Super- and sub-additive envelopes of aggregation functions: interplay between local and global properties, and approximation

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SUPER- AND SUB-ADDITIVE ENVELOPES OF AGGREGATION FUNCTIONS: INTERPLAY BETWEEN LOCAL AND GLOBAL PROPERTIES, AND APPROXIMATION

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Abstract. Super- and sub-additive transformations of aggregation functions have been recently introduced by Greco, Mesiar, Rindone and Šipeky [The superadditive and the subadditive transformations of integrals and aggregation functions, *Fuzzy Sets and Systems* **291** (2016), 40–53]. In this article we give a survey of the recent development regarding the existence of aggregation functions with a preassigned super- and sub-additive transformation, and address approximation of these transformations. The underpinning feature of the presented results is dependence of global properties of super- and sub-additive transformations on local properties of aggregation functions.

1. **Introduction**

Aggregation functions as means of producing a single value out of a multitude of parameters in some controlled and consistent ways have been used for decades in statistics, decision-making, data mining, artificial intelligence, economics, and in numerous other disciplines and aspects of real life. Obviously, there are countless ways of fusing (often inhomogeneous) input data into a single number, but only relatively few of these have found fruitful applications and have been thoroughly studied. We refer the reader for more details about the theory and applications of aggregation functions primarily to the monographs [2] and [3] and references therein, together with a recent collection [11] of retrospects and perspectives of aggregation.

The aim of this short paper is to give a survey of very recent results and challenges in a rather narrow area that emerged only lately, namely, in the so-called super- and sub-additive transformations of aggregation functions. These have been introduced only lately by Greco, Mesiar, Rindone and Šipeky in [5], but appear to be an important addition to the variety of directions in which aggregation functions have been studied. Deferring definitions of all concepts to section 2, on an intuitive level every aggregation function gives rise to a unique super- and sub-additive 'envelope' of the original function; the two 'envelopes' are the super- and sub-additive

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transformations in the terminology of [5]. Our particular interest will be in conditions guaranteeing existence of an aggregation function with preassigned super- and sub-additive 'envelopes'. A closely related area to our focus is determination of exact values of super- and sub-additive transformations of aggregation functions. In both cases emphasis will be given to methods based on investigation of the effect of local properties of aggregation functions near the origin on the global behaviour of their super- and sub-additive transformations.

Despite a relatively short interval between the appearance of [5] and writing of this paper, several contributions to the above topics have been made [6, 7, 8, 9, 13, 14, 15], both in terms of partial answers as well as opening up new and promising directions of research. The reason of birth of this relatively early survey is thus twofold: to sum up the available results (referring to original sources for proofs) in one place and to state a selection of open questions, answers to which would give further impetus to this area of research.

The paper is organized as follows. In section 2 we introduce the fundamental concepts of an aggregation function and its super- and sub-additive transformation. This section also contains a detailed explanation of application-driven motivation for introducing the two transformations. The problem of existence of an aggregation function with a super- and sub-additive transformation given in advance is discussed in section 3. Questions about approximation of values of super- and sub-additive transformations of a given aggregation functions are considered in section 4. Both these sections end with statements of a few related challenge problems, with good potential for further research. The final section 5 contains a handful of remarks together with further challenges related to both the main topics considered in this short survey (an extended abstract of which appeared in [16]).

2. **Super- and Sub-additive Transformations**

From a formal point of view the 'aggregation functions community' divides into two camps: those considering such functions on a compact domain and those preferring non-compact ones. The most favourite domains of the two camps are $[0,1]^n$ and $[0,\infty[^n$, with variations in between the two extremes. In this paper we adhere to the second camp and by an *aggregation function* we will understand an arbitrary mapping $A:\ [0,\infty[^n \to [0,\infty[$ that is increasing in every coordinate and satisfies $A(\mathbf{0}) = A(0,\dots,0) = 0$. However, most of the properties and results presented here are easily transferrable to compact domains.

