The Möbius function of permutations with an indecomposable lower bound

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Abstract

We show that the Möbius function of an interval in a permutation poset where the lower bound is sum (resp. skew) indecomposable depends solely on the sum (resp. skew) indecomposable permutations contained in the upper bound, and that this can simplify the calculation of the Möbius sum. For increasing oscillations, we give a recursion for the Möbius sum which only involves evaluating simple inequalities.

1 Introduction

Let σ and π be permutations of natural numbers, written in one-line notation, with $\sigma = \sigma_1\sigma_2\dots\sigma_m$, and $\pi = \pi_1\pi_2\dots\pi_n$. We say that σ is *contained* in π if there is a sequence $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that for any $r, s \in \{1, \dots, m\}$, $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$. We say that π *avoids* σ if π does not contain σ. The set of all permutations is a poset under the partial order given by containment.

A closed interval $[\sigma, \pi]$ in a poset is the set defined as $\{\tau : \sigma \leq \tau \leq \pi\}$. A half-open interval $[\sigma, \pi)$ is the set $\{\tau : \sigma \leq \tau < \pi\}$. The Möbius function $\mu[\sigma, \pi]$ is defined on an interval of a poset as follows: for $\sigma \not\leq \pi$, $\mu[\sigma, \pi] = 0$; for all λ, $\mu[\lambda, \lambda] = 1$; and for $\sigma < \pi$,

$$\mu[\sigma, \pi] = - \sum_{\lambda \in [\sigma, \pi)} \mu[\sigma, \lambda]. \tag{1}$$

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Our motivation for this paper is to find a *contributing set* $\mathfrak{C}_{\sigma,\pi}$ that is significantly smaller than the poset interval $[\sigma, \pi)$, and a $\{0, \pm 1\}$ weighting function $W(\sigma, \alpha, \pi)$ such that

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi). \tag{2}$$

Plainly, in Equation 2, we could set $\mathfrak{C}_{\sigma,\pi} = [\sigma, \pi)$, and $W(\sigma, \alpha, \pi) = 1$, which is equivalent to Equation 1.

One approach here would be to take a permutation β such that $\sigma < \beta < \pi$. We could then set $\mathfrak{C}_{\sigma,\pi} = \{\lambda : \lambda \in [\sigma, \pi) \text{ and } \lambda \notin [\sigma, \beta]\}$, and $W(\sigma, \alpha, \pi) = 1$, since, from Equation 2, $\sum_{\lambda \in [\sigma,\beta]} \mu[\sigma, \lambda] = 0$. This approach was used in Smith [5], who determined the Möbius function on the interval $[1, \pi]$ for all permutations π with a single descent. Smith's paper is unusual, in that it provides an explicit formula for the value of the Möbius function.

Our approach is different. We identify individual elements (say λ), of the poset that have $\mu[\sigma, \lambda] = 0$. We also show that there are pairs of elements, λ and λ', where $\mu[\sigma, \lambda] = -\mu[\sigma, \lambda']$, and so we can exclude these pairs of elements. Finally, we show that there are quartets of permutations $\lambda_1, \ldots, \lambda_4$ where $\sum_{i=1}^{4} \mu[\sigma, \lambda_i] = 0$; and that we can systematically identify these quartets. By excluding these permutations from $\mathfrak{C}_{\sigma,\pi}$ we can significantly reduce the number of elements in $\mathfrak{C}_{\sigma,\pi}$ compared to the number of elements in the interval $[\sigma, \pi)$. This approach results in the ability to compute $\mu[\sigma, \pi]$, where σ is indecomposable, much faster than evaluating Equation 1. For increasing oscillations, we will show that the elements of $\mathfrak{C}_{\sigma,\pi}$ can be determined using simple inequalities, and that as a consequence $\mu[\sigma, \pi]$ can be determined using inequalities. With this approach, we have computed $\mu[1, \pi]$, where π is an increasing oscillation, up to $|\pi| = 2{,}000{,}000$.

The study of the Möbius function in the permutation poset was introduced by Wilf [9]. The first result in this area was by Sagan and Vatter [4], who determined the Möbius function on intervals of layered permutations. Steingrímsson and Tenner [8] found a large class of pairs of permutations (σ, π) where $\mu[\sigma, \pi] = 0$, as well as determining the Möbius function where σ occurs exactly once in π, and σ and π satisfy certain other conditions. Burstein, Jelínek, Jelínková and Steingrímsson [1] found a recursion for the Möbius function for sum/skew decomposable permutations in terms of the sum/skew indecomposable permutations in the lower and upper bounds. They also found a method to determine the Möbius function for separable permutations by counting embeddings. We use the recursions for decomposable permutations to underpin the first part of this paper. McNamara and Steingrímsson [3] investigated the topology of intervals in the permutation poset, and found a single recurrence equivalent to the recursions in [1].

Many results for $\mu[\sigma, \pi]$ have been obtained by considering ways in which σ can be found in π, which we call an *embedding* of σ in π, although typically only some of the embeddings ("normal embeddings") are counted. In [6], Smith used normal embeddings to determine the Möbius function $\mu[\sigma, \pi]$ when σ and π have the same number of descents.

One problem with embeddings arises in cases such as $\mu[1, 24153]$. Here there are plainly only five ways to embed the permutation 1 into 24153, however $\mu[1, 24153] = 6$, and thus the embedding approach is not sufficient.

One possible solution to this issue is to count the normal embeddings and then add a "correction factor". This approach was used by Smith in [7]. The result applies to all intervals in the permutation poset, although the correction factor is a rather complicated double sum, which still involves $\sum_{\lambda \in [\sigma,\pi)} \mu[\sigma, \lambda]$.

In this paper we show that the Möbius function on intervals with a sum indecomposable lower bound depends only on the sum indecomposable permutations contained in the upper bound. We provide a weighting function that determines which sum indecomposable permutations contribute to the Möbius sum. We then consider increasing oscillations. For these permutations, we show how we can find all of the permutations that contribute to the Möbius sum by applying simple numeric inequalities, which leads to a fast polynomial algorithm for determining the Möbius function.

We start with some essential definitions and notation in Section 2, then in Section 3 we provide a number of preliminary lemmas. We conclude this section with a theorem that gives $\mu[\sigma, \pi]$, where σ is a sum indecomposable permutation, for all π. In Section 4 we consider $\mu[\sigma, \pi]$ where σ is a sum indecomposable permutation, and π is an increasing oscillation. We finish with some concluding remarks in Section 5.

2 Definitions and notation

When discussing the Möbius function, $\mu[\sigma, \pi] = -\sum_{\lambda \in [\sigma,\pi)} \mu[\sigma, \lambda]$, we will frequently be examining the value of $\mu[\sigma, \lambda]$ for a specific permutation λ. We say that this is the *contribution* that λ makes to the sum. If we have a set of permutations $S \subseteq [\sigma, \pi)$ such that $\sum_{\lambda \in S} \mu[\sigma, \lambda] = 0$, then we say that the set S makes *no net contribution* to the sum.

A *sum* of two permutations α and β of lengths m and n respectively is the permutation $\alpha_1, \ldots, \alpha_m, \beta_1 + m, \ldots, \beta_n + m$. We write a sum as $\alpha \oplus \beta$. A *skew sum*, $\alpha \ominus \beta$, is the permutation $\alpha_1 + n, \ldots, \alpha_m + n, \beta_1, \ldots, \beta_n$. As examples, $321 \oplus 213 = 321546$, and $321 \ominus 213 = 654213$, and these are shown in Figure 1. A *sum-indecomposable* (resp. *skew-indecomposable*) permutation is a permutation that cannot be written as the sum (resp. skew-sum) of two smaller permutations.

