingly with changing variables and adding the arbitrary constant after substitution (○ under ‘awareness (misconception)’).

He displayed considerable fluency in language pattern in describing words associated with the topic when he mentioned *integration by-parts, by-substitution, trigonometric identity, definite and indefinite integrals and integration and iterations*. He also recalled words that trigger integration such as *area, regions, sums* and *solving differential equations* (● under ‘language’ and ‘technique’).
Some possible uses of integration did come to the fore as he recalled mechanical problems such as area and mass and differential equations. Possible origins of the topic also surfaced as he mentioned area under a curve using basic functions, starting off with no limits then integrating within constants (○ under ‘uses’ and ‘origins’). His access to this aspect appeared rather comprehensive; perhaps he had been exposed to these aspects.

7b.3.2 The construction task

The construction tasks revealed a different range of Rick’s awareness. In Task 1, he first constructed \( \int_0^3 x - x^2 \, dx \) and remarked that fixing those limits and solving the integral would come out as zero.Constructing subsequent examples, he noted:

Rick: I’m still trying to find the limits .... actually no ... I’ll do integral of \( \sin x \, dx \) from \(-\pi\) to \(\pi\) and work that out by visualising it pictorially because you know sine graph that goes ... it’s an odd function. [...] if you integrate an odd function from \(-a\) to \(a\) it comes out as zero.

Asked to construct another example, Rick generated a particular odd function \( \int_a^x x \, dx \), observing ‘\(x\) is the simplest odd function’. Although initially he varied the limits and the function, he quickly displayed awareness of the object and generalised. He was somewhat articulate in expressing his ideas (○ under ‘awareness’ and ‘technique’ and ○ under ‘language’). He remarked:

Rick: I’ve been taken away from that example because to me that looks more difficult that what I’m doing here [referring to his third example] because that has got two numbers to deal with , you’ve got \(1 - x\), so you have to think of whether you are evaluating that with the limits. You’ve got four numbers to think about
whereas what I’m doing with the odd function is a lot simpler, easier to think about.

In Task 2, Rick instantly recognised the form the expression was given, noting ‘that (third expression) is the differential of that there (second expression)’. He first constructed 

\[
\int x dx = \frac{1}{2} x^2 = \frac{d}{dx} \text{ of } \ldots
\]

and then resorted to 

\[
\int dx = x = \frac{1}{2} \frac{d}{dx} x^2
\].

He noted that the example he had constructed was the same as the one given in that ‘the integral can be expressed as a differential of something else’.

However, he struggled with the more complicated example, taking a long time (more than 3 minutes) to come up with one. He tried \(e^x\) but could not find anything that integrates to it. He observed:

Rick: So let’s go back to logs example and I just took a simple case of \(\frac{d}{dx} x \ln x\) which comes out as \(\ln x + 1\) and I know I can find the integral that’ll come out as \(\ln x\). I can’t integrate it until it comes out to 1 so I just struck out the 1 from there \(\left[ \frac{d}{dx} x \ln x = \ln x + A \right]\).

Although he showed flexibility by starting from the differential part of the expression, he appeared to be tied up by the given example and seemed to have lost track of awareness of the form that he was trying to evaluate (○ under ‘awareness’ and ‘technique’).

Rick’s instant recognition of the form was also perceivable in Task 3, where he spontaneously commented, ‘Its integral of two functions, \(x\) is the differential of \(x^2\), so then that’s just going to be u-substitution’ when asked to describe what came to mind (○ under ‘language’).
He constructed $\int x^2 \, dx$ and $\int \frac{1}{x \ln x} \, dx$ for simpler and more complicated examples, respectively. He suggested that making it easier or more difficult to spot the use of substitution method makes the example simpler or harder. He also demonstrated language fluency (under ‘awareness’, ‘technique’ and ‘language’).

Rick: It’s simpler because it’s easy to spot that. [...] [More complicated] It’s similar because it uses substitution and it’s less obvious that you’re dealing with substitution. It’s easier to spot $\ln$ differential, $x$ is the differential of $\ln x$, probably quite standard, higher level. It’s not a well-known thing and it’s sort of looking the fact that they are higher denominator, you start thinking while $x$ can’t be the differential of $\ln x$ and the custom of writing is $\frac{1}{x}$. That makes it interesting to look at the whole denominator and 1 and to make more difficult once you do make the substitution, its $\frac{1}{u}$ and $\frac{1}{u}$ integrates to $\ln u$ but to look out for that is not as obvious. [...] I think it is harder to spot that as substitution; you’ve got $x \ln x$ in the bottom, it says to some people you do it by-parts because you’ve also got product of two functions.

For simpler and more complicated examples in Task 4, Rick constructed $\int \ln x \, dx$ and $\int \sin xe^x \, dx$, respectively. He pointed out the fact that the simpler example had only one function as opposed to the two functions required for by-parts integration, which was not very obvious. The more complicated example was thought to be so because it involved integration by-parts twice (under ‘awareness’, ‘technique’ and ‘language’).

Because of time constraints, Rick did not attempt Task 5.

The construction task not only drew out Rick’s facility with technique and the robustness of his awareness, it also highlighted his fluency of language. His remarks
revealed a good deal more of these aspects of his understanding compared to what was revealed in the interview. His careful disposition and the intensity of his voice tone suggest willingness to engage in and commitment to the tasks and to detecting and experiencing generality.

The table below summarises Rick’s responses in the interview and the construction tasks.

<table>
<thead>
<tr>
<th>Single M6 Rick</th>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
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<tr>
<td></td>
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<tr>
<td>Interview</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Construction task</td>
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</table>

Table 7b.7: Rick’s responses in the interview and construction tasks

7b.3.3 Making comparison

Rick’s responses in the interview suggest relative fluency in language with an inclination to stress technique. He displayed considerable awareness of connections and misconceptions. His awareness of common errors and access to uses of integration appeared rather comprehensive, so did access to origins of integration.

His responses and the examples he constructed in the construction tasks, however, suggest richer connections and considerable depths in the awareness aspect of his concept image. The examples he constructed suggest the dynamics of his example space and rich understanding. He also displayed strong facility with technique. However, language fluency let him down. However, he did not come across as being very fluent and articulate in expressing his thoughts.
7b.4 Summary

Analysis of students in this group highlights the fact that the six-fold framework as an analytic tool can be used to reveal a good deal about aspects that students are accustomed to stressing and indicating those that they may be in the habit of ignoring.

Students in this group also demonstrated that awareness of connections within the topic was revealed through having language fluency coupled with facility in technique. Technical language was used to express and aid their thinking.
Chapter 8

Main Study: Data Analysis

Interviews with engineering students

8.0 Introduction

Participants in this group were three first-year engineering students at a local university who volunteered to partake in this study. They were all males. Two of them formed a pair and the other was interviewed alone. I chose to interview engineering students to contrast their responses as people who are concerned largely with using mathematics as a tool for engineering purposes. As in the later interviews in Chapter 7b, I have summarised where appropriate what the participants said in the interview and not provided detailed transcript because the space could be better used for analysis of the construction tasks.

8.1 Pair E1 (Robert and Alex)

Alex was a reserved person and was not very articulate in his responses. His partner Robert, on the other hand, was very conversational and articulate. Both of them got an A in A-level Math, an A in Physics and a C in Further Math. As part of their undergraduate courses they took Technological Science 1, Design for Functions, Economy and Structural Industry and an optional Engineering or other broadening subjects such as Philosophy or Politics. Mathematics was part of the Technological Science 1 subject.
8.1.1 The Interview

From the interview, Alex (marked with squares) mentioned meaning of integration as area under a curve and recalled ‘a major thing about it ... important foundation’. Amusingly, he suggested that he was reminded that the integral sign ‘looked really like a snake’ and nothing else (□ under ‘awareness’). Meaning of integration as ‘everything comes together’ and connection to different problems dominated Robert’s (marked with circles) awareness. Accompanied by a sinister laugh, he was reminded ‘to integrate something ... integrate between two limits with respect to something’ when he saw the integral sign (○ under ‘awareness’).

Interviewer: What does the word “integration” mean to you?

Robert: Generally it means everything’s together.

Alex: With math, integration means generally working out the area under a curve.

Robert: Basically I think integration is a method you use to approach different, different problems; not just one particular thing, the connecting part will open different problems.

Interviewer: What comes to mind when you see the sign $\int$?

Robert: To integrate something [laughs] ... integrate between two limits with respect to something.

Interviewer: Do you have any conversation or images in your mind?

Alex: The second I saw you drawing it, that looked really like a snake [laughs] ... other than that ... no.

Interviewer: What does it remind you of?

Alex: Just reminds me of math generally. It’s got a major thing about it ... important foundation.

He also displayed good technical memory in watching out for technique-related mistakes such as deciding on which was $u$ and which was $dv$ for integration by-parts and remembering the double angle formula $\frac{1}{2}\sin\theta$. Alex also pointed out having to remember
which rule to use as an example of his awareness of common mistakes (☐☐ under ‘awareness (misconception)’) (see Table 8.1).

Robert displayed a tendency to use surface characteristics to locate a technique when he stressed trial-and-error and knowing spontaneously ‘what to do’ when he looked at a problem. Alex, picking up on this point, exhibited a particular emotional predisposition with his remarks about mathematics generally as ‘it’s all practice, practice, practice’.

Interviewer: What sorts of things have you discovered you need to watch out for when you are doing integration?

Robert: Basically about ... when you are using the product ... which to use as \( u \) and \( dv \) by \( dn \) ... apart from that ... and about \( \sin \theta \), double angle formula ... about that coz I used to remember and I confuse coz it’s \( \frac{1}{2} \) \( \sin \) and you might divide by 2 ... apart from that ... for me integration is easy and basically that’s the troubles I’ve done.

Alex: ... at the beginning I would look at something and integration for me was tough coz I had no idea what that has to deal with. You have to say ‘Does it fit with this rule? Does it fit with that rule?’ ... in the question you don’t know what to do but you do like one step and decide if you carry it out and you know the answer so ... basically working to get the answer.

Robert: Basically if you look at the problem, there are certain methods you have to follow ... not really but like ... once you work it out, at one point you know whether you are going around or ...

Interviewer: How ... for example?

Robert: Hmm... if it is like a numerical value like ... finding the volume of something and which is about the area and ... you get negative answer. Obviously you know that you have done something wrong ...

Robert: It’s like seeing the problem and like ... picking out the interesting ... for example if we say we have a question \( \frac{x - 3x}{5x^2} \) or something. Basically looking at the denominator ... there is a higher power; we can either obtain partial fraction like a
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quotient. ... We have [seen] a lot of problems; you just look at the problem [and] you
know what to do ... like a proper practice.

Alex: Yes, it’s all practice, practice ...

Words that triggered integration for Robert included volume, area and the reduction
formula. Alex too recalled special words such as product rule, integration by-parts,
inverse of differentiation and differential equations. To these, Robert added integration
factor and $e$ to the power $i\left[e^i\right]$.

Interviewer: What words in a problem/context that tell you that integration is relevant?

Robert: With respect to $x$ ... if they say like volume or area, basically you know you have to
integrate. In differentiation it will be the rate of change of something and you know
you have to differentiate but in integration, for me it will be volume ... and the
reduction formula basically means you have to integrate.

Interviewer: So what are some of the special words/language that you use when you are talking
about integration?

Alex: Technical terms like product rule, integration by-parts, differentiation ... I suppose,
the inverse ... emm ... Differential equation and ...

Robert: Integration factor?

Alex: Yeah.

Robert: $e$ to the $i$ or something $[e^i]$.  

The language fluency appeared to be limited to words and phrases associated with
different techniques. Little reference was made to words related to functions and other
associations (□ under ‘language’ and ‘technique’).

Their remarks about being introduced to integration as the reverse of differentiation
suggested limited access to origins of the topic (□ under ‘origins’). Their examples
of uses of integration included: area problems, moment of inertia, differential equations
and electric filters to look for frequencies and other systems such as vibration. They
seemed to be aware of a range of applications as reasons for being able to integrate (under ‘uses’).

Interviewer: What kinds of problem does integration help to solve?

Robert: Area problems.

Alex: Yeah ... and moment of inertia and differential equations, things like electric filters to look for certain frequency and they use it to calculate what frequency they are ... other systems as well ... vibration.

Their reflections of aspects of the topic that would influence them were evidenced only in their focus on certain techniques of integration. The intensity in voice tones and their relaxed attitude suggested little commitment and involvement in learning integration with a focus on technique.

Table 8.1 below summarises Robert and Alex’s responses in the interview.

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Table 8.1: Summary of Robert and Alex’s response in the interview

8.1.2 The Construction Tasks

The construction tasks revealed a number of important things to suggest something of a balance in emotion, behaviour and awareness for Robert and Alex. In Task 1, Robert seemed confused as he tried to understand the question and appeared to be expecting to find ‘the value of x’. The examples he constructed seemed to confirm this. He first constructed \( \int_0^2 (x - 1) \, dx \), observing that ‘if you change the limits, that thing has a minus
sign ... you multiply by minus ... basically the same thing’. When asked to construct another example, he noted:

Robert: Basically I ... like ... worked it out and then work it backwards ... thinking along that ... I don’t know how to actually ... [inaudible]. Basically I have some values ... what I’m trying to do is like ... find the expression and work it backwards.

Asked to construct another example, he suggested that whole numbers wouldn’t work and it [the answer] had to be a fraction.

Robert: If I get $2x + 1$, it will be just $x^2 + x$ but I don’t think any limit will have that become zero.

Robert appeared to have useful heuristics (to work out backwards) but algebra seemed to let him down. He agreed to suggestions by the interviewer that he decided on a function first and then determined the limits to see if it worked out to zero. Judging by the dimensions he chose to vary in his examples, it seems to me that there was no evidence to suggest graphical thinking. Although he displayed verbal fluency and a sense of the task, he seemed to be caught up in the act of ‘solving problems’ (1) under ‘awareness’ and ‘technique’) (see Table 8.2).

Alex displayed deep awareness of a range of possible functions when he constructed trigonometric functions $\int_{0}^{\pi} \sin x \, dx$ and $\int_{0}^{\pi} \cos x \, dx$. He displayed graphical thinking and showed evidence of having made connection of integration to area (□ under ‘awareness’).

Alex: I just remembered being taught ... emm ... interesting is finding area underneath the graph for sin. It spends as much time underneath the x-axis as well as ... it’s effectively then ... the total area is naught if you’re counting the area above the x-axis as positive and area below x-axis as negative.
However, he did not construct any other examples, which suggests that his sense of possible dimensions that could vary is not very extensive (□ under ‘technique’). The trigonometric examples appeared isolated in his example space. Or else, his manipulation skills were so far in the past that he could not countenance anything more complicated.

Task 2 revealed a good deal more of both Alex and Robert’s awareness of relationship. After commenting on the example as ‘just normal integration’ and seeking clarification from the interviewer about the integral of \( \ln x \), Robert wondered if the function had to be retained. Robert and Alex constructed simple examples,

\[
\int \cosh x \, dx = \frac{e^t + e^{-t}}{2} = \frac{e^t - e^{-t}}{2} + C = \sinh x = \frac{d}{dx} \cosh x \quad \text{and}
\]

\[
\int (1 + x) \, dx = x + \frac{1}{2} x^2 = \frac{d}{dx} \left( \frac{1}{2} x^2 + \frac{1}{6} x^3 \right),
\]

respectively. Robert, however, missed the \( dx \) in rewriting the integral \( \cosh x \, dx \) as integral of \( \frac{e^t + e^{-t}}{2} \). Alex expressed appreciation for Robert’s example, observing that ‘it is a very simple one ... makes sense’.

Robert responded to Alex’s remark and suggested that it was ‘the answer in the middle’ that one should be paying attention to in this task. This remark appears to reinforce his awareness of connections and the relationship between integration and differentiation (□ under ‘awareness’ and ‘technique’).

Robert: Basically it’s the answer, you know the starting point and you know the end point, so it’s the answer in the middle ... so, that won’t be a problem but I think in your case you come up with algebraic equation which ... you have to work this part out ... final answer. I know this part and this part, I just have to find value [in the middle]. I think it might be simpler in that [sense].
Alex constructed \[ \int x \sin x \, dx = \sin x - x \cos x = \frac{d}{dx}(-\cos x - x \sin x + \cos x) \] for the more complicated example, but later realised that he made a mistake (in the sign of the final \( \cos x \)). He was quick to realise his own slip.

Alex: I am being stupid ... that makes it more complicated than I thought. Well ... I thought I've got something more complicated. I've got \( x \sin x \) ... emm ... integrated that ... and ...

Alex suggested that using trigonometric functions and adding an \( x \) in front, which necessitated integration by-parts, made the example more complicated. However, he seemed to struggle with the \( \frac{d}{dx} \) form of the expression, commenting that he 'can’t think of how to pull it ... that form for the last one’. Although he seemed to be aware of the form and connection, the dimensions he chose to vary did not reveal a very rich connection but revealed his awareness of the particular (under 'awareness' and 'technique').

Robert also realised his mistake with the complicated example. He constructed \[ \int \sec x \tan x \, dx = \sec x = \frac{d}{dx} \left[ \int \sec x \right] \] and later commented:

Robert: I think that is wrong. I was trying to derive the integral of \( x^2 \tan x \) is \( x^2 \) but integrating \( x^2 \) so I end up with \( \tan x \) ... I ended up integrating \( \tan x \) ... [inaudible]

[...]

