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On diregular digraphs with degree two and excess two

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Abstract

An important topic in the design of efficient networks is the construction of \((d, k, +\epsilon)\)-digraphs, i.e. \(k\)-geodetic digraphs with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\), where \(M(d, k)\) represents the Moore bound for degree \(d\) and diameter \(k\) and \(\epsilon > 0\) is the (small) excess of the digraph. Previous work has shown that there are no \((2, k, +1)\)-digraphs for \(k \geq 2\). In a separate paper, the present author has shown that any \((2, k, +2)\)-digraph must be diregular for \(k \geq 2\). In the present work, this analysis is completed by proving the nonexistence of diregular \((2, k, +2)\)-digraphs for \(k \geq 3\) and classifying diregular \((2, 2, +2)\)-digraphs up to isomorphism.

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1. Introduction

An important topic in the design of interconnection networks is the directed degree/diameter problem: what is the largest possible order \(N(d, k)\) of a digraph \(G\) with maximum out-degree \(d\) and diameter \(\leq k\)? A simple inductive argument shows that for \(0 \leq l \leq k\) the number of vertices at distance \(l\) from a fixed vertex \(v\) is bounded above by \(d^l\). Therefore, a natural upper bound for the order of such a digraph is the so-called Moore bound \(M(d, k) = 1 + d + d^2 + \cdots + d^k\). A digraph that attains this upper bound is called a Moore digraph. It is easily seen that a digraph \(G\) is Moore if and only if it is out-regular with degree \(d\), has diameter \(k\) and is \(k\)-geodetic, i.e. for any two vertices \(u, v\) there is at most one \(k\)-path from \(u\) to \(v\).

As it was shown by Bridges and Toueg in [1] that Moore digraphs exist only in the trivial cases \(d = 1\) or \(k = 1\) (the Moore digraphs are directed \((k + 1)\)-cycles and complete digraphs \(K_{k+1}\) respectively), much research has been devoted to the study of digraphs that in some sense approximate Moore digraphs. For example, there is an extensive literature on digraphs with maximum out-degree \(d\), diameter \(\leq k\) and order \(M(d, k) - \delta\) for small \(\delta \geq 0\); this is equivalent to relaxing the \(k\)-geodeticity requirement in the conditions for a digraph to be Moore. \(\delta\) is known as the defect of the digraph. The reader is referred to the survey [5] for more information.

In this paper, however, we will consider the following related problem, which is obtained by retaining the \(k\)-geodeticity requirement in the above characterisation of Moore digraphs, but allowing the diameter to exceed \(k\): what is the smallest possible order of a \(k\)-geodetic digraph \(G\) with minimum out-degree \(\geq d\)? A \(k\)-geodetic digraph with minimum out-degree \(\geq d\) and order \(M(d, k) + \epsilon\) is said to be a \((d, k, +\epsilon)\)-digraph or to have excess \(\epsilon\). It was shown in [6] that there are no diregular \((2, k, +1)\)-digraphs for \(k \geq 2\). In 2016 it was shown in [3] that digraphs with excess one must be diregular and that there are no \((d, k, +1)\)-digraphs for \(k = 2, 3, 4\) and sufficiently large \(d\). In a separate paper [7], the present author has shown that \((2, k, +2)\)-digraphs must be diregular with degree \(d = 2\) for \(k \geq 2\). In the present paper, we classify the \((2, 2, +2)\)-digraphs up to isomorphism and show that there are no diregular \((2, k, +2)\)-digraphs for \(k \geq 3\), thereby completing the proof of the nonexistence of digraphs with degree \(d = 2\) and excess \(\epsilon = 2\) for \(k \geq 3\). Our reasoning and notation will follow closely that employed in [4] for the corresponding result for defect \(\delta = 2\).
2. Preliminary results

