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Feynman graphs and related Hopf algebras

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Abstract. In a recent series of communications we have shown that the reordering problem of bosons leads to certain combinatorial structures. These structures may be associated with a certain graphical description. In this paper, we show that there is a Hopf Algebra structure associated with this problem which is, in a certain sense, unique.

1. Introduction

In a relatively recent paper Bender, Brody and Meister [3] introduced a special Field Theory described by a product formula (a kind of Hadamard product for two exponential generating functions - EGF) in the purpose of proving that any sequence of numbers could be described by a suitable set of rules applied to some type of Feynman graphs. Inspired by this idea, we have worked out combinatorial consequences of the product and exponential formulas in a recent series of papers [12, 13, 14, 15, 16, 17, 18].

Here, we consider two aspects of the product formula for formal power series applied to combinatorial field theories. Firstly, we remark that the case when the functions involved in the product formula have a constant term (equal to one) is of special interest as often these functions give rise to substitutional groups. The groups arising from the normal ordering problem of boson strings are naturally associated with explicit vector fields, or their conjugates, in the case when there is only one annihilation operator [14, 17]. We also consider one-parameter groups of operators when several annihilators are present. Secondly, we discuss the Feynman-type graph representation resulting from the product formula. We show that there is a correspondence between the packed integer matrices of the theory of noncommutative symmetric functions and the labelled version of these Feynman-type graphs.

We thus obtain a new Hopf algebra structure over the space of matrix quasi-symmetric functions that is a natural cocommutative Hopf algebra structure on the space of diagrams themselves which originates from the formal doubling of variables in the product formula.

2. Single and double exponentials

2.1. One parameter groups and the connected graph theorem

2.1.1. Substitutions The Weyl algebra W is the $\mathbb{C}$-associative algebra (with unit) defined by two generators a and a^+ and the unique relation $[a, a^+] = 1$. This algebra is of Gelfand-Kirillov dimension 2 and has a basis consisting of the following family $\{(a^+)^k a^l\}_{k,l\geq 0}$.

It is known that it is impossible to represent faithfully a, a^+ by bounded operators in a Banach space, but one often uses the representation $a \mapsto \frac{\mathrm{d}}{\mathrm{dx}};\ a^+ \mapsto x$ as operators acting "on the line" or, better said, on the space of polynomials $\mathbb{C}[x]$. Through this representation (faithful and coined under the name "Bargmann-Fock"), one sees that we can define a grading on W by the weight function $w(a) = -1;\ w(a^+) = 1$.

A homogeneous operator (under this grading) $\Omega \in W$ is then of the form

$$\Omega = \sum_{k,l;\ k-l=e} c(k,l)(a^+)^k a^l \tag{1}$$

According to whether the excess e is positive or negative, the normal ordering of Ω^n reads

$$\mathcal{N}(\Omega^n) = (a^+)^{ne}\left(\sum_{k=0}^{\infty} S_\Omega(n,k)(a^+)^k a^k\right) \text{ or } \left(\sum_{\mathrm{k}=0}^{\infty} \mathrm{S}_\Omega(\mathrm{n},\mathrm{k})(\mathrm{a}^+)^{\mathrm{k}}\mathrm{a}^{\mathrm{k}}\right)(\mathrm{a})^{\mathrm{n}|\mathrm{e}|} \tag{2}$$

We get combinatorial quantities with two indices i.e. an infinite $\mathbb{N} \times \mathbb{N}$ matrix $\{S_\Omega(n,k)\}_{n,k\geq 0}$ which we will call the *generalized Stirling matrix of* Ω. In fact, it is easily checked that, if the coefficients $c(k,l)$ of Ω are non-negative integers, so are the entries $(S_\Omega(n,k))$ of this matrix.

Let us give some examples of these generalized Stirling matrices.

For $\Omega = a^+a$, one gets the usual matrix of the Stirling numbers of the second kind

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3}$$

For $\Omega = a^+aa^+ + a^+$, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 6 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 24 & 36 & 12 & 1 & 0 & 0 & 0 & \cdots \\ 120 & 240 & 120 & 20 & 1 & 0 & 0 & \cdots \\ 720 & 1800 & 1200 & 300 & 30 & 1 & 0 & \cdots \\ 5040 & 15120 & 12600 & 4200 & 630 & 42 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

and for $w = a^+aaa^+a^+$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 60 & 54 & 14 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 144 & 1296 & 2232 & 1296 & 306 & 30 & 1 & 0 & 0 & \cdots \\ 2880 & 40320 & 109440 & 105120 & 45000 & 9504 & 1016 & 52 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{5}$$