I don’t know the details of calculating \( \tan x \) in differentiated form. So basically ... I thought plugging in integral of \( \sec x \tan x \), which will be \( \sec x \) and \( \sec x \) is \( \frac{1}{\cos x} \), which is the integral of \( \cosec x \). So ... basically I’ll find out through [integration by-] parts.

Both of them displayed awareness of connections between integration and differentiation but stumbled in attending to details of the examples.
When invited to describe ways in which the examples he had constructed were like the interviewer’s examples, Alex suggested that not having to do any ‘fancy’ methods of integration made the example simpler and having a sin function and an $x$ made it more complicated. Further probes to describe the dimensions he had varied in his example revealed that he was caught up with the given example and ‘simplified’ it by retaining the form $\int(a + b)dx$.

Interviewer: In what ways are the examples you’ve constructed like my example?
Alex: Emm ... not sure, really ... apart from ...
Interviewer: What is it in your example that is like my example?
Alex: For simple one, it has $2 \ln x + a constant \ldots$ emm ... you use $x$ instead of using [any] variable, not including constant in front of it ... for fractions in the box ...
Interviewer: So you have decided to eliminate all these multiples, fractions in your simple one?
Alex: Yes.
Interviewer: Okay, how is it like my example?
Alex: Err ... apart from the fact that it uses $x$ ... I’m not sure.
Interviewer: Just the fact that it uses $x$?
Alex: Hmm.
Interviewer: Okay. How about the form of representing it? Is it like my example?
Alex: Yes, it just has to.
Interviewer: How about the more complicated example?
Alex: Same form again ...[pause]
Interviewer: Just the form?
Alex: I think ... yes. Emm ... they both got integration by-parts in it ... your example and my complicated one.
Interviewer: Have you used that fact in the construction of this example?
Alex: No, I only just noticed it.

Although he varied a number of dimensions, Alex did not come across to me as articulate enough to express what he had done. He only responded when probed, which suggests
that he might know about the properties but was not explicitly aware of them (□ under ‘language’).

Robert pointed out that he saw the derivative sign and the integral sign, and trigonometric functions came to his mind, because using the trigonometric functions, he knew what to integrate to end up with (the answer) and what he then needed was ‘the middle part, which won’t be that hard’. He suggested that he was ‘sticking to principles’ in choosing $\sec x \tan x$. He admitted that he could have chosen other examples like $x^r$ or $2 \sec x \log_{10} 5x$ but he preferred to use ‘direct, basic trigonometric functions’. However, he acknowledged that his example was incorrect, thinking that integral of $\sec x$ would be $\ln x$.

Robert: Well ... maybe it’s getting used in the final answer and the first answer because even though it’s trig function, I’ve used direct, basic trig function, like $2 \cos x$. It’s just $\cos x$ and in like $\sec x \tan x$ it’s just one integral ... it’s just $\sec x$. So, I’m sticking to basic principles rather than going $x$ squared $x^r$, $2 \sec x \log_{10} 5x$ ... I’m just basically sticking to principles.

[...] Emm... I didn’t think it was complicated but the fact that I’m ending up with the $\ln$ function, I thought the integral of $\sec x$ will be $\ln$, which I don’t know. It’s in the formula booklet, I guess ... so I find the $\ln$ function a bit complicated. Apart from that ...

In my view, Robert was more articulate than Alex in expressing the dimension he had varied in his examples (□ under ‘language’), although his examples did not suggest richer connections and awareness of dimensions-of-possible-variation.

Interviewer: What is it about this example that is like my example?

Robert: Emm... I’m integrating with respect to $x$ in both situations and then it’s basically derivative of another function which is not the same function as this. That is twice the integral of, integrate and integrate twice. But basically the end function is different from the first function and I ended up with the middle part which is totally different from what is given.
Interviewer: How is it different?
Robert: Because I didn’t do a plus sign, it’s just multiplication.
Interviewer: Anything else?
Robert: Apart from that ... the ln is carried throughout the whole thing. In my thing, I guess I end up with different, different thing.

In Task 3, I invited Robert and Alex to describe what came to mind when looking at the integral. The integral seemed to trigger a method of integration for both Robert and Alex. Upon seeing 1, Robert engaged in rapid fire of thoughts when he was reminded of the formula \( \sin^2 x + \cos^2 x = 1 \). Alex referred to the use of the chain rule. Alex consented to Robert’s conjecture on the use of the by-parts methods. However, Robert later corrected himself and suggested integration by substitution (\( \text{under ‘language’} \)).

Robert: It’s just integration by-parts.
Alex: \( x \) equals \( u \).
Robert: Coz basically ... anyway we can’t use \( \sin \) squared because if you integrate that differentiate this, you end up with another \( \sin \) function. So differentiate \( x \) and you are left with 1 so you are integrating one function. Anyway, this is substitution so that is not needed, I guess.

There was no indication of other associations triggered by the integral. Techniques appeared to dominate their attention. When prompted by the interviewer that the integral was usually solved using substitution, Alex commented, ‘that will make a lot more sense’. For the simpler example, Robert constructed \( \int \tan x \, dx \), substituting \( \cos x \) with \( u \) and getting \( \ln \sec x \) as an answer. In the more complicated example, \( [\sin^5 x \cos x \, dx] \), \( u \) was taken as \( \sin x \). Later Robert suggested that the former was the more complicated example and the latter, a simpler example because direct substitution was involved. He was very articulate in describing what was the same and what was different between his examples and my example (\( \text{under ‘awareness’, ‘technique’ and ‘language’} \)).
Interviewer: What is it about the examples that are like my example?

Robert: Basically it is a squared function, not exactly a squared function but ... [it] is to a power and it's two products whereas this is divided by something and then, one single value worked out to get the final answer.

Interviewer: In what ways are they different?

Robert: All my powers are ...the value of \( x \) is single and got it squared and it is completely trig functions, I didn't have another \( x \) coming in anywhere.

Interviewer: How about that one, the more complicated one?

Robert: That's just one value, I guess. There's nothing in common, just that it's a trig function and there's no product or anything, just tan \( x \). Rather than having two terms, I'm having just one term.

The use of lots of 'this', 'that' and 'it' (indefinite pronouns) indicates possible confusion because the referent shifts in mid utterances.

Alex constructed \( f_1 \) as his simpler example, substituting \( x+2 \) as \( u \). However, he did not remember what it would integrate to. He had to be helped by Robert, who suggested it was arc (tan \( x \)). He did not construct a more complicated example (under 'awareness' and 'technique').

In Task 4, Robert constructed \( \int x^2 \sin x \, dx \), suggesting that it was solved using integration by-parts twice, using \( x^2 \) as \( u \) and \( \sin x \) as \( dv/dx \). He pointed out the fact that it was two products and involved trigonometric functions and used by-parts method (twice) constituted the likeness of his example to the given example. The difference, according to him, was \( x \) squared and used the integration by-parts method twice. Alex did not construct any examples (under 'awareness' and 'technique').

Due to time constraints, they did not attempt Task 5.
8.1.3 Making comparison

Both Robert and Alex’s responses in the interview suggest that they had considerable fluency in language when identifying words and special language associated with integration. In the behaviour aspect, Robert displayed a tendency to stress techniques in his approach. Whilst this suggests facility with technique, his remarks also suggested that his attention was dominated by technique, overshadowing other awareness. Both Robert and Alex’s actual fluency with technique was rather limited. Robert did appear to be associative in his thinking when he expressed links of the topic to its other connections (images and associations). Alex, on the other hand, did not give me the same impression. He appeared rather superficial in his associations and awareness of integration. Both displayed only superficial awareness of common obstacles they talked about technical aspects to watch out for when dealing with problems in this topic. The topic was thought to have a wide range of applicability as suggested by the contexts in which they illustrated its use. However, their access to origin of the topic appeared limited. These responses highlighted aspects in the six-fold framework that were stressed (technique) and those that were ignored (connections, scope of uses and origins).

In conjunction with their responses in the interview, the construction tasks revealed differences with aspects that emerged in the interview. Both Robert and Alex displayed considerable facility and flexibility with techniques of integration, judging by the type of examples they constructed and the speed with which the examples were produced and worked out. However, they did not display enough evidence to suggest enriched connections. In a number of instances, the examples constructed duplicated the format of the given examples. I got the impression they were limited by limited facility in technique. What strikes me the most is their apparent inability to make explicit remarks about aspects of integration or the tasks that might suggest fluency and articulacy. Thus, I
was under the impression that they had restricted language fluency, and this was also evident from Alex's hesitancy and Robert's flowing but inarticulate utterances. They came across as highly able individuals with limited facility in techniques of integration and with limited connections and language ease.

Alex was quick to assess and comment on his partner's examples and his own. Responses in the construction tasks also showed significant depths in awareness of their concept image. However, language with which to express understanding is also an important factor, as well as knowledge of techniques. In these aspects, their concept image showed considerable limitations.

The examples they constructed and the dimensions they varied in their examples informed about the nature of their example spaces. Example spaces accessed in this instance could reflect more complex ones held by them unconsciously. It appears that both Robert and Alex's example spaces were well-structured and connected. They saw each particular example as an instance of a general case and the dimensions varied suggest the richness of their example spaces.

Table 8.2 below presents a summary of Pair E1's responses in the interview and the construction tasks. Extracts from the interview that struck me and notes relating them follow. A regularity emerges through use of the six-fold framework discussed in Chapter 2 Section 2.4 in revealing particular aspects of learners' behaviour, awareness and emotion concerned with integration.

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Table 8.2: Robert and Alex's responses in the interview and the construction tasks
8.2 Single E2 (Sam)

Sam was a very cheerful and outspoken person. He obtained an A in A-level Math, an A in Physics, a C in Further Math and a C in Spanish. As part of his module, he took Technological Science 1, Design for Functions, Economy and Structural Industry and Multimedia Technology as his option. Mathematics was part of the Technological Science 1 subject. He was the last student in this group and was interviewed individually.

8.2.1 The Interview

For Sam (marked with circles), integration triggered an association with finding area under graphs. When further probed, he claimed that he related it to differentiation first. He seemed uncertain about his conviction when he said, ‘I think’, backtracking and suggesting that he might be wrong. He commented on the appearance of the integration sign as an ‘S’ but made no reference to what it stood for (unexpressed link to sum). He asserted that he was so used to seeing it for integration and was reminded of nothing else but to integrate (method). The integral sign seemed to trigger technique of integration and an association to area ( unawareness and technique).

When asked about things that he had discovered he needed to watch out for, Sam talked about forgetting to change the limits when doing substitution, to change the sign when dealing with trigonometric functions and sometimes forgetting the arbitrary constant in indefinite integrals ( unawareness (misconception)). Watching out for technical aspects such as these suggests dominance of his attention by technique.

Invited to state what was the same and what was different between \( \int x^2 \, dx \) and \( \int_0^2 x^2 \, dx \), Sam initially commented on definite and indefinite (integrals), having an arbitrary
constant and yielding function and number as answer, respectively. When probed regarding associations of the objects, Sam remarked:

Sam: I’d associate that [definite] with finding area of something and that [indefinite] I’d associate with question to see if you can do integration. I’ve never seen practical use for it except sometimes you can substitute some values afterwards you can work out your constant […] I don’t see many practical uses of that relative to definite integral which is far more useful and neater.

He saw the indefinite integral as a test item and not as a mathematical object. He also displayed emotional disposition when he regarded indefinite integrals as having no practical uses and did not make any remarks about the relationship between the two (under ‘awareness’).

Words that triggered integration for Sam included area, mechanics questions, distance and centre of gravity. He also mentioned limits, with respect to, coefficient, the integral sign $\int$ and limit of sums as examples of the special language he used when talking about integration (under ‘language’). He appears to be quite fluent in the language patterns associated with integration, although his voice tones seemed to somewhat halting.

When asked about how the topic was introduced, Sam recalled doing differentiation and being shown that integration was the reverse process. He displayed surface association of integration to differentiation (under ‘origins’). He also recalled a wide range of uses such as centre of gravity, displacement, acceleration functions and limits of sums and claimed that he ‘would know when [he] saw it’ (under ‘uses’). Sam displayed seeds of emotional engagement with these responses. Also having an awareness of origins and/or potential uses promises a meaningful and hence, motivational context.
8.2.2 The Construction Tasks

The construction tasks exposed, to a greater extent, aspects that were not revealed in the interview. In Task 1, Sam expressed awareness that limits could change when he asked if it had to be the same limits. He displayed awareness that if the limits were kept constant, the function could be varied. He observed that ‘if the limit is the same then any function is going to be zero’. He constructed \( \int_{-a}^{a} \sin x \, dx \) and \( \int_{0}^{2} (4 - 4x) \, dx \) one after another. Asked to construct another example, he seemed compelled to ‘try and think of a good one’ and constructed \( \int_{0}^{2} (x^2 - \frac{1}{x}) \, dx \).

Sam: I was going to do that with a different number or may be do a cos or a tan. Tan is the same, isn’t it? You can substitute the sine with tan and then they’re a bit the same.

He then changed it to \( \int_{0}^{2} \frac{x}{x} \, dx \). He tried to construct an example that maintained the same limits but with a different function by manipulating it. After trying with \( \int_{0}^{2} 2x - \frac{1}{x} \, dx \), he suggested that the example did not work. In the end, he had to be reminded that the integral of \( \frac{1}{x} \) is \( \ln x \), which he then recognised.

Sam’s attention seemed to fluctuate between algebraic and graphical representation of the integral. The dimensions he varied and the range-of-permissible-change he displayed suggest a richness of connections between technique and associations (under ‘awareness’ and ‘technique’). He also appeared very articulate in expressing the dimensions he varied but displayed rather limited facility (under ‘language’).
Sam: That one \[ \int_{0}^{2} (4 - 4x) \, dx \] I just took the constant out because any constant you put in front there is not going to change it. That one \[ \int_{0}^{\pi} \sin x \, dx \] I thought of the graph and worked it out. And that one \[ \int_{0}^{2} x - \frac{1}{x} \, dx \] I went all over my head and tried to work it out but it didn’t make the job.

Interviewer: In what ways are your examples like my example and in what ways are they different?

Sam: That one is different because it has got a trigonometric function there and because the limits aren’t set \[ \int_{-\pi}^{\pi} \sin x \, dx \]. That one is the same because it has the same limit that is different, different coefficient \[ \int_{0}^{2} (4 - 4x) \, dx \] and that one \[ \int_{0}^{2} 2x - \frac{1}{x} \, dx \] is different because it doesn’t work [laughs].

Invited to generalise the situation, he constructed \[ \int_{0}^{2} (a - ax) \, dx = 0, \ a \int_{0}^{2} (1 - x) \, dx \] representing the coefficient \( a \) for the scalar multiplication. He appeared to be ‘dominated’ by technique manipulation (behaviour to be at the foreground of awareness) and suppressed useful connections that could help to get a sense of it. The fact that no reference was made to area substantiates this claim, apart from the single trigonometric example (\( \square \) under ‘awareness’).

In Task 1c, Sam marked the similarities in the functions and the differences in limits in the second and third example. However, he did not display awareness of a relationship between the limits and the function in connection with the integral being zero. The fact that area was mentioned and graphs only mentioned once suggests that there seemed to be virtually no interconnectedness between his verbal associations of integration as area.
in earlier questions and his awareness of it as displayed in this task (○ under 'awareness'). He also displayed awareness of forms and effects of transformation on integrals.

Interviewer: What is the same and what is different in the three integrals?

Sam: I haven’t worked out that one yet \[
\int_1^0 \frac{(1-x)}{2} \, dx = 0 \]
but those two \[
\int_{-1}^{0} (1 + x) \, dx \quad \text{and} \quad \int_{0}^{2} (2 - 2x) \, dx \]
are the same. That one [former] is minus and that one [latter] is plus but the limits swapped around as well. That one is negative and it is going to make the same equation. [...] In the third example the constant doesn’t matter again, it’s between bigger limits but it still equals nothing. The limits are different; [the constant] doesn’t matter.

Interviewer: How are they the same apart from being equal to zero?

Sam: It’s the same as that one but with different limits actually. That is the same as that one [refers to second and third examples] times by a constant of 4 which you can take out. It disappears. If you times that one by 4, you get that one. So if you do times it by 4, it doesn’t change that, so you can do that.

Interviewer: How do you decide on the limits?

Sam: I don’t know. I can’t see a pattern.

Task 2 saw Sam first clarifying whether the integration constants were being ignored. He decided on the ‘answer’ to be \( x \) and constructed \( \int dx = x = \frac{d}{dx} \frac{x^2}{2} \), working out the integral and the differential forms. For a more complicated example, he decided to start with the differential form and used \( \sinh x \) and \( \cosh x \)

\[
\frac{d}{dx} \left( x^2 \sinh x + 4x \right) = x^2 \cosh x + 2x \sinh x + 4 = \int x^2 \sinh x + 2x \cosh x + 2x \cosh x + 2 \sinh x \]
\[
= \int 4x \cosh x + (x^2 + 2) \sinh x
\]
Interviewer: How did you choose that one to be your simpler one?