We will let $G$ stand for a $(2, k, +2)$-digraph for arbitrary $k \geq 2$, i.e. $G$ has minimum out-degree $d = 2$, is $k$-geodetic and has order $M(2, k) + 2$. We will denote the vertex set of $G$ by $V(G)$. By the result of [7], $G$ must be digiregular with degree $d = 2$ for $k \geq 2$. The distance $d(u, v)$ between vertices $u$ and $v$ is the length of the shortest path from $u$ to $v$. Notice that $d(u, v)$ is not necessarily equal to $d(v, u)$. $u \rightarrow v$ will indicate that there is an arc from $u$ to $v$. We define the in- and out-neighbourhoods of a vertex $u$ by $N^−(u) = \{v \in V(G) : v \to u\}$ and $N^+(u) = \{v \in V(G) : u \to v\}$ respectively; more generally, for $0 \leq l \leq k$, the set $\{v \in V(G) : d(u, v) = l\}$ of vertices at distance exactly $l$ from $u$ will be denoted by $N^l(u)$. For $0 \leq l \leq k$ we will also write $T_l(u) = \bigcup_{i=0}^{l} N^i(u)$ for the set of vertices at distance $\leq l$ from $u$. The notation $T_{k-1}(u)$ will be abbreviated by $T(u)$.

It is easily seen that for any vertex $u$ of $G$, there are exactly two distinct vertices that are at distance $\geq k + 1$ from $u$. For any $u \in V(G)$, we will write $O(u)$ for the set of these vertices and call such a set an outlier set and its elements outliers of $u$. Notice that $O(u) = V(G) - T_k(u)$. An elementary counting argument shows that in a digiregular $(2, k, +2)$-digraph every vertex is also an outlier of exactly two vertices. We will say that a vertex $u$ can reach a vertex $v$ if $v \notin O(u)$.

Our proof will proceed by an analysis of a pair of vertices with exactly one common out-neighbour. First, we must show that such a pair exists and deduce some elementary properties of pairs of vertices with identical out-neighbourhoods.

**Lemma 1.** For $k \geq 2$, let $u$ and $v$ be distinct vertices such that $N^+(u) = N^+(v) = \{u_1, u_2\}$. Then $u_1 \in O(u_2)$, $u_2 \in O(u_1)$ and there exists a vertex $x$ such that $O(u) = \{v, x\}$, $O(v) = \{u, x\}$.

**Proof.** Suppose that $u$ can reach $v$ by a $\leq k$-path. Then $v \in T(u_1) \cup T(u_2)$. As $N^+(v) = N^+(u)$, it follows that there would be a $\leq k$-cycle through $v$, contradicting $k$-geodeticity. If $O(u) = \{v, x\}$, then $x \neq v$ and $x \notin T(u_1) \cup T(u_2)$, so that $v$ cannot reach $x$ by a $\leq k$-path. Similarly, if $u_1$ can reach $u_2$ by a $\leq k$-path, then we must have $\{u, v\} \cap T(u_1) \neq \emptyset$, which is impossible. □

**Lemma 2.** For $k \geq 2$, there exists a pair of vertices $u$, $v$ with $|N^+(u) \cap N^+(v)| = 1$.

**Proof.** Suppose for a contradiction that there is no such pair of vertices. Define a map $\phi : V(G) \to V(G)$ as follows. Let $u^+$ be an out-neighbour of a vertex $u$ and let $\phi(u)$ be the in-neighbour of $u^+$ distinct from $u$. By our assumption, it is easily verified that $\phi$ is a well-defined bijection with no fixed points and with square equal to the identity. It follows that $G$ must have even order, whereas $|V(G)| = M(2, k) + 2$ is odd. □

$u$, $v$ will now stand for a pair of vertices with a single common out-neighbour. We will label the vertices of $T_k(u)$ according to the scheme $N^+(u) = \{u_1, u_2\}$, $N^+(u_1) = \{u_3, u_4\}$, $N^+(u_2) = \{u_5, u_6\}$, $N^+(u_3) = \{u_7, u_8\}$, $N^+(u_4) = \{u_9, u_{10}\}$ and so on, with the same convention for the vertices of $T_k(v)$, where we will assume that $u_2 = v_2$.

3. Classification of $(2, 2, +2)$-digraphs

We begin by classifying the $(2, 2, +2)$-digraphs up to isomorphism. We will prove the following theorem.

**Theorem 1.** There are exactly two digiregular $(2, 2, +2)$-digraphs, which are displayed in Figs. 2 and 5.

Let $G$ be an arbitrary digiregular $(2, 2, +2)$-digraph. $G$ has order $M(2, 2) + 2 = 9$. By **Lemma 2**, $G$ contains a pair of vertices $(u, v)$ such that $|N^+(u) \cap N^+(v)| = 1$; we will assume that $u_2 = v_2$, so that we have the situation shown in Fig. 1.

