Icosahedron designs

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A. D. Forbes   T. S. Griggs
Department of Mathematics and Statistics
The Open University
Walton Hall, Milton Keynes MK7 6AA
U.K.

Abstract

It is known from the work of Adams and Bryant that icosahedron designs of order v exist for $v \equiv 1 \pmod{60}$ as well as for $v = 16$. Here we prove that icosahedron designs exist if and only if $v \equiv 1$, 16, 21 or 36 (mod 60), with possible exceptions $v = 21$, 141, 156, 201, 261 and 276.

1 Introduction

The spectrum of integers v for which the complete graph K_v can be decomposed into copies of the graph of one of the Platonic solids is determined for the tetrahedron, octahedron, cube and dodecahedron but only partial results are available for the icosahedron. The current state of knowledge, see also [3], [5], appears to be as follows.

1. Tetrahedron designs are equivalent to Steiner systems $S(2, 4, v)$. The necessary and sufficient condition is $v \equiv 1$ or 4 (mod 12), [9].

2. Octahedron designs are equivalent to Steiner triple systems $S(2, 3, v)$ which can be decomposed into Pasch configurations. The necessary and sufficient condition is $v \equiv 1$ or 9 (mod 24), $v \neq 9$, [8], [1].

3. Cube designs exist if and only if $v \equiv 1$ or 16 (mod 24), [12], [11], [6].

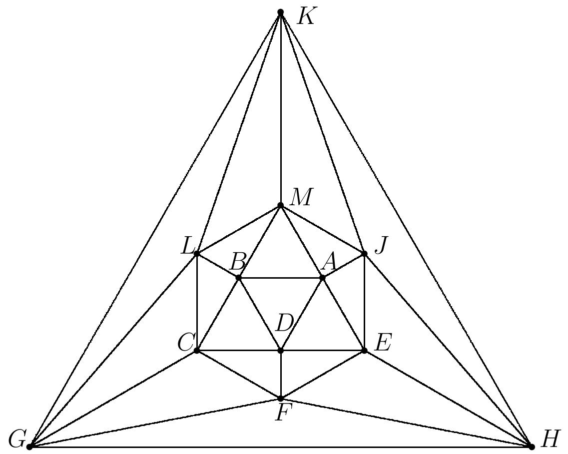
4. Dodecahedron designs exist if and only $v \equiv 1$, 16, 25 or 40 (mod 60) and $v \neq 16$, [2], [3], [4].

5. A necessary condition for the existence of icosahedron designs is $v \equiv 1$, 16, 21 or 36 (mod 60). Prior to this paper, they are known to exist for $v \equiv 1$ (mod 60), [2], and for $v = 16$, [3].

The purpose of this paper is to complete the existence spectrum for the icosahedron, with six possible exceptions. Specifically, we prove the theorem below. Our method is quite general and is applicable for all four residue classes given by the necessary condition. Therefore we include as part of the proof the residue class 1 (mod 60), already done by Adams and Bryant in [2], both for completeness and as an interesting alternative.

Theorem 1.1 *Icosahedron designs exist if* $v \equiv 1 \pmod{60}$, *or if* $v \equiv 16 \pmod{60}$, *or if* $v \equiv 21 \pmod{60}$ *with possible exceptions* $v = 21$, 141, 201 *and* 261, *or if* $v \equiv 36 \pmod{60}$ *with possible exceptions* $v = 156$ *and* 276.

The icosahedron graph has 12 vertices and 30 edges, and we will represent it by an ordered 12-tuple (A, B, C, D, E, F, G, H, J, K, L, M), where the co-ordinates represent vertices, as in the diagram.



Our method of proof uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall that a K-GDD of type u^t is an ordered triple (V, G, B) where V is a base set of cardinality $v = tu$, G is a partition of V into t subsets of cardinality u called *groups* and B is a collection of subsets of cardinalities $k \in K$ called *blocks* which collectively have the property that each pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. When $K = \{k\}$ consists of a single number, we refer to the design as a k-GDD. We will also need K-GDDs of type $u^t w^1$, where $w \neq u$. These are defined analogously, with the base set V being of cardinality $tu + w$ and the partition G being into t subsets of cardinality u and one subset of cardinality w. A *parallel class* in a group divisible design is a subset of the block set in which each element of the base set appears exactly once. A K-GDD is called *resolvable*, and denoted by K-RGDD, if the entire set of blocks can be partitioned into parallel classes.