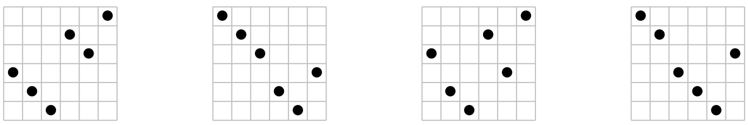
Let α be a permutation, and r a positive integer. Then $\oplus^r \alpha$ is $\alpha \oplus \alpha \oplus \ldots \oplus \alpha \oplus \alpha$, with r occurrences of α. If S is a set of permutations, then $\oplus^r S = \cup_{\lambda \in S} \{\oplus^r \lambda\}$.

The *interleave* of two permutations α and β is formed by taking the sum $\alpha \oplus \beta$, and then exchanging the value of the largest point from α with the value of the smallest point from β. We can also view this as increasing the largest point from α by 1, and simultaneously decreasing the smallest point from β by 1. We write an interleave as $\alpha \oslash \beta$. For example, $321 \oslash 213 = 421536$, see Figure 1.

For completeness, we also define a *skew interleave*, $\alpha \oslash \beta$, which is formed by taking the skew sum $\alpha \ominus \beta$, and then exchanging the smallest point from α with the largest point from β. As an example, $321 \oslash 213 = 653214$, as shown in Figure 1.

The interleave operations, $\obslash$ and $\oslash$, are not associative, as $1 \obslash 1 \obslash 1$ could represent 231 or 312. To avoid this ambiguity, we require that the permutation 1 can either be interleaved to the left or to the right, but not both. It is easy to see that this restriction establishes associativity. We note here that, with this restriction, an expression involving $\oplus$ and $\obslash$ represents a unique permutation regardless of the order in which the operations are applied.

Let α be a permutation with length greater than 1. We will frequently want to refer to permutations that have the form $\alpha \obslash \alpha \obslash \ldots \obslash \alpha \obslash \alpha$. If there are n copies of α being interleaved, then we will write this as $\obslash^n \alpha$, so, for example, we have $\obslash^3(21) = 21 \obslash 21 \obslash 21 = 315264$.



$321 \oplus 213$ $\quad$ $321 \ominus 213$ $\quad$ $321 \obslash 213$ $\quad$ $321 \oslash 213$

Figure 1: Examples of direct and skew sums and interleaves.

For the remainder of this paper, by symmetry it suffices to discuss permutations in relation to sums and interleaves only. For the same reason, references to (in)decomposable permutations may omit the "sum" qualifier.

An *interval* of a permutation is a set of contiguous positions where the set of values is also contiguous. A *simple* permutation is one where the permutation contains no intervals other than those of length one, and the entire permutation.

The *increasing oscillating sequence* is the sequence

$$4, 1, 6, 3, 8, 5, 10, 7, \ldots, 2k + 2, 2k - 1, \ldots.$$

The start of the sequence is depicted in Figure 2. An *increasing oscillation* is a simple permutation contained in the increasing oscillating sequence. For lengths greater than three, there are exactly two increasing oscillations of each length. Let W_n be the increasing oscillation with n elements which starts with a descent, and let M_n be the increasing oscillation with n elements which starts with an ascent. Then

$$\begin{aligned} W_{2n} &= \obslash^n 21, & \qquad & & M_{2n} &= 1 \obslash \left(\obslash^{n-1} 21\right) \obslash 1, \\ W_{2n-1} &= \left(\obslash^{n-1} 21\right) \obslash 1, & \qquad \text{and} & & M_{2n-1} &= 1 \obslash \left(\obslash^{n-1} 21\right). \end{aligned}$$

Note that $W_n = M_n^{-1}$.

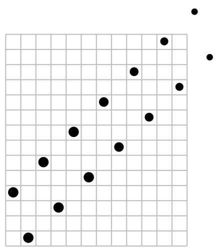


Figure 2: A depiction of the start of the increasing oscillating sequence.

There are instances where, for some permutation α, we are interested in the set of permutations $\{\alpha,\ 1 \oplus \alpha,\ \alpha \oplus 1,\ 1 \oplus \alpha \oplus 1\}$. Given a permutation α, we refer to this set as $\mathcal{F}_{\oplus}(\alpha)$, and we say that this set is the *family* of α. If S is a set of permutations, then $\mathcal{F}_{\oplus}(S) = \cup_{\alpha \in S}\{\mathcal{F}_{\oplus}(\alpha)\}$.

There are some also instances where we are interested in the set of permutations $\mathcal{F}_{\oslash}(\alpha) = \{\alpha,\ 1 \oslash \alpha,\ \alpha \oslash 1,\ 1 \oslash \alpha \oslash 1\}$. Note that every increasing oscillation is an element of $\mathcal{F}_{\oslash}(\oslash^k 21)$ for some $k \geq 1$.

3 Preliminary lemmas and main theorem

In this section our aim is to show that if σ is indecomposable, then for any $\pi \geq \sigma$ there is a $\{0, \pm 1\}$ weighting function $W(\sigma, \alpha, \pi)$ and a set of permutations $\mathfrak{C}_{\sigma,\pi}$, such that

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

If π is the identity permutation $12 \ldots n$ or its reverse, then $\mu[\sigma, \pi]$ is trivial for any σ, and we exclude the identity and its reverse from being the upper bound of any interval under consideration.

As noted earlier, our approach is to show that there are permutations, pairs of permutations, and quartets of permutations in $[\sigma, \pi)$ that make no net contribution to the sum.

We use Proposition 1 and 2, and Corollary 3 from Burstein, Jelínek, Jelínková and Steingrímsson [1]. We start with some required notation. If π is a non-empty permutation with decomposition $\pi_1 \oplus \ldots \oplus \pi_n$, then for any integer i with $0 \leq i \leq n$, $\pi_{\leq i}$ is the permutation $\pi_1 \oplus \ldots \oplus \pi_i$, and $\pi_{>i}$ is the permutation $\pi_{i+1} \oplus \ldots \oplus \pi_n$. An empty sum of permutations is defined as ε, and in particular $\pi_{\leq 0} = \pi_{>n} = \varepsilon$. We can see that $\mu[\varepsilon, \varepsilon] = 1$, $\mu[\varepsilon, 1] = -1$ and $\mu[\varepsilon, \tau] = 0$ for any $\tau > 1$. We now recall the results from Burstein, Jelínek, Jelínková and Steingrímsson:

Proposition 1 (Burstein, Jelínek, Jelínková and Steingrímsson [1, Proposition 1]). *Let σ and π be non-empty permutations with decompositions $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ and $\pi = \pi_1 \oplus \ldots \oplus \pi_n$, with $n \geq 2$. Assume that $\pi_1 = 1$, and let k be the* *largest integer such that $\pi_1, \pi_2, \ldots, \pi_k$ are all equal to* 1. *Let $l \geq 0$ be the largest integer such that $\sigma_1, \sigma_2, \ldots, \sigma_l$ are all equal to* 1. *Then*

$$\mu[\sigma, \pi] = \begin{cases} 0 & \text{if } k-1 > l, \\ -\mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k-1 = l, \\ \mu[\sigma_{>k}, \pi_{>k}] - \mu[\sigma_{>k-1}, \pi_{>k}] & \text{if } k-1 < l. \end{cases}$$