Sam: I just chose $x$ to be in the middle, differentiated it and integrated it. With the more complicated one I started with that $\frac{d}{dx}$ form so I can just differentiate it down.

Interviewer: Why did you start with that?

Sam: Because I just have to differentiate, I didn’t have to integrate. If I start with that one $\frac{d}{dx}$ form I can differentiate to get that and differentiate to get that $\int$ and differentiation is much more fun that integrating [laughs].

[...]

Interviewer: Why did you choose that for the more complicated example?

Sam: Because I made something of similar form to that, a kind of same idea and because it was natural log in the way it worked here. It cancelled a few things out like the $x$’s all cancelled out. But I didn’t think this one would and it didn’t and it did get more complicated. So it wasn’t more complicated than the example. It’s a bit harder.

His remarks and the examples he constructed suggest awareness of form and show facility in technique (under ‘awareness’ and ‘technique’). However, his understanding of form appears to go beyond the relationship between integration and differentiation. He appeared to consider the form in which the integral was written as well $\int (\ln a + b)\,dx$. In the simpler example, he considered the ‘general’ form and in the more complicated example, he appeared to be caught up in complying with the format of the given example as well. He might be using ‘example’ to refer to the form rather than what was to be done to the form.

Interviewer: In what ways are the examples you’ve constructed like my example?

Sam: That one (simpler example) isn’t. I just made it up.

Interviewer: It is not like this at all?

Sam: No, except it is written in the same form but that would be the idea.

Interviewer: Yeah.

Sam: It has nothing to do with that.
Interviewer: You used the $x$, I used the $x$.

Sam: Yeah but your $x$ came by accident, I chose the $x$. This one I modelled on that one $[d/dx \text{form}]$, took the similar form then I added something on by own invention.

In expressing his awareness, he came across as being fairly articulate (óż under ‘language’). There may be a possibility that Sam had noticed and ‘internalised’ the dimensions he had kept constant but the fact that he was not very explicit or specific about what he had done suggests a limited awareness of multiple dimensions or the role they played in the integration.

Interviewer: In what ways are the examples you’ve constructed like my example?

Sam: It’s the same in that it’s $x^2$ multiplied by ... product of $x^2$ and a function of some kind. I just chose a different function at random.

Interviewer: sinh?

Sam: Yeah and then I added random amount of $x$’s.

Interviewer: How is it the same then?

Sam: Because I chose this product, I just picked a different function.

In Task 3, Sam displayed attention to technique and expressed awareness of associations (imagery) when probed.

Sam: If I was gonna be solving it, I would be thinking I’m gonna have to use ... substitution.

Interviewer: You are deciding on the method to do that?

Sam: It’s a function multiplied by a derivative, isn’t it? I think. It’s just $x^2$ as $u$ and then $d/dx$ ...

Interviewer: How are you seeing it?

Sam: I’m looking for patterns like the $x$’s of derivatives and squared to decide what method to use.

Interviewer: Do you have any conversation or images? What comes to your mind?

Sam: I didn’t imagine it like picture what the graph would look like. I didn’t do anything like that. Just looked at it and saw what I recognized and tried to get it into a form that I can deal with.
He displayed a disposition to engage with form. When asked to construct simpler and more complicated examples that used substitution, Sam remarked that ‘any integral can be done by substitution’ and suggested that even a really simple one like $x^3$ could be done by setting it to $x$ times $x^2$, although he hinted that one would not normally do that. He then constructed $\int (\sin x \cos x) \, dx$ for a simpler example, suggesting that the obvious $\text{function times derivative}$ structure he had chosen. For the more complicated example, he first constructed

$$f(\sin 5x - \cos 3x) \, dx = f(\sin 5x - \cos x - \sin 2x) \, dx = \sin 7x \, dx - f\cos x \, dx$$

and then realised that it was incorrect. Algebra seems to let him down (● under ‘awareness’ and ○ under ‘technique’).

Sam: $u$ would be $\sin x$ again. You have to separate the cos into cos times cos squared. No, that can’t be right. I don’t do the substitution yet, I’d rearrange it first because cos squared times cos cubed is the same as cos times $(1 - \sin^2 x)$ minus that so you get $1 - u^2 + \sin^2 x - \cos x$ equals $\sin 7x$. Can you substitute for that? [Pause] That doesn’t work, does it because it is minus; we need a product of the functions [Long pause]. I don’t think that’s going to work. That’s not true anyway, $\sin 5x + \sin 3x$ doesn’t mean equal $\sin 7x$.

After spending a long time trying to work out the example, upon request by the interviewer, he constructed $3 \int x^2 \ln x$ and remarked that ‘if we bring the 3 inside, make it $x^3$ and then it is like that one (given example), $u$ is $x^3 \left[ \int x^2 \ln(x^3) \right]$. He suggested that although it was simpler $x$’s, one would not have seen it quite as fast because ‘once it’s ($x$’s) inside the function, it makes it seem more complicated than it is’. He appeared to have decided to make the method (substitution) less obvious, although he did not make explicit reference to this. Again, he appeared to be caught up in the act of using the same format $\int x f(x^3) \, dx$ as in the given example.
Sam: That one is the same, I just took that (the example given), changed and I took the 3 outside to look as if different but I made it up by myself. [Pause] That one is more like your example.

He displayed considerable fluency in language, although he was not very articulate in being specific about the dimensions he had varied and those he had kept constant (under ‘language’).

Interviewer: What is it about this example that is like my example?

Sam: (refers to \( \int (\sin^3 x - \cos^3 x) \, dx \)) [It] uses trig functions, aside from that ... it uses powers of functions, not powers of \( x \).

Interviewer: I have a product here and you don’t.

Sam: No.

Interviewer: What is it about this example that is like my example (the simpler one)?

Sam: This one \( \int \sin x \cos x \, dx \) is a product; they are both function times derivative so they are quite easy to solve.

Interviewer: How are they different?

Sam: Different functions, slightly raised powers but it makes no difference really. That could be \( x^{30} \) or something as long as that one is 1 less. I just used a different function, picked one at random.

Task 4 invited subjects to construct a simpler and a more complicated example of an integral that was usually solved using by-parts method. Sam constructed \( \int \ln x \, dx \) for a simpler example because ‘one of your parts is just 1’ and ‘1 is easy to manipulate’. He constructed \( \int \sin^2 x \, dx \) for a more complicated example, observing that ‘if you run it through, you get the integral of \( \cos^2 x \) and you do that one by-parts again and then you get \( \sin^2 x \) and then you equate them to each other and take them across and you get twice .... because you have to do it by-parts. It’s more complicated because you have to do it twice’. He showed facility with use of integration by-parts (under ‘technique’).
Sam asserted that he recalled the integrals from past experience and did not base them on the given example to construct simpler and more complicated examples. He did display awareness of form of the example, but was not very articulate about it (☐ under ‘awareness’ and ☐ under ‘language’).

Sam: I’ve never thought of them in terms of yours for simpler and harder. Based on... because these are ones I’ve seen in the past, they’re not like I’ve just invented them now. They are all products as they have to be for integration by-parts, so they are all going to be like that. They can’t be solved any other way.

My probes revealed his specific reference to dimensions he had changed in his examples. However, the use of ‘this’, ‘that’ and ‘it’ to refer to the integrals suggests shifts in reference and so it is unclear which integral he was referring to at any time.

Interviewer: What is different [in your examples]?

Sam: That one (simpler) is different because there are two things being multiplied together ... the same like something squared. And that one (more complicated) is different because it’s not completely clear what the two things are.

Interviewer: Why is it simpler?

Sam: Because you are integrating a simpler function than it is in that because that (example given) you are integrating the \( x \) whereas mine you are just integrating 1.

Interviewer: How is it the same?

Sam: They are both functions, they are both ... actually they’re not because that 1 is not. I was going to say you are reducing it, you are differentiating the \( x \) down and making it simpler until it becomes 1 and disappears. That is quite the opposite actually because you are making the simple thing more complicated; it’s only because you can’t integrate that whereas that one you can integrate without it getting any more complicated.

Interviewer: The complicated one, how is it like my example?

Sam: It doesn’t reduce because you end up with an equally complicated answer, more complicated because of the \( uv \) then you have to do it again and then it becomes simple.
Interviewer: Whereas this one reduces?
Sam: By itself.
Interviewer: How is it the same?
Sam: Product of two functions, [involves] trigonometric function ... aren't that many similarities.

Task 5 \[\int(x+3)(x^2+6x)^{17} \, dx\] further substantiated Sam’s facility and flexibility with technique. He made a remark that the example involved substitution and had function-derivative form. He decided to ‘copy’ the example given and constructed \[\int x(2+x^2)^{17} \, dx\].

He made a remark that ‘it’s the same but one degree lower’ [referring to the given example]. He observed that he had ‘reduced the power of the \(x\) so just differentiated it again and got that one \((x^2 \text{ in } x^2 + 6x)\) down to just a constant and that one \((3 \text{ in } x + 3)\) disappeared’. For the more complicated example, he constructed \[\int(cos \, x + \frac{1}{x})(sin \, x + ln \, x)^{17} \, dx\]. He observed:

Sam: Similarly like I said it’s a function times its derivative; it’s the same kind of layout. It’s different because it uses more complicated functions, makes it harder, power is higher which really doesn’t make much difference in the hardness because it’s just going to mean dividing by it and increasing it. I still say that’s (complicated example) harder than that one. That one (example given) if you don’t do it as a function-derivative, you can multiply that out whereas that one you couldn’t, well you could but you’d be there for a while.

Again, Sam did not display any richer connections than before but merely awareness of the particular (\(\bigcirc\) under ‘awareness’ and ‘technique’). He seemed to be caught up with details in the given example. Although he displayed awareness of form, the dimensions he varied and those he kept constant revealed restricted connections.
8.2.3 Making comparison

Sam’s responses in the interview suggested relative fluency in language and some facility in technique. However, his remarks also suggest prevalence of attention to technique. He did display considerable instances of connections, although his awareness of common errors associated with the topic appeared to be rather simplistic. Although his access to uses seemed comprehensive, he did not display rich access to origins.

From his responses, and remarks he made and dimensions he varied in his examples, I get the impression that Sam’s facility and flexibility with technique of integration was quite strong. In many instances, he displayed enriched connections and associations. However, he did not come across as being very fluent and articulate in using language to explicitly express his thoughts and make useful remarks. At times, he appeared to have ‘internalised’ aspects that might have augmented his awareness had he made explicit reference to them.

Sam’s responses in the construction tasks suggest considerable depths in his awareness aspect of his concept image. Again, language fluency let him down. The variety of examples he constructed also demonstrated the richness and connectedness of his accessed example space.

Table 8.3 below summarises Sam’s responses in the interview and the construction tasks.

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td>Interview</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Construction</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>task</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 8.3: Sam’s responses in the interview and construction tasks
8.3 Summary

In the interview, students in this group displayed considerable fluency in aspects related to integration. Questions in the interview shed light on the structure of their concept image. Their responses highlighted considerable depths in their concept image of integration. However, responses in the construction tasks generally revealed limitations in Robert and Alex’s concept images of integration as shown in the dimensions-of-possible-variation they were aware of in the examples they constructed. Sam, however, demonstrated quite a rich concept image for integration. They were better in form than in manipulative detail in engaging in the tasks. However, students in this group committed lots of mistakes and were held back by these.
Chapter 9

Summary of results

9.0 Introduction

In this chapter, I present a summary of results from the main study. The main purpose of this chapter is to bring together the results from Chapters 6, 7a, 7b and 8 in order to make comparisons across all the groups of participants and to say something about the similarities and differences in the aspects probed in the interview and the construction tasks. Sections 9.1 and 9.2 present a summary of the findings, both responses in the interview and in the construction tasks. I have opted to do the latter by including tables showing the full range of the examples constructed by the participants in each task and have indicated which ones are appropriate for the task and which are not. Section 9.3 presents a summary of density tables that record presentation of data both from the interview and the construction tasks. I have opted to do this by juxtaposing the density tables so that the visual method of recording can be used to see the differences. This is done to get an overview of the data and to compare across groups. Finally in Section 9.4, I discuss a summary of the results.

9.1 Responses in the interview

Questions in the interview centred around the meaning triggered by “integration” and the integral sign $\int$, knowledge of techniques, language patterns and common misconceptions associated with integration, access to uses and origins that gave rise to the topic in the first place. There was an indication that some aspects become the focus of attention in certain groups of learners; this was inferred from the things they said. Some
learners displayed deep understanding of integration and associated integrals as objects for calculating area and integration as the reverse process of differentiation. Integration also triggered many other associations such as techniques and methods. Others showed surface understanding simply by relating integration with the reverse process of differentiation. I introduced the term *superficial understanding* to signal surface connections, *deep understanding* to indicate robust connections, and an intermediate, emergent sophistication in connections, which I have termed *immersed state of understanding* in order to distinguish the different depths in learners' understanding.

In particular, the PGCE students displayed restricted awareness of integration (with the partial exception of Jon and Rob). Most of them displayed a propensity to become caught up in techniques, while awareness of common errors related to integration appeared technique-related and was weak or non-existent. I took this to mean that their attention was more focused on procedures for carrying out rules and techniques and not so much on other connections. In terms of origins or uses of integration, none of them showed much knowledge of the kind that could have contributed to a richer understanding and appreciation. Superficial understanding was more common in this group of students.

Associations that are triggered in the students in Chapter 7a (average mathematics students) suggest they were largely dominated by surface connections, with the exception of Marlene, Simon, Matt and Beth who demonstrated richer awareness in relating integration with details of triggered associations and in identifying common misconceptions they needed to guard against. Most of them related integration only to area or to anti-differentiation and showed a predisposition to associate integration with techniques. Some of them displayed an immersed state of language patterns when identifying words related to integration (Marlene and Simon). Except for Simon, access to uses and origins appeared superficial.
As might be expected, students in Chapter 7b (strong mathematics students) displayed a more comprehensive view of integration in the interview. All of them displayed immersed connections, in which integration triggered more than just notions of area and anti-differentiation. Students in this group displayed facility with the language of functions, graphs, limits, etc. They also demonstrated informed behaviour in terms of fluent language patterns and facility with techniques, with the exception of Kim and Darren who, at times, showed fragmented facility. Access to origins and uses in which integration was used was displayed at the level of immersed understanding for all except Harry and Danny who otherwise demonstrated deep knowledge.

Some of the engineering students (Robert and Sam) displayed rich awareness in associating integration with multiple uses. However, all of them displayed at best an immersed state of awareness of common errors. All three of them were conversant with language patterns related to integration, though they were largely technique-oriented. In terms of uses, Sam and, to an extent, Andy displayed rich understanding, while none of them displayed any knowledge of or familiarity with origins which acted as a source for integration as a topic.

9.2 Responses in the construction tasks

In this section, I present a summary of examples constructed by the participants in this study. I have compiled the examples in order to make a comparison between them. I have opted to present the summary task by task in order to look for similarities and differences within the group and across the group. The tables present the tasks followed by the participants' examples and a brief commentary of each example constructed. Where appropriate, I have indicated in the commentary section of the table whether the
participants have displayed awareness of structure in the examples they constructed (variety and scope of dimensions varied, summarised as structural awareness).

9.2.1 Task 1

Task 1 offered subjects the opportunity to vary dimensions in the object which represents integration as area under the graph. Asking subjects to construct another and another example of an integral which gives zero as the answer could encourage them to vary more dimensions and come up with different, more sophisticated examples. Table 9.1 below summarises the examples constructed by the different subjects.

<table>
<thead>
<tr>
<th>Task 1</th>
<th>Given $\int_{0}^{2} (1-x)dx = 0$. Can you find another example like this where the answer is 0?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>Example 2</td>
</tr>
<tr>
<td>Marlene</td>
<td>$\int_{0}^{3} (1-x)dx$</td>
</tr>
<tr>
<td>Craig</td>
<td>$\int_{0}^{1} (1-x)dx$</td>
</tr>
<tr>
<td>Matt</td>
<td>$\int_{0}^{2} (2-x)dx$</td>
</tr>
<tr>
<td>Beth</td>
<td>$\int_{0}^{2} (2-x)dx$</td>
</tr>
<tr>
<td>Simon</td>
<td>$\int_{0}^{2} (1-x)dx$</td>
</tr>
<tr>
<td>Sarah</td>
<td></td>
</tr>
<tr>
<td>Darren</td>
<td>$\int_{0}^{2} (1-x)^2 dx$</td>
</tr>
<tr>
<td>Kim</td>
<td>$\int_{0}^{3} \left(1-x^3\right) dx$</td>
</tr>
</tbody>
</table>

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Chapter 9

Summary of results

Harris
\[ \int_{0}^{\pi/2} \sin x \, dx \]
Structural awareness but incorrect second example.

Danny
\[ \int_{0}^{2} (2 - 2x) \, dx \]
Attempts at structural generalisation; missed limits in second example.

Rick
\[ \int_{0}^{\pi/3} 2x - x^3 \, dx \]
Structural awareness of form (odd functions).

Robert
\[ \int_{0}^{1} (x - 1) \, dx \]
Minor changes but changes nevertheless.

Alex
\[ \int_{0}^{\pi} \sin x \, dx \]
Classic familiars; structural awareness of form.

Sam
\[ \int_{0}^{\pi} \sin x \, dx \]
Correct examples except third example; structural awareness of form.