A *Steiner system* $S(2,k,v)$, also called a balanced incomplete block design (BIBD) with parameters $(v,k,1)$, is an ordered pair (V,B) where V is the base set and B is the block set of a k-GDD of type 1^v. Observe that if $x \in V$ and B_x is the set of blocks containing x, then $(V \setminus \{x\}, \{b \setminus \{x\} : b \in B_x\}, B \setminus B_x)$ is a k-GDD of type $(k-1)^{(v-1)/(k-1)}$. Moreover, if the Steiner system has a parallel class G, say, then $(V, G, B \setminus G)$ is a k-GDD of type $k^{v/k}$. As is well known, a Steiner system $S(2,k,k^2)$, also called an *affine plane* of order k, is resolvable and exists whenever k is a prime power. More generally, a k-GDD of type n^k exists whenever there exist $k-2$ mutually orthogonal Latin squares (MOLS) of side n.

2 The main construction

The principal result of this section is Proposition 2.1, below. Here we are able to prove Theorem 1.1 with a relatively small number of possible exceptions most of which are disposed of individually in Section 3. Our first lemma is a summary of known results. This is followed by new decompositions that will be used as ingredients for our main construction.

Lemma 2.1 *(i) (Adams and Bryant) Icosahedron designs exist for all* $v \equiv 1 \pmod{60}$. *Moreover, the complete* 4*-partite graph* $K_{20,20,20,20}$ *can be decomposed into* 80 *icosahedra and the complete* 5*-partite graph* $K_{15,15,15,15,15}$ *can be decomposed into* 75 *icosahedra.*
(ii) (Adams, Bryant and Buchanan) There exists a decomposition of K_{16} *into* 4 *icosahedra.*

Proof. See [2] and [3]. For the three graphs mentioned in the statement of the lemma we give here our own icosahedron decompositions aligned to the diagram in Section 1.

K_{16}. Let the vertex set of the graph be Z_{16}. The decomposition consists of the icosahedra

$(7,6,11,8,10,0,14,2,12,13,9,5)$,

$(8,14,3,5,2,1,0,6,4,13,7,15)$,

$(0,4,3,9,12,8,13,1,15,11,10,5)$,

$(1,4,11,7,9,2,3,15,10,6,12,14)$.

$K_{20,20,20,20}$. Let the vertex set of the graph be Z_{80} partitioned according to residue classes modulo 4. The decomposition consists of the icosahedron

$(0,1,3,6,55,76,49,10,33,75,12,62)$

under the action of the mapping $i \mapsto i+1 \pmod{80}$.

$K_{15,15,15,15,15}$. Let the vertex set of the graph be Z_{75} partitioned according to residue classes modulo 5. The decomposition consists of the icosahedron

$(0,1,3,7,18,66,27,35,61,64,55,42)$

under the action of the mapping $i \mapsto i+1 \pmod{75}$. □

Lemma 2.2 *There exists an icosahedron design of order* 81.

Proof. Let the vertex set of the complete graph K_{81} be Z_{81}. The decomposition consists of the icosahedra

$(24, 14, 6, 40, 56, 18, 1, 10, 71, 34, 3, 75)$,

$(9, 61, 29, 75, 38, 40, 41, 66, 28, 27, 48, 72)$,

$(17, 33, 27, 0, 26, 5, 50, 8, 64, 22, 37, 58)$,

$(27, 4, 34, 7, 14, 65, 38, 71, 76, 69, 16, 68)$

under the action of the mapping $i \mapsto i + 3 \pmod{81}$. $\square$

Lemma 2.3 *The complete* 4*-partite graph* $K_{15,15,15,15}$ *can be decomposed into* 45 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{60} partitioned according to residue classes modulo 4. The decomposition consists of the icosahedra