Let $\Omega = \sum_{k,l\geq 0} c(k,l)(a^+)^k a^l$ (finite supported sum) be a general term of W in normal form and let us call a *dominant term*, the sum of monomials with maximum length $k+l$. It is not difficult to prove that, if Ω is homogeneous, the dominant term consists of a single monomial $c(k_0,l_0)(a^+)^{k_0}a^{l_0}$. Thus, the dominant term of Ω^n must be $c(k_0,l_0)^n(a^+)^{nk_0}a^{nl_0}$. Then, for example in the case when $e = k_0 - l_0 \geq 0$, in the generalized Stirling matrix of Ω, the rightmost non-zero coefficient of the line n has address $(n, n.l_0)$ and bears the coefficient $c(k_0,l_0)^n$. All these matrices are row-finite and triangular iff $l_0 = 1$ (which means that no monomial possesses more than one a).

Remark 2.1 *i) There is a beautiful combinatorial expression of the normal form of w^n in case w is a string in a and a^+. The normal form of w is*

$$\mathcal{N}(w) = \sum_{k\geq 0} r(B,k)(a^+)^{r-k}a^{s-k} \tag{6}$$

where $r(B,k)$ is the kth rook number of a certain board B constructed after w (see [13, 25], and $r = |w|_{a^+}$; $s = |w|_a$ are the number of occurences of a^+ and a in w.
ii) To each matrix $M \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ of this kind and more generally "row finite" matrices (which means that, for each n, the family $(M(n,k))_{k\in\mathbb{N}}$ is finite supported), one can associate a transformation of EGFs (see [14, 17]*) $f \mapsto \hat{f}$ such that, if $f = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ then $\hat{f} = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ (with $b_n = \sum_{k\geq 0} M(n,k)a_k$).*
iii) It can be shown that, if no monomial of Ω possesses more than one a, the action of the transformation induced by Ω (through the Bargmann-Fock representation) can be expressed in terms of vector fields or their conjugates, thus the one-parameter group $e^{\lambda\Omega}$ acts by substitutions and products [14, 17].

2.1.2. Combinatorial matrices and one-parameter groups One can also draw generalized Stirling matrices from another source, namely from the combinatorial graph theory.
Let $\mathcal{C}$ be a class of graphs such that

$$\Gamma \in \mathcal{C} \text{ iff every connected component of } \Gamma \text{ is in } \mathcal{C} \tag{7}$$

For these classes of graphs, one has the exponential formula [9, 23, 21] saying roughly that

$$\text{EGF(all graphs)} = e^{\text{EGF(Connected Graphs)}} \tag{8}$$

This implies, in particular, that the matrix

$$M(n,k) = \text{ number of graphs with } n \text{ vertices and having } k \text{ connected components} \tag{9}$$

is the matrix of a substitution (see [14, 17]). One can prove, using a Zariski-like argument (a polynomial vanishing for every integer vanishes everywhere), that, if M is such a matrix (with identity diagonal) then, all its powers (positive, negative and fractional) are substitution

matrices and form a one-parameter group of substitutions, thus coming from a vector field on the line which can be computed.
But no nice combinatorial principle seems to emerge.
For example, beginning with the Stirling substitution $z \mapsto e^z - 1$, we know that there is a unique one-parameter group of substitutions $s_\lambda(z)$ such that, for λ integer, one has the value ($s_2(z) \leftrightarrow$ partition of partitions)

$$s_2(z) = e^{e^z-1} - 1 \ ; \ s_3(z) = e^{e^{e^z-1}-1} - 1 \ ; \ s_{-1}(z) = ln(1+z) \tag{10}$$

but we have no nice description of this group nor of the vector field generating it.

2.2. A product formula
The Hadamard product of two sequences $(a_n)_{n\geq 0}$; $(b_n)_{n\geq 0}$ is given by the pointwise product $(a_n b_n)_{n\geq 0}$. We can at once transfer this law on EGFs by

$$\Big(\sum_{n\geq 0} a_n \frac{x^n}{n!}\Big) \ \odot_{exp} \ \Big(\sum_{n\geq 0} b_n \frac{x^n}{n!}\Big) := \sum_{n\geq 0} a_n b_n \frac{x^n}{n!} \tag{11}$$

In the following, we will omit the subscript (in $\odot_{exp}$) as this will be the only kind of Hadamard product under consideration.