Jon
\[ \int_{0}^{1} (3 - x^3) \, dx \]
Correct examples; generalisation based on empirical evidence.

Denise
\[ \int_{0}^{1} (x - 1) \, dx \]
Correct examples but with minor changes.

Rob
\[ \int_{0}^{1} x^3 - 4 \, dx \]
Awareness of class but not of many examples; missed \( dx \).

Hanna
\[ \int_{0}^{1} (x^2 + 5) \, dx \]
Incorrect example; no awareness of form.

<table>
<thead>
<tr>
<th>Engineering students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harris</td>
</tr>
<tr>
<td>Danny</td>
</tr>
<tr>
<td>Rick</td>
</tr>
<tr>
<td>Robert</td>
</tr>
<tr>
<td>Alex</td>
</tr>
<tr>
<td>Sam</td>
</tr>
<tr>
<td>Jon</td>
</tr>
<tr>
<td>Denise</td>
</tr>
<tr>
<td>Rob</td>
</tr>
<tr>
<td>Hanna</td>
</tr>
</tbody>
</table>

Table 9.1: Examples constructed for Task 1

From the table we can see that with the exception of Simon, most of the average mathematics students were concerned with manipulating the given example and varying limited dimensions in the example. They stuck to numbers and linear functions. Changes made were minor and demonstrated almost no awareness of structure. There was no evidence of thinking in terms of area in the examples they constructed.

The strong mathematics students displayed awareness of area in the examples they constructed. Kim showed sophistication in the general example she constructed, although the details were incorrect. Generally, participants in this group varied more dimensions...
and varied extensively. They produced non-linear functions and displayed awareness of structure.

The engineering students also displayed awareness of area. They produced classic familiar examples and displayed awareness of form.

Most of the PGCE students displayed a tendency to manipulate the given example, with the exception of Rob who displayed awareness of area. Although Denise’s examples were correct, she did not make any reference to area.

9.2.2 Task 2

Task 2 invited subjects to construct simpler and more complicated examples compared to an expression relating integration and differentiation. The purpose of this task was to reveal subjects’ awareness of dimensions-of-possible-variation in the structure by varying dimensions in the example to make it simpler and more complicated. Table 9.2 below summarises the examples they constructed.
### Task 2

Given that \( 2 \int (\ln x + \frac{3}{2}) \, dx = 2x \ln x + x = \frac{d}{dx} x^2 \ln x \). Construct another integral with its two corresponding expressions, which is simpler, and one which is more complex.

<table>
<thead>
<tr>
<th>Average mathematics students</th>
<th>Simpler example</th>
<th>Complicated example</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marlene 2 ( \int x^2 , dx = \frac{2x^3}{3} = \frac{d}{dx} \frac{x^4}{4} )</td>
<td>sin and cos and log</td>
<td>Partially correct examples; semi-structural awareness of form.</td>
<td></td>
</tr>
<tr>
<td>Craig ( \int \ln x \sin x , dx )</td>
<td>Wrong example; no appreciation of form.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Matt ( \int \frac{1}{2} x^2 = \frac{d}{dx} \frac{1}{6} x^3 )</td>
<td>( 2x + \sin x = x^2 - \cos x = \frac{d}{dx} \frac{x^3}{3} - \sin x )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Beth ( \int (1 - x) , dx )</td>
<td>Incorrect second example; semi-structural awareness of form.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simon ( \int 2x^4 , dx = 0.4x^5 = \frac{d}{dx} 0.4 \frac{x^6}{6} )</td>
<td>Incorrect second example; semi-structural awareness of form.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sarah ( \int \cos x \sec x , dx = \ldots = \frac{d}{dx} \ldots )</td>
<td>Incorrect second example; semi-structural awareness of form.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strong mathematics students</th>
<th>Simpler example</th>
<th>Complicated example</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Darren ( 2 \int dx = 2x = \frac{d}{dx} x^2 )</td>
<td>( (-2 \sin x \cos x - 2 \cos x \sin x + \frac{1}{x} - \frac{1}{x^2}) , dx = \cos x \sin x - \sin x , x + \log x - \frac{1}{x} )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Kim ( \int x , dx = \frac{x^2}{2} = \frac{d}{dx} \frac{x^3}{6} )</td>
<td>( \frac{d}{dx} \sin x \cos x = 3 \sin^2 x \cos x \sin x - \sin^2 x , x = \int (6 \sin x \cos^2 x - 10 \sin^2 x \cos x , x) , dx )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Harris ( \int (1 + x)^3 , dx = \frac{d}{dx} \frac{(1 + x)^4}{4} + C )</td>
<td>( xe^x , dx = xe^x - \int e^x = \frac{d}{dx} xe^x - 2e^x )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Danny ( \int x , dx = \frac{x^2}{2} + C = \frac{d}{dx} \frac{(-x^3)}{6} + C )</td>
<td>( 2 \int \sin x e^{\sin x} - (2 \cos^2 x) e^{\sin x} , dx = 2(\cos x) e^{2 \sin x} = \frac{d}{dx} (2e^{\sin x}) )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Rick ( \int dx = x = \frac{d}{dx} x^2 )</td>
<td>( \frac{d}{dx} x \ln x = \ln x + 1 )</td>
<td>Correct examples; structural awareness and appreciation of form.</td>
<td></td>
</tr>
<tr>
<td>Engineering students</td>
<td>Robert</td>
<td>$\int \cosh x , dx = \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} + C$</td>
<td>$\int \sec x \tan x , dx = \sec x = \frac{d}{dx} \left[ \ln</td>
</tr>
<tr>
<td>----------------------</td>
<td>--------</td>
<td>-------------------------------------------------</td>
<td>-------------------------------------------------</td>
</tr>
<tr>
<td></td>
<td>Alex</td>
<td>$\int (1 + x) , dx = x + \frac{1}{2} x^2$</td>
<td>$\int x \sin x , dx = \sin x - x \cos x = \frac{d}{dx} \left( \cos x - x \sin x + \cos x \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= \frac{d}{dx} \left( \frac{1}{2} x^2 + \frac{1}{6} x^3 \right)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sam</td>
<td>$\int dx = x = \frac{d}{dx} \frac{x^2}{2}$</td>
<td>$\int \left( x^2 \sinh x + 4x \right) , dx = x^2 \cosh x + 2x \sinh x + 4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= \int x^2 \cosh x + 2x \cosh x + 2x \cosh x + 2 \sinh x$</td>
<td>$= \int 4x \cosh x + (x^2 + 2) \sinh x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PGCE students</td>
<td>Jon</td>
<td>$\int e^x + x^2 = e^x + \frac{x^3}{3} = \frac{d}{dx} \left( e^x + \frac{1}{12} x^4 \right)$</td>
<td>$\int \frac{1}{\sqrt{x^2 + x^4 + x^6}} , dx$</td>
</tr>
<tr>
<td></td>
<td>Denise</td>
<td>$\int (\ln x + 1) , dx = x \ln x - x + x$</td>
<td>$\int (8\ln x + x^7) , dx = 8(x \ln x - 8x) + x^8$</td>
</tr>
<tr>
<td></td>
<td>Rob</td>
<td>$2 \int (x^2 + 2x) , dx = \frac{d}{dx} \ldots$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Hanna</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 9.2: Examples constructed for Task 2
Examples constructed by the average mathematics students showed that most of them constructed a correct simpler example but struggled with the more complicated example, except Matt who constructed a complicated example that included a trigonometric function. Sarah displayed awareness of structural form but did not complete her more complicated example. With the exception of Craig who showed almost no awareness of structure in this task and Matt who showed structural awareness of form, others in this group showed some structural awareness.

The strong mathematics students displayed more sophistication in their examples they constructed. Most of them produced a very simple example $\int dx$ or $\int \sin(x)dx$ and quite complex complicated example. They produced quite complex examples and revealed structural awareness of form.

The engineering students also displayed sophistication in the examples they constructed. They used trigonometric and hyperbolic functions and revealed awareness of form.

Some of the PGCE students constructed correct simpler examples but struggled with the more complicated examples. Some of them varied more dimensions than others. Others did not construct any example. Most of them showed no structural awareness of form.

Generally, there was appreciation of form for many but only a few saw they could start with any twice differentiable function in the third expression.

9.2.3 Task 3

Task 3 invited subjects to construct simpler and more complicated examples to an integral usually solved using substitution. Table 9.3 summarises the examples they constructed.
Chapter 9  
Summary of results

Task 3: Given that \( \int_{0}^{\pi} x \sin(x^2) \, dx = 1 \). Construct an integral which uses the same idea, which is simpler, and one which is more complex.

<table>
<thead>
<tr>
<th>Student Type</th>
<th>Student Name</th>
<th>Simpler example</th>
<th>Complicated example</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average mathematics students</td>
<td>Marlene</td>
<td>( \int 2 \sin x^2 , dx )</td>
<td>( \int \ln x^2 )</td>
<td>Minor changes in first example; incorrect examples; no structural awareness.</td>
</tr>
<tr>
<td></td>
<td>Craig</td>
<td>-</td>
<td>( \int x \sin(2x - 4) , dx )</td>
<td>Incorrect example; no structural awareness.</td>
</tr>
<tr>
<td></td>
<td>Matt</td>
<td>( \int \cos 2x )</td>
<td>( \int \frac{\cos x}{\sin 6x} )</td>
<td>Correct example; structural awareness.</td>
</tr>
<tr>
<td></td>
<td>Beth</td>
<td>-</td>
<td>-</td>
<td>Incorrect example; no structural awareness.</td>
</tr>
<tr>
<td></td>
<td>Simon</td>
<td>-</td>
<td>-</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Sarah</td>
<td>-</td>
<td>-</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td>Strong mathematics students</td>
<td>Darren</td>
<td>( \int \frac{1}{x^2 + 1} , dx )</td>
<td>( \int \frac{1}{\sqrt{x(x+1)}} , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Kim</td>
<td>( \int \frac{2x}{x^2 + 1} , dx )</td>
<td>( \int \frac{2}{\sqrt{1 - 3x^2}} , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Harris</td>
<td>( \int x e^{x^2} , dx )</td>
<td>( \int \frac{1}{\cos^{999} x \sin^{100} x} , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Danny</td>
<td>( \int \frac{1}{u \ln u} , du )</td>
<td>( \int \frac{1}{\sqrt{1 - x^2}} , dx )</td>
<td>Correct examples; structural awareness of form (method) but used a different kind of substitution.</td>
</tr>
<tr>
<td></td>
<td>Rick</td>
<td>( \int x x^2 , dx )</td>
<td>( \int \frac{1}{x \ln x} , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Robert</td>
<td>( \int \tan x , dx )</td>
<td>( \int \sin^5 x \cos x , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td>Engineering students</td>
<td>Alex</td>
<td>( \int \frac{1}{(x+2)^2 + 1} , dx )</td>
<td>( \int \frac{1}{x} , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Sam</td>
<td>( \int (\sin x \cos x) , dx )</td>
<td>( \int x^2 \ln(x^3) , dx )</td>
<td>Correct example; semi-structural awareness of form; missing ( dx ).</td>
</tr>
<tr>
<td></td>
<td>Jon</td>
<td>( \int \frac{1}{\sqrt{x^2 + 1}} , dx )</td>
<td>( \int \sin x , dx )</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Sam</td>
<td>( \int \frac{1}{x - x^2} , dx )</td>
<td>( \int \frac{1}{x^4 - x^2} , dx )</td>
<td>Incorrect examples; no structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Jon</td>
<td>( \int x , dx )</td>
<td>( \int x^3 , dx )</td>
<td>Incorrect example; no structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Denise</td>
<td>( \int \frac{x}{\sqrt{x^4 + 3}} , dx )</td>
<td>( \int \frac{1}{x^4 - x^2} , dx )</td>
<td>Correct example; structural awareness.</td>
</tr>
<tr>
<td></td>
<td>Rob</td>
<td>( \int x \sin x , dx )</td>
<td>( \int \frac{1}{x^4 - x^2} , dx )</td>
<td>Incorrect example; no structural awareness of form.</td>
</tr>
<tr>
<td></td>
<td>Hanna</td>
<td>( \int \frac{x}{\sqrt{x^4 + 3}} , dx )</td>
<td>( \int \frac{1}{x^4 - x^2} , dx )</td>
<td>Incorrect example; no structural awareness of form; missing ( dx ).</td>
</tr>
</tbody>
</table>

Table 9.3: Examples constructed for Task 3
Most of the average mathematics students did not come up with correct examples. Attention appeared to be focused on simplifying and complexifying certain parameters and not on this method of integration.

The strong mathematics students displayed sophistication and awareness of the substitution method of integration in the examples they constructed, suggesting a structural awareness which they could then make more complicated.

The engineering students also displayed awareness of what is important in this method of integration. The examples they constructed showed sophistication in understanding.

With the exception of Denise, the PGCE students constructed examples that did not reflect robust understanding of this method of integration. They did not appear to have appreciated the structure or format of this substitution method.

9.2.4 Task 4

Task 4 invited subjects to construct simpler and more complicated examples to an integral usually solved using by-parts method. Table 9.4 summarises the examples they constructed.

<table>
<thead>
<tr>
<th>Task 4</th>
<th>Simpler example</th>
<th>Complicated example</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average mathematics students</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marlene Craig Matt Beth Simon Sarah Darren</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\int \ln x , dx$</td>
<td>$\int \cos x \tan x , dx$</td>
<td>Correct first examples; some structural awareness of form.</td>
<td></td>
</tr>
<tr>
<td>$\int x \sin(x) , dx$</td>
<td></td>
<td>Minor changes; semi-structural awareness of form</td>
<td></td>
</tr>
<tr>
<td>$\int e^x , dx$</td>
<td>$\int \log x , dx$</td>
<td>Correct examples; structural awareness of form.</td>
<td></td>
</tr>
</tbody>
</table>
With the exception of Beth who made minor changes, most of the average mathematics students did not construct any examples. Those who did constructed examples that did not reflect structural awareness of form.

The strong mathematics students constructed simpler and more complicated examples that reflected awareness of dimensions-of-possible-variation. They used trigonometric functions, logarithms and exponentials. They displayed more complex variation and revealed structural awareness of form, even if one was over adventurous!

The engineering students who did produce examples gave correct examples that reflected structural awareness of form. They showed some complex variation.
Most of the PGCE students did not produce the more complicated example, except Denise, one of whose examples was over adventurous. The simpler examples they constructed did not reveal awareness of form.

9.2.5 Task 5

Task 5 invited subjects to construct simpler and more complicated examples to an integral of functions of the form \( f'/f \). Table 9.5 below summarises the examples they constructed.
Given $\int (x+3)(x^2+6x)^3 \, dx$. Construct an integral which uses the same idea, which is simpler, and one which is more complex.

<table>
<thead>
<tr>
<th>Student</th>
<th>Simpler example</th>
<th>Complicated example</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marlene</td>
<td>$\int (x+1) \sin^3 x , dx$</td>
<td>Longer polynomials with higher power.</td>
<td>No evidence of structural awareness.</td>
</tr>
<tr>
<td>Craig</td>
<td>$\int \tan x , dx$</td>
<td>$\int \frac{x^n}{(1 + x^2)^n} , dx$</td>
<td>Incorrect examples; no structural awareness of form.</td>
</tr>
<tr>
<td>Matt</td>
<td>$\int (x^2 + \sec x) , dx$</td>
<td></td>
<td>Incorrect example; no structural awareness of form.</td>
</tr>
<tr>
<td>Beth</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sarah</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Darren</td>
<td>$\int (2x+1)^2 (4x^2 + 2x) , dx$</td>
<td>$\int (42x^{16} + 18x^{12} + 96x^8)^{21} (7x^{19} + 3x^{15} + 16x^9)^{19} , dx$</td>
<td>Correct examples; structural awareness of form; some errors.</td>
</tr>
<tr>
<td>Kim</td>
<td>$\int (x+1) , dx$</td>
<td>$\int (3x^3 + 4) (17x + 1)^3 , dx$</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td>Harris</td>
<td>$\int 2x(1 + 2x)^2 , dx$</td>
<td>$\int (e^x + 3)(e^{2x} + 6e^x)^3 e^x , dx$</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td>Danny</td>
<td>$\int (2x + 6)(x^2 + 6x)^3 , dx$</td>
<td>$\int \left( \sum_{x=0}^{n} x^n \right) \left( \sum_{x=0}^{n} x^n \right) , dx$</td>
<td>Correct examples; structural awareness of form.</td>
</tr>
<tr>
<td>Rick</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Robert</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Alex</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>PGCE students</td>
<td>Sam</td>
<td>$\int (2 + x^2)^3 , dx$</td>
<td>$\int (\cos x + \frac{1}{x})(\sin x + \ln x)^7 , dx$</td>
</tr>
<tr>
<td>---------------</td>
<td>-----</td>
<td>--------------------------</td>
<td>-----------------------------------------------</td>
</tr>
<tr>
<td>Jon</td>
<td></td>
<td>$\int \frac{z^3 + 1}{z^3 + d} , dz$</td>
<td></td>
</tr>
<tr>
<td>Denise</td>
<td></td>
<td>$\int (x)(2x + 4)^2 , dx$</td>
<td>$\int (x + 1)^2 (2x^2 + 4)^3 , dx$</td>
</tr>
<tr>
<td>Rob</td>
<td></td>
<td>$\int (x + 1)(x + 1)^3 , dx$</td>
<td>$\int x^2 (2x^2 + 5) , dx$</td>
</tr>
<tr>
<td>Hanna</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 9.5: Examples constructed for Task 5
For this task, the examples constructed by the average mathematics students did not reflect understanding of this method of integration. They varied dimensions that did not match the given example. There was no evidence of structural awareness of form.