There have been a vast variety of properties of aggregation functions that have been studied in the literature; we refer the interested reader to the monograph [3]. Here we will focus on two of these that also appear to be most recent [5], called super- and sub-additivity. Formally, an aggregation function $A:\ [0,\infty[^n \to [0,\infty[$ is *super-additive* if $A(\mathbf{u}+\mathbf{v}) \geq A(\mathbf{u}) + A(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in [0,\infty[^n$. Similarly, such an aggregation function A is *sub-additive* if $A(\mathbf{u}+\mathbf{v}) \leq A(\mathbf{u}) + A(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in [0,\infty[^n$. Besides resemblance of the two properties to issues in the theory of measure and integration we give another type of motivation arising from applications in economics. At the same time we will explain how one arrives at the two fundamental concept of super- and sub-additive transformations of aggregation functions, loosely following [5] in a somewhat adapted form.

Imagine a manufacturer producing a certain type of goods, and assume that the production can be described in terms od a vector $\mathbf{x} \in [0, \infty[^n$ whose coordinates $x_1, x_2, \ldots, x_n$ represent 'production factors'. The factors may include raw material, costs of machine acquiring and repairs, transportation logistic, labour, and so on, and are assumed to cumulatively represent the investment to produce a certain amount of output associated with $\mathbf{x}$. Now, suppose that the *maximum* revenue the manufacturer obtains by selling the goods represented by $\mathbf{x}$ to *one* retailer from a set of available dealers is given by $A(\mathbf{x})$ for some aggregation function A. For the sake of this model example, suppose also that the manufacturer knows the values of A for a certain range of production factor vectors. Knowing A, the manufacturer may consider splitting the production factor vector $\mathbf{x}$ in the form of a sum of, say, k 'smaller' production vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)} \in [0, \infty[^n$ and sell the corresponding smaller amounts to k retailers within the given set for revenues $A(\mathbf{x}^{(1)}), \ldots, A(\mathbf{x}^{(k)})$. If he finds out that the sum of these revenues *exceeds* the original expected revenue $A(\mathbf{x})$, then selling the k smaller amounts would obviously be the preferred option. This leads to the advice for the manufacturer to try to maximize the values of the sums $A(\mathbf{x}^{(1)}) + \ldots + A(\mathbf{x}^{(k)})$ over the decompositions $\mathbf{x} = \mathbf{x}^{(1)} + \ldots + \mathbf{x}^{(k)}$ described above.

Let us now reverse the roles and consider a retailer who would like to buy a portfolio of n products from a set of available manufacturers. A portfolio may be represented by a vector $\mathbf{x} \in [0, \infty[^n$ whose i-th coordinate indicates the amount of i-th product. Suppose that the *minimum* price at which this deal can be realised within the given set of producers is given by the value $A(\mathbf{x})$ of some aggregation function A. Assuming that the retailer knows A for a range of inputs, she may look at splitting the portfolio into, say, k 'smaller' parts $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)} \in [0, \infty[^n$ summing to $\mathbf{x}$ and buy these at prices $A(\mathbf{x}^{(1)}), \ldots, A(\mathbf{x}^{(k)})$ from a subset of k manufacturers. If the sum of these prices is *smaller* than the earlier expected price $A(\mathbf{x})$, then one would naturally go for this saving option. And, as above, this suggests that the retailer should try to find the minimum of the sums $A(\mathbf{x}^{(1)}) + \ldots + A(\mathbf{x}^{(k)})$ over decompositions $\mathbf{x} = \mathbf{x}^{(1)} + \ldots + \mathbf{x}^{(k)}$ into vectors from $[0, \infty[^n$ to minimize her expenses.

Abstracting from possible real-life limitations regarding the decompositions of the above vectors $\mathbf{x}$, the two situations lead naturally to the following definition. For an aggregation function $A : [0, \infty[^n \to [0, \infty[$, the *super-additive* and *sub-additive* transformations, A^* and A_*, of the function A, are functions $[0, \infty[^n \to [0, \infty]$ given by

$$A^*(\mathbf{x}) = \sup \{\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \ ; \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \leq \mathbf{x}\} \ , \tag{1}$$

$$A_*(\mathbf{x}) = \inf \{\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \ ; \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \geq \mathbf{x}\} \ . \tag{2}$$

Thus, in the ideal world, the advice for the manufacturer in the first example would be to follow the strategy given by values of A^*, while the advice for the retailer in the second example would be to apply the strategy suggested by the values of A_*. We reiterate that in practice one would have to take into the account restrictions on realistic decompositions of the vector $\mathbf{x}$ into summands (which may assume values in certain ranges only, for example), but in what follows we will keep to the 'ideal model' as given by (1) and (2).