But, it is not difficult to check that the family

$$\left(\frac{(y\frac{\partial}{\partial x})^n}{n!} \ \frac{x^m}{m!}\right)_{n,m\in\mathbb{N}} \tag{12}$$

is summable in $\mathbb{C}[[x,y]]$ (the space of formal power series in x and y) as we have

$$\frac{(y\frac{\partial}{\partial x})^n}{n!} \ \frac{x^m}{m!} = \begin{cases} 0 & \text{if } n > m \\ \frac{y^n x^{m-n}}{n!(m-n)!} & \text{otherwise} \end{cases} \tag{13}$$

and therefore, for $F(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $G(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ one gets the *product formula*

$$(F \odot G)(x) := F(x\frac{\partial}{\partial y})G(y)\big|_{y=0} = \sum_{n\geq 0} a_n b_n \frac{x^n}{n!} \tag{14}$$

With this product, the set of series forms a commutative associative algebra with unit, which is actually the product algebra $\mathbb{C}^{\mathbb{N}}$.

2.3. The double exponential formula
The case $F(0) = G(0) = 1$ will be of special interest in our study. Every series with constant term 1 can be represented by an exponential $exp(\sum_{n\geq 1} L_n \frac{x^n}{n!})$ which can be expanded using Bell polynomials and Faà di Bruno coefficients. Let us now recall some facts about these combinatorial notions.
We still consider the alphabet $\mathbb{L} = \{L_1, L_2, \cdots\} = \{L_i\}_{i\geq 1}$, then the complete Bell polynomials [7] are defined by

$$exp(\sum_{m\geq 1} L_m \frac{x^m}{m!}) = \sum_{n\geq 0} Y_n(\mathbb{L}) \frac{x^n}{n!} \tag{15}$$

We will denote alternatively $Y_n(L_1, \cdots L_n)$ for $Y_n(\mathbb{L})$ as this polynomial is independent from the subalphabet $(L_m)_{m>n}$. We know [7] that

$$Y_n(\mathbb{L}) = Y_n(L_1, \cdots L_n) = \sum_{||\alpha||=n} ((\alpha))\mathbb{L}^\alpha = \sum_{||\alpha||=n} ((\alpha)) L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_n^{\alpha_n} \tag{16}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$ is an integral vector, $||\alpha|| := \sum_{j=1}^{m} j\alpha_j$, $\mathbb{L}^\alpha = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_n^{\alpha_n}$ is the multiindex standard notation and

$$((\alpha)) = \frac{||\alpha||!}{(1!)^{\alpha_1}(2!)^{\alpha_2}\cdots(n!)^{\alpha_n}(\alpha_1)!\cdots(\alpha_n)!} \tag{17}$$

is the Faà di Bruno coefficient [8, 20] which will be interpreted, in the next section, as enumerating structures called *set partitions*.
Combining (15) and (16) one gets

$$exp\left(\sum_{m\geq 1} L_m \frac{(y\frac{\partial}{\partial x})^m}{m!}\right) exp\left(\sum_{n\geq 1} V_n \frac{x^n}{n!}\right)\Big|_{x=0} = \sum_{k\geq 0}\frac{y^k}{k!}\left(\sum_{||\alpha||=||\beta||=k} ((\alpha))((\beta))\mathbb{L}^\alpha\mathbb{V}^\beta\right) \tag{18}$$

Formula (18) will be called in the sequel the *double exponential formula.*

2.4. Monomial expansion of the double exponential formula
In this paragraph, we will use unordered and ordered set partitions. By an unordered partition P of the set X we mean a finite subset $P \subset (\mathfrak{P}(X) - \{\emptyset\})$ ($\mathfrak{P}(X)$ is the set of all subsets of X [7]) such that

$$\bigcup_{Y\in P} Y = X \text{ and } (Y_1, Y_2 \in P,\ Y_1 \neq Y_2 \Longrightarrow Y_1 \cap Y_2 = \emptyset) \tag{19}$$

this explains why *without any convention* the classical Stirling number of the second kind $S(0,0)$ equals 1. The elements of P are called *blocks.*
Following Comtet ([8] p 39), we will say that a partition P is of type $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ iff there is no j-block for $j > m$ and α_j j-block(s) for each $j \leq m$. This implies in particular that the set X is of cardinality $||\alpha|| := \sum_{j=1}^{m} j\alpha_j$.

Here one can see easily that the number of blocks of a partition of type α is $|\alpha| = \sum_{j=1}^{m} \alpha_j$. An ordered partition of type α of the set X is just a partition in which the blocks are labelled from 1 to $|\alpha|$.
In other words, one could say that an ordered partition is a *list* of subsets and an unordered partition is a *set* of subsets.
To every ordered partition $P = (B_1, B_2, \cdots, B_{|\alpha|})$ corresponds an unordered one $\Phi_p(P) = \{B_1, B_2, \cdots, B_{|\alpha|}\}$ where Φ_p is the "forgetful" function which forgets the order. Now to a pair $(P^{(1)}, P^{(2)})$ of ordered partitions of the same set (call it X)