We can immediately deduce some information on the possible positions of $v$ and $v_1$ in $T_2(u)$.

**Lemma 3.** If $v \notin O(u)$, then $v \in N^+(u_1)$. If $v_1 \notin O(u)$, then $v_1 \in N^+(u_1)$.

**Proof.** $v \notin T(u_2)$ by 2-geodesity. $v \neq u$ by construction. If we had $v = u_1$, then there would be two distinct $\leq 2$-paths from $u$ to $u_2$. Also $v_1 \notin \{u\} \cup T(u_2)$ by 2-geodesity and by assumption $u_1 \neq v_1$. □
Since \( v \) and \( v_1 \) cannot both lie in \( N^+(u_1) \) by 2-geodecity, we have the following corollary.

**Corollary 1.** \( O(u) \cap \{v, v_1\} \neq \emptyset \).

We will call a pair of vertices \((u, v)\) with a single common out-neighbour bad if at least one of

\[
O(u) \cap \{v_1, v_3\} = \emptyset, O(u) \cap \{v_1, v_4\} = \emptyset, O(u) \cap \{u_1, u_3\} = \emptyset, O(u) \cap \{u_1, u_4\} = \emptyset.
\]

holds. Otherwise such a pair will be called good.

**Lemma 4.** There is a unique \((2, 2, +2)\)-digraph containing a bad pair.

**Proof.** Assume that there exists a bad pair \((u, v)\). Without loss of generality, \( O(u) \cap \{v_1, v_3\} = \emptyset \). By Lemma 3 we can set \( v_1 = u_3 \). By 2-geodecity \( v_3 = u \). We cannot have \( v_4 = v_3 = u \), so \( u_4 \) must be an outlier of \( u \). By Corollary 1 it follows that \( O(u) = \{v, v_4\} \).

Consider the vertex \( u_1 \). By Lemma 3, if \( u_1 \notin O(v) \), then \( u_1 \in N^+(v_1) \). However, as \( v_1 = u_3 \), there would be a 2-cycle through \( u_1 \). Hence \( u_1 \notin O(v) \). As \( O(u) = \{v, v_4\} \), we have \( V(G) = \{u, u_1, u_2, u_3 = v_1, u_4, u_5, u_6, v, v_4\} \) and \( O(v) = \{u_1, u_4\} \). As neither \( u \) nor \( v \) lies in \( T(u_1) \), we must also have \( u_2 \notin O(u_1) \). As \( u_1 \) can reach \( u_1, u_4, u \) and \( v_4 \), it follows that without loss of generality we either have \( O(u_1) = \{u_2, v_4\} \) and \( N^+(u_4) = \{u_5, u_6\} = N^+(u_2) \) or \( O(u_1) = \{u_2, u_6\} \) and \( N^+(u_4) = \{v, u_5\} \). In either case, \((v, u_1)\) is a good pair.

Suppose firstly that \( N^+(u_2) = N^+(u_4) \). Then \( v \) is an outlier of \( u \) and \( u_2 \). As each vertex is the outlier of exactly two vertices, \( v_1 \) must be able to reach \( v \) by a \( \leq 2 \)-path. Hence \( v_4 \rightarrow v \). Likewise \( u_2 \) can reach \( v \), so without loss of generality \( u_5 \rightarrow u \). Suppose that \( O(u_2) \cap \{u, u_1\} = \emptyset \). As \( u \) and \( v \) have a common out-neighbour, we must have \( u_6 \rightarrow u \). Since \( u \rightarrow u_1 \), by 2-geodecity we must have \( u_5 \rightarrow u_1 \). However, this is a contradiction, as \( v \) and \( u_1 \) also have a common out-neighbour. Therefore, at least one of \( u, u_1 \) is an outlier of \( u_2 \). By Lemma 1 \( u_4 \) is an outlier of \( u_2 \). Therefore either \( O(u_2) = \{u, u_4\} \) or \( O(u_2) = \{u_1, u_4\} \). If \( O(u_2) = \{u, u_4\} \), then \( u_2 \) must be able to reach \( u_1, v_1 \) and \( u_4, u_5 \rightarrow v \) and \( v \rightarrow v_1 \), so \( v_1 \in N^+(u_6) \). As \( u_1 \rightarrow v_1 \), we must have \( N^+(u_5) = \{v, u_1\} \). As \( v \) and \( u_1 \) have a common out-neighbour, this violates 2-geodecity. Hence \( O(u_2) = \{u_1, u_4\} \) and \( u_2 \) can reach \( u_1 \) and \( u_4 \). As \( v \rightarrow v_1 \), \( v_1 \in N^+(u_6) \). As \( v_1 \rightarrow v_4 \), it follows that \( N^+(u_5) = \{v, u_4\} \). However, \( u_4 \rightarrow v \), so this again violates 2-geodecity.

