Designing teacher education for pre-university mathematics: articulating and operationalising pedagogic messages

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Abstract

Reviews of mathematics teacher development point to a problematic and under-theorised relationship between learning mathematics and learning about mathematics pedagogy. We adopt a Bernsteinian perspective to the design and delivery of a course preparing early career teachers to teach pre-university mathematics. The iterative development and underlying rationale of five 'key pedagogic messages' is outlined as a process of tactical design. Ongoing evaluation showed the pedagogic messages provided a critical pivot in managing transitions between thinking about the mathematical content and thinking about how to teach that content. In particular, they provided a means for us, as teacher educators, to make explicit our own thinking on mathematics pedagogy and its operationalisation, and hence render it available for reflection and critique. We argue for a design principle of articulating how teacher educators operationalise a few, explicit pedagogic messages; that pre-university mathematics presents a critical site for mathematics teacher education because it facilitates this principle; and that Bernsteinian notions of framing and knowledge structures are useful in designing as well as characterising pedagogic settings.

1. Introduction

Education policy in England is moving towards a position where every 16-year-old should study some mathematics in preparation for work or university, requiring more mathematics teachers to be trained (Smith, 2017). At present, the standard mathematics course for this age group is A-level mathematics, taken by 12% of the cohort and preparing them for mathematics-rich university subjects. However, within initial teacher education (ITE), A-level teaching receives lower priority than other teaching experiences. This paper contributes to mathematics teacher education by arguing that learning to teach is not independent of the context of target mathematical content and learners.

Rather, A-level mathematics is a setting that offers distinctive experiences and opportunities to 'pivot' between reflecting as a learner and as a teacher, and thus should have a higher profile in teacher education.

The paper arises from our thinking while iteratively designing a stand-alone programme preparing beginner teachers for A-level Mathematics teaching in early career (AMTEC), reaching 500 participants over 16 course runs. It articulates the "knowledge from practice" gained by teacher educators' self-study (Loughran, 2014) as we considered whether, and how, the nature of the context affected what participants were learning about teaching, what they should learn, and how our teaching supported that learning. This is informed by Bernsteinian notions of interacting pedagogies for mathematics and for mathematics-teaching. We discuss the resulting design principle of articulating how our practice operationalised a few, explicit pedagogic messages pertinent to pre-university mathematics teaching, with the aim of focusing on details of the relationship between teaching and learning. Others have offered such guidance, for example asking "What are the key aspects of mathematics pedagogy for teaching that form the foundations for Initial Teacher Education?" (JMC, 2017) but without examining how those key aspects are realised in teacher education practice.

2. Our orientation to teachers' knowledge

In this section, we introduce our theoretical orientation to teachers' knowledge, the language we use to frame our design and the research that contributed to our thinking. We see becoming a mathematics teacher as assuming a beginner teacher identity and developing specialised knowledge (Shulman, 1986), including knowledge of *mathematics* (M) and of *mathematics pedagogy* (MP). This last is focussed on schools, concerned with the practices of teaching and learning appropriate to school-age students and school mathematics and the pedagogic thinking behind those. Within ITE we expect teachers to develop both these forms of knowledge, since teachers' mathematics knowledge underpins mathematics pedagogy (Baumert et al., 2009). At the same time, teacher educators are enacting their own pedagogic knowledge, encompassing how teachers learn to teach; how they learn to teach mathematics, and the thinking behind all these (Loughran, 2014). This is the *mathematics teaching* pedagogic knowledge (MTP) that underpins curriculum planning and enactment in teacher education. So within any teacher education setting, there can be two pedagogies at play: the mathematics pedagogy intended for enactment in a school context and the mathematics teaching pedagogy that frames beginner teachers' professional learning (Hordern, 2015). The principle of course design that we discuss is a matter of mathematics teaching pedagogy, and it arose from considering how teacher participants engaged with mathematics and mathematics pedagogy in the professional development course.

As with much ITE in England, our course is based on McIntyre's conception of "practical theorising." This recognises that experienced teachers have knowledge-in-use: a repertoire of activities and strategies, with criteria and habits for examining and developing these that go "well beyond common sense" (Hagger & McIntyre, 2006, p. 68). McIntyre argues that teachers learn by reflecting on goals and strategies when immersed in pedagogic situations, but considers that beginner teachers do not yet have the range of experiences and habits for full participation in professional self-development. They need to learn from other people's practice and ideas (including those of teachers, educational scholars and researchers). Indeed, experienced teachers' learning has also been characterised as developing through informal, but purposeful, professional conversations about practice (Horn, 2010).