$$P^{(1)} = (B_1^{(1)}, B_2^{(1)}, \cdots, B_{k_1}^{(1)}) \qquad\qquad P^{(2)} = (B_1^{(2)}, B_2^{(2)}, \cdots, B_{k_2}^{(2)}) \tag{20}$$

one can associate the intersection matrix $IM_o(P^{(1)}, P^{(2)})$ such that the entry of address (i, j) is the number of elements of the intersection of the block i of the first partition and the block j of the second. For example with partitions of $X = \{1, 2, 3, 4, 5, 6\}$, and specifying

$$P^{(1)} = (\{1, 2, 5\}, \{3, 4, 6\}) \qquad\qquad P^{(2)} = (\{1, 2\}, \{3, 4\}, \{5, 6\}),$$

one gets

	$\{1,2\}$	$\{3,4\}$	$\{5,6\}$
$\{1,2,5\}$	2	0	1
$\{3,4,6\}$	0	2	1

and hence the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \tag{21}$$

Formally, $IM_o(P^{(1)}, P^{(2)})$ is the matrix of size $k_1 \times k_2$ such that

$$IM_o(P^{(1)}, P^{(2)})[i,j] = \mathrm{card}(\mathrm{B}_\mathrm{i}^{(1)} \cap \mathrm{B}_\mathrm{j}^{(2)}) \tag{22}$$

The matrices obtained in such a way form the set of packed matrices defined in [19] as, indeed, one sees that every packed matrix can be obtained through the matching procedure illustrated above.
If we consider now a pair of unordered partitions $(Q^{(1)}, Q^{(2)})$, we cannot associate to them a single matrix but rather a class of matrices obtained from the preimages of $(Q^{(1)}, Q^{(2)})$ under $\Phi_p \times \Phi_p$. In a compact formulation, the set of matrices so obtained is

$$\left\{ IM_0(P^{(1)}, P^{(2)}) \right\}_{\Phi_p(P^{(1)})=Q^{(1)};\ \Phi_p(P^{(2)})=Q^{(2)}} \tag{23}$$

For example, with

$$(Q^{(1)}, Q^{(2)}) = (\,\{\{1,2,5\},\{3,4,6\}\}\,, \{\{1,2\},\{3,4\},\{5,6\}\}\,) \tag{24}$$

one gets the 12 preimages $(P_i^{(1)}, P_j^{(2)})$, where $P_i^{(1)}$ are among the two preimages of $Q^{(1)}$ and $P_j^{(2)}$ are among the 6 preimages of $Q^{(2)}$. Explicitely

$$P_1^{(1)} = (\{1,2,5\},\{3,4,6\}) \qquad P_2^{(1)} = (\{3,4,6\},\{1,2,5\})$$

are the preimages of $Q^{(1)}$ and

$$\begin{array}{ll} P_1^{(2)} = (\{1,2\},\{3,4\},\{5,6\}) & P_2^{(2)} = (\{1,2\},\{5,6\},\{3,4\}) \\ P_3^{(2)} = (\{3,4\},\{1,2\},\{5,6\}) & P_4^{(2)} = (\{3,4\},\{5,6\},\{1,2\}) \\ P_5^{(2)} = (\{5,6\},\{1,2\},\{3,4\}) & P_6^{(2)} = (\{5,6\},\{3,4\},\{1,2\}) \end{array}$$

are the preimages of $Q^{(2)}$.
The set of matrices so obtained reads

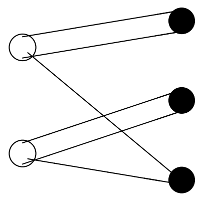
$$\begin{aligned} IM_u(M) \quad = \quad & \left\{ \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \right\}. \end{aligned} \tag{25}$$

This is the orbit of one of them under permutation of lines and columns.
The correspondence which, to a pair of unordered partitions, associates a class of matrices (under permutations of lines and columns) will be denoted IM_u.

Thus, one gets a commutative diagram of mappings

$$\begin{array}{ccc} \text{Pairs of ordered partitions} & \xrightarrow{\Phi_p \times \Phi_p} & \text{Pairs of unordered partitions} \\ {\scriptstyle IM_o}\downarrow & & \downarrow{\scriptstyle IM_u} \\ \text{Packed matrices} & \xrightarrow{Class} & \text{Classes of packed matrices} \\ {\scriptstyle Dg_o}\downarrow & & \downarrow{\scriptstyle Dg_u} \\ \text{Labelled diagrams} & \xrightarrow{\Phi_d} & \text{Diagrams} \end{array} \tag{26}$$

The scheme presented above shows how to associate to a pair of ordered (resp. unordered) set partitions, a packed matrix (resp. a class of packed matrices). The packed matrices can be alternatively represented by *labelled diagrams* which are bipartite multigraphs built from two sets of vertices being a column of white spots (WS) and column of black spots (BS) as shown below.