The strong mathematics students showed sophistication in their examples (although with some errors). They reflected understanding of this method of integration and displayed structural awareness.

Only Sam had time to try this task in the engineering group. He produced examples that showed understanding of this method of integration and some form of sophistication. There was evidence for structural awareness.

The examples produced by the PGCE students were mostly incorrect and did not reflect understanding of this method of integration. There was no evidence of structural awareness.

9.3 Summary of density diagrams

The density diagrams show the different depths of responses in the aspects probed in the interview and the construction tasks in the framework components, namely awareness, behaviour and motivation. They highlight the different aspects of integration that dominate learners’ attention and those that were not expressed. Also, they record the different depths in participants’ understanding in the aspects probed based on the coding system that I developed, described in Chapter 5. Because all the participants were given the same opportunity to reveal the same aspects of the topic, variation in what was revealed is of significance. Often, there was a significant discrepancy between what was said in the interview and what was revealed through the construction tasks. The density diagrams made it possible for this contrast to be appreciated visually.
It needs to be pointed out that the multiplicity of the density shapes in these tables does not correspond directly to length of utterances in the interviews, but rather to the number of different elements detected in utterances and actions. The tables code the presence of different aspects and depths in the participants' responses in terms of awareness, behaviour and emotion. Sometimes a transcript goes on and on where little was added to the learners' understanding. Other times, what the learners said or did contributed significantly to the indications of depth of understanding in comparison to what was said or done previously.

The density diagrams show that in the interview, the PGCE group displayed limited language fluency and limitations in access to origin and uses of integration. They also displayed overall, relatively limited awareness of connections and a domination of technique in the interview. The construction tasks, however, revealed more depth in Rob's awareness of integration but were not very revealing for others.

The 'average' mathematics students also demonstrated limited language fluency and limited awareness of connections and access to origin and uses in the interview. The construction tasks were much more revealing in behaviour and awareness, though not naturally, in misconceptions. Some of them (Sarah and Simon) revealed more depth of awareness of integration than the others.

As expected, most of the 'strong' mathematics students displayed rich language fluency and awareness of misconceptions in the interview, although their access to origin and uses was not as rich as it could be. The construction tasks provided opportunity to reveal more depth in language fluency, technique in context, connections and structural awareness. For some, the construction tasks revealed more depth than the interview.
The engineering students displayed limited language fluency and access to origin but rich access to uses of integration in the interview. Either as students or subsequently in their professional practice, they are sufficiently concerned with uses for them to readily come to mind in the interview. The construction tasks revealed more depth in Sam’s awareness but not for others.

The density diagrams are presented below.
## Table 6.1: Jon and Denise’s responses in the interview and construction tasks

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td><strong>Interview</strong></td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td><strong>Construction tasks</strong></td>
<td>○ ○ ○ ○ ○</td>
<td>○ ○ ○ ○ ○</td>
<td>○ ○ ○ ○ ○</td>
</tr>
</tbody>
</table>

## Table 6.2: Rob and Hanna’s responses in the interview and construction tasks

<table>
<thead>
<tr>
<th></th>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td><strong>Interview</strong></td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td><strong>Construction tasks</strong></td>
<td>○ ○ ○ ○ ○</td>
<td>○ ○ ○ ○ ○</td>
<td>○ ○ ○ ○ ○</td>
</tr>
</tbody>
</table>

---

324
<table>
<thead>
<tr>
<th>Pair</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marlene &amp; Craig</td>
<td>Interview</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Construction tasks</td>
<td>○○○○○</td>
<td>○○○</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>○</td>
<td></td>
</tr>
<tr>
<td>M2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sarah &amp; Simon</td>
<td>Interview</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Construction tasks</td>
<td>○○</td>
<td>○○○○○</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>○</td>
<td></td>
</tr>
</tbody>
</table>

Table 7a.3: Marlene and Craig’s responses in the interview and the construction tasks

<table>
<thead>
<tr>
<th>Pair</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beth &amp; Matt</td>
<td>Interview</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Construction tasks</td>
<td>○○○○○</td>
<td>○○○○○</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>○</td>
<td></td>
</tr>
</tbody>
</table>

Table 7a.6: Sarah and Simon’s responses in the interview and the construction tasks

<table>
<thead>
<tr>
<th>Pair</th>
<th>Language fluency</th>
<th>Facility with technique</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beth &amp; Matt</td>
<td>Interview</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Construction tasks</td>
<td>○○○○○</td>
<td>○○○○○</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>○</td>
<td></td>
</tr>
</tbody>
</table>

Table 7a.9: Matt and Beth’s responses in the interview and the construction tasks
Table 7b.3: Kim and Darren’s responses in the interview and the construction tasks

Table 7b.6: Harris and Danny’s responses in the interview and the construction tasks

Table 7b.6: Rick’s responses in the interview and construction tasks
### Table 8.2: Robert and Alex's responses in the interview and the construction tasks

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td>Interview</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 8.3: Sam's responses in the interview and the construction tasks

<table>
<thead>
<tr>
<th>Behaviour</th>
<th>Awareness</th>
<th>Emotion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Language fluency</td>
<td>Facility with technique</td>
<td>Connectedness</td>
</tr>
<tr>
<td>Interview</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Single</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
9.4 Summary of results

Responses to the direct probes (interview) show that different aspects of integration dominated the participants' attention. In general, the PGCE group displayed limited language fluency and limitations in access to origin and uses of integration. They also displayed overall, relatively limited awareness of connections and a domination of technique in their responses to direct questioning. The 'average' mathematics students also demonstrated limited language fluency and limited awareness of connections and access to origin and uses. As expected, most of the 'strong' mathematics students displayed rich language fluency and awareness of misconceptions, although their access to origin and uses was not as rich as it could be. The engineering students displayed limited language fluency and access to origin but rich access to uses of integration. Either as students or subsequently in their professional practice, they are sufficiently concerned with uses for them to readily come to mind in the interview.

What is striking about these students' conceptions is that their concept image of integration varied considerably between them. Emphasis on components that make up the structure was different; some aspects were stressed more than others. As anticipated, the engineering students were most technique-oriented but with knowledge of uses, while the stronger the students mathematically, the less dependent they were on techniques and the more adventurous in their examples. The PGCE students were also technique-oriented but with some sense of connections. Stressing one aspect of a topic and ignoring others may render learners' experience in attempting to understand impoverished. However, what was stressed in the interview is what dominated learners' attention at any moment in that situation and not necessarily a true reflection of the complexity of the nature of their understanding. I argued in the aims of this study that a way of revealing the extent of learners' understanding which incorporated not only the displayed behaviour but also
Chapter 9 Summary of results

their awareness, was needed, so that teachers and researchers alike could appreciate more of the complexity of learners’ understanding.

My research reinforces Gattegno’s (1987) view that learning involves educating awareness so as to trigger actions through becoming aware of their awareness. One such awareness which was investigated in this study was awareness of dimensions-of-possible-variation (Marton and Booth, 1997; Watson and Mason, 2005) in mathematical examples. By asking learners to construct examples meeting specified constraints, I conjectured that what they chose to vary would indicate something of the extent to which they were aware of possible dimensions that could vary. In this way, the examples constructed would provide insight into the structure of learners’ understanding while at the same time offering opportunity to educate their awareness. In the event, this conjecture was borne out. Most of those interviewed revealed more complex and sophisticated awareness during the construction tasks than emerged in the interviews. The ‘strong’ mathematics students and the application oriented engineering students displayed a more sophisticated awareness than the other students.

The example-construction tasks did in fact reveal aspects of learners’ understanding which were not revealed in the interview. Examples constructed by the learners revealed a focus of their attention in the midst of thinking which was often different from what they asserted in the interview. The examples also reflected learners’ awareness of variant and invariant features in integration problems and the extent to which they saw given examples as representatives of a wider class.

The examples constructed by the participants in this study are taken to reflect the nature of their understanding of the different methods of integration. Some varied more dimensions and showed structural awareness of form while others were more concerned with minor details of the examples. Generally, the ‘average’ mathematics students and
the PGCE students were more technique-based while the ‘strong’ mathematics students had more structural awareness. The engineering students also showed some structural awareness of form. Even the ‘strong’ mathematics students made overly adventurous conjectures about the potential use of the techniques probed. It is an open question as to whether a more sophisticated appreciation enables greater playfulness and variation, or whether getting learners to be more playful in example construction could lead to greater sophistication, or both.

In particular, responses to the example construction tasks gave evidence of:

- the nature of learners’ awareness-attention, behaviour and emotion,
- learners’ sense of generality and example space.

I discuss each of these in Chapter 10.

In interviewing the ‘strong’ mathematics students about their examples, I was alerted to aesthetic elements of example-construction. However, because of the difficulties to be sufficiently precise about what constitutes mathematical aesthetic and to operationalise it, I decided not to include this dimension of example-construction in the analysis. I have mentioned this observation in Chapter 10 as something that can be pursued in future research.

In relation to concept image, the framework provided a structure with which to reveal fine detail of learners’ concept images together with other aspects not normally associated with concept images. By means of that structure, aspects of the topic that were not focused upon and caused difficulties were revealed and diagnosed. Aspects of the framework that did not feature in many of the subjects’ utterances were awareness of misconceptions and the motivational origins and uses of integration.

In the next chapter, I show how these results address my initial research questions.
Chapter 10

Conclusions and Implications

10.0 Introduction

In this chapter, I summarise the findings of this study of learners’ understanding of integration by drawing upon the aims identified in Chapter 1. I also discuss some limitations of the study and implications of the findings for learning.

The next two sections present a review of the findings in relation to the aims of the study. Section 10.1 reports on findings comparing the understanding displayed in interviews with what was revealed by engaging in example-construction tasks and a discussion of issues that emerged. A consideration of the limitations of the study is presented in Section 10.2. This is followed by a discussion of the implications of the study for future research in Section 10.3 and a reflection on the effect of the study on me in Section 10.4.

10.1 Summary of findings

My aims in the study, as identified in Chapter 1, were to:

- expose the range of responses to be expected from learners from different backgrounds in relation to their understanding of integration using the ‘Structure of a Topic’ framework,
- reveal learners’ awareness and sense of generality through example-construction tasks,
- use learners’ awareness of “dimensions-of-possible-variation” (Watson and Mason, 2005) to say something about their awareness and understanding,
• suggest ways of teaching more effectively by explicitly directing learners’ attention to “dimensions-of-possible-variation” in the examples used and by encouraging them to articulate them clearly, thereby enriching their awareness.

The specific research questions that I wanted to address were:

• Can the ‘Structure of a Topic’ framework be used effectively as the basis for probing students’ understanding of a topic (in this case, integration)? If so, what particular aspects does it reveal?

• What is the same and what is different about using construction tasks and probes based on the ‘Structure of a Topic’ framework for revealing students’ understanding?

• What similarities and differences are revealed about engineering students, mathematics students and PGCE mathematics students by using the two forms of probes (interview and construction tasks)?

I believe understanding is a complex process extending beyond learners’ ability to reproduce mathematical techniques in familiar or slightly modified situations (Skemp, 1976; Mason, 2002a; Mason and Johnston-Wilder, 2004). I see understanding as not only the basis for using knowledge to solve routine problems correctly but as the basis for acting creatively in unfamiliar situations and so needs to be considered not only based on what is displayed through actions when working out problems, but also on what is revealed about their general awareness.

When probed in an interview, what a learner says is what is triggered and comes to mind when being asked questions, but this cannot be taken as all that the learner knows because they may suppress some things, and they may not be cued into accessing other things. They may in fact have richer awareness of more aspects of a topic than that expressed during the interview. This was borne out in the study. Asking learners to
construct examples meeting specified constraints was found to be a complementary way to reveal other aspects of their awareness. This has both a research dimension, and a pedagogic dimension, since construction tasks could influence learners by giving them access to richer, and more informed behaviour for use in the future. To this end, responses from the interview were contrasted with responses in the construction tasks to say something about and to explore the nature of learners' awareness and thus, their understanding.

Although studies reported by Hong and Thomas (1997), Rasslan and Tall (2002) and Sealey (2006) have found that learners could not cope with integration in slightly modified situations, similar to my findings, these studies have focused on exploring understanding through observing behaviour alone. Little was done to investigate root causes of learners' awareness. My research reveals that observing behaviour alone through behaviour in slightly modified tasks does not give a full view of learners' understanding. By prompting learners to construct examples, what they choose to vary in the examples reveals further dimensions, depth and nature of their awareness, and thus of their understanding. As I have discussed in Chapter 2, awareness plays a vital role in development of understanding. Learners' ability to discern dimensions-of-possible-variation in mathematical examples can reveal dynamics and depths of their awareness. By becoming aware of features of a topic not previously at the focus of their attention, learners in fact revealed to themselves aspects of the concept that had not previously been salient to them. Their awareness is altered.

In this study, the 'Structure of a Topic' framework was used as a framework around which to organise and to generate probes to investigate learners' understanding. Integration was the topic chosen for this study and as such, a spectrum of learners meeting integration at various stages of their studies was used, namely A-level students,
mathematics students (average and strong), engineering students and PGCE students. My results in this study have allowed me to address my initial research questions.

I found that of the constructs reviewed in Chapter 2, the ones that are most significant in the analysis of data were Gattegno’s (1987) claims about awareness, Marton and Booth’s (1997) dimensions-of-[possible]-variation, Watson and Mason’s (2005) example space and Tall and Vinner’s (1981) *concept image*. These constructs helped me distinguish different forms of solution to example construction tasks and appreciate learners’ structural awareness.

10.1.1 The ‘Structure of a Topic’ framework

In contrast to previous use as a pedagogic device to inform teaching, in this study the ‘Structure of a Topic’ framework is used as a research tool to reveal strengths and weaknesses in learners’ understanding and appreciation of integration. The framework incorporates other aspects beyond concept image proposed by Tall and Vinner (1981), to include awareness of common errors, and emotion (as discussed in Chapter 2). It offered learners the opportunity to reveal the depth of their grasp of integration. The framework was used in this study both to generate research probes and to analyse the responses. Responses from the interview were contrasted with responses in the construction tasks to say something about and to explore the nature of learners’ awareness and understanding. Often, there was a notable contrast between what learners said in interview, and what they actually attended to during construction tasks, so the framework is usefully augmented by construction tasks. These results allow me to address my fourth aim and my first and second research questions.

My use of the framework as a *structure* by means of which to reveal aspects of learners’ understanding and appreciation of integration demonstrates that it has potential for use
not only for lesson preparation, which was its original intent, but for constructing research probes and for analysing learners' responses to probes. Not only were aspects of concept images revealed, but other aspects as well. The dimensions-of-possible-variation in the examples they constructed provided insight into the scope of their understanding. This in turn could be used to inform the choice of pedagogic strategies to enrich understanding in the future. I conclude that the 'Structure of a Topic' framework provides a useful structure for identifying aspects of a concept image that cause difficulty for learners and for probing those and other aspects.

The use of the framework in this study provided a way of distinguishing between students having different backgrounds, approaches, and uses for mathematical topics. This was accomplished by providing opportunity for all the subjects in this study to reveal a range of specific and pertinent aspects of their understanding. As such, some of them revealed more depth in their responses than others in the aspects probed. It did indeed turn out to be the case that the probes used, which were based on different aspects of the structure, did serve to reveal not only the aspects intended, but other aspects as well. For example, a probe used to reveal awareness also provided evidence for techniques. In using the framework, I was on the look out for aspects outside the framework and two aspects which emerged near the end of my analysis were evidence of appreciating a mathematical aesthetic and displaying a disposition to engage, neither of which are in previous articulations of the framework or the notion of concept image. If I were doing the study again, I would incorporate these and I will include them in any further use of the 'Structure of a Topic' framework.

From a teaching perspective, thinking in terms of the six components of the framework could enable teachers to recognise aspects which need strengthening so that learners have a more balanced and informed understanding. In particular, where there is a propensity to
stress techniques, teachers can direct attention, through example-construction, to missing elements. Being pushed beyond mere use of techniques to get answers proved to be challenging for participants in this study because of the novelty of the tasks, in which case familiarity with the usual contexts of integration was compromised. However, for some of them, the nature of the tasks themselves afforded a shift in seeing integration in its totality. For others, explicit directing of attention to realise this shift was needed. And for yet others, the dominance of techniques alone prevented them from appreciating such a shift.

10.1.2 Learners' awareness—attention, behaviour and emotion

The construction tasks were designed to prompt learners to take initiative and so to reveal those aspects of the topic which came to mind in an unusual situation. Thus construction tasks offer a different window on student understanding and appreciation, but the 'Structure of a Topic' framework provides a structure for making sense of the constructions learners make. The examples constructed by learners provided insight into the focus of their attention in the first instance, which says something about their awareness. The kinds of example the learners constructed also suggest their awareness of possible variations in the given example. Results presented in this section help address my first aim and my second and third research questions.