Proposition 2 ([1, Proposition 2]). *Let σ and π be non-empty permutations with decompositions $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_m$ and $\pi = \pi_1 \oplus \ldots \oplus \pi_n$, with $n \geq 2$. Assume that $\pi_1 \neq 1$, and let k be the largest integer such that $\pi_1, \pi_2, \ldots, \pi_k$ are all equal to π_1. Then*

$$\mu[\sigma, \pi] = \sum_{i=1}^{m} \sum_{j=1}^{k} \mu[\sigma_{\leq i}, \pi_1] \mu[\sigma_{>i}, \pi_{>j}].$$

Corollary 3 ([1, Corollary 3]). *Let σ and π be as in 2. Suppose that σ is sum indecomposable, so $m = 1$. Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \pi_1] & \text{if } \pi = \oplus^k \pi_1, \\ -\mu[\sigma, \pi_1] & \text{if } \pi = \left(\oplus^k \pi_1\right) \oplus 1, \\ 0 & \text{otherwise,} \end{cases}$$

A simple consequence of Propositions 1 and 2 is the identification of some intervals of permutations where the value of the Möbius function is zero.

Lemma 4. *Let $\pi \in \{1 \oplus 1 \oplus \tau, \tau \oplus 1 \oplus 1, \mathcal{F}_{\oplus}((\oplus^r \alpha) \oplus \tau')\}$, where τ is any permutation, r is maximal, α is sum indecomposable, and τ' is any permutation greater than* 1. *Let σ be a sum indecomposable permutation. Then $\mu[\sigma, \pi] = 0$.*

Proof. Consider $\pi = 1 \oplus 1 \oplus \tau$. We use Proposition 1. If $\tau_1 = 1$, then $k \geq 3$, and $l \leq 1$, and the result follows immediately. Now assume that $\tau_1 \neq 1$. Then $k = 2$. If $\sigma > 1$, then again the result follows immediately. If $\sigma = 1$, then we have $\mu[\sigma, \pi] = -\mu[\sigma_{>k-1}, \pi_{>k}] = -\mu[\varepsilon, \tau] = 0$. The case for $\pi = \tau \oplus 1 \oplus 1$ follows by symmetry.

Now consider $\pi = \mathcal{F}_{\oplus}((\oplus^r \alpha) \oplus \tau')$. If $\pi = (\oplus^r \alpha) \oplus \tau$, or $\pi = (\oplus^r \alpha) \oplus \tau \oplus 1$, then we use Proposition 2. In that context we have $m = 1$ and $k = r$, and so $\mu[\sigma, \pi] = \sum_{j=1}^{r} \mu[\sigma, \pi_1] \mu[\varepsilon, \pi_{>j}]$ For every value of j, $\pi_{>j}$ is non-empty and greater than 1, and so $\mu[\varepsilon, \pi_{>j}] = 0$ for all j, and hence every term in the sum is zero. If $\pi = 1 \oplus (\oplus^r \alpha) \oplus \tau$ or $\pi = 1 \oplus (\oplus^r \alpha) \oplus \tau \oplus 1$, then we use Proposition 1, which reduces to one of the previous cases. □

We now turn to identifying pairs and quartets of permutations that make no net contribution to the Möbius sum. We start by showing that if σ and α are indecomposable, and $r \geq 1$, and with $\pi \in \mathcal{F}_{\oplus}(\oplus^r \alpha)$, then $\mu[\sigma, \pi]$ and $\mu[\sigma, \alpha]$ have the same magnitude.

Lemma 5. *Let $\pi \in \mathcal{F}_{\oplus}(\oplus^r \alpha)$, where $r \geq 1$ and $\alpha > 1$ is sum indecomposable. Let σ be a sum indecomposable permutation. Then*

$$\mu[\sigma, \pi] = \begin{cases} \mu[\sigma, \alpha] & \text{if } \pi = \oplus^r \alpha \text{ or } 1 \oplus (\oplus^r \alpha) \oplus 1, \\ -\mu[\sigma, \alpha] & \text{if } \pi = 1 \oplus (\oplus^r \alpha) \text{ or } (\oplus^r \alpha) \oplus 1. \end{cases}$$

As a consequence, $\mathcal{F}_\oplus(\oplus^r\alpha)$ makes no net contribution to $\mu[\sigma,\pi]$ if $\mathcal{F}_\oplus(\oplus^r\alpha) \subseteq [\sigma,\pi)$.

Proof. If $\pi = \oplus^r\alpha$ or $\pi = (\oplus^r\alpha) \oplus 1$, then this is immediate from Corollary 3. If $\pi = 1 \oplus (\oplus^r\alpha)$ or $\pi = 1 \oplus (\oplus^r\alpha) \oplus 1$, then we use Proposition 1.

For the net contribution of $\mathcal{F}_\oplus(\oplus^r\alpha)$, $\sum_{\lambda \in \mathcal{F}_\oplus(\oplus^r\alpha)} \mu[\sigma,\lambda] = 0$. □

We now have a lemma that adds a further restriction to the permutations that have a non-zero contribution to the Möbius sum.

Lemma 6. *If $\sigma \leq \pi$, and $\alpha \in [\sigma,\pi]$ is sum indecomposable, and r is the smallest integer such that $1 \oplus (\oplus^r\alpha) \oplus 1 \not\leq \pi$, then $\mathcal{F}_\oplus(\oplus^k\alpha) \subseteq [\sigma,\pi)$ for all $k \in [1,r)$.*

Proof. For any $k < r$, $\sigma \leq \oplus^k\alpha < 1 \oplus \left(\oplus^k\alpha\right) \oplus 1 \leq \pi$. Note that by Lemma 5 the net contribution of the family $\mathcal{F}_\oplus(\oplus^k\alpha)$ to $\mu[\sigma,\pi]$ is zero. □

Observation 7. *Using the same terminology as Lemma 6, if $k > r+1$ then we must have $\oplus^k\alpha \not\leq \pi$. As a consequence, for each indecomposable $\alpha \in [\sigma,\pi]$, the only families of α that can have a non-zero net contribution to $\mu[\sigma,\pi]$ are $\mathcal{F}_\oplus(\oplus^r\alpha)$ and $\mathcal{F}_\oplus(\oplus^{r+1}\alpha)$.*

We now eliminate two specific permutations from the Möbius sum.

Lemma 8. *If π is any permutation with $|\pi| > 3$ apart from the identity permutation and its reverse, and σ is sum indecomposable, then the permutations 1 and $1 \oplus 1$ make no net contribution to the Möbius sum $\mu[\sigma,\pi]$.*

Proof. If $\sigma = 1$, then the interval contains both 1 and $1 \oplus 1$. Since $\mu[1,1] = 1$ and $\mu[1,12] = -1$, there is no net contribution to $\mu[\sigma,\pi]$. If $\sigma > 1$, then $\sigma \neq 12$, and so neither 1 not 12 is in the interval. □

Before we present the main theorem for this section, we formally define the weight function and the contributing set. Let α be a sum indecomposable permutation. The weight function, $W(\sigma,\alpha,\pi)$, is defined as

$$W(\sigma,\alpha,\pi) = \begin{cases} 1 & \text{If} \begin{cases} \sigma \leq \oplus^r\alpha \leq \pi \text{ and} \\ 1 \oplus (\oplus^r\alpha) \not\leq \pi \text{ and} \\ (\oplus^r\alpha) \oplus 1 \not\leq \pi, \end{cases} \\ -1 & \text{If} \begin{cases} \sigma \leq \oplus^r\alpha \leq \pi \text{ and} \\ 1 \oplus (\oplus^r\alpha) \leq \pi \text{ and} \\ (\oplus^r\alpha) \oplus 1 \leq \pi \text{ and} \\ \oplus^{r+1}\alpha \not\leq \pi, \end{cases} \\ 0 & \text{Otherwise,} \end{cases} \tag{3}$$

where r is the smallest integer such that $1 \oplus (\oplus^r\alpha) \oplus 1 \not\leq \pi$.