The AMTEC course was designed for early career teachers, taken as those in their qualifying or first years of teaching so we could assume some such knowledge-in-use, albeit limited. The course lies outside the natural, school setting for professional learning since it is run either at a university or online, with the participants positioned as learners. One reason for this setting is our focus on A-level mathematics. All the course participants have previously studied this content, but seek to revisit the material in preparation for teaching. Lowrie and Jorgensen (2016, p. p211) report a growing international emphasis on addressing teachers' mathematical knowledge, and characterise some teachers' knowledge suitable for passing examinations as too "fragmented" for teaching. Our vision for AMTEC was that the course would move between re-learning (or learning) mathematical topics and discussing teaching repertoires, i.e. rehearsing when, where, how and why to use particular classroom approaches (Feiman-Nemser, 2001). This is recognised as an effective professional development approach: reviews of externally-provided courses (Stoll, Harris, & Handscomb, 2012) show that experienced teachers benefit from being learners again, exploring and examining what motivates, influences and hinders them, in order to better understand student learning.

This raises the first issue that we considered in our planning, described by Adler (2015, p. 144) as the requirement for dual attentions whenever professional development involves mathematical activity: "in mathematics teacher education, both ‘mathematics’ and ‘teaching’ are objects of learning. At different times each will be privileged or backgrounded”. Adler writes from a South African perspective about programmes that explicitly aim to teach some mathematics, and so are similar, in one key respect, to our own context of revisiting participants’ school knowledge of A-level mathematics. We see a connection here to the critique in Back et al. (2009)’s review of mathematics-specific professional development courses: that teacher learning is usually conceived as “doing mathematics and thinking about connections within it” (p3). They recommend that courses go further and are explicit about connections to teaching and students’ thinking, in order to relate experiences of learning to classroom practice. In our terms, this signals the importance of considering how course content *balances* attentions between mathematics (M) and mathematics pedagogy (MP); thus AMTEC participants engage with both target areas of knowledge. This also raised the question of how, in practice, tutors should support participants to make *shifts* in attention, pivoting between reflection on M and on MP.

The second theoretical issue we considered in our course design is that teachers’ knowledge is practice-based and often tacit. That is, much professional knowledge is not articulated in its natural use and yet it is recognised and realised by experienced teachers (Loughran, 2014; Winch, Oancea, & Orchard, 2015). Hagger & McIntyre (2006) warn that the complexity of classrooms means it is not reasonable to expect beginner teachers to learn tacit knowledge by observation alone. Instead, they learn when such knowledge is (intentionally or unintentionally) modelled – that is, demonstrated as part of teaching practice - then brought to their attention and spoken about. Even then, there is a body of work (e.g. Coles, 2013) showing convincingly that when teachers observe practice they construct differing accounts of/for it. From this perspective, in our setting where both mathematics and mathematics pedagogy are being learnt, it is we as teacher educators who are modelling pedagogic practice. Following the advice of Back et al (2009), we aim to bring this practice to participants’ attention, provide a language to speak about it, and provide a means of organising that discussion. This leads to two questions regarding course design: firstly, how to support participants to notice and articulate what is normally hidden and tacit? And secondly, how to select what to emphasise among the many possible and valid accounts that can be made? We also note that as teacher educators, crucially, we enact MTP focused on developing teachers’ content knowledge, and it is as part of this wider practice that we model MP as appropriate to schools. Our interpretation requires course designers to consider how their MTP (supporting participants doing mathematics for the purpose of teaching) is similar or different to MP (supporting students learning mathematics) and whether, and how, to make this relationship explicit when participants are reflecting on pedagogy.

One way of thinking about these issues is through Bernstein's theoretical work (1999, 2003) on the ways that knowledge is socially constructed and circulated. This has been used to study school and teacher education settings (Bourne, 2003; Morais, 2002; Parker & Adler, 2014) and to argue that mathematics and mathematics education are different forms of knowledge (Lerman, Xu, & Tsatsaroni, 2002). Bernstein suggests that an area of knowledge derives a structure from the social ways in which it is produced and – importantly – how it is taught and learnt. Bernstein characterises pedagogies by their *recontextualisation* rules, regarding how knowledge is changed when relocated from its substantive practice into the pedagogic context, their *evaluative* rules, regarding what counts as relevant, legitimate knowledge, and *distribution* rules, regarding who can access knowledge. Mathematics is recontextualised from its field of academic production by being selected, re-ordered and focused into school mathematics. Bernstein (1999) does not discount the kinds of everyday, functional, maybe tacit knowledge that use mathematics, such as in knitting or shopping, but he sees these as lying within *horizontal* knowledge discourses passed on largely through local, personal networks. They are recognisable as mathematical precisely because mathematics is constructed as a context-independent, *vertical* discourse with strong social and epistemic norms, and hence organising power over the empirical world. Bernstein characterises mathematics as having a "strong grammar" in that its specialised language can provide "an explicit conceptual syntax capable of relatively precise empirical descriptions" (1999, p. 164). He describes learning as acquiring the *recognition* rules - or evaluative criteria/habits - for what constitutes legitimate mathematical knowledge, and the *realisation* rules of how to produce this knowledge. Within classrooms, a teacher is typically positioned as having access to knowledge and thus able, and required, to respond to students' legitimating appeals (Morais, 2002). However, it is a feature of mathematics that students can appeal to mathematics itself in their attempts to fix meanings - and in some pedagogies students are invited to do this (Parker & Adler, 2014). For example, mathematics teachers can step back from evaluating students' mathematics, point them to recognised ways of checking reasoning, and invite them to determine legitimacy from within mathematics itself.