For Jon and Rob (PGCE), working on constructing examples revealed richer awareness compared to that displayed in the interview. Although Rob appeared to have a negative attitude towards integration as a topic, the examples he constructed revealed a rich appreciation. For Denise and Hanna (PGCE) who were predisposed to technique manipulation, the construction tasks did not reveal richer awareness. They did display greater facility with technique than came out in the interview but limited fluency with
technical terms. In articulating what they perceived to be similarities and differences between the given example and the examples they constructed, it was evident that the PGCE students focused on techniques of integration and the extent to which they varied dimensions in their examples was limited, even after their attention was directed specifically to the idea of possible variations. They did not appear to appreciate the shift in attention and their approach remained technique-oriented.

For the average mathematics students, the construction tasks revealed, to some extent, an enriched awareness (in relation to what was revealed in the interview) in treating the examples as instances of a general case. This translated into observable behaviour in articulating ideas and in using techniques, especially for Marlene, Simon and Sarah. Others seemed to be caught up with rote learning techniques and did not display deep associations or comprehensive understanding. Their attention needed to be directed specifically to the gist of the tasks in order to get them thinking about possible variations. Even so, the dimensions they chose to vary were fairly limited.

All of the students in Chapter 7b (strong mathematics students) showed a richer sense of awareness, language fluency, technique mastery and positive attitude towards integration evidenced through the construction tasks than in the interview. Except for Kim who struggled a little with the example-construction tasks initially, other students in this group displayed an enriched appreciation and understanding of integration beyond what came from the interview, and beyond what students from other groups displayed. Evidence from my research suggests that most of the students in this group had a richer awareness in terms of imagery and associations compared to students with other backgrounds, providing access to facility with techniques and fluency with language to express ideas and to speak about integration. These students perceived structural similarities in the examples and were able to articulate similarities and differences explicitly. Some of them
(Kim and Rick) appeared to have fully internalised the language and techniques, while others (Darren, Harris and Danny) even appeared to have begun to absorb the idea of example-construction into their way of working on mathematical problems. Even though they had not explicitly engaged in example-construction previously, they quickly adapted to it, and even recognised the value of it for future use. They remained confident and their sensitivity to notice features in mathematical examples characterised their positive attitude and disposition.

The engineering students, who appeared to have rich understanding in terms of techniques and awareness of integration in the interview, did not display full, comprehensive understanding in the construction tasks. Their awareness of integration appeared constrained by their attention to techniques rather than to approaching the tasks holistically. Among them, Sam showed a more enriched awareness of dimensions-of-possible-variation and facility with technique in the construction tasks. The dimensions to vary chosen by Robert and Alex indicated limitations in their sense of generality, although they proved to be resourceful in the interview. Inviting them to articulate what they perceived to be similarities and differences in the examples confirmed that their attention was dominated by techniques.

10.1.3 Learners’ sense of generality and the nature of their example spaces

In inviting learners to construct examples meeting specified constraints, the aim was to investigate the extent to which they saw mathematical examples as particular cases of generality. In constructing these examples, what they chose to vary revealed something about their awareness of dimensions-of-possible-variation and their sense of generality. The examples constructed by the participants in this study also reflected their grasp of different methods of integration. The findings of this study indicate that for some
learners, shifting their attention from details of an example to what was generalised afforded a comprehensive and appreciative view of integration. The example-construction tasks provided access to such a shift for some. Others appeared to view the tasks as yet another problem to solve: they tried to access memorised techniques and failed to engage in the tasks creatively. These results help address the second and third aims and second research question as described in Section 10.1.

In discussing learners’ grasp of integration and sense of generality, I found it useful to talk about their example spaces, introduced by Watson and Mason (2005). In particular, the PGCE students (Jon and Denise, and Rob and Hanna) focused on techniques of integration, despite attempts made to shift their attention to see what was exemplified. Examples they constructed implied that their example spaces appeared fragmentary and isolated. With the exception of Rob (in some instances), the others relied on techniques to get them through the construction tasks and overlooked other aspects such as dimensions-of-possible-variation and other useful associations.

Techniques also dominated attention of the ‘average’ mathematics students. The dimensions they varied in their examples reflected considerable limitations in their awareness. Most of them struggled to cope with the novelty of the construction tasks and failed to engage creatively. Even after their attention was explicitly directed towards becoming aware of dimensions-of-possible-variation, they did not show significant appreciation of the shift to experience generality. Marlene was the only one of these participants who explicitly expressed some appreciation, but even so, she displayed limited awareness of the range-of-permissible-change for the dimensions she varied. The construction tasks also suggested the isolated nature of elements in some learners’ example spaces.
The ‘strong’ mathematics students seemed to have a comprehensive understanding of integration. Their facility with techniques of integration matched a rich awareness and sense of connection. However, not all of them appreciated the shift in focus of attention to experience generality. Learners in this group displayed a strong sense of awareness of dimensions-of-possible-variation and rich example spaces. Perhaps because constructing examples was habitual and occurred almost naturally, some of them did not regard construction tasks as anything novel. Even so their expressions of generality were not always mathematically correct.

The engineering students focused on techniques and use of integration and had quite an over-arching or encompassing view of the topic. The examples they constructed suggested richness in their example spaces, at least to some extent. The dimensions they varied were nevertheless rather limited and they were not very articulate in expressing their understanding. Their sense of generality always seemed to be overshadowed by attention to technique.

10.1.4 Mathematical aesthetic

The emotional strand of the ‘Structure of a Topic’ is a reminder of the affective affordances offered by or available in a mathematical topic, including aspects such as origins and use, which could influence or motivate the students. It was not my intention in this study to make conclusions about the emotional state of the individuals or to study what motivated them when studying a topic. Rather, I was looking for discernable behaviour that mirrored or reflected aspects of the topic that could have influenced behaviour.

However, amongst the ‘strong’ students, there was evidence or reflections of aesthetic elements (Dreyfus and Eisenberg, 1986; Sinclair, 2004) in the behaviour both in the
interview and in the construction tasks in judging the merits of their own examples or
other's examples. Evidence of an emerging mathematical aesthetic was evident in
learners producing simpler or more complicated examples when they valued some
examples or rejected something as being unacceptable. In the absence of words that
indicate emotions such as 'nasty', 'nice', 'evil' and so on, it is hard to categorise the
aesthetic component but the examples they constructed suggest that they had ruled out
some examples and included others.

On returning to the data to look for evidence of a mathematical aesthetic, there was not
enough evidence nor was there a clear way to delineate different aesthetic sensibilities.
This is an aspect of the emotional strand of the 'Structure of a Topic' framework which
deserves further study.

10.1.5 Unintentional impact of constructing examples

While research is commonly conceived of as an exocentric process where one person
probes another, I discerned that my study actually had an impact on student learning in
terms of revealing to the learners themselves aspects of the topic that previously they had
not focused upon. Participants in this study displayed a range of awareness of the effect
of the construction tasks on their understanding of integration. The tasks that were
intended to be research probes in fact influenced some of the learners' thinking about
integration and changed their perception, at least at the time. Some of them became aware
of the change and were able to articulate the change explicitly and express their
appreciation. Others focused on the details of the tasks and did not appear to notice the
change or were unable to or chose not to express appreciation clearly.

Asked whether constructing examples had changed her perception of integration,
Marlene (average mathematics student) observed:
I think it helps you discern what’s in front of you in the sense that you saying *what is it that makes it what it is*. Once you’ve isolated that you can then identify other things which are similar either more complex or less complicated but still the same in similarities. I’ve never tried to make easier or more complicated ones virtually the same thing or not certainly with integration but it made me look at the function being integrated and consider the method you would need and the characteristics of the function itself to try and *discern some parameters, put it in a box*, deprive it in some way to make it possible to *make similar the deviant.*

The construction tasks seemed to have afforded a shift in Marlene’s attention from focusing on rules of the method to use to discerning properties in the examples. By tinkering with the example to construct simpler and more complicated examples, Marlene acknowledged that constructing examples meeting specified constraints made her focus on exemplariness of the examples rather than on the manipulative details. The above extract shows Marlene’s sensitisation to and appreciation of the structure in the example once she had identified the method to use.

Similarly, Kim (strong mathematics student) noted that what she said and what she brought to the conscious mind when working with integration problems were not necessarily the same thing. She observed that constructing examples had made her realise that. Kim said:

> It [constructing examples] has probably made me realise that when I was saying integration as area under the graph, actually I don’t really think about it as area under the graph that much. It’s more about just kind of applying some sort of transformation on the function in a way, like if you are asked to integrate something it’s sort of applying the set of rules you have for polynomials and for sines and cos etc to what you’ve got using parts and substitution in different matters like that in order to come with the integral for.

She continued to say,

> We don’t normally have situations where you are told to give examples; you are just given things to do. That’s normally the other way round rather than you actually thinking of the
examples. I suppose as students in school, we haven’t really had a chance to create many ideas like this for ourselves. We have always been given ones to derive or to evaluate ourselves.

Kim’s observation suggests that the kinds of exposure to integration that learners brought had placed emphasis on applying rules rather than appreciating the examples for what they exemplified. Kim also observed that constructing simpler and more complicated examples had made her think about what made a complex problem appear hard.

It has made me think quite a lot about how something can be made harder. I think it sometimes makes you think why is that harder and then does it necessarily need to be harder or is it just because of a particular occurrence of some sort of function within the equation, like trigonometric function.

Beth (average Mathematics student) observed that she had not been exposed to this much variation to integration. Constructing examples made her become aware of deeper aspects of understanding of integration. It also made her wonder about the depth of her own understanding of integration.

[It] made me think that I really don’t know much about integration. … [It is] hardly anything to do with integration, just completely different. I don’t think I’ve ever seen physically seen this much stuff on integration or whatever. So it’s quite different. …It’s made me sort of understand them a little bit more because I’m actually having to figure out physically what’s going on to actually know what’s being done because [I] used to think integration [is] either divide by the power and add one and it’s not as simple as that.

Sam (Engineering student) also expressed clearly the way he discerned properties in the examples and constructed new examples. He remarked:

See patterns between similarities between different functions, makes it easy to work out to solve things. […] You just change the … identify the bits you need for it to be the same, like same style. For it to be substitution reaction, what qualities it needs and just make those values easier or more complicated. It’s like [retaining] the general formula, like the function times the
derivative thing for the substitution, take out that bit because that is key to it and then make the
function more complicated or simpler.

Other subjects were caught up in the details of the tasks and did not articulate any change
in perception from having constructed examples meeting constraints.

Robert (Engineering student) noted:

Just many problems that are in fact the same in dimension although they look different.

Similarly, Craig (average mathematics student) remarked:

I'm starting to understand what are classes of different integrals because we haven't got ..oh,
that's going to be hard or that's going to be of essence. Just standard old-fashioned exam ... is it
worth pondering about or is it worth ... is it going to be easy marks if it looks like that or is it
going to be vivid if it is like that.

Sarah (average mathematics student) also commented on the details of the tasks. She
observed:

It has kind of made me think more about where I should may be clear things as a linear factor
and if I wanted it to be simple and may be put like some powers of x if I wanted to make it more
complicated.

For Matt (average mathematics student), the experience of constructing examples was
like A-level revision and it did not change his perception, except now he knew how to
integrate by parts or substitution.

It appears that these latter subjects regarded the tasks as 'yet another problem to solve'
and not as belonging to a certain class of examples. Hanna (PGCE mathematics student)
observed that she never regarded examples as having many 'forms' and that constructing
examples was not part of 'the concept' she understood.

It's really hard when you say, "Can you think of a simpler one" because you just think of the
concept as the concept, don't you? You don't think of it as having that many forms.
Darren (strong mathematics student) displayed deep connections to the topic and demonstrated rich facility with technique. However, he suggested that the thought processes of constructing and working out examples were the same for him. He remarked:

These things were almost identical to the thought processes I go through to just solve the things.

They weren’t very different for me. … What I tended to do was I tended to start with the rules and then work from the rules of the method to create something that is [required].

In distinguishing simpler and more complicated examples, he suggested that:

the actual complexity has no bearing in … or the method you use to solve a certain problem is independent of how complex the thing actually is.

He commented on the tasks themselves and displayed little or no awareness of any effect on his understanding. Perhaps his concept image of integration incorporates awareness of dimensions-of-possible-variation in examples, and constructing examples occurred naturally in his studying.

10.1.6 Change in attitude and thinking about integration

Becoming aware of change in one’s perception and developing the ability to express appreciation of such a change requires learners to become sensitised to that change. A change in perception afforded by the example-construction tasks is particularly important since learners become sensitised to notice structure in mathematical examples. Awareness of what can change and what must remain constant helps learners discern form from details in examples. By becoming aware of features not previously at the focus of their attention, learners who were expecting to be ‘tested’ about their knowledge in fact revealed to themselves aspects that were not previously salient to them. While some learners expressed appreciation of the revealed awareness, others focused on details of the tasks themselves and did not articulate any awareness of this change.
10.2 Limitations of the study

This study aimed at exploring different ways of understanding integration based upon semi-structured interviews with learners ranging from A-level students to graduates and a spectrum of first year undergraduates in between. These students were not selected on the basis of ability, gender or cultural background, although these variables might have some influence on an individual’s ideas about integration and how these are expressed. Rather, these students were selected based on their discipline at university and on volunteering to their teacher or course lecturer to participate in this study. However, the nature of the school or university where the sample was from indicates a general pointer to their mathematical sophistication and maturity. In general, it is likely that subjects who volunteered to participate were those who were prepared to share their views and ideas. There could be further differences to be observed in more reserved learners who are less articulate in expressing their ideas.

The sample for the main study consisted of 18 students who were interviewed mostly in pairs but two of them individually. The interviews were specific questions and were based on several tasks in which learners were asked to construct examples. Because of the nature of the tasks and because the examples constructed by the interviewees were considered individually, the number of interviews that I could manage at any time and the number of tasks that could be developed and used in the interview were limited. As the study progressed, ideas emerged for other construction tasks, but unfortunately too late! Constraints in finding students meant that I had access to a limited and uncontrollable number of students to participate in the study. As a result, I ended up with a variation in the number of students in the various groups. A more representative sample would enrich the findings of the study, but is unlikely to show any startling differences. Despite the size of the ‘sample’, the range of student experience which came to light in this study
gives some indication of the range of types of understanding students are likely to experience, either at different times in their studies, or through different orientations to mathematics in general and integration in particular.

10.3 Implications for future research

Two important issues for further research emerged from this study. The first concerned exploration of the understanding of integration reported here, which was controlled to see the extent to which it varied with learners’ academic background. The second was to investigate further the impact of developing the habit of constructing examples in order to develop a sense of generality and appreciation of integration as a topic. In the following sections, I discuss each of these.

A third possibility emerged during analysis of the data. It is quite likely that the aspects of a topic which remain with a learner reflect a mixture of the aspects stressed by their teacher, and their own particular propensities and interests. It would be interesting to use the ‘Structure of a Topic’ framework to compare how teachers structure lessons on a topic, and what learners bring to the surface later when their understanding and appreciation is probed via interviews and construction tasks.

10.3.1 Varying learners’ background

This study was conducted with students from four different academic backgrounds to see the extent to which understanding of integration varied, and how it developed from the time it was introduced at school to undergraduate studies in which learners began to specialise and graduates studies. Future research could benefit from a much wider variety of learners’ backgrounds in which a range of sophistication in understanding might be observed. My study suggests that, to an extent, learners’ background manifested itself in the nature of their understanding and appreciation of integration. It would be useful to
explore the impact of a richer and more broadly based use of the ‘Structure of a Topic’ framework in a wider variety of student backgrounds.

Teaching could engage students with different intentions and in going beyond mere concern with techniques. It would also be of both theoretical and practical interest to explore the extent to which students described as ‘high attaining’ differed in respect of engaging with and making use of examples, and whether ‘low attaining’ students could benefit from a richer and more balanced approach to topics, making use of example-construction.

If opportunity arose to do the research differently, I would plan to include construction tasks that invite aesthetic judgements during example construction. Requests to construct a ‘nice’ example, an ‘ugly’ example and the like could reveal aspects related to motivational and appreciative aspects of a topic, as part of the emotional strand of the ‘Structure of a Topic’ framework. In asking learners to construct such examples, I discovered that it is important to ask ‘In what ways is this an example of ...?’ because often the dimensions or features they choose to change are obscure. Where time permits, it would be valuable to prompt learners to ‘check their example’ to make sure it does what they expect it to do.

### 10.3.2 Longitudinal impact of constructing examples

Although several authors such as Weber and Alcock (2004), Zaslavsky and Lavie (2005) and Zazkis and Leikin (2007) have explored the use of construction tasks, I wanted to explore their use in a variety of other topics, on the basis of the effects in my study. Constructing examples meeting specified constraints has proved useful in revealing learners’ awareness and their appreciation of integration under the constraints of the interview situation and in the presence of the interviewer and his/her probes. The probes
directed learners' attention and scaffolded their experience so that they revealed to themselves (and to the interviewer) more aspects of their awareness and ways of thinking about the topic than might otherwise be the case. My emphasis in this study had been to investigate learners' awareness of dimensions-of-possible-variation in examples. Consequently, the kinds of examples learners constructed and their appreciation of integration came about as a result of the controlled nature of the interview and interview probes. Watson and Mason (2005) observed that "exemplification is individual and situational" (p. 50). Therefore, learners' awareness of integration and the evoked example spaces could be situated.