We are left with the case \( O(u_1) = \{u_2, u_6\} \) and \( N^+(u_4) = \{v, u_5\} \). Then \( v_1 \in O(u_2) \), as neither \( v \) nor \( u_1 \) lies in \( T(u_2) \). Observe that \( u_2 \) and \( u_4 \) have a single common out-neighbour, so by Corollary 1 \( O(u_2) \cap \{u_4, v\} \neq \emptyset \). Therefore either \( O(u_2) = \{u_1, u_4\} \) or \( O(u_2) = \{v_1, v\} \). Suppose firstly that \( O(u_2) = \{v_1, u_4\} \). Then \( N^+(u_2) = \{u, v, u_1, v_4\} \). As \( N^+(u_4) = \{v, u_5\} \), \( u_5 \rightarrow v \), so \( u_6 \rightarrow v \). As \( N^+(u) \cap N^+(v) \neq \emptyset \). \( u_5 \rightarrow u \). \( u \rightarrow u_1 \), so necessarily \( u_6 \in N^+(u_5) \). However, \( v \in N^+(u_1) \cap N^+(v) \), contradicting 2-geodecity.

Hence \( O(u_2) = \{v_1, v\} \) and \( N^+(u_2) = \{u, u_1, u_4, v_4\} \). As \( u_4 \rightarrow u_5, u_5 \rightarrow u_4 \), Thus \( u_6 \rightarrow u_4 \). Now \( u_4 \rightarrow u_5 \) and \( u_5 \rightarrow u_1 \) implies that \( N^+(u_5) = \{u_1, v_4\} \) and \( N^+(u_6) = \{u_4\} \). Finally we must have \( N^+(u_4) = \{v, u_6\} \). This gives us the \((2, 2, +2)\)-digraph shown in Fig. 2. \( \square \)

We can now assume that all pairs given by Lemma 2 are good. Let us fix a pair \((u, v)\) with a single common out-neighbour. It follows from Corollary 1 and the definition of a good pair that \( v_1 \in O(u) \); otherwise \( O(u) \) would contain \( v, v_3 \) and \( v_4 \), which is impossible. Likewise \( u_1 \in O(v) \).
Considering the positions of \( v_3 \) and \( v_4 \), we see that there are without loss of generality four possibilities: (1) \( u = v_3, u_4 = v_4 \), (2) \( u = v_3, u_4 = v_4 \), (3) \( N^+(u_1) = N^+(v_3) \) and (4) \( u_3 = v_3, O(u) = \{v_1, u_4\} \). A suitable relabelling of vertices shows that case 4 is equivalent to case 1.a below, so we will examine cases 1 to 3 in turn.

**Case 1:** \( u = v_3, u_4 = v_4 \)

Depending upon the position of \( v \), we must either have \( O(u) = \{v_1, v\} \) and \( O(v) = \{u_1, u_3\} \) or \( v = u_3 \) (see Fig. 3).

**Case 1.a:** \( O(u) = \{v_1, v\}, O(v) = \{u_1, u_3\} \)

In this case \( V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\} \). Without loss of generality, \( O(u_1) \) and \( O(v_1) \) have a single common out-neighbour, namely \( u_4 \), so as we are assuming all such pairs to be good, we have \( u_3 \in O(v_1) \), \( u \in O(u_1) \). By 2-geodecity, \( N^+(u_4) \subset \{u_5, u_6, v\} \), so without loss of generality either \( N^+(u_4) = \{u_5, u_6\} \) or \( N^+(u_4) = \{u_5, v\} \).