$1 \oplus (\oplus^r \alpha)$	$(\oplus^r \alpha) \oplus 1$	$\oplus^{r+1}\alpha$	Möbius contribution
$\leq \pi$	$\leq \pi$	$\leq \pi$	0
$\leq \pi$	$\leq \pi$	$\not\leq \pi$	$-\mu[\sigma, \alpha]$
$\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\leq \pi$	$\not\leq \pi$	0
$\not\leq \pi$	$\not\leq \pi$	$\not\leq \pi$	$\mu[\sigma, \alpha]$

Table 1: Möbius contribution from family members.

The contributing set $\mathfrak{C}_{\sigma,\pi}$ is defined as

$$\mathfrak{C}_{\sigma,\pi} = \left\{ \alpha : \begin{array}{l} \alpha \in [\sigma, \pi), \\ \alpha \text{ is sum indecomposable, and} \\ W(\sigma, \alpha, \pi) \neq 0 \end{array} \right\}.$$

We have one last lemma before we move on to the main theorem.

Lemma 9. *If σ and α are sum indecomposable, then for any permutation π, $\mu[\sigma, \alpha]W(\sigma, \alpha, \pi)$ gives the contribution of the set of families $\mathcal{F}_\oplus(\oplus^r \alpha)$ to the Möbius sum, where r is any positive integer.*

Proof. By Observation 7, we only need consider the contribution made by $\oplus^r \alpha$ and $\oplus^{r+1}\alpha$, where r is the smallest integer such that $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$.

If $\sigma \not\leq \oplus^r \alpha$, or $\oplus^r \alpha \not\leq \pi$, then $\mathcal{F}_\oplus(\oplus^r \alpha)$ makes no net contribution to the Möbius sum. Now assume that $\sigma \leq \oplus^r \alpha \leq \pi$. First, we can see that if $1 \oplus (\oplus^r \alpha) \not\leq \pi$, or $(\oplus^r \alpha) \oplus 1 \not\leq \pi$ then $\oplus^{r+1}\alpha \not\leq \pi$. We can also see that if $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$ then $1 \oplus \left(\oplus^{r+1}\alpha\right) \not\leq \pi$ and $\oplus^{r+1}\alpha \not\leq \pi$. The possibilities remaining are itemised in Table 1, where the Möbius contribution is determined by applying Lemma 5. We can see that in every case $W(\sigma, \alpha, \pi)$ provides the correct weight for the Möbius function $\mu[\sigma, \alpha]$. □

We are now in a position to present the main theorem for this section.

Theorem 10. *If σ is a sum indecomposable permutation, and $|\pi| > 3$, then*

$$\mu[\sigma, \pi] = - \sum_{\alpha \in \mathfrak{C}_{\sigma,\pi}} \mu[\sigma, \alpha] W(\sigma, \alpha, \pi).$$

Proof. Let $\alpha \leq \pi$ be an indecomposable permutation.

Using Lemmas 4 and 8 we can see that any permutations not in the set $\mathcal{F}_\oplus(\oplus^r \alpha)$ can be excluded from $\mathfrak{C}_{\sigma,\pi}$, as these permutations make no net contribution to the Möbius sum.

For every α, by Lemma 9, $\mu[\sigma, \alpha]W(\sigma, \alpha, \pi)$ provides the contribution to the Möbius sum of all families $\mathcal{F}_\oplus(\oplus^r \alpha)$, where r is a positive integer. □

Theorem 10 reduces the number of permutations that need to be considered as part of the Möbius sum. We can see that the largest permutation in $\mathfrak{C}_{\sigma,\pi}$ must have length less than $|\pi|$, and so we can apply Theorem 10 recursively to the permutations in $\mathfrak{C}_{\sigma,\pi}$ to determine their Möbius values. In this recursion, if we are attempting to determine $\mu[\sigma, \lambda]$, we can stop if $|\sigma| = |\lambda|$ or $|\sigma| = |\lambda| - 1$, as in these cases $\mu[\sigma, \lambda]$ is $+1$ and -1 respectively.

4 Increasing oscillations

We now move on to increasing oscillations. Given an indecomposable permutation σ, and an increasing oscillation π, our aim in this section is to describe $\mathfrak{C}_{\sigma,\pi}$ in precise terms. We will find a sum for the Möbius function, $\mu[\sigma, \pi]$, which only requires the evaluation of simple inequalities.

If π is an increasing oscillation with length less than 4, then $\mu[\sigma, \pi]$ is trivial to determine for any σ. For the remainder of this section we assume that π has length at least 4.

We partition the set of increasing oscillations with length greater than 1 into five disjoint subsets. These subsets are $\{21\}$, $\{\oslash^{k+1}21\}$, $\{1 \oslash (\oslash^k 21)\}$, $\{(\oslash^k 21) \oslash 1\}$, and $\{1 \oslash (\oslash^k 21) \oslash 1\}$, where k is a positive integer. If two increasing oscillations are in the same subset, then we say that they have the same *shape*.

We now determine what permutations contained in an increasing oscillation have a non-zero contribution to the Möbius sum.

Lemma 11. *Let π be an increasing oscillation, and let $\sigma \leq \pi$ be sum indecomposable. Let S be the subset of the permutations in the interval $[\sigma, \pi)$ that can be written in the form $\mathcal{F}_\oplus(\oplus^r \mathcal{F}_\oslash(\oslash^k 21))$ for some $k, r \geq 1$. If $\lambda \in [\sigma, \pi)$, and $\lambda \notin S$, then $\mu[\sigma, \lambda] = 0$.*

We note here that $\mathcal{F}_\oslash(\oslash^k 21)$ is a set containing only increasing oscillations.

Proof. We start by showing that if π is an increasing oscillation, and $\lambda = \lambda_1 \oplus \ldots \oplus \lambda_m \leq \pi$, where each λ_i is sum indecomposable, then every λ_i is an increasing oscillation. This is trivially true if λ is itself an increasing oscillation, thus it is sufficient to show that if λ is an increasing oscillation, then deleting a single point results in either an increasing oscillation, or a permutation that is the sum of two increasing oscillations.

If $k = 1$, then we can see that deleting a single point results in a permutation with the required characteristic.

Now assume that $k > 1$. Let $\lambda = 1 \oslash (\oslash^k 21)$. Deleting the leftmost point gives $\oslash^k 21$, and deleting the rightmost point gives $1 \oslash (\oslash^{k-1} 21) \oplus 1$. Deleting the second point gives $21 \oplus (\oslash^{k-1} 21)$, and deleting the last-but-one point gives $1 \oslash (\oslash^{k-1} 21) \oslash 1$. Deleting any even point $2t$ except the second or second-to-last results in $(1 \oslash (\oslash^{t-1} 21) \oslash 1) \oplus ((\oslash^{k-t} 21))$. Finally, deleting any odd point

$2t+1$ apart from the first or last results in $\big(1 \ominus (\ominus^{t-1}21)\big) \oplus \big(1 \ominus (\ominus^{k-t}21)\big)$. Thus if $\lambda = 1 \ominus \big(\ominus^{k}21\big)$, then deleting a single point from λ results in either an increasing oscillation, or a permutation that is the sum of two increasing oscillations.