For completeness we note that further bits of motivation for introducing the above transformations is given in [4] by way of sub-decomposition and super-decomposition integrals based on aggregation functions.

It is easy to show that the (aggregation) functions A^* and A_* are super-additive and sub-additive, respectively, giving rise to the introduced terminology. One clearly has $A^* = A$ if A is super-additive, and $A_* = A$ if A is sub-additive. The two transformations A^* and A_* may equivalently be introduced as super- and sub-additive envelopes of A and we will leave the details to the reader.

3. **Aggregation Functions with Given Super- and Sub-additive Transformations**

Following the scenario outlined in the Introduction, we will focus on relations of an aggregation function to its super- and sub-additive transformations. In particular, we will address the following fundamental question from the point of view of theory:

Given a super-additive function f and a sub-additive function g, both from $[0,\infty[^n$ into $[0,\infty[$, with $f(\mathbf{0}) = g(\mathbf{0}) = 0$, and such that $f(\mathbf{x}) \geq g(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty[^n$, does there exist an aggregation function $A:\ [0,\infty[^n \to [0,\infty[$ such that $A^* = f$ and $A_* = g$?

In order to formulate strongest available results to date in this direction we need to introduce stronger versions of super- and sub-additivity. In the terminology of e.g. [12], a function $h:\ [0,\infty[^n \to [0,\infty[$ is *strictly super-additive* and *strictly sub-additive*, respectively, if $h(\mathbf{y}) + h(\mathbf{z}) < h(\mathbf{y}+\mathbf{z})$ for every pair of points $\mathbf{y}, \mathbf{z} \in [0,\infty[^n \backslash \mathbf{0}$, and, analogously, $h(\mathbf{y}) + h(\mathbf{z}) > h(\mathbf{y}+\mathbf{z})$ with the same quantifiers. (It appears that such functions have not been studied in detail *per se*.) Also, let $\mathbf{e_i}$ be the i-th unit vector in $[0,\infty[^n$, where $i \in \{1, 2, \ldots, n\}$. Finally, for an aggregation function $A:\ [0,\infty[^n \to [0,\infty[$ let ∇_A and ∇^A denote the n-dimensional vector whose i-th component $(\nabla_A)_i$ and $(\nabla^A)_i$ is equal to $\liminf_{t\to 0^+} A(t\mathbf{e_i})/t$ and $\limsup_{t\to 0^+} A(t\mathbf{e_i})/t$, respectively.

Equipped with this notation and terms we offer the following (and in our view somewhat surprising) result, proved in [9]; dots in the statement represent the standard dot product in $[0,\infty[^n$.

Theorem 3.1. *Let $A:\ [0,\infty[^n \to [0,\infty[$ be an aggregation function. If A^* is continuous and strictly super-additive, then $A^*(\mathbf{x}) = A(\mathbf{x})$ and $A_*(\mathbf{x}) = \nabla_A \cdot \mathbf{x}$ for every $\mathbf{x} \in [0,\infty[^n$. Dually, if A^* is continuous and strictly sub-additive, then $A_*(\mathbf{x}) = A(\mathbf{x})$ and $A^*(\mathbf{x}) = \nabla^A \cdot \mathbf{x}$ for every $\mathbf{x} \in [0,\infty[^n$.*

The contrapositive form of Theorem 3.1 may be used to provide a negative partial answer to our fundamental question about the existence of aggregation functions with given sub- and super-additive transformations.

Corollary 3.2. *Let $f, g : [0,\infty[^n \to [0,\infty[$ be continuous functions such that $f(\mathbf{0}) = g(\mathbf{0}) = 0$ and $f(\mathbf{x}) \geq g(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty[^n$. If*

(a) *f is strictly super-additive and g is sub-additive but not linear, or*

(b) *g is strictly sub-additive and f is super-additive but not linear,*

then there is no aggregation function $A : [0,\infty[^n \to [0,\infty[$ with $A^* = f$ and $A_* = g$.