Suppose that \( N^+(u_4) = \{u_5, u_6\} \). By elimination, \( O(v_1) = \{v, u_3\} \). As \( G \) is diregular, every vertex is an outlier of exactly two vertices; \( v \) is an outlier of \( u \) and \( v_1 \), so both \( u_1 \) and \( u_2 \) can reach \( v \) by a \( \leq 2 \)-path. Hence \( v \in N^+(u_3) \). As \( v \rightarrow v_1 \), we see that \( v_1 \) is an outlier of \( u_1 \); as \( u \) is also an outlier of \( u_1 \), we have \( O(u_1) = \{u, v_1\} \) and \( N^+(u_3) = \{v, u_2\} \). As \( u \rightarrow u_2 \), this is impossible.

Now consider \( N^+(u_4) = \{u_5, v\} \). We now have \( O(v_1) = \{u_3, u_6\} \). Thus \( u_3 \in O(v) \cap O(v_1) \), \( u_3 \in T_2(u_4) \). \( v \) is not adjacent to \( u_3 \), \( u_3 \in N^+(u_5) \), \( u_2 \) and \( u_4 \) have \( u_5 \) as a unique common out-neighbour, so \( u_6 \in O(u_4) \), \( v \in O(u_2) \). As \( u_6 \in O(v_1) \cap O(u_4) \), \( u_1 \) can reach \( u_6 \). Hence \( u_6 \in N^+(u_3) \). Neither \( u \) nor \( v \) lie in \( T(u_4) \), \( u_2 \in O(u_1) \). Therefore either \( O(u_1) = \{u, u_2\} \) or \( O(u_1) = \{u_2, v_1\} \). If \( O(u_1) = \{u, u_2\} \), then \( N^+(u_2) = \{u_6, v_1\} \). \( u_2 \) cannot reach \( v_1 \), since \( u_3 \in T(u_2) \), \( u_2 \in O(u_2) \). As \( u_6 \in O(v_1) \cap O(u_4) \), \( u_1 \) and \( u_3 \) have a common out-neighbour, namely \( u_4 \). By elimination, \( v \) and \( v_1 \) also have in-neighbours in \( \{u_5, u_6\} \). As \( u_1 \) and \( v_1 \) have a common out-neighbour, we have \( N^+(u_5) = \{u_3, v_1\} \), \( N^+(u_6) = \{u_1, v\} \). However, both \( u_3 \) and \( v_1 \) are adjacent to \( u \), violating 2-geodecity.

**Case 1.b: \( v = u_3 \)**

There exists a vertex \( x \) such that \( V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, x\} \). \( O(u) = \{v_1, x\} \) and \( O(v) = \{u_1, x\} \). As \( x \in O(u) \cap O(v) \), \( u_1 \) and \( u_2 \) can reach \( x \), so without loss of generality \( x \in N^+(u_4) \cap N^+(u_5) \). As \( u_5 \) and \( u_4 \) have a common out-neighbour, \( u_6 \in O(u_4) \). Also, \( u_1 \) and \( v_1 \) have a common out-neighbour, so \( u \in O(u_1) \) and \( O(u_1) = \{u_4, u_5\} \). Thus \( N^+(u_4) = \{x, u_6\} \). Observe that \( u_2 \) and \( u_4 \) have the out-neighbour \( u_6 \) in common. Thus \( x \in O(u_2) \), whereas we already have \( x \in O(u) \cap O(v) \), a contradiction.

**Case 2:** \( u = v_3, O(u) = \{v_1, u_4\} \)

As \( v \) is not equal to \( v_1 \) or \( u_4 \), \( v \) must lie in \( T_2(u) \). Without loss of generality, \( v = u_3 \). Hence \( V(G) = \{u, u_1, u_2, v, u_4, u_5, u_6, v_1, v_4\} \) and \( O(v) = \{u_1, u_4\} \). We have the configuration shown in Fig. 4. Hence \( u_1 \) can reach
Taking into account adjacencies between members of $u_1, v, u_4, u_2$ and $v_1$, so we have without loss of generality one of the following: a) $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$, b) $O(u_1) = \{u, u_5\}$, $N^+(u_4) = \{u_6, v_4\}$, c) $O(u_1) = \{u_5, u_6\}$, $N^+(u_4) = \{u, v_4\}$ or d) $O(u_1) = \{u_5, v_4\}$, $N^+(u_4) = \{u, u_6\}$.