$(23, 21, 55, 20, 41, 42, 16, 27, 34, 49, 30, 40)$,

$(25, 23, 57, 22, 43, 44, 18, 29, 36, 51, 32, 42)$,

$(0, 37, 14, 31, 2, 44, 1, 7, 57, 30, 32, 27)$

under the action of the mapping $i \mapsto i + 4 \pmod{60}$. $\square$

Lemma 2.4 *The complete* 8*-partite graph* K_{15^8} *can be decomposed into* 210 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 119\}$ partitioned into $\{i+7j : j = 0, 1, \ldots, 14\}$, $i = 0, 1, \ldots, 6$, together with $\{105, 106, \ldots, 119\}$. The decomposition consists of the icosahedra

$(24, 97, 5, 115, 91, 10, 7, 75, 41, 32, 36, 109)$,

$(82, 0, 15, 72, 13, 109, 98, 5, 24, 44, 45, 18)$

under the action of the mapping $i \mapsto i + 1 \pmod{105}$ for $i < 105$, $i \mapsto (i + 1 \pmod{15}) + 105$ for $i \geq 105$. $\square$

Lemma 2.5 *The complete* 9*-partite graph* K_{15^9} *can be decomposed into* 270 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{135} partitioned according to residue classes modulo 9. The decomposition consists of the icosahedra

$(25, 98, 95, 40, 63, 124, 58, 23, 4, 115, 28, 111)$,

$(50, 40, 26, 18, 114, 20, 70, 130, 97, 77, 82, 51)$

under the action of the mapping $i \mapsto i + 1 \pmod{135}$. $\square$

Lemma 2.6 *The complete* 12*-partite graph* $K_{15^{12}}$ *can be decomposed into* 495 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0,1,\ldots,179\}$ partitioned into $\{i+11j : j=0,1,\ldots,14\}$, $i=0,1,\ldots,10$, together with $\{165,166,\ldots,179\}$. The decomposition consists of the icosahedra

$(34,29,11,135,57,26,93,107,77,61,175,137)$,

$(99,73,176,80,171,141,113,137,139,21,121,100)$,

$(88,79,5,85,157,17,80,46,140,10,117,130)$

under the action of the mapping $i \mapsto i+1 \pmod{165}$ for $i<165$, $i \mapsto (i+1 \pmod{15})+165$ for $i \geq 165$. □

Lemma 2.7 *The complete* 13*-partite graph* $K_{15^{13}}$ *can be decomposed into* 585 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{195} partitioned according to residue classes modulo 13. The decomposition consists of the icosahedra

$(93,87,108,19,180,79,86,69,18,106,83,173)$,

$(146,87,51,23,64,121,60,18,6,62,105,135)$,

$(171,75,40,102,179,55,93,174,155,62,125,76)$

under the action of the mapping $i \mapsto i+1 \pmod{195}$. □

Lemma 2.8 *The complete* 5*-partite graph* $K_{20,20,20,20,15}$ *can be decomposed into* 120 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0,1,\ldots,94\}$ partitioned into $\{i+4j : j=0,1,\ldots,19\}$, $i=0,1,2,3$, together with $\{80,81,\ldots,94\}$. The decomposition consists of the icosahedra

$(16,25,78,15,84,20,53,14,63,12,75,62)$,

$(83,23,49,4,57,46,20,81,39,45,88,8)$,

$(71,0,69,10,33,67,2,86,78,64,85,5)$

under the action of the mapping $i \mapsto i+2 \pmod{80}$ for $i<80$, $i \mapsto (i-80+3 \pmod{15})+80$ for $i \geq 80$. □

Lemma 2.9 *There exist icosahedron designs of order v for $v=61$, 76, 121, 136, 181 and 196.*

Proof. Our proofs for $v=61$, 121 and 181 are provided for completeness and as alternatives to those given in [2].

Construct a complete graph K_{61} as follows. Take the complete 4-partite graph K_{15^4}, add an extra point, ∞, and overlay a complete graph K_{16} on each partition augmented by ∞. Since we have decompositions into icosahedra of K_{16} from Lemma 2.1 and K_{15^4} from Lemma 2.3, the required decomposition of K_{61} is achieved. The constructions for $v=76$, 121, 136, 181 and 196 are similar and use the decompositions of K_{15^5}, K_{15^8}, K_{15^9}, $K_{15^{12}}$ and $K_{15^{13}}$ from Lemmas 2.1, 2.4, 2.5, 2.6 and 2.7 respectively. □

We are now ready to state and prove the main result of this section.