A similar argument applies to the other three cases, which we omit for brevity.

To complete the proof, we now see that by Lemma 8, we can ignore $\lambda = 1$ and $\lambda = 1 \oplus 1$. If $\lambda = \lambda_1 \oplus \lambda_2 \oplus \ldots \oplus \lambda_m \leq \pi$, then by the argument above, every λ_i is an increasing oscillation. Applying Lemma 4 completes the proof. $\square$

Following Observation 7, it is clear that, if $\alpha \in \mathcal{F}_{\ominus}(\ominus^{k}21)$, then for any family $\mathcal{F}_{\oplus}(\oplus^{r}\alpha)$, we only need consider the cases $\oplus^{r}\alpha$ and $\oplus^{r+1}\alpha$ where r is the smallest integer such that $1 \oplus (\oplus^{r}\alpha) \oplus 1 \not\leq \pi$.

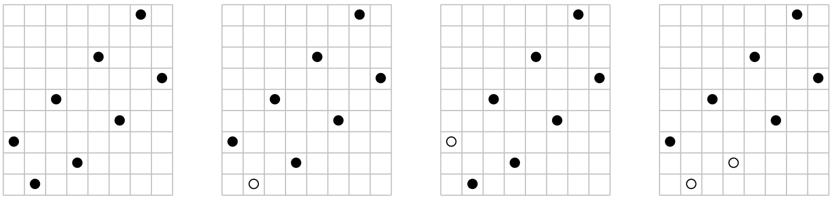
Given some $\pi = W_n$ or M_n, we will find inequalities that relate n, r and k and the shape of α that will allow us to find the values that contribute to the Möbius sum. We know from Lemma 11 the shape of the permutations that contribute to the Möbius sum. For each of the four types of increasing oscillation (W_{2n}, W_{2n-1}, M_{2n} and M_{2n-1}), we can examine how each shape can be embedded so that the unused points at the start of the increasing oscillation are minimised. Figure 3 shows examples of embeddings into W_{2n}. This gives us an inequality relating to the start of the embedding. Similarly, we can find inequalities for the end of the embedding. We can also find inequalities that relate to the interior (when $r > 1$), and Figures 4 and 5 show examples of this. We can use these inequalities to determine what values of k will allow the shape to be embedded. For each allowable value of k, we can then determine the maximum value of r such that $1 \oplus (\oplus^{r}\alpha) \oplus 1 \not\leq \pi$. This then means that, by evaluating inequalities alone, we can identify the specific permutations that could contribute to the Möbius sum.

We first have two lemmas that examine inequalities at the start and end of an embedding.

Lemma 12. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of an element λ of $\mathcal{F}_{\oplus}(\oplus^{r}\alpha)$ into π, the minimum number of unused points at the start of π depends on the start of λ, and on π, and is as shown below:*

Start of λ	$\pi = W_{2n}$	$\pi = W_{2n-1}$	$\pi = M_{2n}$	$\pi = M_{2n-1}$
$21\ldots$	0	0	0	0
$\ominus^{k+1}21\ldots$	0	0	1	1
$1 \ominus \big(\ominus^{k}21\big)\ldots$	1	1	0	0
$1 \oplus 21\ldots$	1	1	1	1
$1 \oplus \big(\ominus^{k+1}21\big)\ldots$	1	1	2	2
$1 \oplus 1 \ominus \big(\ominus^{k}21\big)\ldots$	2	2	1	1

Proof. It is clear that if we minimise the number of points at the start of an embedding, then the number of unused points depends on π, and the start of



$\obslash^4 21$ $\qquad$ $1 \obslash (\obslash^3 21)$ $\qquad$ $1 \oplus (\obslash^3 21)$ $\qquad$ $1 \oplus 1 \obslash (\obslash^2 21)$

Figure 3: Embedding the start of α in W_{2n}.

α. The values in Lemma 12 are found by considering each of the possibilities. We illustrate some of these cases in Figure 3. $\square$

Lemma 13. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of an element λ of $\mathcal{F}_{\oplus}(\oplus^r \alpha)$ into π, the minimum number of unused points at the end of π depends on the end of λ, and on π, and is as shown below:*

End of λ	$\pi = W_{2n}$	$\pi = W_{2n-1}$	$\pi = M_{2n}$	$\pi = M_{2n-1}$
$\ldots 21$	0	0	0	0
$\ldots \obslash^{k+1} 21$	0	1	1	0
$\ldots (\obslash^k 21) \obslash 1$	1	0	0	1
$\ldots 21 \oplus 1$	1	1	1	1
$\ldots (\obslash^{k+1} 21) \oplus 1$	1	2	2	1
$\ldots (\obslash^k 21) \obslash 1 \oplus 1$	2	1	1	2

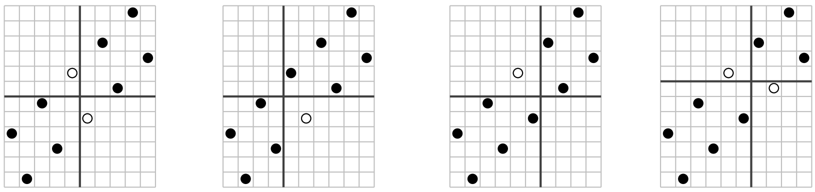
Proof. We examine all the possibilities as we did in Lemma 12. $\square$

We now consider how closely copies of some sum indecomposable α can be embedded into π. This leads to two inequalities that relate α, π and the maximum number of copies of α that can be embedded in π. Where $\alpha \neq 21$, the shape of α fixes the way the two copies can be embedded in an increasing oscillation. If $\alpha = 21$, then we will see that there are choices for the embedding.

Lemma 14. *If π is an increasing oscillation, and $\alpha \neq 21$, and $\alpha \leq \pi$ is sum indecomposable, then in any embedding of $\oplus^r \alpha$ into π, the minimum number of points between the start and end of $\oplus^r \alpha$ depends on α, and is as shown below:*

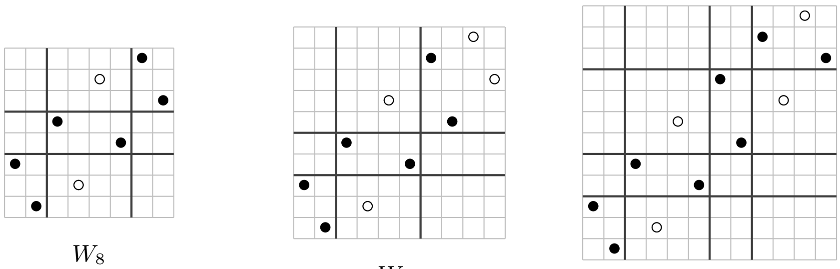
Shape of α	*Points in $\oplus^r \alpha$*	*Unused points*	*Minimum points*
$\obslash^{k+1} 21$	$2kr$	$2r - 2$	$2kr + 2r - 2$
$1 \obslash (\obslash^k 21)$	$2kr + r$	$r - 1$	$2kr + 2r - 1$
$(\obslash^k 21) \obslash 1$	$2kr + r$	$r - 1$	$2kr + 2r - 1$
$1 \obslash (\obslash^k 21) \obslash 1$	$2kr + 2r$	$2r - 2$	$2kr + 4r - 2$

Proof. If $r = 1$, then there are no unused points, and so the minimum number of points depends solely on the points in α, and the table reflects this.