**Case 2.a: $O(u_1) = \{u, v_4\}$, $N^+(u_4) = \{u_5, u_6\}$**

As $v_4 \in O(u) \cap O(u_1)$, $u_2$ can reach $v_4$ and without loss of generality $v_4 \in N^+(u_4)$. $N^+(u_2) = N^+(u_4)$, so by Lemma 1 $u_2 \in O(u_4)$, $u_4 \in O(u_2)$, $u_5 \in O(u_6)$ and $u_6 \in O(u_5)$. Hence $u_4 \in O(v) \cap O(u_2)$, so $v_1$ can reach $u_4$, so $u_4 \in N^+(v_4)$. Neither $u_5$ nor $u_6$ lies in $N^+(u_1)$, so $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{u_4, u_5\}$. Hence $O(v_4) = \{u, u_1\}$. Observe that $N^+(u_1) = N^+(u_4)$, so that $v \in O(u_4)$. Therefore $v \notin N^+(u_5) \cup N^+(u_6)$, yielding $O(u_2) = \{u_4, u_5\}$, $N^+(u_2) = \{u_4, v_1, u, u_1\}$. As $v_1 \rightarrow v_4$ and $N^+(u_2) = N^+(u_4)$, we must have $N^+(u_5) = \{v, u\}$, $N^+(u_6) = \{v_1, u_1\}$. This yields the digraph in Fig. 5. Unlike the digraph in Fig. 2, this digraph contains pairs of vertices with identical out-neighbourhoods, so the two are not isomorphic.

**Case 2.b: $O(u_1) = \{u, u_5\}$, $N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \rightarrow v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, so $u_4$ can reach $u_4$. As $u_4 \rightarrow u_6$, we must have $u_5 \rightarrow u_4, u_2$ and $u_4$ have $u_6$ as a common out-neighbour, so $u_4 \in O(u_2), u_5 \in O(u_4)$. Therefore $v_4 \in O(u) \cap O(u_2)$, so that $u_6$ can reach $v_4$, but $v_4 \notin T(u_6)$, so $N^+(u_6)$ contains an in-neighbour of $v_4$. $u_4 \notin N^+(u_6)$, so we have $u_6 \rightarrow v_4$. We have $u_5 \in O(u_4) \cap O(u_1)$, so $v_1$ can reach $u_5$ and hence $v_4 \rightarrow u_5, u_1$ cannot reach $u_6$, as $u_2, u_4 \notin T(v_1)$, so $O(v_1) = \{u_4, u_6\}$, $N^+(u_5) = \{u_5, v\}$. Now $u_2$ and $u_4$ have $u_6$ as a unique common out-neighbour, so $u_6 \in O(v_4)$, $v \in O(u_2)$. Thus $O(u_2) = \{v, v_4\}$ and $N^2(u_2) = \{u_4, v_1, u, u_1\}$. Taking into account adjacencies between members of $N^2(u_2)$, it follows that $N^+(u_6) = \{u_4, u_1\}$, $N^+(u_6) = \{u_1, v_1\}$. However, $(u_2, u_4)$ now constitutes a bad pair, contradicting our assumption.

**Case 2.c: $O(u_1) = \{u, u_5\}$, $N^+(u_4) = \{u_6, v_4\}$**

As $u_4 \rightarrow v_4$, $u_4 \notin N^+(v_4)$, so $u_4 \in O(v_1)$. Hence $u_4 \in O(v) \cap O(v_1)$, implying that $u_2$ can reach $u_4$. Without loss of generality, $u_5 \rightarrow u_4$. There are three possibilities: (i) $O(v_1) = \{u_4, u_6\}$, $N^+(u_4) = \{v, u_5\}$, (ii) $O(v_1) = \{u_4, u_5\}$, $N^+(u_4) = \{v, u_6\}$ and (iii) $O(v_1) = \{u_4, v\}$, $N^+(u_4) = \{u_5, u_6\}$.

(i) $O(v_1) = \{u_4, u_6\}$, $N^+(u_4) = \{v, u_5\}$

$u_1$ and $v_4$ have $v$ as a unique common out-neighbour, so $u_4 \in O(v_4)$. However, this contradicts $v_4 \rightarrow u_5 \rightarrow u_4$.