Labelled diagram of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

Let us explain how to associate to a (drawn) diagram a packed matrix. The white (resp. black) spots are labelled from 1 to r (resp. 1 to c) from top to bottom and the number of lines from the i-th white spot to the j-th black spot is exactly the entry a_{ij} of the matrix. Conversely, a packed matrix of dimension $r \times c$ being given, one draws r white spots (resp. c black spots) and (with the labelling as above) join the i-th white spot to the j-th black by a_{ij} lines. This gives exactly the one-to-one correspondence between (drawn) diagrams and packed matrices.

In the sequel, we set $Diag_u := Dg_u \circ IM_u$ and $Diag_o := Dg_o \circ IM_o$ for the mappings which associate diagrams to pairs of partitions. Now, the multiplicity of a diagram $\mathcal{D}$ is the number of pairs $(P^{(1)}, P^{(2)})$ of unordered partitions such that $Dg_u(IM_u(P^{(1)}, P^{(2)})) = \mathcal{D}$.
Let us call bitype of a diagram $\mathcal{D}$ the pair $(\alpha(P^{(1)}), \alpha(P^{(2)}))$ where $Dg_u(IM_u(P^{(1)}, P^{(2)})) = \mathcal{D}$ (remark that it does not depend on the choosen premiage inside the formula) and we will refer it as the bitype $(\alpha(\mathcal{D}), \beta(\mathcal{D})))$ of $\mathcal{D}$. In a similar way $\alpha(\mathcal{D})$ (resp. $\beta(\mathcal{D})$) will be called the left (resp. the right) type of $\mathcal{D}$ and $|\mathcal{D}|$ $(= |\alpha(\mathcal{D})| = |\beta(\mathcal{D})|)$ will denote the number of edges of $\mathcal{D}$.
The product formula now reads

$$exp\Big(\sum_{m\geq 1} \frac{L_m}{m!}\left(y\frac{\partial}{\partial x}\right)^m\Big)exp\Big(\sum_{n\geq 1}\frac{V_n}{n!}x^n\Big)\Big|_{x=0} =$$

$$\sum_{n\geq 0}\frac{y^n}{n!}\Big(\sum_{\substack{\mathcal{D}\text{diagram}\\ |\mathcal{D}|=\text{n}}} mult(\mathcal{D})\mathbb{L}^{\alpha(\mathcal{D})}\mathbb{V}^{\beta(\mathcal{D})}\Big) = \sum_{\mathcal{D}\ \text{diagram}} \frac{mult(\mathcal{D})}{|\mathcal{D}|!} m(\mathcal{D}, \mathbb{L}, \mathbb{V}, y) \tag{27}$$

with

$$m(\mathcal{D}, \mathbb{L}, \mathbb{V}, y) := \mathbb{L}^{\alpha(\mathcal{D})}\mathbb{V}^{\beta(\mathcal{D})}y^{|\mathcal{D}|}. \tag{28}$$

3. Diagrammatic expansion of the double exponential formula
The main interest of the expansion (27) is that we can impose (at least) two types of rules on the diagrams

- on the diagrams themselves (selection rules) : on the outgoing degrees, ingoing degrees, total or partial weights (the graph is supposed oriented from white to black spots)
- on the set of diagrams (composition and decomposition rules): product and coproduct on the space of diagrams.

We have already such a structure on the space of monomials (i.e. the polynomials). The (usual) product of polynomials is well known and amounts to the addition of the multidegrees. The (usual) coproduct is given by the substitution of a “doubled” variable to each variable [4, 20]. For example, with $P = x^2y^3$, we first form $(x_1 + x_2)^2(y_1 + y_2)^3$, expand and then separate (on the left) the “1” labelled variables and (on the right) the “2” labelled. As

$$\begin{aligned} P = {} & x_1^2y_1^3 + 3x_1^2y_1^2y_2 + 3x_1^2y_1y_2^2 + x_1^2y_2^3 + 2x_1y_1^3x_2 + 6x_1y_1^2x_2y_2 + \\ & 6x_1y_1x_2y_2^2 + 2x_1x_2y_2^3 + y_1^3x_2^2 + 3y_1^2x_2^2y_2 + 3y_1x_2^2y_2^2 + x_2^2y_2^3 \end{aligned} \tag{29}$$

one gets, with Δ the coproduct operator,

$$\begin{aligned} \Delta(P) = {} & x^2y^3 \otimes 1 + 3x^2y^2 \otimes y + 3x^2y \otimes y^2 + x^2 \otimes y^3 + \\ & 2xy^3 \otimes x + 6xy^2 \otimes xy + 6xy \otimes xy^2 + 2x \otimes xy^3 + \\ & y^3 \otimes x^2 + 3y^2 \otimes x^2y + 3y \otimes x^2y^2 + 1 \otimes x^2y^3. \end{aligned} \tag{30}$$

The space of polynomials with product and coproduct (and other items like neutrals, coneutrals and antipode, which will be made more precise in the next paragraph) is endowed with the structure of a Hopf algebra.
The last consideration suggests the following question:

Is it possible to structure the (spaces of) diagrams into a Hopf algebra ?