$21 \oslash 21 \oplus$ $21 \oslash 21$ $21 \oslash 21 \oplus$ $1 \oslash 21 \oslash 21$ $21 \oslash 21 \oslash 1 \oplus$ $21 \oslash 21$ $21 \oslash 21 \oslash 1 \oplus$ $1 \oslash 21$

Figure 4: Packing α as close as possible when $\alpha \neq 21$.



W_8 W_{10} W_{12}

Figure 5: Examples of unused points when embedding $\oplus^r 21$.

Assume now that $r > 1$. If $\alpha \neq 21$, then we can see that the interleave fixes the layout of each copy of α, so we simply pack each copy as close as possible. This packing clearly depends on the start and end of α, and it is simple to examine the four possibilities. Examples are shown in Figure 4. □

We now turn to the case where $\alpha = 21$. This is more complex than the previous cases. We can see that there must be at least one point between each copy of α. We can insert each copy of 21 in two ways, one where the points are horizontally adjacent, and one where the points are vertically adjacent. These alternatives can be seen in Figure 5. Alternating these means that there will be exactly one point between each copy of α, so this embedding minimises the number of points between the start and end of $\oplus^r \alpha$. The complication in this case relates to how we start and end the embedding. We illustrate this by showing, in Figure 5, maximal embeddings where we are embedding into W_8, W_{10} and W_{12}. A detailed examination of each possible case gives us our second inequality.

Lemma 15. *If π is an increasing oscillation, and $\alpha = 21$ then for $\oplus^r \alpha$ to be contained in π we must have $3r - 1 \leq 2n$ for $\pi \in \{W_{2n}, M_{2n}\}$, and $3r \leq 2n$ for $\pi \in \{W_{2n-1}, M_{2n-1}\}$.*

Proof. In every case we start by embedding the first 21 into the first two elements of the permutation. Thereafter, we embed each successive 21 as close as possible to the preceding 21. The minimum number of elements to embed r copies of 21 will be $2r$ elements to hold the points of the 21s, and $r - 1$ intermediate empty elements. For W_{2n} and M_{2n}, this then gives $3r-1 \leq 2n$, and for W_{2n-1} and M_{2n-1} we obtain $3r-1 \leq 2n-1$. □

We now have a complete understanding of the number of points required to embed any permutation that contributes to the Möbius sum into an increasing oscillation. The following Lemma summarises the situation.

Lemma 16. *If π is an increasing oscillation, and $\alpha \in \mathcal{F}_{\oslash}(\oslash^k 21) \leq \pi$ (so α is sum indecomposable), then for $\oplus^r \alpha$ to be contained in π, the inequality in the table below must be satisfied, where $k \geq 1$.*

π	*Shape of α*	*Inequality*
W_{2n}, M_{2n}	21	$3r-1 \leq 2n$
W_{2n-1}, M_{2n-1}	21	$3r \leq 2n$
W_{2n}	$\oslash^{k+1}21$	$2kr+2r-2 \leq 2n$
W_{2n-1}	$1 \oslash \left(\oslash^k 21\right)$	$2kr+2r+2 \leq 2n$
M_{2n-1}	$\left(\oslash^k 21\right) \oslash 1$	$2kr+2r+2 \leq 2n$
M_{2n}	$1 \oslash \left(\oslash^k 21\right) \oslash 1$	$2kr+4r-2 \leq 2n$
$W_{2n}, W_{2n-1}, M_{2n-1}$	$1 \oslash \left(\oslash^k 21\right) \oslash 1$	$2kr+4r \leq 2n$
All other cases		$2kr+2r \leq 2n$

Proof. We apply Lemmas 12, 13, 14 and 15 to the possibilities for π and α. □

As a consequence of Lemmas 12 and 13 we can define a relationship between the minimum number of points required to embed some $\oplus^r \alpha$, and the minimum number of points required to embed $1 \oplus (\oplus^r \alpha)$, $\oplus^r \alpha \oplus 1$ and $1 \oplus (\oplus^r \alpha) \oplus 1$.

Corollary 17. *If π is an increasing oscillation, and $\alpha \leq \pi$ is sum indecomposable and if the minimum number of points required to embed $\oplus^r \alpha$ into π is C, then the minimum number of points required to embed $1 \oplus (\oplus^r \alpha)$ into π is $C+2$, the minimum number of points required to embed $\oplus^r \alpha \oplus 1$ into π is $C+2$, and the minimum number of points required to embed $1 \oplus (\oplus^r \alpha) \oplus 1$ into π is $C+4$.*

Proof. We can see from Lemmas 12 and 13 that adding $1 \oplus$ at the start of a permutation increases the number of points required by two – one for the new point, and one that is unused. Similarly, adding $\oplus 1$ at the end increases the points required by two. □

Lemma 16 gives us inequalities that any $\oplus^r \alpha$ must satisfy to ensure that $\oplus^r \alpha \leq \pi$. Further, Corollary 17 gives us inequalities that, for a given $\oplus^r \alpha$ allow us to determine if $1 \oplus (\oplus^r \alpha) \leq \pi$, $(\oplus^r \alpha) \oplus 1 \leq \pi$ and $1 \oplus (\oplus^r \alpha) \oplus 1 \leq \pi$. We can therefore determine what values of r and k will result in $\mathcal{F}_{\oplus}(\oplus^r \alpha)$ contributing to the Möbius function. We now consider inequalities that relate σ and α, so that we can determine if $\sigma \leq \alpha$ using an inequality.

Lemma 18. *If $\sigma > 1$ is an increasing oscillation, and $\alpha \in \mathcal{F}_{\oslash}(\oslash^k 21)$ for some k, then for σ to be contained in α the inequality in the table below must be satisfied, where $k \geq 1$.*

σ	*Shape of* α	*Inequality*
$W_{2n-1}, M_{2n}, M_{2n-1}$	21	*False*
W_{2n-1}	$\left(\ominus^k 21\right) \ominus 1$	$k \geq n-1$
M_{2n-1}	$1 \ominus \left(\ominus^k 21\right)$	$k \geq n-1$
$W_{2n-1}, M_{2n}, M_{2n-1}$	$1 \ominus \left(\ominus^k 21\right) \ominus 1$	$k \geq n-1$
M_{2n}	$\ominus^{k+1} 21$	$k \geq n+1$
All other cases		$k \geq n$

Proof. We examine all possible cases. □

We are now nearly ready to present the main theorem for this section. Informally, for each possible shape of permutation α, we will first find the minimum and maximum values of k such that $\sigma \leq \alpha \leq \pi$, as any other values of k result in α being outside the interval. For each α and each k, we then determine the minimum value of r such that $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$. We can then use this value of r (assuming it is non-zero) to determine the weight to be applied to $\mu[\sigma, \alpha]$. The set of αs with a non-zero weight is a contributing set $\mathfrak{C}_{\sigma,\pi}$. At this point we can substitute a value for any $\mu[\sigma, \alpha]$ where $|\sigma| \leq |\alpha| - 1$. We then use the same process recursively to determine the contributing set for the remaining elements of $\mathfrak{C}_{\sigma,\pi}$.