However, whether learners develop the habit of constructing examples for themselves rather than in response to probes is a crucial issue. Traditionally, examples have been used merely to illustrate methods of carrying out procedures to resolve problems and to illustrate concepts. I am drawn back to Vygotsky’s sense of development, in which the ability to act ‘in oneself’ when triggered by a relative expert, develops into an ability to act ‘for oneself’ (van der veer and Valsiner, 1991, p. 334) thus giving the learner greater control over initiating action. Rather than simply being triggered or cued into working on examples, learners who engage in constructing examples for themselves are likely to appreciate generality in mathematical examples. Therefore, having been exposed to constructing examples and having appreciated the shift in attention to become aware of dimensions-of-possible-variation in mathematical examples and to experience generality, it is plausible that learners would in future perceive examples as instances of generality. A longitudinal study which looks at changes in learners’ perception and appreciation of a topic such as integration as a result of developing a habit of constructing examples would reinforce and extend the findings of this study.
Habitual construction of examples by learners would both reveal aspects of their understanding and help them gain richer, more comprehensive appreciation of mathematical topics. Based on my research findings, I am confident that learners who develop the habit of constructing examples for themselves would appreciate dimensions-of-possible-variation in mathematical examples and discern them as variable dimensions in order to put aspects of mathematical problems into perspective and act accordingly. Becoming aware of aspects related to associations and connections of a topic whilst attending to aspects that require mastery in techniques, stimulated many of the subjects in this study to engage creatively in the tasks I set them, so there is every reason to expect that this would be the case if construction tasks were integrated into teaching practices.

It might be possible to use the ‘Structure of a Topic’ framework in conjunction with the Pirie-Kieren model of understanding (Pirie and Kieren, 1989) to add extra richness to the various aspects of understanding while they are in development, and also to see how construction tasks becoming a standard feature of the pedagogy might impact on the outward and folding-back movements of student understanding.

10.4 Reflecting on the effect of the study on me

The probes used in this study had the unusual effect of revealing to learners themselves, aspects about their awareness of the topic as revealed through their construction of mathematical examples. I had not expected that the tasks that were intended as research probes would have such a noticeable impact on some learners’ thinking about the topic and about themselves.

As much as I learned about the learners, the study also developed my sensitivity to notice aspects of learners’ understanding in their responses. I learned a good deal about interviewing, and the tensions between researching and the desire to teach subjects
instead of interview them! On completing this study, I realised that I have become more sensitive to what learners focus their attention on and to the things that dominate their attention. In inviting learners to construct examples, I also became attuned to notice what they were aware of when working mathematical examples. As such, I acknowledge the importance of probes in the interviews and my sensitivity to notice aspects of learners' understanding has inspired me to engage in future research with careful planning and preparation of interviews and with confidence.
BIBLIOGRAPHY


Bibliography


Bibliography


Appendix A: Pilot Study - Students’ responses in the interview

Interviewer: What does the word “integration” mean to you?

Jonathan: Finding the **area under a curve** … **Opposite of differentiation**.

Alex: Math …, infinitely small bits chunked to get rough area finely chopped … use of powers … depends on equations. Does not mean much in real life. In the classroom, only in very, very specific mathematical job, I can’t see it being used much in a normal, day to day real life situation. Like I’m going to differentiate the cartons of milk I have, you know…

Sam: Finding **area under a curve**, trapezium … Simpson’s rule…

Clara: You can use it to solve **volume of revolution** … mid-ordinate …

Geoff: **Opposite of differentiation**, make something bigger …, when I think of integration, like integrating into society… you wanna put things together, people from different backgrounds coming together.

Heidi: Increase power, divide by new power, multiply by constant.

Adam: Something to do with Math and it’s about finding **area under graph**. It can be between limits and can be specifically defined. You also use it basically to find out what the answer if something is divided by zero whether it is zero. Impossible kind of thing because when you have integrated, the curve you can divide into different sections and answer what is something divided by zero.

John: **Area underneath a curve, opposite of differentiation**, power goes up over power, between limits, have C.

Abby: Increase power (index), divide with new power.

Table 4.1: Responses of students to the question on meaning of integration
Appendix A

Pilot study responses: Qualitative

Interviewer: What comes to mind when you see the sign $\int$?

Jonathan: Integration ... missing something (limits), triggers $\frac{dx}{dy} \left[ \frac{dy}{dx} \right]$, looks Greek, take numbers [and]

turn into powers. You get the will to take numbers in front of an $x$ and put them up as a power

and ... it's just integration, really.

Alex: Doesn't give me the will to do anything. It's like posters on the doors; you just go in and start
doing.

Jonathan: It starts you thinking in certain lines with what you want to do with the equation ... You probably

know that is something you are calculating the area. ... What is it called when you cut those thin

sections?

Alex: $\frac{dy}{dx}$. All I can think of is $\frac{dy}{dx}$, it comes straight to mind ... It comes to mind because it's related to

the whole area.

Jonathan: That's the form for differentiation and you're doing the opposite when you're integrating.

Sam: Lower and upper limits, just integrate.

Clara: Integrate something with respect to $x$ ... $dx$ at the end. It gets difficult as you move on.

Geoff: Integrate; in front of something to integrate.

Heidi: Integrate, substitute numbers, looking for differences, will have limits, squiggly S, integrate ...
tells you limits.

Adam: Integration, finding area under a curve, increasing power and dividing by original power...

finding area.

John: Integration, usually have limits, raise power over new power, add C.

Abby: In front of function, $dx$, limits, integrate.

Table 4.2: Responses of students to question on integral sign
Interviewer: What sorts of things have you discovered you need to watch out for when you are doing integration?

Jonathan: [Be] careful that you don’t start confusing the numbers, that you do the right thing when you are taking numbers and putting somewhere else, when you are adding when you are suppose to multiply. You make sure you don’t do it too much or do it in the wrong place or start doing differentiation by mistake … completely not the answer you’d expect. You have to watch out for negatives when you are trying to find areas between upper and lower limits. It takes it from being … taking numbers away, things can start increasing as well as decreasing.

Alex: Numbers in front stuck it on top as a power.

Sam: Plus C, negative powers when it’s increased it goes up to -2, oh +1 so it goes to 0, ain’t it and logs as well, you need … if its power is -1, you need log… Make sure you divide the n\(^{th}\) power. When you are solving for area under a curve, you have to have units as well like area has to be unit\(^2\).

Clara: Sometimes you are given something like that, like log 1 over something _\frac{1}{something}_, you have to solve them by partial fractions or in the form of _\frac{A}{something}_, you have to solve it by partial fractions. So if you get log _\frac{1}{whatever}_, the formula _\frac{1}{something}_, solve it by partial fractions as well.

Probably I can get a negative area. Sometimes you put negative when the question says positive.

When you are taking them both away, remember when you are ‘minus’ing from there you end up positive.

Geoff: Negative powers, fractional powers because change answers.

Heidi: Increasing power and dividing with new power and not old power, limits (do top first not the other way round) – that’s what we’re suppose to do, the general rule) doing subtraction smallest from the biggest (limits), finding area under a curve, you have bit above and bit below.

Adam: Powers of fraction, usually make mistake when raising power for integration, I’d follow differentiation rule. What type of integration depending upon how complicated the formula … by-parts, substitution… You have to think of all the ones you have and which would be most suitable towards it so you look at it, can’t do it the easiest way… too difficult… just go through and try and improve.

John: Look at limit, sign (negative/positive), quadratic equations, swapping sign and times outside limits.
Limits ... where integrate to and from, if there is C you need to find, for \( e^x \), remember to divide with 2, difference to rule, \( e^x \) stays the same.

**Table 4.3: Responses of students to question on things to watch out for**

Interviewer: What are some of the special words/language that you use when you are talking about integration?

Jonathan: Splitting the area into small sections ... the arbitrary constant ...

Sam: \( dx \), integral, limit, area under a curve, constant.

Geoff: \( f(x) \), \( dx \), different letters ...

Heidi: Integrals, function of \( x \), coefficient, constant, power.

Adam: Volume of revolution, different ways of integrating, trapezium, Simpson’s rule.

John: Integration by-parts, substitution, function \([in y'y'' form]\) that sign or limits.

Abby: Limits, plus constant, powers for function.

Jay: \( dx \), limits.

**Table 4.4: Responses of students to question on technical terms**
Interviewer: What words in a problem/context that tell you that integration is relevant?

Jonathan: Besides integrate ... show by using approximation technique, algebraic technique or something.

Alex: They can also say "This has been differentiated, put it back where it was". ... Find area under a curve.

Sam: If it was a diagram, area or something, you know you have to integrate ... Sometimes if there is the graph and a line and find the area between these two equations ... The word function f(x) ... they probably give you the formula, you have to integrate. Sometimes the f(x) stands for y ... see the next question.

Clara: Given integral, find the original formula ... Integrate by substitution, by-parts ... they normally ask you to integrate that. Use integration by-parts to ...

Geoff: Integrate, find maximum/minimum.

Heidi: Find equation, if one is area, find equation of the curve ... you substitute limit number into equation. Is that differentiation or integration?

Adam: Apart from integrate ... find the area, shaded area.

John: Integrate, points of inflection, turning points, between limits, showing area, unknown constant you have to find.

Abby: Area under a curve with limits, volume of revolution. In mechanics, given v, find speed or position vector, given distance find speed or vice versa.

Table 4.5: Responses of students to question on words that trigger integration
Appendix A Pilot study responses Qualitative

Interviewer: What are some of the techniques of integration that you can think of?

Sam: Substitution, parts, adding to power and divide by new power ... We've done loads with u and v, integration by substitution.

Clara: I'll try and simplify a bit and then integrate it and if there are brackets its [a] whole lot easier because you have each one to integrate.

Geoff: Just do it ... not apply to real world ... increase power by 1, divide by new power; if complicated functions (by parts?) simplify – multiply function and do the same (inc. power) or do it individually then multiply.

Heidi: Increasing power, substitution ... 

Adam: Increasing power and dividing by new power, by parts, splitting formula apart, substitution sometimes differentiation (differentiating bottom to equal the top), normally inverse log.

Abby: Between limits, substitution, adding C, integration by-parts (complicated), two bits of function (integration by substitution, something the question tells, by practice.

Table 4.6: Responses of students to question on techniques
Interviewer: What kinds of problems does integration help to solve?

Jonathan: Varying levels within containers, maximum/minimum.

Alex: Find area under that curve ... Architectural use to work out how much you can use something, how big your holes are going to be for water pipes by scientists. I can't imagine most people having heard of it that much, let alone know how to use it.

Sam: In Physics, statistics, probability, density, variables, standard deviation, expected values.

Geoff: Minimum/maximum.

Heidi: Max volume, finding volume of how much coke in a can, minimize to not waste material. Is that differentiation? Graph, although not in real world, speed.

Adam: Area, volume of revolution, a lot of engineering thing, working out building features, tunnels. They need to work out area, central forces.

In Mechanics, velocity, distribution, statistics – probability, density distribution, experimental mean ... very theoretical. Integration in statistics, theorem sample not what you could get this is what you should get then get the observed value which is going to differ. I think integration is more used towards getting something theoretical, it's not something that perhaps could work in real life.

John: Integration by-parts/substitution, segments under the graph, Ralphson.

Abby: Area, volume, mechanics (speed, distance).

Table 4.7: Responses of students to question on use of integration
Interviewer: What differences are there between $\int x^2 dx$ and $\int_0^2 x^2 dx$?

Jonathan: Limits, this is calculating area between two points on a curve and this is more general, you can’t find an answer because you haven’t slotted any numbers. You add some things ... this already has numbers ... This can give you an integration value, this can’t ... unless you add something to it, you kind of choose $x$ to be something.

Sam: One has a limit, one hasn’t ... Both [have] got integral sign, both [use] same formula.

Clara: That one when integrated you have a constant and that one won’t. You get a numerical value for that and $\frac{x^3}{3}$ for that. ... Area underneath the curve $x^2$ from 0 to 2, that one just integrating function.

Jack: Limits, this one is general ... whole area under the curve and this one ... specific, defined area between 2 and 0.

Andy: Finding area, taking formula for integrating and when you put limits you can find area.

John: Limits ... make sure plus C at the end, apart from that... same. Both integrating $x^2$. You are making $x^2$ curve straight so you can easily work out area underneath between limits. This one you are just working out the straight line. You need straight line to work out the area because slightly different area at each point so if you make it straight you could estimate the area. you’re effectively working out what infinity times zero is because anything times zero equals zero and anything times infinity is infinity so you are working out what that sign is.

Heidi: One has limits and one has no limits. Can’t do anything. Just the same, need limits to work it out.

Geoff: Dimension analysis, this one ... area. that one ... is the same.

Lara: One has limits, C cancels out because you put limit in it. This one ... no limit, be itself (function).

It’s the same thing but more steps to work it out. This one you get a number, this one you end up with an equation.

Table 4.8: Responses of students to question on differences between definite/indefinite integrals
### Appendix B: Pilot Study – Questionnaire responses

<table>
<thead>
<tr>
<th></th>
<th>Integration means ...</th>
<th>The integration sign ( \int ) reminds you of ...</th>
<th>Things to watch out for when doing integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>The summation of rectangles of infinitesimal width, and some given height (given by the function). Eventually ending up with the area under some graph between given intervals of ( x ) of a given function.</td>
<td>The letter S, which to me represents 'Summation'.</td>
<td>Integrals, integrands, arbitrary constants, with respect to, definite, indefinite, anti-derivative.</td>
</tr>
<tr>
<td>S2</td>
<td>To me, Integration means the area underneath a graph, it means making equations order higher, it means calculations involving numerous equations, it means the fundamental theorem of calculus.</td>
<td>This brings to mind a few lines of Partial Fractions, indefinite work to do! Brings to mind calculations, and formulas that I've used to integrate things before.</td>
<td></td>
</tr>
<tr>
<td>S3</td>
<td>Finding a quantity be it area, volume etc. of a shape in any dimension of space given boundaries to integrate between.</td>
<td>Find the indefinite integral</td>
<td>Integration by parts, reduction formulae, differentials, sequences, limits.</td>
</tr>
<tr>
<td>S4</td>
<td>The opposite of differentiation</td>
<td>Integrate the function that follows with respect to whichever differential is used</td>
<td>Differentials, limits, integrate with respect to,</td>
</tr>
<tr>
<td>S5</td>
<td>Finding areas below curves opposite of differentiation</td>
<td>Integral</td>
<td>Arbitrary constant, derivatives, differentials</td>
</tr>
<tr>
<td>S6</td>
<td>Mathematical term meaning to do the opposite of differentiation</td>
<td>Integrate, indefinite integral</td>
<td>Substitution, integration by parts, definite/indefinite integral</td>
</tr>
<tr>
<td>S7</td>
<td>Going back to your original function from differentiating</td>
<td>Sign means integration, but without limits</td>
<td>Definite integral, indefinite integral, integration by parts, reduction formulae</td>
</tr>
<tr>
<td>S8</td>
<td>Opposite of differentiation</td>
<td>Integration</td>
<td>Substitution, parts, area under curve</td>
</tr>
<tr>
<td>S9</td>
<td>It is a mathematical term which is the opposite of differentiation</td>
<td>Integration</td>
<td>Substitution, integration by parts,</td>
</tr>
<tr>
<td>Appendix B</td>
<td>Pilot study responses_Quantitative</td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>----------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S10 The thingy you do to equations which brings them up to the higher level, either undoing a differential or working out areas.</td>
<td>Integration</td>
<td>With respect to, $dx$ - differential indefinite definite by parts</td>
<td></td>
</tr>
<tr>
<td>S11 Opposite to differentiation. finding an area under a curve.</td>
<td>Integrate, find an area under a curve, indefinite integral</td>
<td>Definite integral, indefinite integral constant, substitution, by parts</td>
<td></td>
</tr>
<tr>
<td>S12 Finding areas under lines.integrating areas under lines, reverse differentiation - easily mixed up!</td>
<td>Integration (once) - possibility of Separation of variables, limits, derivative, implicit/explicit, differential equations, first/second order, by parts, substitution, trig identities, reduction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S13 Using a higher powered function to calculate the area beneath the curve of a given function.</td>
<td>Integrating a function with respect to given variable</td>
<td>Sum over a range for definite, infinitesimal change in variable, arbitrary constant for indefinite integrals</td>
<td></td>
</tr>
<tr>
<td>S14 Integration means a process (any process) by which one would obtain the original equation from the original equation's derivative.</td>
<td>Integration</td>
<td>Integral, constant of integration, improper integral, limit of integration.</td>
<td></td>
</tr>
<tr>
<td>S15 Integration is the process of finding the area below a curve, and if a definite integral bounded by the curve and two points on the x-axis. Also the general formula for simple integrals $(1/(n+1))\times k\times x^{n+1}$</td>
<td>This sign suggests that whatever follows will need to be integrated, with respect to a particular variable, denoted by $dx$, for example if with respect to $x$.</td>
<td>Integral; particular integral; definite integral; auxiliary equation; integration by parts; etc..</td>
<td></td>
</tr>
<tr>
<td>S16 Integration is the opposite of differentiation and is used to find out things such as the area under a graph and certain volumes.</td>
<td>This is the symbol for integration and hence means the same as what I mentioned above.</td>
<td>Definite integral indefinite integral with respect to</td>
<td></td>
</tr>
<tr>
<td>S17 It is the opposite of differentiation.</td>
<td>The integral of a certain function, possibly between two set values</td>
<td>Integrand, limits, anti-derivative</td>
<td></td>
</tr>
</tbody>
</table>
Appendix B  Pilot study responses Quantitative

given at the extremities of that sign.