Is it possible that this structure be compatible, in some sense, with the mapping $(\mathcal{D}, \mathbb{L}, \mathbb{V}, y) \to m(\mathcal{D}, \mathbb{L}, \mathbb{V}, y)$ *?*

Answer is yes. To establish it, we have to proceed in three steps.

- First Step : Define the space(s)
- Second Step : Define a product
- Third Step : Define a coproduct

3.1. Algebra structure

First step. — Let $Diag_{\mathbb{C}}$ (resp. $LDiag_{\mathbb{C}}$) be the $\mathbb{C}$-vector space freely generated by the diagrams (resp. labelled diagrams) i.e.

$$Diag_{\mathbb{C}} := \bigoplus_{d \text{ diagram}} \mathbb{C} \qquad\qquad LDiag_{\mathbb{C}} := \bigoplus_{d \text{ labelleddiagram}} \mathbb{C}d \tag{31}$$

at this stage, we have a linear mapping (linear arrow) $LDiag_{\mathbb{C}} \to Diag_{\mathbb{C}}$ provided by the linear extension of Φ_d and an arrow (linear, by construction) $m(., \mathbb{L}, \mathbb{V}, z) : Diag_{\mathbb{C}} \to \mathbb{C}[\mathbb{L} \cup \mathbb{V} \cup \{z\}]$ provided by the linear extension of $m(., \mathbb{L}, \mathbb{V}, z)$.

Second step. — We remark that, if

$$d_1 \star d_2 = \boxed{\begin{array}{c} d_1 \\ d_2 \end{array}} \tag{32}$$

denotes the superposition of the diagrams, then

$$m(d_1 \star d_2, \mathbb{L}, \mathbb{V}, z) = m(d_1, \mathbb{L}, \mathbb{V}, z)m(d_2, \mathbb{L}, \mathbb{V}, z). \tag{33}$$

The law (32) makes sense as well for labelled and unlabelled diagrams. In the first case, it amounts to computing the blockdiagonal product of packed matrices. Indeed, for $M_1,\ M_2$ being packed matrices, one has

$$Dg_o\left(\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\right) = Dg_o(M_1) \star Dg_o(M_2). \tag{34}$$

This product yields the product of monomials in the following way. From $\mathcal{D}$ a diagram and all the other parameters fixed, with the setting of (28), we get a polynomial.

The product (32) is associative with unit (the empty diagram), it is compatible with the arrow Φ_d and so defines the product on *Diag* which, in turn is compatible with the product of monomials.

$$\begin{array}{ccccc} \text{Labelled diagrams}^2 & \xrightarrow{\Phi_d\times\Phi_d} & \text{Diagrams}^2 & \xrightarrow{m(?,\mathbb{L},\mathbb{V},z)\times m(?,\mathbb{L},\mathbb{V},z)} & \text{Monomials}^2 \\ \text{product}\big\downarrow & & \text{product}\big\downarrow & & \text{product}\big\downarrow \\ \text{Labelled diagrams} & \xrightarrow{\Phi_d} & \text{Diagrams} & \xrightarrow{m(?,\mathbb{L},\mathbb{V},z)} & \text{Monomials} \end{array} \tag{35}$$

Remark 3.1 *One sees easily that the set of labelled diagrams (resp. diagrams) is closed under products and therefore is a monoid. The spaces $LDiag_{\mathbb{C}}$ and $Diag_{\mathbb{C}}$ are thus algebras of these monoids [1, 2, 4].*

3.2. Admissible coproducts

For the coproduct on *LDiag*, we have several possibilities:

(i) split with respect to the white spots (two ways : by intervals and by subsets)

(ii) split with respect to the black spots (two ways : by intervals and by subsets)

(iii) split with respect to the edges

The discussion goes as follows:

a) (iii) does not give a nice identity with the monomials (when applying $d \mapsto m(d, ?, ?, ?)$) nor do (ii) and (iii) by intervals.

b) (i) and (ii) are essentially the same (because of the WS $\leftrightarrow$ BS symmetry).

In fact (i) and (ii) by subsets give a good representation and, moreover, they are appropriate for several physics models.