These features of mathematical knowledge stand in contrast to knowledge about mathematics pedagogy. There is considerable diversity within what is considered knowledge about teaching mathematics, and there are multiple ways of articulating that and of establishing claims to legitimacy, even within regulated fields such as research journals (Lerman et al., 2002) or teacher education curricula (Tatto & Hordern, 2017). These authors suggest that in Bersteinian terms, mathematics education is a vertical discourse with a relatively weak grammar. It is unlike mathematics, but like sociology, because it allows different constituent languages to coexist and offer meanings to instances of experience. When teachers construct differing accounts of practice, this is not necessarily a matter of inexperience but a feature of the form of knowledge. Thus the third issue that influenced us was the idea that mathematics and mathematics pedagogy are different *forms* of knowledge that need to be balanced within the course.

Bernstein uses two concepts to analyse the power and control relations of a social knowledge structure. Classification refers to the degree of maintenance of boundaries between categories of knowledge. The discussion above suggests that, although mathematics itself is strongly classified, there is a weaker boundary between mathematics and mathematics pedagogy. Indeed, there are arguments that mathematics pedagogic knowledge is part of mathematical knowledge (Watson & Barton, 2011). Framing is concerned with social relations - who in the pedagogic relation controls what. Teacher education settings are usually characterised by strongly framed selection of content: the teacher educator, with higher status in the pedagogic relation, controls the focal activities in each encounter. In contrast there is relatively weak framing of sequencing, pace and hierarchical relations; for example participants can ask questions, divert the topic and choose where to sit (Ensor, 2004; Morais, 2002). Evaluative rules may be strongly or weakly framed, depending on how the criteria for successful performance are open to negotiation by participants.

Parker and Adler (2014) analysed framing in a mathematics teacher education setting whose focus on mathematical knowledge is similar to the AMTEC course. They found similarities in the two pedagogies of mathematics and mathematics teaching that reflect an underlying constructivism. Strong framing of content, coupled with weaker sequencing, pace and hierarchical relations suggested that learning was the organisation of individual, local experiences into knowledge structures. Thus, the mathematics pedagogy started with experiences of manipulating mathematical objects and symbols, while the mathematics teaching pedagogy started with participants observing/taking part in model classroom practices. They found differences, however, in the framing of evaluative rules. Teacher educators made explicit the mathematical reasoning that defined valid mathematical knowledge, modelling a mathematics pedagogy which uses the organising power of mathematics. They weakened framing, asking participants to articulate rules of mathematical reasoning that legitimated empirical instances, thus realising mathematical knowledge in the setting. In contrast, the evaluative rules for mathematics teaching were either left implicit within the model of teaching (as in Back et al., 2009) or announced as 'best' practice, with less time given for articulation or critique. We agree with these authors' argument that this difference matters. When the possibility of learning is structured primarily in the form of self-directed reflection, then "knowledge with respect to mathematics teaching is likely to be differentially distributed across [teachers] in this pedagogic context: those with access to the evaluative criteria (through their backgrounds, cultural capital) are more likely to be able to recognise what it is that is being privileged, and (re)produce it in the various forms of assessment" (Parker & Adler, 2014, p. 216). What we take from this discussion is that, within such dual settings, the emerging evaluative rules of mathematics pedagogy may appear less visible and negotiable to participants than those of mathematics. The implication for our planning of the AMTEC course was that our design should emphasise the articulation and discussion of these rules.

3. Designing the AMTEC course

Burkhardt (2009) characterizes design as having technical, tactical and strategic aspects, broadly moving from specifics to system-wide concerns. We concentrate here on the tactical design, which Burkhardt describes as concerned with overall internal structure including: selecting learning goals; specifying sequences and connections within the materials; and balancing linear coherence with diverse multiple connections. These relate strongly to the questions of curricular selection, balance and shifts that arose from our ongoing study of literature. Course design was an iterative process, informed by participants' feedback and evaluation and discussion amongst the team of experienced teacher educators. For 11 of the 16 course runs, two tutors attended each session, one leading and one observing implementation and outcomes.