Proposition 2.1 *(i) There exist icosahedron designs of order v for $v = 180t+15w+61$ if $w \in \{0,1,4,5,8,9\}$ and $t \geq w/4$.*
(ii) There exist icosahedron designs of order v for $v = 240t + 15w + 81$ if $w \in \{0,1,4,5,8,9,12,13\}$ and $t \geq w/4$.

Proof. If $t = 0$, then $w = 0$ and (i) and (ii) follow from Lemmas 2.9 and 2.2 respectively. So we assume henceforth that $t \geq 1$. There exists a 4-RGDD of type 4^{3t+1} for $t \geq 1$, [10], see also [7], and a simple computation establishes that it has $4t$ parallel classes of $3t + 1$ blocks each. Let $w \in \{0,1,4,5,8,9,12,13\}$ and assume that $w \leq 4t$. If $w > 0$, add a new group of w points, associate with each new point a distinct parallel class and extend each of its blocks by adding the point to them, thus creating $w(3t + 1)$ five-element blocks. This is possible since w does not exceed the number of parallel classes of the 4-RGDD. Thus we have created:

nothing new if $w = 0$,

a $\{4,5\}$-GDD of type $4^{3t+1}w^1$ if $1 \leq w < 4t$,

a 5-GDD of type $4^{3t+1}w^1$ if $w = 4t$.

Take this design as the GDD for Wilson's construction. Let i be a positive integer. Replace each point that was in the base set of the original 4-RGDD by i elements (i.e. inflate by a factor of i). Inflate each new point by a factor of 15. Add a further point, ∞. Lay a complete graph K_{4i+1} on each of the original, i-inflated groups together with ∞ and lay a complete graph K_{15w+1} on the new, 15-inflated group together with ∞. If $w < 4t$, replace each remaining original 4-element block by a complete 4-partite graph $K_{i,i,i,i}$. If $w > 0$, replace each 5-element block containing four original points and one new point by a complete 5-partite graph $K_{i,i,i,i,15}$. If icosahedron decompositions of all of the relevant graphs exist, then this construction yields a design of order $12it + 4i + 15w + 1$ for $t \geq w/4$.

To prove part (i), we set $i = 15$ and use the icosahedron decompositions of K_{16} and K_{15^5} from Lemma 2.1, K_{61}, K_{76}, K_{121} and K_{136} from Lemma 2.9 and $K_{15,15,15,15}$ from Lemma 2.3. For part (ii), we set $i = 20$ and the additional icosahedron decompositions needed are of K_{81} from Lemma 2.2, $K_{20,20,20,20}$ from Lemma 2.1, $K_{20,20,20,20,15}$ from Lemma 2.8 and K_{181} and K_{196} from Lemma 2.9. □

Table 1 gives the details of how Proposition 2.1 can be used to find an icosahedron design of order v for each $v \equiv 1$, 16, 21 or 36 (mod 60), except for those values stated as missing. To complete the proof of Theorem 1.1, we have only to deal with the relevant missing values.

3 The missing values

In this section we simply work our way through as many as possible of the missing values stated in Table 1. We already have an icosahedron decomposition of K_v for

Table 1: Proposition 2.1

i	w	minimum t	$12it + 4i + 15w + 1$	missing values
15	0	0	$180t + 61$	–
15	4	1	$180t + 121$	121
15	8	2	$180t + 181$	1, 181, 361
15	1	1	$180t + 76$	76
15	5	2	$180t + 136$	136, 316
15	9	3	$180t + 196$	16, 196, 376, 556
20	0	0	$240t + 81$	–
20	4	1	$240t + 141$	141
20	8	2	$240t + 201$	201, 441
20	12	3	$240t + 261$	21, 261, 501, 741
20	1	1	$240t + 96$	96
20	5	2	$240t + 156$	156, 396
20	9	3	$240t + 216$	216, 456, 696
20	13	4	$240t + 276$	36, 276, 516, 756, 996

$v = 16$ from Lemma 2.1 as well as for $v = 76$, 121, 136, 181 and 196 from Lemma 2.9, and the empty set provides the icosahedron design of order 1. In the residue class 1 (mod 60), this leaves only $v = 361$ unresolved. It is contained in [2] but repeated here for completeness.