In the next section, we develop the possibility (i) and (ii) by subsets.

4. Hopf algebra structures associated with Δ_{WS} and Δ_{BS}

4.1. The philosophy of bi- and Hopf algebras thru representation theory

Let $\mathcal{A}$ be a k-algebra (k is a field as $\mathbb{R}$ or $\mathbb{C}$). In this paragraph, we consider associative algebras with unit (AAU). A representation of $\mathcal{A}$ is here a pair (V, ρ_V) where V is a k-vector space and $\rho_V : \mathcal{A} \to End_k(V)$ a morphism of k-algebras (AAU).

One can make operations with representations as direct sums and quotient of a representation by a sub-representation (a sub-representation is a subspace which is closed under the action of $\mathcal{A}$). In general, one does not know how to endow the tensor product (of two representations) and the dual (of a representation) with the structure of representation.

It is however classical in two cases: groups and Lie algebras.

If G is a group, a representation of G is a pair (V, ρ_V) where V is a k-vector space and $\rho_V : G \to Aut_k(V)$ a morphism of groups. If G is a Lie algebra, a representation of G is a pair (V, ρ_V) where V is a k-vector space and $\rho_V : G \to End_k(V)$ a morphism of Lie algebras

(i.e. $\rho_V([u,v]) = \rho_V(u)\rho_V(v) - \rho_V(v)\rho_V(u)$). These two cases enter the scheme of (AAU) as a representation of a group can be extended uniquely as a representation of its algebra kG and a representation of a Lie algebra as a representation of $\mathcal{U}_k(G)$, its envelopping algebra. These two constructions (kG and $\mathcal{U}_k(G)$) are (AAU).
For the sake of readibility let us denote in all cases $\rho_V(g)(u)$ by $g.u$ ($g \in G$ and $u \in V$).
If G is a group and V, W two representations, we construct a representation of G on $V \otimes W$ by

$$g.(u \otimes v) = g.u \otimes g.v \tag{36}$$

If G is a Lie algebra and V, W two representations, we construct a representation of G on $V \otimes W$ by

$$g.(u \otimes v) = g.u \otimes v + u \otimes g.v \tag{37}$$

This can be rephrased in saying that the action of g in the first case (group) is $g \otimes g$ and in the second (Lie algebra) $g \otimes 1 + 1 \otimes g$ (1 is here for the appropriate identity mapping). In the two cases, it amounts to give a linear mapping $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which will be called a coproduct.
One can show [6] that, if we want that this new operation enjoy "nice" properties (associativity of the tensor product etc...), one has to suppose that this coproduct is a morphism of (AAU) ($\mathcal{A} \otimes \mathcal{A}$ has received the structure of - non twisted - tensor product of algebras), that it is coassociative with a counit [6]. Let us make these requirements more precise.
The first says that for all $x, y \in \mathcal{A}$ one has $\Delta(xy) = \Delta(x)\Delta(y)$, the second that the two compositions

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\Delta \otimes 1_{\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \text{ and } \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{1_{\mathcal{A}} \otimes \Delta} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \tag{38}$$

are equal, the third says that there is a mapping (linear form) $\varepsilon : \mathcal{A} \to k$ such that the compositions

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\Delta \otimes \varepsilon} \mathcal{A} \otimes k \xrightarrow{\text{nat}} \mathcal{A} \text{ and } \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\varepsilon \otimes \Delta} k \otimes \mathcal{A} \xrightarrow{\text{nat}} \mathcal{A} \tag{39}$$

(where nat is for the natural mappings) are equal to the identity $1_{\mathcal{A}}$.
An algebra (AAU) together with a coproduct Δ and a counity ε which fulfills the three requirements above is called a **bialgebra**.

If, moreover one wants to have a nice dualization of the representations (i.e. nice structures for the duals $V^* = Hom(V, k)$), it should exist an element of $Hom(\mathcal{A}, \mathcal{A})$ such that the compositions

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\alpha \otimes 1_{\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A} \text{ and } \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{1_{\mathcal{A}} \otimes \alpha} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A} \tag{40}$$

are equal to $e_A \varepsilon$ (where e_A denotes the unit of A). When a bialgebra possesses such an element (unique), it is called the **antipode** and the bialgebra a **Hopf algebra**. For more details and connections to physics, one can consult [6].

One can prove that the bialgebras constructed below possess an antipode and then are Hopf algebras.