Proofs of Theorem 3.1 and Corollary 3.2 given in [9] evolved from a number of findings established in five earlier papers [13, 14, 15, 6, 7] devoted to the study of global behaviour of transformations of an aggregation function A depending on its local behaviour near zero, as captured by the vectors ∇_A and ∇^A. We will sum up the two most important steps in this development, as they lead to interesting open questions. To explain the situation we need to introduce a few more concepts.

A function $h : [0,\infty[^n \to [0,\infty[$ will be said to be *strictly directionally convex* if $h(\mathbf{x}) + h(\mathbf{y}) < h(\mathbf{u}) + h(\mathbf{v})$ for every 4 points $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in [0,\infty[^n$ such that $\mathbf{u} < \mathbf{x}, \mathbf{y} < \mathbf{v}$ and $\mathbf{u}+\mathbf{v} = \mathbf{x}+\mathbf{y}$. The analogous notion of strict directional concavity is obtained by reversing the very first equation. Directional convexity (in its non-strict version, that is, with the '$\leq$' sign in the first inequality) is also known as ultra-modularity [10]. This concept can be defined by non-decreasing increments; it is known to imply continuity and to be equivalent to super-modularity in conjunction with coordinate-wise convexity [1]. It is easy to see [15] that (strict) directional convexity implies (strict) super-additivity; the modification for concavity and sub-additivity is obvious.

Let us continue with another concept, introduced in [6, 7]. We way that a function $f : [0,\infty[^n \to [0,\infty[$ has the *overrunning property* if there exists a super-additive function h on $[0,\infty[^n$ such that the ratio f/h is strictly increasing in every coordinate on $]0,\infty[^n$. The *underrunning property* is defined analogously by replacing super-additivity with sub-additivity and the property of being strictly increasing by being strictly decreasing. The overrunning (underrunning) property easily implies strict super-additivity (strict sub-additivity), cf. [6, 7].

We now pass onto the announced explanation. In the paper [15], preceded by [13, 14], the conclusion of Corollary 3.2 was proved by assuming strict directional convexity of f in part (a) and strict directional concavity of g in part (b), with all other items unchanged. On the other hand, in [6, 7] the conclusion of Corollary 3.2 was established by assuming the overrunning property of f in part (a) and the underrunning of g in part (b), again with all the remaining items unaltered. As already stated, strict directional convexity implies continuity and strict super-additivity, and the overrunning property implies super-additivity; we omit the straightforward 'dual' versions of these statements.

The results presented above generate obvious challenges as well, and we explicitly state a few which, in our opinion, may lead to a fruitful research and interesting results.

In this connection it would be interesting to *completely* clarify the relationship between strict directional convexity, the overrunning property, and strict super-additivity (and their dual versions), with continuity involved as well.

Challenge 1. Establish non-trivial sufficient conditions for pairs of super- and sub-additive functions f, g on $[0,\infty[^n$ with $f \geq g$ and zero value at the origin, for which there is an aggregation function A on $[0,\infty[^n$ such that $A^* = f$ and $A_* = g$.

We emphasise non-triviality here to avoid observations of the type 'if f is super-additive and $g(x) = \nabla_f \cdot \mathbf{x}$, then one can take $A = f$, or 'if g is sub-additive and $f(x) = \nabla^g \cdot \mathbf{x}$, then one can take $A = g$, as such statements are just obvious twists of Theorem 3.1.

Challenge 2. Prove further sufficient conditions for pairs of super- and sub-additive f, g
on $[0,\infty[^n$ with $f \geq g$ and zero value at the origin in order not to admit an aggregation function A on $[0,\infty[^n$ such that $A^* = f$ and $A_* = g$.

We saw examples of such conditions above – strict directional convexity, overrunning, and strict super-additivity, but it would be interesting to have more, as a necessary and sufficient contition appears to be out of sight. We also discussed a few relations among the properties just listed, which call for:

Challenge 3. Clarify completely the relationship between strict directional convexity, the overrunning property, and strict super-additivity (and their dual versions), with continuity involved as well.

4. Approximation of Super- and Sub-additive Transformations of Aggregation Functions

The other interesting question brought up in the Introduction is that of determining values of super-and sub-additive transformations of a given aggregation function. As the definitions (1) and (2) of the transformations involve calculation of limits, there is no hope for a way to compute their *exact* values in general. A more realistic approach is to try to apply procedures that would lead to approximate values of the two transformation, that is, their determination up to a preassigned precision.