In the pilot stage of the course (8 months, 2 runs with 54 participants), initial design decisions concerned the overarching mathematical themes and related activities. We started with an over-arching theme for each day of the course: Making sense of algebraic expressions; Calculus and graphs; and Modelling with mathematics. A-level mathematics topics were selected within those themes. Their rationale is discussed in Smith, Bretscher and Golding (2017). In making a thematic selection we were influenced by our longstanding appreciation of the work of Geoff Faux (1998) who argued for the power of emphasising "big ideas in mathematics" (such as "equivalence" or "a lot for a little") as helpful both for students, and for teachers to reflect on when improving their teaching. We also drew on Lerman and Murphy's design of a course for experienced mathematics teachers, new to A-level, themed on big ideas such as "proving" and "infinity" (Kuntze et al., 2011). However we were aware that when this research collaboration evaluated a similar approach in ITE, the participants struggled to identify instances of the big ideas within the school curriculum. We interpret this as a further intimation that intended curricular connections are not necessarily evident to participants.

Within the topics, we selected activities that either supported participants in learning mathematics or modelled activities that we considered valuable examples of A-level mathematics pedagogy, often both. The first type of activity was exemplified by a task where participants used graphing software to represent the region under a curve, setting the left limit as $x = 0$. Varying the right limit and measuring the resulting area led to conjecturing an area function, and hence the definite integral (or indefinite integral if they consider how varying the left limit adds a constant term). This activity was mentioned frequently on feedback forms as "something that developed your own mathematical understanding" by connecting geometric area with integral functions. However, the activity would be beyond the boundaries of most A-level courses as it synthesises prior awareness of different meanings and notations for integration. For the second type, we chose activities whose use with participants could closely mirror the way we would use them with A-level students, although within MTP this could be speeded up, curtailed or extended. Often, we introduced an element of role-play into sessions as we asked participants to allow us to behave as if they were year 12 students. This allowed detailed discussion of our own teacher-ly actions that they had experienced as a group of learners. In addition we included sessions on assessment, students' behaviour and participation, compared with younger students with whom they had more experience, to assist participants in reflecting on the nature of A-level classes.

Initially, we allocated short periods to reflection on pedagogy at the end of each task and each session. We intended to encourage a broad scope of pedagogic reflections, and hoped these would be raised by participants in a discussion led by tutors (thus with strong framing of evaluative criteria). They included: details of how participants had thought, felt and learnt during a task, comparisons with year 12 students' thinking, feeling and learning; details of how we had directed activities, unpicking why those choices were made, differences in how they/we would orchestrate the same activities with students; comparisons with teaching younger students. Although we experienced some success in supporting participants to notice and articulate aspects of pedagogy, the observing tutors noted that they appeared to prefer speaking about their new understandings of the mathematics. They used the weaker framing of sequencing and pace within teacher education settings to keep the whole-group conversations on such topics. In addition, when participants moved to discuss pedagogy, they needed longer periods of time to review what had happened and then to consider the rationale behind that.

In the second phase (8 months, 6 runs with 152 participants, we became more explicit and thematic about the mathematics pedagogy content of the course. To do this we considered the framing of evaluative criteria within MP, and how we could use this to support participants in not just recognising but realising such pedagogy in their own classrooms. Our approach was two-fold: we articulated 'messages' about good mathematics pedagogy, and we gave time to explain and discuss how we had operationalised these messages in our mathematics teaching pedagogy, including when and how we had varied them because the location was teacher education and not school. The rationale for articulating these messages was to provide a common language that explicitly described what we valued. We argued that this articulation was necessarily limited: not only does the weak grammar of mathematics pedagogy preclude explicit organisation of experience, but, used alone, it risks producing a "professional argot" that could provide participants with rules for recognising best practice but lacks opportunities for realisation (Ensor 2004, 225). The second aspect is therefore the most important: relating those messages to tutors' decisions, actions and compromises. In order to do this, we allocated fifteen minutes at the end of each hour to reflect on the preceding activities and how a few messages had been operationalised locally across them,as exemplified in the next section. We changed who was initially responsible for presenting statements of legitimate MP knowledge: tutors now took a few minutes to draw attention to their own decisions and actions, promoting their valued pedagogic messages.

Since participants proved highly cognitively engaged in making sense of their own, and others', mathematical learning, we intended that the shift to thinking about teaching should feel just as immediate and be equally concerned with noticing and making sense of planning decisions. Participants were then invited to discuss in groups and reflect individually what they thought was significant in how the pedagogic messages had been translated into practice, thereby giving them more control over producing and evaluating MP knowledge.

We describe this second phase as thematic, because once we had adopted the responsibility for selecting target MP content as well as mathematical content within each session, we appreciated this had to be organised. Given the complexities of articulating MP knowledge, participants needed examples of contexts in which the same pedagogic messages could be operationalised in different ways. This entailed selecting amongst the different pedagogic messages that we ourselves enacted –often tacitly - in our MTP knowledge-in-use. We used a bottom-up process: reviewing previous iterations of the course and considering which aspects of MP were central to the MTP activities or had provoked rich discussion of A-level teaching. This resulted in five "key pedagogic messages" (KPM) that already acted informally as over-arching themes for the course. For each session we then analysed and articulated which KPM was best instantiated in the associated activities, and how. We return to the selection of key messages in the discussion section.