4.2. Bialgebra structures on LDiag and Diag
The space spanned by the packed matrices has already received a structure of Hopf algebra, the algebra **MQSym** [19]. We briefly review the structure of this Hopf algebra.
We describe in details Δ_{WS} as the other coproduct is actually got by the same process but applied on the columns instead of the lines. Let M be a packed matrix of dimensions $k_1 \times k_2$ for every subset $X \in [1..k_1]$ we consider the matrix $\pi_X(M) := pack(M[X, [1..k_2]])$, the restriction

to the lines of X and then packed (with this restriction to the lines, we only need to perform a horizontal packing). Thus, the coproduct Δ_{WS} reads

$$\Delta_{WS}(M) = \sum_{X+Y=[1..k_1]} \pi_X(M) \otimes \pi_Y(M) \tag{41}$$

To avoid confusion we will call the supporting space $\mathcal{H}_{WS}$ (= **MQSym**). We keep the (total) grading of **MQSym** by the total weight (i.e. the sum of the coefficients) of the matrices. The packed matrices are a linear basis of $\mathcal{H}_{WS} = \mathbf{MQSym}$, thus every element expresses uniquely

$$x = \sum_{M \text{ packed}} \lambda_M(x) M \tag{42}$$

The coproduct above is cocommutative and with counit $\lambda_\emptyset$ where $\emptyset$ is the void matrix corresponding to the void diagram. This particular matrix will be denoted $\mathbf{1}_{\mathcal{H}_{WS}}$.
For example, with the packed matrix above one has

$$\begin{aligned}\Delta_{WS}(\begin{pmatrix}2&0\\0&2\\1&1\end{pmatrix}) \quad = \quad & \begin{pmatrix}2&0\\0&2\\1&1\end{pmatrix} \otimes \mathbf{1}_{\mathcal{H}_{WS}} + \begin{pmatrix}2\end{pmatrix} \otimes \begin{pmatrix}0&2\\1&1\end{pmatrix} + \begin{pmatrix}2\end{pmatrix} \otimes \begin{pmatrix}2&0\\1&1\end{pmatrix} + \\ & \begin{pmatrix}1&1\end{pmatrix} \otimes \begin{pmatrix}2&0\\0&2\end{pmatrix} + \begin{pmatrix}2&0\\0&2\end{pmatrix} \otimes \begin{pmatrix}1&1\end{pmatrix} + \\ & \begin{pmatrix}2&0\\1&1\end{pmatrix} \otimes \begin{pmatrix}2\end{pmatrix} + \begin{pmatrix}0&2\\1&1\end{pmatrix} \otimes \begin{pmatrix}2\end{pmatrix} + \mathbf{1}_{\mathcal{H}_{WS}} \otimes \begin{pmatrix}2&0\\0&2\\1&1\end{pmatrix}\end{aligned}$$

This coproduct is compatible with the usual coproduct on the monomials for the constant alphabet $\mathbb{V} = \mathbf{1}_{\mathbb{N}}$ defined by $V_n = 1$ for all $n \geq 0$. Then, using Sweedler's notation, for this particular $\mathbb{V}$, if $\Delta_{WS}(d) = \sum d_{(1)} \otimes d_{(2)}$, one has

$$m(d, \mathbb{L}' + \mathbb{L}'', \mathbf{1}_{\mathbb{N}}, z) = \sum m(d_{(1)}, \mathbb{L}', \mathbf{1}_{\mathbb{N}}, z) m(d_{(2)}, \mathbb{L}'', \mathbf{1}_{\mathbb{N}}, z) \tag{43}$$

Thus, one sees easily that, with this structure (product with unit, coproduct and the counit), $LDiag_{\mathbb{C}}$ is a bialgebra graded in finite dimensions and then a Hopf algebra.
The arrow $LDiag_{\mathbb{C}} \to Diag_{\mathbb{C}}$ endows $Diag_{\mathbb{C}}$ with a structure of Hopf algebra.

5. Conclusion

The structure of the Hopf algebras $LDiag_{\mathbb{C}}, Diag_{\mathbb{C}}$, by a theorem of Cartier, Milnor and Moore [5, 22], is that of envelopping algebras of their primitive elements ($Diag_{\mathbb{C}}$, being commutative, is thus an algebra of polynomials).
Moreover, it appears that the structure described above is the starting point for a series of connections with mathematical and physical Hopf algebras. The coproduct Δ_{WS} is the cristallisation ($q = 1$) of a one-parameter deformation of coproducts (all coassociative) on $LDiag_{\mathbb{C}} \simeq \mathbf{MQSym}$, the other end ($q = 0$) being an infinitesimal coproduct isomorphic to $\Delta_{\mathbf{MQSym}}$. Recently, **FQSym** (a subalgebra of **MQSym**) has been established by Foissy [10] as a case in a family of Hopf algebras of decorated planar trees which is strongly related to other Hopf algebras like Connes-Kreimer's and Connes-Moscovici's [10, 11].

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