Families of complementary distributions

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SUPPLEMENTARY MATERIAL: proofs of results in Section 3

Using

$$E_H(T^2) = 1 - 2\int_0^1 tH(t)\,dt \quad \text{and} \quad \int_0^1 tH^{-1}(t)\,dt = E_H\{TH(T)\},$$

gives rise to the following proof:

`Proof of Theorem 3.5`

$$\begin{aligned}
E_{G_p}(X^2) &= 1 - 2\int_0^1 xG_p(x)dx \\
&= 1 - 2\left\{(1-p)/3 + (2p-1)\int_0^1 xM_p^{-1}(x)dx\right\}/p \\
&= 1 - 2\left[(1-p)/3 + (2p-1)E_{M_p}\{SM_p(S)\}\right]/p \\
&= 1 - 2\,((1-p)/3 + (2p-1)\times \\
&\qquad \left[p^2E_F\{VF(V)\} + p(1-p)\{1+E_F(V^2)\}/2 + (1-p)^2/3\right])/p \\
&= p(2p+1)/3 - 2p(2p-1)E_F\{VF(V)\} - (2p-1)(1-p)E_F(V^2).
\end{aligned}$$

The theorem results by elementary methods.

Using

$$E_H(T^3) = 1 - 3\int_0^1 t^2H(t)\,dt \quad \text{and} \quad \int_0^1 t^2H^{-1}(t)\,dt = E_H\{TH^2(T)\}$$

the proof of Theorem 3.6 ensues:

`Proof of Theorem 3.6`

$$\begin{aligned}
E_{G_p}(X^3) &= 1 - 3\int_0^1 x^2G_p(x)dx \\
&= 1 - 3\left\{(1-p)/4 + (2p-1)\int_0^1 x^2M_p^{-1}(x)dx\right\}/p \\
&= 1 - 3\left[(1-p)/4 + (2p-1)E_{M_p}\{SM_p^2(S)\}\right]/p \\
&= 1 - 3\big((1-p)/4 + (2p-1)\times\big[(1-p)(p^2+2p+3)/12 \\
&\qquad + p^3E_F\{VF^2(V)\} + p^2(1-p)E_F\{V^2F(V)\} \\
&\qquad + p(1-p)^2E_F(V^3)/3\big]\big)/p \\
&= p(1+p+2p^2)/4 - (2p-1)\times\big[3p^2E_F\{VF^2(V)\} \\
&\qquad + 3p(1-p)E_F\{V^2F(V)\} + (1-p)^2E_F(V^3)\big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{G_p}(X) &= p(1+p+2p^2)/4-(2p-1)\times\left[3p^2E_F\{VF^2(V)\}\right.\\
&\quad \left.+3p(1-p)E_F\{V^2F(V)\}+(1-p)^2E_F(V^3)\right]\\
&- 3\{p-(2p-1)E_F(V)\}\times\left[p(2p+1)/3\right.\\
&\qquad\qquad \left.-(2p-1)\left\{2pE\{VF(V)\}+(1-p)E_F(V^2)\right\}\right]\\
&+ 2\left[p^3-3p^2(2p-1)E_F(V)+3p(2p-1)^2\{E_F(V)\}^2\right.\\
&\qquad\qquad \left.-(2p-1)^3\{E_F(V)\}^3\right]
\end{aligned}$$

which can be rearranged to give the result of the theorem.

`Proof of Theorem 3.7`

$$\begin{aligned}
L_{2;G_p} &= \int_0^1 \left[p(1-p)u-(1-p)^2u^2-2(2p-1)(1-p)uM_p^{-1}(u)\right.\\
&\qquad \left.+p(2p-1)M_p^{-1}(u)-(2p-1)^2\{M_p^{-1}(u)\}^2\right]/p^2du\\
&= \left[p(1-p)/2-(1-p)^2/3-2(2p-1)(1-p)E_{M_p}\{SM_p(S)\}\right.\\
&\qquad \left.+p(2p-1)E_{M_p}(S)-(2p-1)^2E_{M_p}(S^2)\right]/p^2\\
&= \Big(p(1-p)/2-(1-p)^2/3-2(2p-1)(1-p)\times\\
&\qquad \left[p^2E_F\{VF(V)\}+p(1-p)\{1+E_F(V^2)\}/2+(1-p)^2/3\right]\\
&\qquad +p(2p-1)\left\{pE_F(V)+1-p/2\right\}\\
&\qquad -(2p-1)^2\left\{pE_F(V^2)+(1-p)/3\right\}\Big)/p^2.
\end{aligned}$$

Further manipulation completes the proof.

`Proof of Theorem 3.8`

$$\begin{aligned}
L_{3;G_p} &= \int_0^1 \left[3p(1-p)^2u^2-2(1-p)^3u^3-p^2(1-p)u\right.\\
&\qquad +6p(2p-1)(1-p)uM_p^{-1}(u)-6(1-p)^2(2p-1)u^2M_p^{-1}(u)\\
&\qquad -6(1-p)(2p-1)^2u\{M_p^{-1}(u)\}^2+3p(2p-1)^2\{M_p^{-1}(u)\}^2\\
&\qquad \left.-2(2p-1)^3\{M_p^{-1}(u)\}^3-p^2(2p-1)M_p^{-1}(u)\right]/p^3\,du\\
&= \left[p(1-p)^2-(1-p)^3/2-p^2(1-p)/2\right.\\
&\qquad +6p(2p-1)(1-p)E_{M_p}\{SM_p(S)\}-6(1-p)^2(2p-1)E_{M_p}\{SM_p^2(S)\}\\
&\qquad -6(1-p)(2p-1)^2E_{M_p}\{S^2M_p(S)\}+3p(2p-1)^2E_{M_p}(S^2)\\
&\qquad \left.-2(2p-1)^3E_{M_p}(S^3)-p^2(2p-1)E_{M_p}(S)\right]/p^3
\end{aligned}$$

$$
\begin{aligned}
= \; & \big((1-p)(4p-4p^2-1)/2 + 6p(2p-1)(1-p)\left[p^2 E_F\{VF(V)\}\right. \\
& \quad + p(1-p)\{1+E_F(V^2)\}/2 + (1-p)^2/3\big] - 6(1-p)^2(2p-1)\times \\
& \quad \big[(1-p)(p^2+2p+3)/12 + p^3 E_F\{VF^2(V)\} + p^2(1-p)E_F\{V^2F(V)\} \\
& \quad + p(1-p)^2 E_F(V^3)/3\big] - 6(1-p)(2p-1)^2\left[p^2 E_F\{V^2F(V)\}\right. \\
& \quad \left. + p(1-p)\{1+2E_F(V^3)\}/3 + (1-p)^2/4\right] \\
& \quad + 3p(2p-1)^2\left\{(1-p)/3 + pE_F(V^2)\right\} \\
& \quad - 2(2p-1)^3\left\{(1-p)/4 + pE_F(V^3)\right\} \\
& \quad - p^2(2p-1)\left\{(1-p)/2 + pE_F(V)\right\}\big)/p^3 \\
= \; & \big[p^4(1-p)(1-2p)/2 + 6p^3(2p-1)(1-p)E_F\{VF(V)\} \\
& \quad + 3p^4(2p-1)E_F(V^2) - 6p^3(1-p)^2(2p-1)E_F\{VF^2(V)\} \\
& \quad - 6p^4(1-p)(2p-1)E_F\{V^2F(V)\} - 2p^5(2p-1)E_F(V^3) \\
& \quad - p^3(2p-1)E_F(V)\big]/p^3
\end{aligned}
$$

and the result follows almost immediately.