(ii) $O(v_1) = \{u_4, u_5\}$, $N^+(u_4) = \{v, u_6\}$

Neither $u_4$ nor $v_1$ lie in $T(u_2)$, so $v_4 \in O(u_2)$. Now observe that $u_2$ and $u_4$ have $u_6$ as unique common out-neighbour, so $v \in O(u_2)$, yielding $O(u_2) = \{v_4, v_4\}$ and $N^2(u_2) = \{u_4, u_1, u, v_1\}$. As $u_4 \rightarrow v$ and $u \rightarrow u_1$, we must have $N^+(u_5) = \{u_4, u_1\}$, $N^+(u_6) = \{u, v_1\}$, a contradiction, since $u_1 \rightarrow u_4$.

(iii) $O(v_1) = \{u_4, v\}$, $N^+(u_4) = \{u_5, u_6\}$

We now have $N^+(u_2) = N^+(u_4)$, so $u_2 \in O(u_4), v_4 \in O(u_2), u_5 \in O(u_6), u_6 \in O(u_5)$. Also $N^+(u_4) = N^+(u_1)$, so $u_4 \in O(v_1), v_1 \in O(u_4)$ and $u \in O(u_4)$. $u \in O(u_2)$ implies that $u \notin N^+(u_2) \cap N^+(u_6)$, so we see that $u \in O(u_2)$ and hence $O(u_2) = \{u, v_4\}$ and $N^2(u_2) = \{u_4, u_1, v, v_1\}$. As $u_1 \rightarrow u_4$ and $u_4 \rightarrow v$, we have $N^+(u_5) = \{u_4, v\}$, $N^+(u_6) = \{u_1, v_1\}$. It is not difficult to show that this yields a $(2, 2, +2)$-digraph isomorphic to that in Fig. 5.

**Case 2.d: $O(u_1) = \{u_4, v_4\}$, $N^+(u_4) = \{u, u_6\}$**

In this case $v_4 \in O(u) \cap O(u_1)$, so $u_2$ can reach $v_4, u_4$ and $v_1$ have unique common out-neighbour $u$, so $v_4 \in O(u_4), u_6 \in O(v_1)$. If $u_6 \rightarrow v_4$, then we would have $u_4 \rightarrow u_6 \rightarrow u_4$, contradicting $v_4 \in O(u_4)$, so $u_5 \rightarrow v_4$. This also implies that $u_5 \notin N^+(u_4)$, so $u_5 \in O(v_1)$, yielding $O(v_1) = \{u_5, u_6\}$ and $N^+(u_4) = \{v, u_4\} = N^+(u_1)$. Now $v_4, u_1 \notin T(u_2)$, so $O(u_2) = \{v, u_4\}$.
and \( N^2(u_2) = \{v_1, v_4, u, u_1\} \). As \( v_1 \to v_4 \) and \( v_1 \to u \), it follows that \( N^+(u_5) = \{v_4, u\} \), \( N^+(u_6) = \{u_1, v_1\} \). However, we now have paths \( u_4 \to u \to u_1 \) and \( u_4 \to u_6 \to u_1 \), which is impossible.

**Case 3:** \( N^+(u_1) = N^+(v_1) \)

It is easy to see by 2-geodecity that \( V(G) = \{u, u_1, u_2, u_3, u_4, u_5, u_6, v, v_1\} \), \( O(u) = \{v, v_1\} \) and \( O(v) = \{u, u_1\} \). As \( u_1, v_1 \not\in T(u_2) \), we have \( \mathcal{O}(u_2) = \{u_3, u_4\} \) and \( N^2(u_2) = \{u, u_1, v, v_1\} \). Without loss of generality, \( N^+(u_5) = \{u_1, v_1\} \), \( N^+(u_6) = \{v, u_1\} \). \( u \) and \( v \) have in-neighbours apart from \( u_5 \) and \( u_6 \) respectively, so without loss of generality \( u_3 \to u_4 \to v \). Likewise, \( u_5 \) and \( u_6 \) have in-neighbours other than \( u_2 \), so, as \( u_3 \to u \) and \( u_6 \to v \), we must have \( N^+(u_3) = \{u, u_6\} \), \( N^+(u_4) = \{v, u_5\} \). But now we have paths \( u_3 \to u \to u_1 \) and \( u_3 \to u_6 \to u_1 \), violating 2-geodecity.