Finally, the third phase of the design (20 months, 8 runs with 305 participants) was concerned with refinement and scaling. Inducting new tutors led to debating and refining the wording of the pedagogic messages, as each tutor compared them against their own values. Guidance notes for each session set out how we intended each 'message' to be operationalised. New tutors suggested changes to sessions and inevitably modified their implementation. These were tested against the strategic design and improvements incorporated. The resulting AMTEC curriculum was organised around a tactical selection of which mathematics <u>and</u> which mathematics pedagogy would be addressed, and how.

In the remainder of the paper, we introduce the five messages chosen to fill the role of explicit evaluative criteria and illustrate how each was operationalised in AMTEC activities. In doing so we give a flavour of the tutors' contribution that precedes participant reflection in each session. It is worth repeating that the articulation of evaluative criteria is not the most important of these aspects. We do not claim that these are an ideal set of messages that characterise good A-level teaching. Rather, we argue that they form a sufficient and important set of pedagogic messages that can be instantiated and explained in the context of learning and teaching A-level mathematics, and that this is a useful design principle within mathematics teaching pedagogy. Finally we discuss issues around this principle of emphasising MP through articulating the operationalisation of selected pedagogic messages.

4. Key Pedagogic Messages

In this section we outline the resulting pedagogic messages, considering why each message was considered to communicate good practice within A-level mathematics pedagogy and how it was operationalised in the sessions.

Table 1 Mapping themes, content and key pedagogic messages

Theme	Mathematical focus	Key Pedagogic messages				
		Progres-sion	Lan-guage	Engage-ment	Connec-tions	Tools
Making sense of algebraic expressions	Polynomials	X	X	X	X	
	Binomial Theorem		X	X	X	
	Trigonometric Functions	X		X	X	
Calculus and graphs	Graphs & Change		X	X	X	X
	Differentiation	X	X		X	X
	Integration		X		X	X
Modelling with Mathematics	Modelling with graphs			X	X	X
	Newton's Laws	X	X	X		
	Data & Distributions	X		X	X	
No theme	Students' behaviour and participation; Assessing problem-solving					

4.1. A-level planning starts by understanding progression from GCSE;

This pedagogic message emphasises the need to take into account students' prior knowledge when introducing new mathematical ideas. This message is likely to be familiar to participants from their experiences of teaching mathematics to younger students. However, it seems to take on new salience in the context of teaching A-level. It is easy for those new A-level teachers to over-estimate the maturity of students beginning year 12, both in mathematical and behavioural terms. In addition, they may assume that A-level students, having chosen the subject, are a mathematically homogenous group. Understanding progression from GCSE is operationalised within AMTEC sessions by modelling pedagogy that acknowledges the different mathematical backgrounds students bring to A-level study and that students may be beginning their A-level course with unstable or superficial understandings of GCSE mathematics. For example, the session on differentiation begins with a reminder of how to find the gradient of curve at a point by calculating the gradient of the tangent to the curve at that point.

An equally important value underpinning this pedagogic message is the coherence of mathematics. We see this operationalized in teaching as communicating a concise narrative of how ideas develop within school mathematics. For example, the session entitled ‘Polynomials and the Factor theorem’ (based upon a published resource, Standards Unit, 2009) addresses the conceptual similarities between sketching graphs of linear, quadratic and finally cubic functions in terms of finding the y- and x-intercepts by setting x=0 and y=0 respectively. Similarly, the session entitled ‘Representing Trigonometric Functions’ begins by recapping ratios in right-angled triangles as a means to pose the question, ‘How do we define trigonometric functions for angles larger than 90 degrees?’, and introduces the unit circle as a new representation. Participants have reported that emphasizing progression between prior and new knowledge has prompted them to re-think how they present mathematical ideas before age 16 to provide better A-level preparation for their students.