To stick to the theme of this article, which is the interplay between local and global properties of transformations of aggregation functions, we present two approximation results for their values. The first will be base on the behaviour of a given aggregation function near the origin, while the second will reflect what happens if similar considerations are applied arbitrarily far from the origin.

Let us begin with the first approach, which is based on the study of the behaviour of an aggregation function $A: [0,\infty[^n \to [0,\infty[$ near the origin and in specified directions. The choice of directions is motivated primarily by the fact that A may grow much faster along certain vectors than in the directions given by coordinate axes.

To develop this, we let L be a non-empty set of linearly independent unit vectors from $[0,\infty[^n$; clearly $1 \le |L| \le n$. In analogy with the definitions of ∇^A and ∇_A before the statement of Theorem 3.1, for each unit vector $\mathbf{u} \in L$ we define

$$s_{\mathbf{u}} = \limsup_{t\to 0^+} \frac{A(t\mathbf{u})}{t} \quad \text{and} \quad i_{\mathbf{u}} = \liminf_{t\to 0^+} \frac{A(t\mathbf{u})}{t} \ . \tag{3}$$

Observe that in general one may have $s_{\mathbf{u}} \in [0,+\infty]$. Obviously, (3) is a generalization of the definition of a directional derivative in multivariate calculus; hence the use of *unit* vectors $\mathbf{u} \in L$. Further, we let $\mathbf{s}$ and $\mathbf{i}$ be, respectively, the vector $(s_{\mathbf{u}})_{\mathbf{u}\in L}$ and $(i_{\mathbf{u}})_{\mathbf{u}\in L}$.

We are now ready to present estimates on A^* and A_* with the help of (3) that generalize the findings of [13]. By $\mathrm{Span}^+(L)$ we denote the set of all linear combinations of vectors from L with non-negative real coefficients. If $\mathbf{x} = \sum_{\mathbf{u}\in L} c_{\mathbf{u}}\mathbf{u} \in \mathrm{Span}^+(L)$ we let $[\mathbf{x}]_L$ denote the coordinate vector $(c_{\mathbf{u}})_{\mathbf{u}\in L}$. The standard dot product of two vectors $\mathbf{v}$ and $\mathbf{w}$ is simply denoted $\mathbf{v}\cdot\mathbf{w}$. In this notation, the following estimates are proved in [8].

Proposition 4.1. *If $\mathbf{x} \in \mathrm{Span}^+(L)$, then $A^*(\mathbf{x}) \ge \mathbf{s}\cdot[\mathbf{x}]_L$ and $A_*(\mathbf{x}) \le \mathbf{i}\cdot[\mathbf{x}]_L$.*

As already pointed out, the bounds of Proposition 4.1 were derived in [8] by looking at the growth of an aggregation function A near the origin in directions specified by the unit vectors $\mathbf{u} \in L$. But if t will not be allowed to assume (positive) values that are arbitrarily close to zero, values of $A(t\mathbf{u})/t$ appearing in (3) may be quite different from $s_{\mathbf{u}}$ and $i_{\mathbf{u}}$. In the paper [8] the authors deal with this situation as well and present another type of estimates for A^* and A_* in terms of the ratio $A(t\mathbf{u})/t$ but taken at preassigned non-zero values of t. Assuming L to be as before, choose a collection $t_{\mathbf{u}}$ ($\mathbf{u} \in L$) of positive real numbers and define

$$\alpha_{\mathbf{u}} = \frac{A(t_{\mathbf{u}}\mathbf{u})}{t_{\mathbf{u}}} \ . \tag{4}$$

We also define $\boldsymbol{\alpha} = (\alpha_{\mathbf{u}})_{\mathbf{u}\in L}$ to be the vector with coordinates $\alpha_{\mathbf{u}}$ for $\mathbf{u} \in L$, and we let $\mathbf{1}$ be the $|L|$-dimensional vector with all coordinates equal to 1. With this in hand we are ready to present another result of [8].