S18  Integrate something... I don’t Integration  Differentiation
       know!

Table 4.9: Responses to questions on awareness/misconceptions in the interview
<table>
<thead>
<tr>
<th>Scenario</th>
<th>Words that trigger integration</th>
<th>Technical terms associated with integration?</th>
<th>Techniques</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Integrate, cumulative distribution function, areas under graphs</td>
<td>Integrals, Integrands, arbitrary constants, with respect to, definite, indefinite, anti-derivative.</td>
<td>Integrating by separation of variables, substitution, by parts, partial fractions. Approximate an integral by the trapezium rule (depending on the difficulty of the function in question).</td>
</tr>
<tr>
<td>S2</td>
<td>Area, separation of variables, reduction formulae</td>
<td>Partial Fractions, indefinite integral, definite integral, derivatives, integrand</td>
<td>Integral of something is the same thing to one higher power, and over that higher power.</td>
</tr>
<tr>
<td>S3</td>
<td>Find the area of this cross section, find the volume of this cube, find the displacement of the particle with respect to time etc.</td>
<td>Integration by parts, reduction formulae, differentials, sequences, limits.</td>
<td>Differentiation, trigonometric formulae etc.</td>
</tr>
<tr>
<td>S4</td>
<td>Area</td>
<td>Differentials, limits, integrate with respect to,</td>
<td>Integration by parts, substitution</td>
</tr>
<tr>
<td>S5</td>
<td>Find the area, probabilities</td>
<td>Arbitrary constant, derivatives, differentials</td>
<td>Substitution, integration by parts</td>
</tr>
<tr>
<td>S6</td>
<td>Area under a curve</td>
<td>Substitution, integration by parts, definite/indefinite integral</td>
<td>Integration by parts</td>
</tr>
<tr>
<td>S7</td>
<td></td>
<td>Definite integral, indefinite integral, integration by parts, reduction formulae</td>
<td>Integration by parts or substitution</td>
</tr>
<tr>
<td>S8</td>
<td>Integrate</td>
<td>Substitution, parts, area under curve</td>
<td>Parts</td>
</tr>
<tr>
<td>S9</td>
<td>Integrate</td>
<td>Substitution, integration by parts</td>
<td>Integration by parts or substitution</td>
</tr>
<tr>
<td>S10</td>
<td>Integrate, find the area under the graph, find the area between these two lines, this is a differential, work out the equation.</td>
<td>With respect to $dx$ - differential indefinite definite by parts</td>
<td>Integral $u , dv = u , v - \int v , du$ maple formula sheet just know it (for simple ones)</td>
</tr>
<tr>
<td>S11</td>
<td>Find an area under a curve, by using a substitution, by parts, with respect to.</td>
<td>Definite integral, indefinite integral, constant, substitution, by parts.</td>
<td>By parts, substitution, identities</td>
</tr>
<tr>
<td>S12</td>
<td>Finding area under graph, differential equation, separating variables.</td>
<td>Separation of variables, limits, derivative, implicit/explicit, differential equations, first/second order, by parts, substitution, trig identities, reduction</td>
<td>By parts, identities, substitution, reverse differentiation</td>
</tr>
<tr>
<td>S13</td>
<td>Sum, average values in probability questions, areas, volumes.</td>
<td>Sum over a range for definite, infinitesimal change in variable, arbitrary constant for indefinite integrals</td>
<td>Substitution (incl. trig/hyperbola), partial fractions, integration by parts</td>
</tr>
<tr>
<td>S14</td>
<td>For example if the question states that to find the equation of the curve we must integrate, or if the question asks to find the total amount by adding small bits (or area under curve) by integration.</td>
<td>Integral, constant of integration, improper integral, limit of integration.</td>
<td>For simple integrals I first add the power to the variable and then decide to find the original coefficient to which the original power was multiplied with to get the new coefficient in the integral.</td>
</tr>
<tr>
<td>S15</td>
<td>If you are given, for example the rate of change, then you may be asked to solve for a particular value, then separation of variables followed by integration is required. Or simply having the words ordinary differential equation would suggest integration will be required.</td>
<td>Integrand; particular integral; definite integral; auxiliary equation; integration by parts; etc.</td>
<td>Integration by parts, power series integration, reduction formulae, etc</td>
</tr>
<tr>
<td>S16</td>
<td>Find the area under the graph, integrate, with respect to, reduction formulae.</td>
<td>Definite integral indefinite integral with respect to</td>
<td>Integration by parts, substitution</td>
</tr>
<tr>
<td>S17</td>
<td>Find the area of..., differential equation, mechanics problems</td>
<td>Integrand, limits, anti-derivative</td>
<td>Integration by parts, substitutions, trigonometric identities</td>
</tr>
</tbody>
</table>
Table 4.10: Responses to questions on technique/language patterns in the *interview*

| S18 | Finding area under curves | Differentiation | Mathematical techniques! |
**Int:** What differences are there between $\int x^2 \, dx$ and $\int_0^2 x^2 \, dx$?

<table>
<thead>
<tr>
<th>Student</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>The first one is an indefinite integral, i.e. no evaluation takes place so an arbitrary constant is added to the end: The second one is a definite integral in which you evaluate the integral between 2 and 0 (in this case), no constant added:</td>
</tr>
<tr>
<td>S2</td>
<td>First one is indefinite, and you get an equation, second is definite, and you get a numbered answer</td>
</tr>
<tr>
<td>S3</td>
<td>First is indefinite second is definite</td>
</tr>
<tr>
<td>S4</td>
<td>The second has been integrated with limits, giving an answer of $8/3$, while the first can only be integrated to $(x^3/3)+c$ where $c$ is an integration constant</td>
</tr>
<tr>
<td>S5</td>
<td>The second integral will be evaluated to a number and the first will be left in terms of $x$</td>
</tr>
<tr>
<td>S6</td>
<td>The first is an indefinite integral and will contain a constant, the second will be an exact figure</td>
</tr>
<tr>
<td>S7</td>
<td>One is indefinite the other definite</td>
</tr>
<tr>
<td>S8</td>
<td>The first is an indefinite integral having an arbitrary constant- an expression, the second will be a number</td>
</tr>
<tr>
<td>S9</td>
<td>The first is indefinite and the second has limits so it is exact</td>
</tr>
<tr>
<td>S10</td>
<td>The first gives an equation $(x^3/3)+c$ (indefinite) the second gives a number ($8/3$) (definite)</td>
</tr>
<tr>
<td>S11</td>
<td>The second example has boundaries between 0 and 2 along the x-axis. it is a definite integral</td>
</tr>
<tr>
<td>S12</td>
<td>One is evaluated between limits so has an exact numerical answer - the other is not between limits so will just be an expression</td>
</tr>
<tr>
<td>S13</td>
<td>LHS is indefinite and give the general solution as a result, RHS is definite and gives the area under the curve between 0 and 2</td>
</tr>
<tr>
<td>S14</td>
<td>One of the integrals has limits the other doesn't.</td>
</tr>
<tr>
<td>S15</td>
<td>The first is indefinite, and will give an equation, in this case, $(1/3)(x^3)+c$, and the second is a definite integral giving a numerical answer (and no $c$ at any time in the answer), in this case, $8/3$ square units.</td>
</tr>
<tr>
<td>S16</td>
<td>I'm unable to view the 2nd formula here so I've no idea!!</td>
</tr>
<tr>
<td>S17</td>
<td>The first is indefinite and will have an unknown constant of integration. The second is definite and will have a set value ($8/3$)</td>
</tr>
<tr>
<td>S18</td>
<td>The first one don't have limits, and it needs to add ' + C' (a constant) at the end of the answer, however the second has limits, which gives exact answer ...</td>
</tr>
</tbody>
</table>

**Table 4.11: Responses of students to question on differences between definite/indefinite integrals**
### Appendix B Pilot study responses - Quantitative

Int: What kinds of problems does integration solve?

<table>
<thead>
<tr>
<th>S1</th>
<th>Differential equations in applied mathematics Finding probability &amp; cumulative distribution functions, expectations of random variables in probability theory. Could possibly be used to approximate a given series.</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2</td>
<td>Area under graph, that’s all I can think of!</td>
</tr>
<tr>
<td>S3</td>
<td>Kinematics, finding quantities of mass, space etc.</td>
</tr>
<tr>
<td>S4</td>
<td>Integration problems</td>
</tr>
<tr>
<td>S5</td>
<td>Area, probabilities (normal distribution)</td>
</tr>
<tr>
<td>S6</td>
<td>Areas under curve, volumes of revolution, mechanical maths</td>
</tr>
<tr>
<td>S7</td>
<td></td>
</tr>
<tr>
<td>S8</td>
<td>Areas under curve, volumes of revolution, differential equations</td>
</tr>
<tr>
<td>S9</td>
<td>Area under curves</td>
</tr>
<tr>
<td>S10</td>
<td>Areas</td>
</tr>
<tr>
<td>S11</td>
<td>Calculus problems?</td>
</tr>
<tr>
<td>S12</td>
<td>Areas under curves - solving differential equations i.e. analyzing changes with respect to time</td>
</tr>
<tr>
<td>S13</td>
<td>Many differential equations,</td>
</tr>
<tr>
<td>S14</td>
<td>Finding total charge inside a closed surface, finding area under curves on graph.</td>
</tr>
<tr>
<td>S15</td>
<td>It is used in solving things like population growth models, and in other areas of mechanics as well to give solutions to differential rate change equations, etc</td>
</tr>
<tr>
<td>S16</td>
<td>Ones involving graphs</td>
</tr>
<tr>
<td>S17</td>
<td>Mechanics problems, differential equations, area under graph problems, energy used problems</td>
</tr>
<tr>
<td>S18</td>
<td>Almost everything?!</td>
</tr>
</tbody>
</table>

Table 4.12: Responses to questions on motivation in the **interview**
<table>
<thead>
<tr>
<th>Int:</th>
<th>Given $\int_0^1 (1-x)dx = 0$. What are the things in the integral that you can change and still get the answer zero?</th>
<th>Can you give me an example?</th>
<th>Can you give me another example?</th>
<th>Can you give me another example?</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>You could change the limits of integration, you could change the integrand signs, or you could actually change the integrand completely. You could multiply (a definite integral which is actually equal to 0) it by a constant.</td>
<td>To get an answer 0, you could evaluate it between 2 and 0: $\int_0^2 (1-x)dx = x(1-x/2) = 0$</td>
<td>You could change the integrand to $(x-1)$, and evaluate between 2 and 0: $\int_2^0 (x-1)dx = x(x/2-1) = 0$</td>
<td>You could multiply it by some constant: $\int_0^1 k(1-x)dx = kx(1-x/2) = 0$</td>
</tr>
<tr>
<td>S2</td>
<td>Upper limit of integration, lower limit of integration, the one that is in the brackets, multiple of $x$.</td>
<td>$\int_0^1 (1-x)dx$</td>
<td>$\int_0^1 (5-x)dx$</td>
<td>$\int_{-1}^1 (7-5x)dx$</td>
</tr>
<tr>
<td>S10</td>
<td>Multiplying by zero? Also, modify the integral numbers, 1 to any number ($n$) and the 0 to 2 $-n$.</td>
<td>0 * equation above</td>
<td>top number = 5</td>
<td>top number = 6 bottom number = -3 number = -4</td>
</tr>
<tr>
<td>S12</td>
<td>Separate</td>
<td>$\int_0^1 dx - \int_0^1 xdx = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S13</td>
<td>If the number 1 can be changed to any number $A$ as long as the coefficient of $x$ is $-A$, you will get 0 (multiplication by a constant $A$ will not change answer as $A*0 = 0$).</td>
<td>$\int_0^1 (340 - 340x)dx$</td>
<td>$\int_0^1 (999 - 999x)dx$</td>
<td>$\int_0^1 (450 - 450x)dx$</td>
</tr>
<tr>
<td>S14</td>
<td>Change the higher limit of 1 to 0.</td>
<td>$\int_0^1 x^2 + 1dx$</td>
<td>$\int_0^1 x^3 - 3x^3 dx$</td>
<td>$\int_0^1 f(x^2)dx$</td>
</tr>
<tr>
<td>S15</td>
<td>You can change the bounds or the main equation ...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4.13: Questionnaire responses to the construction task

<table>
<thead>
<tr>
<th>S18</th>
<th>You could change the top limit to 0 or the bottom limit to 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\int_{-\infty}^{\infty} x dx$</td>
</tr>
</tbody>
</table>

*Appendix B  Pilot study responses_Quantitative*
### Fluency with Language

**Use**
- Mentions integrate, area under a curve or volume of revolution or surface features of an integral.
- Mentions technique and applications of integration.
- Mentions integrate, limits, definite/indefinite, arbitrary constant, function, graphs, areas, solve differential equation.
- Uses language that shows lack of fluency in displaying awareness of structure, often stumbling, incomplete thoughts.
- Uses language fluently but not robust enough to demonstrate awareness of structure often but not always relevant.
- Uses language fluently to demonstrate awareness of structure and usually relevant.

**Recall**
- Recalls specific technique and/or difficulties with it (technical aspects).
- Recalls techniques and displays understanding of some techniques.
- Mentions a few techniques and/or displays broader sense of understanding of the techniques.
- Shows limited facility with technique. Uses technique inappropriately or incompletely.
- Shows restricted facility but limited flexibility with technique. Uses technique appropriately but not always efficiently or gets diverted. Corrects own slips.
- Demonstrates facility and flexibility with use of technique. Uses technique appropriately and efficiently. Deals with unexpected answers.

**Awareness/Connections**
- Does not mention or display awareness of connection/association such as area, volume.
- Displays awareness of association with area/volume but does not display links with images or other connections.
- Mentions association with area/imagery/other associations and/or demonstrates awareness of this connection.
- Refers to techniques inappropriately or inarticulately.
- Refers to techniques appropriately.
- Refers to techniques appropriately and matter-of-factly.

**Misconceptions**
- Mentions limits, arbitrary constant.
- Mentions misconceptions related to technique.
- Mentions association with continuous functions, area (positive/negative), images.
- Makes classic errors.
- Displays awareness of own slips and mistakes.
- Displays awareness of other’s slips or possibilities of them.

**Scope of Context**
- States limited scope of use of concept. Mentions reverse process of differentiation.
- Mentions a substantial number of uses of concept. Mentions limits, reverse process of differentiation and rules, use in a few related contexts.
- Mentions common mistakes related to sense-of the concept including continuous functions, area (positive/negative), images.
- Indicates awareness of what problems integration solves beyond exercises. Limited to one of anti-differential, area or differential equations.

**Scope of Root problems**
- Indicates awareness of sources which lead to integration as a topic.

### Facility with Technique

**Use**
- Shows limited facility with technique. Uses technique inappropriately or incompletely.
- Shows restricted facility but limited flexibility with technique. Uses technique appropriately but not always efficiently or gets diverted. Corrects own slips.
- Demonstrates facility and flexibility with use of technique. Uses technique appropriately and efficiently. Deals with unexpected answers.

**Recall**
- Recalls specific technique and/or difficulties with it (technical aspects).
- Recalls techniques and displays understanding of some techniques.
- Mentions a few techniques and/or displays broader sense of understanding of the techniques.

**Awareness/Connections**
- Mentions association with area/imagery/other associations and/or demonstrates awareness of this connection.
- Refers to techniques inappropriately or inarticulately.
- Refers to techniques appropriately.
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**Scope of Root problems**
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### Awareness/Connections

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- Does not mention or display awareness of connection/association such as area, volume.
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### Categorisation of density shapes

<table>
<thead>
<tr>
<th>Scope of Context</th>
<th>Scope of Root problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Limits, reverse process of differentiation</td>
<td>Conclusive account of origins of integration in quadrature and differential equations and anti-differentiation.</td>
</tr>
<tr>
<td>Mentions a substantial number of uses of concept.</td>
<td>Indicates awareness of sources which lead to integration as a topic.</td>
</tr>
<tr>
<td>Mentions common mistakes related to sense-of the concept including continuous functions, area (positive/negative), images.</td>
<td></td>
</tr>
<tr>
<td>Makes classic errors.</td>
<td></td>
</tr>
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<td></td>
</tr>
<tr>
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<td></td>
</tr>
</tbody>
</table>

### Appendix C

#### Misconceptions

- Mentions limits, arbitrary constant.
- Mentions misconceptions related to technique.
- Mentions common mistakes related to sense-of the concept including continuous functions, area (positive/negative), images.

#### Scope of Context

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#### Scope of Root problems

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#### Categorisation of density shapes

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---

Table 5: Categorisation of density shapes

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<tr>
<th>Fluency with Language</th>
<th>Facility with Technique</th>
<th>Awareness/Connections</th>
<th>Misconceptions</th>
<th>Scope of Context</th>
<th>Scope of Root problems</th>
</tr>
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<td></td>
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