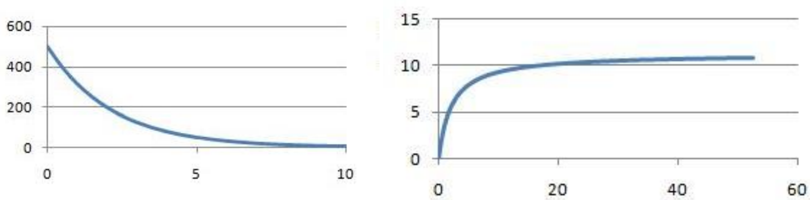
4.2. Use precise language and ‘unpick’ notation

A-level mathematics introduces students to a wide range of mathematical vocabulary, sometimes adding a precise meaning to everyday words and sometimes with overlapping or subtly different meanings (such as derivative/differential). Students are also expected at A-level to interpret and use conventional symbolic notations, develop more fluent algebraic manipulation and reason about mathematical expressions. This KPM was aimed firstly at alerting participants to the range of new vocabulary and symbols. For many participants this was an encounter with A-level material after some time, so that vocabulary and symbols were temporarily unfamiliar and they were well placed to recognise students’ need to spend time making sense of terminology. We exemplify ‘unpicking’ notation by referring to expressions of the binomial expansion: asking participants to describe its many patterns, its hidden coefficients of 1 and indices of zero. We emphasise the power of using consistent language across multiple representations, as in the example given for progression: reiterating “the y-intercept is the value of y when x is zero”.

A second aim was for individuals to articulate mathematical reasoning that would usually be tacit, a feature of teachers’ practice, preparing themselves to communicate this better to students, and to notice what is done but not said.

In doing so, participants explored mathematical language. This is emphasised in our calculus sessions, where we take time to discuss terms such as gradient, slope, curve and what it makes sense to say about them. One such question that has led to fruitful discussion about precise language includes the graphs shown in Figure 1, foregrounding the distinction between describing a graph as “increasing”, having “a positive gradient” or “an increasing gradient”. We have phrased this as precision over rigour, treating the latter as mathematical while the former can apply to both mathematics and MP. Teachers may want to vary the rigour of their talk according to circumstance but should choose their language precisely to have this effect.

Figure 1: For which graphs is the gradient increasing?



A regular pedagogic question that arises from participants is whether similar language activities should be used with students: where do we stop modelling MP and turn to MTP? Our reply is that we aim first for participants to rehearse and model consistent precise use of language themselves, and invite participants to think of circumstances when they would, or would not, correct student talk or ask students to refine their language use.

4.3. Prioritise students’ engagement through and in mathematical reasoning

This KPM aimed to expose the decision-making behind our selection and adaptation of the AMTEC mathematical activities. We had explicitly aimed to align selection of content within both mathematics and mathematics teaching pedagogies, finding tasks that would engage participants as learners and inspire them to teach A-level. Equally we wanted to promote the message that studying A-level mathematics should be enjoyable and intellectually stimulating for students. The KPM made the strongly framed selection of content into a point of discussion, bringing attention to how tutors had planned and, importantly, compromised contingently to ensure time for participants to consolidate mathematical learning.

Our first attempt at articulating this pedagogic message as “prioritise student engagement” raised the question within the design team of what kind of engagement we thought was important. Further reflection and negotiation identified our focus as supporting mathematical reasoning. In planning sessions, we selected resources that would engage students (and participants) *in* mathematical reasoning because we believe thinking mathematically is what makes the subject intellectually stimulating and underpins the development of conceptual fluency. We also prioritised engagement *through* mathematical reasoning because we believe the satisfaction of achieving success in reasoning mathematically is what makes mathematics enjoyable. In this way, we aimed to model and promote an engaging pedagogy for teaching A-level mathematics by highlighting how the selected resources supported reasoning and how we had managed them to maximise such opportunities.

An instance of operationalising this KPM is explaining how we have adapted resources to make them more accessible or manageable, whilst retaining features that engage pupils through and in mathematical reasoning. For example, we explain how and why we adapted a task that involves matching descriptions of scientific processes with their graphical representations (Nrich, no date) by reducing the number of graphs and simplifying the written descriptions into a common format that emphasises the role of variables. In this way, we made the task more mathematically accessible and readable for students, and likely to fit in a single lesson, whilst retaining the underlying rationale of using mathematical reasoning to model scientific processes. In another task, a pile of paper cards is used to form plausible factorisations of a cubic polynomial. Here we ask the participants to consider whether the tactile quality of selecting and arranging three factor cards was important in supporting their own (and students') mathematical reasoning, and whether it merits the time spent producing and managing multiple sets.

4.4. Make connections between mathematical topics and representations.

A-level mathematics is rich in internal connections, both between topics and within them, and progresses knowledge from the earlier school curriculum. Connections in mathematics allow students to reason by analogy, a form of disciplinary reasoning that features strongly in A-level as students work with graphical and algebraic representations of functions. Teachers can point to connections in mathematics as evidence that ideas and reasoning met in one context can solve problems in another, returning us to Faux's (1998) big idea that mathematics offers "a lot for a little". In instantiating this KPM we focused on connections between apparently distinct topics rather a progression of learning or abstraction (see 4.1). We exemplify these connections through three problems featuring the binomial coefficient $\binom{n}{r}$: a situated counting problem, counting the instances of $a^r b^{n-r}$ in the expansion of $(a+b)^n$ and a probability problem. We also show connections between representations in the same topic: in this same session, we look at different written ways of expanding algebraic brackets and discuss the equivalences between them. In the trigonometry session, participants use both graphs and a unit circle approach to solve trigonometric equations and discuss which features of the problem are made more visible by each representation.