We first define some supporting functions. Let $\operatorname{RawMinK}(\sigma, \alpha)$ be the minimum value of k that satisfies the inequality in Lemma 18. For the first inequality, which is always false, we set $k = |\pi|$, as this will force the sum, defined later in Theorem 19, to be empty.

Let $\operatorname{MinK}(\sigma, \alpha)$ be defined as

$$\operatorname{MinK}(\sigma, \alpha) = \begin{cases} 1 & \text{If } \sigma = 1 \text{ and } \alpha \neq \ominus^{k+1} 21, \\ 2 & \text{If } \sigma = 1 \text{ and } \alpha = \ominus^{k+1} 21, \\ \operatorname{RawMinK}(\sigma, \alpha) & \text{otherwise.} \end{cases}$$

Observe that for any $k < \operatorname{MinK}(\sigma, \alpha)$, we have $\alpha < \sigma$, and so $\mathcal{F}_{\oplus}(\oplus^k \alpha)$ makes no net contribution to the Möbius sum.

Let $\operatorname{MaxK}(\alpha, \pi)$ be defined as the maximum value of k that satisfies the inequality in Lemma 16, if the shape of α and the shape of π are different; and one less than the maximum value of k that satisfies the inequality if the shape of α and the shape of π are the same. For the first two inequalities, which do not involve k, we set $\operatorname{MaxK}(\alpha, \pi) = 1$ if the inequality is satisfied, and $\operatorname{MaxK}(\alpha, \pi) = 0$ if not. Observe here that for any $k > \operatorname{MaxK}(\alpha, \pi)$ we have $\alpha \not< \pi$, and so $\mathcal{F}_{\oplus}(\oplus^k \alpha)$ makes no contribution to the Möbius sum.

We define the weight function for increasing oscillations, $W_{io}(\sigma, \alpha, \pi)$, as

$$W_{io}(\sigma, \alpha, \pi) = \begin{cases} 1 & \text{If } (\oplus^r \alpha) \oplus 1 \not\leq \pi, \\ -1 & \text{If } (\oplus^r \alpha) \oplus 1 \leq \pi \text{ and } \oplus^{r+1} \alpha \not\leq \pi, \\ 0 & \text{Otherwise,} \end{cases}$$

where r is the smallest integer such that $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$. These conditions are simpler than those given in the weight function (3) for Theorem 10 as, by Corollary 17, if $(\oplus^r \alpha) \oplus 1 \not\leq \pi$ then $1 \oplus (\oplus^r \alpha) \not\leq \pi$ and vice-versa. Furthermore, we will see that this weight function is only used when $\sigma \leq \oplus^r \alpha \leq \pi$.

We are now in a position to state our main theorem for this section. In this theorem, we consider the contribution to the Möbius sum of each possible shape of some sum indecomposable α. There are five possible shapes, and, given that the expression for each shape is identical, we abuse notation slightly by writing our theorem as a sum over the shapes, thus the first sum in Theorem 19 is over the possible shapes of α, where four of the shapes have a parameter k. For each shape, the limits on the interior sum determine the minimum and maximum values of k, using the summation variable v. We use the notation α_v to represent the actual permutation that has the shape α, where the parameter k has been set to the value of v. As an example, if $\alpha = 1 \oslash \left(\oslash^k 21\right)$, and $v = 2$, then $\alpha_v = 1 \oslash \left(\oslash^2 21\right) = 24153$.

Theorem 19. *Let π be an increasing oscillation, and let $\sigma \leq \pi$ be sum indecomposable. Then*

$$\mu[\sigma, \pi] = \sum_{\alpha \in \mathcal{S}} \sum_{v=\mathrm{MinK}(\sigma,\alpha)}^{\mathrm{MaxK}(\alpha,\pi)} \mu[\sigma, \alpha_v] W_{io}(\sigma, \alpha_v, \pi)$$

where the first sum is over the possible shapes of a sum indecomposable permutation contained in an increasing oscillation, so $\mathcal{S} = \{21,\ \oslash^{k+1}21,\ 1 \oslash \left(\oslash^k 21\right),\ \left(\oslash^k 21\right) \oslash 1,\ 1 \oslash \left(\oslash^k 21\right) \oslash 1\}$.

Proof. By Lemma 11 the only sum-decomposable permutations contained in an increasing oscillation that contribute to the Möbius sum are $\mathcal{F}_{\oplus}(\oplus^r \alpha)$, where $\alpha \in \mathcal{S}$.

If we set $r = 1$, then for each α in $\mathcal{S}$ Lemma 18 provides the smallest value of k such that $\sigma \leq \alpha$. If there is no such value of k, then we use $|\pi|$, as the maximum value of k must be smaller than this, and so the sum is empty.

Again setting $r = 1$, for each α in $\mathcal{S}$ Lemma 16 provides the maximum value of k such that $\alpha \leq \pi$. If there is no value of k that satisfies the inequality, then we set $\mathrm{MaxK}(\alpha, \pi) = 0$, thus forcing the sum to be empty.

Thus the permutations α_v in the sum

$$\sum_{\alpha \in \mathcal{S}} \sum_{v=\mathrm{MinK}(\sigma,\alpha)}^{\mathrm{MaxK}(\alpha,\pi)}$$

are those that could contribute to the Möbius sum, and for any α_v not included in the sum, $\mathcal{F}_{\oplus}(\oplus^r \alpha_v)$ has a zero contribution to the Möbius sum for any r.

Further, we can see from the construction method that any α_v included in the sum has $\sigma \leq \oplus^r \alpha_v \leq \pi$ for at least one value of r, as if this was not the case, then we would have $\mathrm{MinK}(\sigma, \alpha) > \mathrm{MaxK}(\alpha, \pi)$, and so the sum would be empty.

We have therefore shown that the α_v-s included in the sum form a contributing set, and we could therefore set $\mathfrak{C}_{\sigma,\pi}$ to be those α_v-s, and use Theorem 10. We now show that the increasing oscillation weight function $W_{io}(\sigma, \alpha, \pi)$ is equivalent to $W(\sigma, \alpha, \pi)$ as defined in the general case.

By Corollary 17, if $(\oplus^r \alpha) \oplus 1 \not\leq \pi$ then $1 \oplus (\oplus^r \alpha) \not\leq \pi$ and vice-versa, and so the condition for $(\oplus^r \alpha) \oplus 1$ also covers $1 \oplus (\oplus^r \alpha)$. As discussed above, we know that there is at least one value of r such that $\sigma \leq \oplus^r \alpha \leq \pi$, and so $W_{io}(\sigma, \alpha, \pi)$ does not need to include this condition. Thus the increasing oscillation weight function $W_{io}(\sigma, \alpha, \pi)$ is equivalent to $W(\sigma, \alpha, \pi)$ as defined in the general case. □

4.1 Example of Theorem 19

As an example of Theorem 19 in action, we show how to determine

$$\mu[3142, 315274968] = \mu[\oslash^2 21, \oslash^4 21 \oslash 1].$$

We start by considering each possible shape of α, setting $r = 1$, and then using the inequalities in Lemmas 16 and 18 to determine the minimum and maximum values of k. This gives us

Shape of α	Minimum k	Maximum k
21	1	1
$\oslash^k 21$	2	4
$1 \oslash (\oslash^k 21)$	2	3
$(\oslash^k 21) \oslash 1$	2	3
$1 \oslash (\oslash^k 21) \oslash 1$	2	3