Lemma 3.1 *There exists an icosahedron design of order* 361*.*

Proof. Create a 5-GDD of type 4^6 by removing a point from an affine plane of order 5. Inflate each point by a factor of 15 and add an extra point, ∞. On each inflated group together with ∞ place the icosahedron design of order 61 from Lemma 2.9 and replace each block of the 5-GDD by the icosahedron decomposition of $K_{15,15,15,15,15}$ from Lemma 2.1. □

We next deal with the residue class 16 (mod 60). There are three unresolved values: $v = 316$, 376 and 556.

Lemma 3.2 *There exists an icosahedron design of order* 316*.*

Proof. There exists a 5-GDD of type 1^u if $u \geq 5$ and $u \equiv 1$ or 5 (mod 20), [9]. In particular, a 5-GDD of type 1^{21} exists. Inflate each point by a factor of 15 and adjoin a further element, ∞. On each inflated group together with ∞ place the icosahedron design of order 16 from Lemma 2.1. Replace each block by the icosahedron decomposition of $K_{15,15,15,15,15}$ from Lemma 2.1. □

Lemma 3.3 *There exist icosahedron designs of order v for $v = 376$ and* 556*.*

Proof. There exists a 4-GDD of type 1^{12t+1}, $t \geq 1$, [9]. Inflate each element of the base set by a factor of 15 and adjoin a further element, ∞. On each inflated group together with ∞ place the icosahedron design of order 16 from Lemma 2.1. Replace each block by the icosahedron decomposition of $K_{15,15,15,15}$ from Lemma 2.3. This construction actually creates icosahedron designs of order $180t+16$ for $t \geq 1$ of which we require only cases $t = 2$ and 3. □

Lemmas 3.2 and 3.3 complete the proof of Theorem 1.1 for residue class 16 modulo 60. In order to deal with some of the unresolved values in the remaining two residue classes we need several further decompositions.

Lemma 3.4 *The complete* 5*-partite graph* $K_{20,20,20,20,30}$ *can be decomposed into* 160 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 109\}$ partitioned into $\{i + 4j : j = 0, 1, \ldots, 19\}$, $i = 0, 1, 2, 3$, together with $\{80, 81, \ldots, 109\}$. The decomposition consists of the icosahedra

$(47, 93, 25, 2, 104, 11, 89, 53, 74, 24, 50, 13)$,

$(24, 41, 35, 34, 73, 82, 68, 71, 108, 6, 109, 19)$

under the action of the mapping $i \mapsto i + 1 \pmod{80}$ for $i < 80$, $i \mapsto (i - 80 + 3 \pmod{30}) + 80$ for $i \geq 80$. □

Lemma 3.5 *The complete* 6*-partite graph* $K_{2,2,2,2,2,2}$ *can be decomposed into two icosahedra.*

Proof. Let the vertex set of the graph be Z_{12} partitioned according to residue classes modulo 6. The decomposition consists of the icosahedra

$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$,

$(0, 2, 4, 9, 5, 1, 6, 8, 10, 3, 11, 7)$. □

Lemma 3.6 *The complete* 6*-partite graph* $K_{4,4,4,4,4,4}$ *can be decomposed into* 8 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{24} partitioned according to residue classes modulo 6. The decomposition consists of the icosahedra

$(19, 10, 3, 5, 15, 2, 16, 7, 12, 8, 6, 17)$,

$(23, 14, 7, 9, 19, 6, 20, 11, 16, 12, 10, 21)$,

$(3, 18, 11, 13, 23, 10, 0, 15, 20, 16, 14, 1)$,

$(0, 1, 8, 21, 13, 5, 9, 16, 17, 18, 10, 2)$

under the action of the mapping $i \mapsto i + 12 \pmod{24}$. □

Lemma 3.7 *The complete* 6*-partite graph* $K_{10,10,10,10,10,10}$ *can be decomposed into* 50 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 59\}$ partitioned into $\{i + 5j : j = 0, 1, \ldots, 9\}$, $i = 0, 1, \ldots, 4$, together with $\{50, 51, \ldots, 59\}$. The decomposition consists of the icosahedron

$(41, 32, 29, 13, 45, 50, 0, 2, 3, 39, 6, 55)$

under the action of the mapping $i \mapsto i+1 \pmod{50}$ for $i < 50$, $i \mapsto (i+1 \pmod{10})+50$ for $i \geq 50$. □