4.5. Deliberate use of tools so that students can make and share predictions

Several of our sessions model how tools can be used to support students in making and sharing predictions. Computers provide one such tool and, the session on differentiation begins by using GeoGebra to gather empirical data on the gradient of the tangent to the curve $y = x^2$ for a series of x-coordinates, then plot the corresponding (*x*, *gradient at x*) points. We ask participants to predict an expression for the gradient function based on their empirical data and graphic display. They confirm their prediction through graphing their suggestions and later seek a proof through differentiation by first principles. In another session, participants are asked to watch a video (http://graphingstories.com) relating two variables and simultaneously sketch a graph of the relationship. They share their predictions, reflecting on aspects of the graph which need more careful attention before watching the video again at half-speed, leading to the improvement of their original sketch-graphs. The way these activities are sequenced promotes participants' engagement in making and sharing predictions. We use prediction because these are specific conjectures that participants can verify through the subsequent use of tools.

In later phases, the focus of discussion on this KPM shifted to emphasise *how* tools are used by the tutor (or *how* their use by participants is structured) to engage participants in making and sharing predictions. This shift was reflected by including the word 'deliberate' to indicate careful and planned use of tools that focuses attention on mathematical concepts. For example, in the session on differentiation we draw attention to how we slow down dynamic variation by using arrow keys, rather than the mouse, to control sliders to step incrementally along the x-axis. This deliberate sequence allows the tutor to stop and discuss with participants what happens to the tangent and the numerical gradient of $y = x^2$ as the x coordinate increases. Visual complexity is reduced by introducing new features of the graphic display one at time and addressed by discussing each as or even before they are revealed.

One other aspect of this KPM is the referent of 'tool' in its wording. Initially, the exemplification of deliberate tool use in our sessions centred on technology and emphasised that this provided valuable representations for exploring

A-level mathematics. In later discussions, tutors reflected that other classroom tools required equally deliberate consideration of learners and mathematics and concluded that this KPM has wider reference than the four sessions indicated in Table 1. In practice, many participants comment on the use of open tasks and mini-whiteboards as tools to structure mathematical exploration. In a post-session interview, one participant characterised a matching activity as “it kind of gave you the parameters of which to think, which I think was quite a nice structure, give students the confidence to explore more challenging thoughts without being overwhelmed.” These comments switch between noticing herself as learner in the setting and imagining herself as teacher in school, and back again as she generalized: “So I think that really clicked – modelling how you can teach actually any subject I suppose, any topic like that…. And the whiteboards as well helped, because you could erase and do things again.”

4. Discussion

This paper has outlined how a course designed to prepare early career teachers for teaching A-level was structured around mathematical themes and, more innovatively, key pedagogic messages. The selection of pedagogic messages was pragmatic and grounded in the design process. Through this process, we selected the objectives and activities which shaped participants’ learning, both of mathematics and of mathematics teaching, and simultaneously emphasised/ rejected aspects of our own approach to directing those activities. The pedagogic messages arose from the design process and guided it.

We conceptualise these pedagogic messages in three ways. Firstly they articulate for tutors and for participants some criteria for what we consider to be good practice, identified in our critical discussions as relevant to A-level classroom settings and A-level teacher education settings. We are aware that selecting the KPM necessarily foregrounds our values in discussions where we want participants to articulate their own. Nevertheless we argue that our being prepared to justify our values in this way, and communicate how they are put into practice, provides a defined, public position that participants can critique, which is lacking when values are left implicit. We liken this to the Bernsteinian notion of radical visible pedagogy: visible because tutors are “explicit in acknowledging responsibility for taking up a position of authority” and radical because learning is structured as “a collective endeavour rather than a neutral and individual attainment”(Bourne, 2003, p. 511).

Secondly, we articulate, as above, how the KPM have been instantiated in tutors' own decisions and actions, thus bringing tacit elements of practice into focus as elements that should have the attention of teachers. Thirdly, in the teacher education setting, the messages act as 'pivots' between participants acting as a learner and reflecting as a teacher. They offer a means of managing the strongly-framed engagement in mathematical reasoning and the more weakly-framed learning about mathematics teaching in order to balance these two objectives. One contribution of this paper is an argument about mathematics teaching pedagogy in ITE: that messages about what is good teaching need as careful planning, articulation and enactment as messages about what is legitimate mathematics.