Theorem 4.2. *Let $A:\ [0,\infty[^n\to [0,\infty[$ be a continuous aggregation function and let the quantities $t_{\mathbf{u}}$ for $\mathbf{u} \in L$ be as defined in (4). Then, for every $\varepsilon > 0$ there exist positive real numbers $\gamma_{\mathbf{u}}$ for $\mathbf{u} \in L$ such that for every $\mathbf{x} = \sum_{\mathbf{u}\in L} c_{\mathbf{u}}\mathbf{u}$ with $c_{\mathbf{u}} \ge \gamma_{\mathbf{u}}$ one has*

$$A^*(\mathbf{x}) \ge (\boldsymbol{\alpha} - \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L \quad \text{and} \quad A_*(\mathbf{x}) \le (\boldsymbol{\alpha} + \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L \ . \tag{5}$$

The form of the bounds of Theorem 4.2 is suitable for comparing them with those of Proposition 4.1. Both are bounding the values of $A^*(\mathbf{x})$ and $A_*(\mathbf{x})$ in terms of a function that is linear in $\mathbf{x}$, except that the estimates from Theorem 4.2 are applicable only for arguments that are sufficiently away from the origin. The error term built into the statement of Theorem 4.2 appears unavoidable when restricting to linearity of bounds. In practice, however, the bounds of Theorem 4.2 may be much better than those of Proposition 4.1 for the simple reason that away from zero one may well have $\boldsymbol{\alpha} - \varepsilon\mathbf{1} > \mathbf{s}$ and $\boldsymbol{\alpha} + \varepsilon\mathbf{1} < \mathbf{i}$ for suitable ε, L and a choice of $t_{\mathbf{u}}$ for $\mathbf{u} \in L$.

in principle, one could take advantage of a plethora of available approximation methods (just consider, say, those for evaluation of integrals) and try to adapt these to furnish approximations of super- and sub-additive transformations of aggregation functions. Another way could consist of taking an ε-approximation of an aggregation function to start with, say, by a piecewise linear function with a finite number of linear pieces on every bounded subset of $[0, \infty[^n$, and then proceed by calculating exact values of such an approximation by converting the task to a linear programming problem (alas, of a potentially huge dimension).

As in section 3, in this place we also formally state a few challenges the current state-of-the-art in this area presents.

Challenge 4. Develop a theory of approximation of the super- and sub-additive transformations of aggregation functions.

Specifically, in the context of the indicated piecewise linear approximations, it would be interesting to answer the following question.

Challenge 5. What is the complexity of determining $A^*(\mathbf{x})$ and $A_*(\mathbf{x})$ for a piecewise linear aggregation function $A : [0, \infty[^n \to [0, \infty[$ with a finite number of linear pieces on every bounded subset of its domain?

5. **Conclusion**

In this short survey we presented the most recent results regarding the questions of existence of aggregation functions with given super- and sub-additive transformations, and the question of numerical approximation of these transformations. Our methodological focus was on the study of dependence of global behaviour of the two transformations on the local behaviour of an aggregation function near the origin. We emphasise that the concepts of super- and sub-additive transformations of an aggregation function are very recent. This implies that, on the one hand, it would not be realistic to expect a huge number of results at this point; on the other hand, however, this area f research appears to be fruitful and calling for further study. We have emphasized this by including several challenge problems in sections 3 and 4 that may serve as directions of future research connected with the study of interplay between local and global properties of aggregation functions.

To conclude with we present another two research problems related to the material presented. The first combines the ideas brought up in sections 3 and 4:

Challenge 6. Investigate necessary and sufficient condition for a given pair of super- and sub-additive functions $f, g : [0, \infty[^n \to [0, \infty[$ with $f \geq g$ and zero value at the origin, and for a given $\varepsilon > 0$, to admit an aggregation function $A : [0, \infty[^n \to [0, \infty[$ such that $|f - A^*| < \varepsilon$ and $|g - A_*| < \varepsilon$?

Our last challenge addresses possible transfer and further development of the ideas and results presented to the case of a compact domain of aggregation functions. More concretely, in the light of defining aggregation functions as in the survey [11], we state:

Challenge 7. Develop and strengthen the ideas of sections 3 and 4 to aggregation functions $[0,1]^n \to [0,1]$.

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