Lemma 3.8 *The complete* 7*-partite graph* K_{10^7} *can be decomposed into* 70 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{70} partitioned according to residue classes modulo 7. The decomposition consists of the icosahedron

$(0, 1, 3, 6, 10, 56, 16, 64, 41, 53, 35, 26)$

under the action of the mapping $i \mapsto i + 1 \pmod{70}$. □

Lemma 3.9 *The complete* 7*-partite graph* $K_{10^6,5}$ *can be decomposed into* 60 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 64\}$ partitioned into $\{i + 6j : j = 0, 1, \ldots, 9\}$, $i = 0, 1, \ldots, 5$, together with $\{60, 61, \ldots, 64\}$. The decomposition consists of the icosahedron

$(10, 23, 13, 45, 54, 50, 21, 35, 37, 60, 24, 3)$

under the action of the mapping $i \mapsto i+1 \pmod{60}$ for $i < 60$, $i \mapsto (i+1 \pmod 5)+60$ for $i \geq 60$. □

Lemma 3.10 *The complete* 7*-partite graph* $K_{10^6,25}$ *can be decomposed into* 100 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 84\}$ partitioned into $\{i + 5j : j = 0, 1, \ldots, 9\}$, $i = 0, 1, \ldots, 4$, together with $\{50, 51, \ldots, 59\}$ and $\{60, 61, \ldots, 84\}$. The decomposition consists of the icosahedra

$(52, 24, 2, 64, 35, 14, 69, 38, 73, 46, 33, 47)$,

$(62, 3, 46, 59, 19, 35, 2, 68, 23, 54, 74, 5)$

under the action of the mapping $i \mapsto i+1 \pmod{50}$ for $i < 50$, $i \mapsto (i+1 \pmod{10})+50$ for $50 \leq i < 60$, $i \mapsto (i - 60 + 1 \pmod{25}) + 60$ for $i \geq 60$. □

Lemma 3.11 *The complete* 8*-partite graph* $K_{10^7,30}$ *can be decomposed into* 140 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 99\}$ partitioned into $\{i + 7j : j = 0, 1, \ldots, 9\}$, $i = 0, 1, \ldots, 6$, together with $\{70, 71, \ldots, 99\}$. The decomposition consists of the icosahedra

$(87, 2, 73, 3, 7, 41, 36, 90, 19, 45, 18, 21)$,

$(50, 71, 28, 13, 0, 82, 17, 62, 40, 86, 34, 11)$

under the action of the mapping $i \mapsto i + 1 \pmod{70}$ for $i < 70$, $i \mapsto (i - 70 + 3 \pmod{30}) + 70$ for $i \geq 70$. □

Lemma 3.12 *The complete* 12*-partite graph* $K_{5^{12}}$ *can be decomposed into* 55 *icosahedra.*

Proof. Let the vertex set of the graph be $\{0, 1, \ldots, 59\}$ partitioned into $\{i + 11j : j = 0, 1, \ldots, 4\}$, $i = 0, 1, \ldots, 10$, together with $\{55, 56, \ldots, 59\}$. The decomposition consists of the icosahedron

$$(38, 24, 0, 1, 8, 43, 55, 14, 48, 12, 16, 21),$$

under the action of the mapping $i \mapsto i+1 \pmod{55}$ for $i < 55$, $i \mapsto (i+1 \pmod 5)+55$ for $i \geq 55$. □

Lemma 3.13 *The complete* 13*-partite graph* $K_{5^{13}}$ *can be decomposed into* 65 *icosahedra.*

Proof. Let the vertex set of the graph be Z_{65} partitioned according to residue classes modulo 13. The decomposition consists of the icosahedra

$$(0, 1, 3, 6, 10, 17, 51, 29, 50, 13, 33, 42)$$

under the action of the mapping $i \mapsto i + 1 \pmod{65}$. □

There are seven values unresolved in the residue class 21 (mod 60). We are able to construct icosahedron designs for three of these values, leaving four possible exceptions.