For each shape of α, and each value of k, we then use the inequalities in Lemma 16 to determine the minimum value of r such that $1 \oplus (\oplus^r \alpha) \oplus 1 \not\leq \pi$, and we then calculate the weight using this value of r. This gives

α	r	Weight
21	No possibilities	
$\oslash^2 21$	2	1
$\oslash^3 21$	1	-1
$\oslash^4 21$	1	-1
$1 \oslash (\oslash^2 21)$	1	-1
$1 \oslash (\oslash^3 21)$	1	-1
$(\oslash^2 21) \oslash 1$	2	1
$(\oslash^3 21) \oslash 1$	1	-1
$1 \oslash (\oslash^2 21) \oslash 1$	1	-1
$1 \oslash (\oslash^3 21) \oslash 1$	1	-1

This leads to the following initial expression:

$$\begin{aligned}\mu[\oslash^2 21, \oslash^4 21 \oslash 1] =& \mu[\oslash^2 21, \oslash^2 21] - \mu[\oslash^2 21, \oslash^3 21] - \mu[\oslash^2 21, \oslash^4 21] \\ & - \mu[\oslash^2 21, 1 \oslash (\oslash^2 21)] - \mu[\oslash^2 21, 1 \oslash (\oslash^3 21)] \\ & + \mu[\oslash^2 21, \oslash^2 21 \oslash 1] - \mu[\oslash^2 21, \oslash^3 21 \oslash 1] \\ & - \mu[\oslash^2 21, 1 \oslash (\oslash^2 21) \oslash 1] - \mu[\oslash^2 21, 1 \oslash (\oslash^3 21) \oslash 1]\end{aligned}$$

We know that $\mu[\oslash^2 21, \oslash^2 21] = 1$, and that

$$\mu[\oslash^2 21, 1 \oslash (\oslash^2 21)] = \mu[\oslash^2 21, \oslash^2 21 \oslash 1] = -1.$$

Applying Theorem 19 recursively to the other intervals eventually yields

$$\mu[\oslash^2 21, \oslash^4 21 \oslash 1] = -6.$$

5 Concluding remarks

The results in [1] provide two recurrences to handle the case where π is decomposable. This work handles the case where σ is indecomposable. It overlaps with [1] when σ is indecomposable and π is decomposable. This leaves the case where σ is decomposable and π is indecomposable for further investigation.

We can see that by symmetry $\mu[\sigma, W_n] = \mu[\sigma^{-1}, M_n]$. If we consider the value of the principal Möbius function, $\mu[1, \pi]$, where π is either W_n or M_n, then it is simple to show that the absolute value of the principal Möbius function is bounded above by 2^n. The weight function for increasing oscillations can be ± 1, and we can see no obvious reason why there should not be two distinct values, i and j, with the same parity, such that the signs of $\mu[1, W_i]$ and $\mu[1, W_j]$ were different. We have experimental evidence, based on the values of W_n and M_n for $n = 1 \ldots 2{,}000{,}000$ that suggests that $\mu[1, W_{2n}] < 0$, and that $\mu[1, W_{2n-1}] > 0$.

Figure 6 is a log-log plot of the absolute values of $\mu[1, W_{2n}]$ from $n = 8{,}000$ to $n = 10{,}000$. As can be seen, there seems to be some evidence of "banding", and we have confirmed that this pattern continues for all values examined. Examination of the values of $\mu[1, W_{2n-1}]$ reveals the same patterns.

Following discussions at Permutation Patterns 2017, Vít Jelínek [2] provided the following conjecture (rephrased to reflect our notation).

Conjecture 20 (Jelínek [2])**.** Let $M(n)$ denote the absolute value of the Möbius function $\mu[1, W_n] = \mu[1, M_n]$. Then for $n > 50$ we have

$$\begin{aligned} M(2n) = n^2 &\iff n+1 \text{ is prime and } n \equiv 0 \pmod 6 \\ M(2n) = n^2 - 1 &\iff n+1 \text{ is prime and } n \equiv 4 \pmod 6 \\ M(2n+1) = n^2 - n &\iff n+1 \text{ is prime and } n \equiv 0 \pmod 6 \\ M(2n+1) = n^2 - n - 1 &\iff n+1 \text{ is prime and } n \equiv 4 \pmod 6 \end{aligned}$$

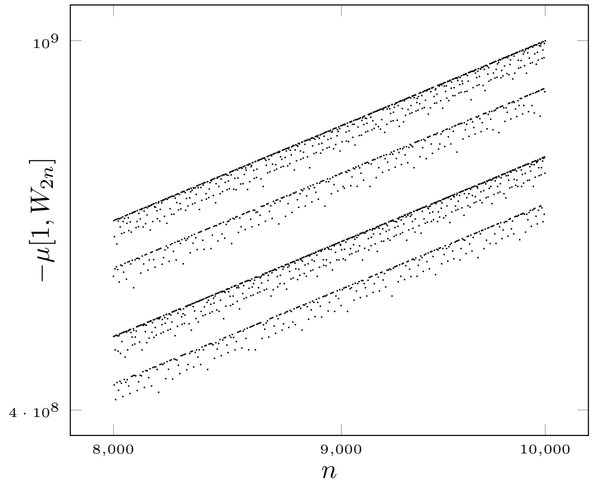


Figure 6: Log-Log plot of $|W_{2n}|$.

Further, Jelínek notes that there does not seem to be any other "small" constant k such that $M(n) = (n^2 - k)/4$ infinitely often.

We also have the following conjecture relating to the "banding" of the values.

Conjecture 21. Let $M(n)$ denote the absolute value of the Möbius function $\mu[1, W_n] = \mu[1, M_n]$. Let $E(n) = M(n)/(n^2)$, and let $O(n) = M(n)/(n^2 + n)$. Then, with $n \geq 1$, there exist constants $0 < a < b < c < d < e < f < g < 1$ such that

$$\begin{aligned} E(12n+10) &\in [a,b] & \qquad O(12n+11) &\in [a,b] \\ E(12n+2) &\in [c,d] & O(12n+3) &\in [c,d] \\ E(12n+6) &\in [c,d] & O(12n+7) &\in [c,d] \\ E(12n+4) &\in [e,f] & O(12n+5) &\in [e,f] \\ E(12n+8) &\in [g,1] & O(12n+9) &\in [g,1] \\ E(12n) &\in [g,1] & O(12n+1) &\in [g,1] \end{aligned}$$

Examining the first 1,500,000 values of $\mu[1, W_n]$ gives the following estimates for the constants.

a	b	c	d	e	f	g
0.615	0.680	0.692	0.760	0.821	0.896	0.923

The *complete nearly-layered* permutations are formed by interleaving descending permutations. Formally, a complete nearly-layered permutation has the form

$$\alpha_1 \oslash \alpha_2 \oslash \ldots \oslash \alpha_{k-1} \oslash \alpha_k$$

where each α_i is a descending permutation, with $\alpha_i > 1$ for $i = 2, \ldots, k-1$. If we set $\alpha_i = 21$ for $i = 2, \ldots, k-1$, and $\alpha_1, \alpha_k \in \{1, 21\}$, then we obtain the increasing oscillations.

The computational approach taken for increasing oscillations could, we think, be adapted to complete nearly-layered permutations. It is clear that the equivalent of the inequalities in Lemmas 16 and 18 would be somewhat more complex than those found here, but we believe that it should be possible to define an algorithm that could determine the Möbius function for complete nearly-layered permutations where the lower bound is sum indecomposable.

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