In terms of our Bersteinian analytic tools, by introducing KPM as an element of tactical design we have produced stronger framing of selection. This now covers both the learning of mathematics, through choosing the classroom activities, and the learning about mathematics teaching, in choosing how to model those activities and the features of them that are signalled as worthy of discussion. Correspondingly, tutors have had to exert stronger control over pace and social relations. Participants are often engaged with mathematical activities and want to spend time developing their recognition and realisation of legitimate mathematics language and practices. In our design, tutors curtail this engagement to shift attention to teaching. They thus perturb the expectations that pace and social relations in ITE will be weakly framed. One possible critique is that this shift in expectation positions participants as passive learners rather than developing professionals who are able to select from such experiences what is needed to develop their practice. While aware of this risk, we consider that our use of KPM is an example of "practical theorising" in the sense of Hagger and McIntrye (2006, p68) aimed at empowering teachers in developing a repertoire of practical ideas that have been "effectively evaluated against a variety of criteria." Our contribution here has been to design a thematic approach in the teacher education setting that foregrounds criteria we consider pertinent to A-level teaching. Far from devaluing teachers' expertise we are explicitly making time and providing a model for such evaluation in order to show its importance as part of the teacher role.

Ensor's work (2004) in unpicking the framing of mathematics pedagogy and mathematics teaching pedagogy has been influential to our thinking. She considered that the strong framing of location in teacher education meant that messages about mathematics teaching remained implicit, unrealized in practice, with participants learning mathematics in ways they attributed to university classrooms. She argued for a different modality of teacher education with more flexibility between settings. Our work supports her analysis of the potential tensions in the interacting pedagogic situations. However we argue instead that learning to teach A-level is a setting in which this flexibility can exist, because participants can be positioned, and position themselves, as both learners and teachers of mathematics. We have aimed to show, in our setting out of KPM above, how the operationalisation of mathematics teaching pedagogy in the ITE setting is close to the articulated messages about mathematics pedagogy in school. This is enabled by the participants' appreciation for developing robust knowledge of mathematics A-level topics that meets the critical demands of planning and teaching, and goes beyond what was sufficient to succeed in mathematics examinations. With this goal, participants are authentically engaged in the activities as learners, or can at least observe that others are. The KPM shift attention between noticing the effects of particular activities as learners and noticing how teaching supports these. In participants' reflection sheets, we have found some evidence that they are using what they notice about the ITE setting and "rehearsing" how to adapt this for school classrooms.

We explicitly asked participants to accept the element of role play in this setting, which allowed a strong framing of social relations and pace. However the flexible framing of location derives not only from this implied consent. Designing the course with attention to the interaction of pedagogies has the effect of providing more authenticity in the pivot between reflecting as a learner and as a teacher, and more balance in the treatment of the evaluative rules of mathematics and mathematics pedagogy. These alignments support participants and tutors in effecting the change of perspective. More authenticity is provided by the combination of articulating KPM and selecting content precisely to focus attention onto those aspects that are closely related. More balance is provided by treating practical theorising of mathematics teaching in a similar way to practical theorising of mathematics.

The KPM embody principles of teaching, and participants are invited to discuss their instantiation in our practice and their own in the same way that Parker and Adler (2014) describe teachers calling on articulations of correct mathematical reasoning in order to critique and realize their own. The difference is that mathematics has a strong grammar so that its modes of reasoning can be agreed. In contrast the key pedagogic messages are, by the nature of mathematics education, contestable in their articulation of values and in their application to practice. Our key pedagogic messages themselves are neither original or controversial, bearing comparison with other such attempts (JMC, 2017). They derive their value as MTP tools in the way that they can be related to the empirical particularities of A-level activities and our teacher education setting, and this is the design principle that we propose, and have used, for other teacher education settings. In this way, this paper extends Bersteinian critiques of teacher education to show how the same concepts can be used to consider pedagogic design. Thinking about our framing of implicit and explicit evaluative messages, selection, pace and social relations has provided theoretical structure to design deliberations, helping make our own pedagogic beliefs and practices explicit and open to critique.

An implication of the arguments above is that pre-university mathematics presents a critical site for mathematics teacher education precisely because mathematical knowledge is a focus for participants. Hence the mathematics teaching pedagogy can be moved closer to desirable mathematics pedagogy of the classroom and become a focus of critical attention. Similar situations may arise when learning topics that usually appear earlier in the curriculum, and with younger students. In these cases, however, teacher educators sometimes have to pick tasks that make the familiar strange, so that beginning teachers are provoked to interrogate other ways of understanding mathematics. Such tasks are not necessarily those that they would use with school-age students. This risks the participants switching off because of apparent irrelevance to observed school practice, or mistakenly thinking that this mathematics teaching pedagogy is directly modelling desirable mathematics pedagogy, then feeling demoralized when they try and fail to live up to these expectations. We have found that even the most familiar A-level topics, such as introducing calculus, offer opportunities for teachers to be unsure, to ask questions, to articulate developing understandings and to test them against the understandings of others and articulations of good practice, whether those concern mathematics or its teaching.

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