Lemma 3.14 *There exists an icosahedron design of order* 441*.*

Proof. There exists a 5-GDD of type 4^5, i.e. a complete set of MOLS of side 4. Replace each point of one of the groups by 30 elements, replace all other points by 20 elements and add an extra point, ∞. On each inflated group of the 5-GDD together with ∞ place either the icosahedron design of order 121 from Lemma 2.1 or the icosahedron design of order 81 from Lemma 2.2, and replace each block by the icosahedron decomposition of $K_{20,20,20,20,30}$ from Lemma 3.4. □

Lemma 3.15 *There exists an icosahedron design of order* 501*.*

Proof. Take an affine plane of order 7 and remove a point to obtain a 7-GDD of type 6^8. Remove a further point. This creates a $\{6, 7\}$-GDD of type $6^7 5^1$.

Inflate each point in the seven 6-element groups by a factor of 10. In the 5-element group inflate three points by a factor of 10 and two points by 25. Add an extra point, ∞. On each inflated group of the $\{6, 7\}$-GDD together with ∞ place either the icosahedron design of order 61 from Lemma 2.1 or the icosahedron design of order 81 from Lemma 2.2. Replace each 6-element block by the icosahedron decomposition of K_{10^6} from Lemma 3.7. Replace each 7-element block by either the icosahedron decomposition of K_{10^7} from Lemma 3.8 or the icosahedron decomposition of $K_{10^6 25}$ from Lemma 3.10. □

Lemma 3.16 *There exists an icosahedron design of order* 741*.*

Proof. Take an affine plane of order 8 and select a parallel class as the groups of an 8-GDD of type 8^8. Remove two points from one group. This creates a $\{7,8\}$-GDD of type $8^7 6^1$.

Inflate each point in the seven 8-element groups by a factor of 10, inflate each point in the 6-element group by 30 and add ∞. On each inflated group of the $\{7,8\}$-GDD together with ∞ place either the icosahedron design of order 81 from Lemma 2.2 or the icosahedron design of order 181 from Lemma 2.1. Replace each 7-element block by the icosahedron decomposition of K_{10^7} from Lemma 3.8 and replace each 8-element block by the icosahedron decomposition of $K_{10^7 30}$ from Lemma 3.11. □

Lemmas 3.14–3.16 complete the proof of Theorem 1.1 for residue class 21 modulo 60. The main omission is of course 21 itself. Indeed, together with a few known decompositions in addition to those presented in this paper, the existence of an icosahedron design of order 21 would suffice to prove that icosahedron designs exist for all $v \equiv 21 \pmod{60}$. However, we have been unable to decide whether or not a decomposition of K_{21} exists.

Finally, we construct icosahedron designs for all but two of the missing values given in Table 1 for the residue class 36 (mod 60).

Lemma 3.17 *There exists an icosahedron design of order* 36.

Proof. Let the vertex set of the complete graph K_{36} be Z_{36}. The decomposition consists of the icosahedra

$(23, 26, 8, 30, 5, 16, 14, 17, 33, 15, 31, 0)$,
$(15, 1, 23, 3, 6, 4, 17, 0, 29, 22, 34, 2)$,
$(15, 30, 0, 34, 9, 20, 18, 21, 25, 19, 35, 4)$,
$(19, 5, 15, 7, 10, 8, 21, 4, 33, 14, 26, 6)$,
$(19, 34, 4, 26, 1, 12, 22, 13, 29, 23, 27, 8)$,
$(23, 9, 19, 11, 2, 0, 13, 8, 25, 18, 30, 10)$,
$(0, 3, 8, 12, 21, 17, 20, 13, 16, 4, 11, 7)$

under the action of the mapping $i \mapsto i + 12 \pmod{36}$. □

Lemma 3.18 *There exists an icosahedron design of order* 96.

Proof. Construct an 8-GDD of type 8^8 from an affine plane of order 8 (as in Lemma 3.16) and remove two entire groups to obtain a 6-GDD of type 8^6. Inflate each point by a factor of 2. On each inflated group place the icosahedron design of order 16 from Lemma 2.1 and replace each block by the icosahedron decomposition of K_{2^6} from Lemma 3.5. □

Lemma 3.19 *There exists an icosahedron design of order* 216 *which contains a sub-design of order* 36.

Proof. There exists a 6-GDD of type 9^6, i.e. four MOLS of side 9. Inflate each point by a factor of 4, on each inflated group place the icosahedron design of order 36 from Lemma 3.17 and replace each block by the icosahedron decomposition of K_{4^6} from Lemma 3.6. □

Lemma 3.20 *There exists an icosahedron design of order* 396.

Proof. There exists a 5-GDD of type 4^5, i.e. a complete set of MOLS of side 4. In four of the groups inflate each point by a factor of 20. In the remaining group inflate three points by 15 and one point by 30. Add an extra point, ∞. On each inflated group together with ∞ place either the icosahedron design of order 81 from Lemma 2.2 or the icosahedron design of order 76 from Lemma 2.9. Replace each block by either the icosahedron decomposition of $K_{20,20,20,20,15}$ from Lemma 2.8 or the icosahedron decomposition of $K_{20,20,20,20,30}$ from Lemma 3.4. □

Lemma 3.21 *There exists an icosahedron design of order* 456.

Proof. Take the 7-GDD of type 6^8 from Lemma 3.15. In seven of the groups inflate each point by a factor of 10. In the remaining group inflate five points by 5 and one point by 10. Add an extra point, ∞. On each inflated group together with ∞ place either the icosahedron design of order 61 from Lemma 2.1 or the icosahedron design of order 36 from Lemma 3.17. Replace each block by either the icosahedron decomposition of K_{10^7} from Lemma 3.8 or the icosahedron decomposition of $K_{10^6,5}$ from Lemma 3.9. □

Lemma 3.22 *There exists an icosahedron design of order* 516.

Proof. Construct an 8-GDD of type 8^8 from an affine plane of order 8 (as in Lemma 3.16). Remove one entire group and one further point to obtain a $\{6,7\}$-GDD of type $8^6 7^1$. In the six 8-element groups inflate each point by a factor of 10 and in the 7-element group inflate each point by 5. Add an extra point, ∞. On each inflated group together with ∞ place either the icosahedron design of order 81 from Lemma 2.2 or the icosahedron design of order 36 from Lemma 3.17. Replace each 6-element block by the icosahedron decomposition of K_{10^6} from Lemma 3.7, and replace each 7-element block by the icosahedron decomposition of $K_{10^6,5}$ from Lemma 3.9. □

Lemma 3.23 *There exists an icosahedron design of order* 696.

Proof. Construct a $\{4,5\}$-GDD of type $4^7 5^1$ from a 4-RGDD 4^7 as in Proposition 2.1. Inflate each point of the seven 4-element groups by a factor of 20 and in the 5-element group inflate four points by 30 and one point by 15. Add an extra point, ∞. On each inflated group together with ∞ place either the icosahedron design of order 81 from Lemma 2.2 or the icosahedron design of order 136 from Lemma 2.9. Replace each 4-element block by the icosahedron decomposition of $K_{20,20,20,20}$ from Lemma 2.1, and replace each 5-element block by either the icosahedron decomposition of $K_{20,20,20,20,15}$

from Lemma 2.8 or the icosahedron decomposition of $K_{20,20,20,20,30}$ from Lemma 3.4. □

Lemma 3.24 *There exists an icosahedron design of order* 756.

Proof. There exists a 4-GDD of type 12^4, i.e. a pair of MOLS of side 12. Inflate each element by a factor of 15 and adjoin a further 36 elements. On each inflated group together with the extra 36 elements place the icosahedron design of order 216 from Lemma 3.19 ensuring that one of its sub-designs of order 36 overlays the adjoined points. Replace each block by the icosahedron decomposition of $K_{15,15,15,15}$ from Lemma 2.3. □

Lemma 3.25 *There exists an icosahedron design of order* 996.

Proof. Construct a 16-GDD of type 16^{16} from an affine plane of order 16. Remove three groups entirely and a further nine points from one of the remaining groups to obtain a $\{12, 13\}$-GDD of type $16^{12}7^1$. Inflate each element by a factor of 5 and adjoin ∞. On each inflated group together with ∞ place either the icosahedron design of order 81 from Lemma 2.2 or the icosahedron design of order 36 from Lemma 3.17. Replace each block by either the icosahedron decomposition of $K_{5^{12}}$ from Lemma 3.12 or the icosahedron decomposition of $K_{5^{13}}$ from Lemma 3.13. □

With Lemmas 3.17–3.25, the proof of Theorem 1.1 is complete.

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