**Corollary 2.** There is a unique \((2, 2, +2)-\)digraph containing no bad pairs.

This completes our analysis of diregular \((2, 2, +2)-\)digraphs. As it was shown in [7] that there are no non-diregular \((2, 2, +2)-\)digraphs, \((2, 2, +2)-\)digraphs are now classified up to isomorphism. These conclusions have been verified computationally by Erskine [2]. It is interesting to note that neither of the \((2, 2, +2)-\)digraphs are vertex-transitive, for in each case there are exactly three vertices contained in two 3-cycles. However, there does exist a Cayley \((2, 2, +5)-\)digraph (on the alternating group \(A_4\)), so it would be interesting to determine the smallest vertex-transitive \((2, 2, +\epsilon)-\)digraphs.

### 4. Main result

We can now complete our analysis by showing that there are no diregular \((2, k, +\epsilon)-\)digraphs for \( k \geq 3 \). Let \( G \) be such a digraph. By **Lemma 2**, \( G \) contains vertices \( u \) and \( v \) with a unique common out-neighbour. In accordance with our vertex-labelling convention, we have the situation in **Fig. 6**. A triangle based at a vertex \( x \) represents the set \( T(x) \).

We now proceed to determine the possible outlier sets of \( u \) and \( v \).

**Lemma 5.** \( v \in N^{k-1}(u_1) \cup O(u) \) and \( u \in N^{k-1}(v_1) \cup O(v) \). If \( v \in O(u) \), then \( u_2 \in O(u_1) \) and if \( u \in O(v) \), then \( u_2 \in O(v_1) \).

**Proof.** \( v \) cannot lie in \( T(u) \), or the vertex \( u_2 \) would be repeated in \( T_u \). Also, \( v \not\in T(u_2) \), or there would be a \( \leq k \)-cycle through \( v \). Therefore, if \( v \not\in O(u) \), then \( v \in N^{k-1}(u_1) \). Likewise for the other result. If \( v \in O(u) \), then neither in-neighbour of \( u_2 \) lies in \( T(u_1) \), so that \( u_2 \in O(u_1) \). \( \square \)

**Lemma 6.** Let \( w \in T(v_1) \), with \( d(v_1, w) = l \). Suppose that \( w \in T(u_1) \), with \( d(u_1, w) = m \). Then either \( m \leq l \) or \( w \in N^{k-1}(u_1) \). A similar result holds for \( w \in T(u_1) \).

**Proof.** Let \( w \) be as described and suppose that \( m > l \). Consider the set \( N^{k-m}(w) \). By construction, \( N^{k-m}(w) \subseteq N^{k}(u_1) \), so by \( k \)-geodecity \( N^{k-m}(w) \cap T(u_1) = \varnothing \). At the same time, we have \( l + k - m \leq k - 1 \), so \( N^{k-m}(w) \subseteq T(v_1) \). This implies that \( N^{k-m}(u_1) \cap T(u_2) = \varnothing \). As \( V(G) = \{u \} \cup T(u_1) \cup T(u_2) \cup O(u) \), it follows that \( N^{k-m}(w) \subseteq \{u \} \cup O(u) \). Therefore \( |N^{k-m}(w)| = 2^{k-m} \leq 3 \). By assumption \( 0 \leq m \leq k - 1 \), so it follows that \( m = k - 1 \). \( \square \)

**Corollary 3.** If \( w \in T(v_1) \), then either \( w \in \{u \} \cup O(u) \) or \( w \in T(u_1) \) with \( d(u_1, w) = k - 1 \) or \( d(u_1, w) \leq d(v_1, w) \).

**Proof.** By \( k \)-geodecity and **Lemma 6**. \( \square \)

**Corollary 4.** \( u_1 \in N^{k-1}(u_1) \cup O(u) \) and \( u_1 \in N^{k-1}(v_1) \cup O(v) \).

**Fig. 6.** Configuration for \( k \geq 3 \).
**Proof.** We prove the first inclusion. By **Corollary 3**, $v_1 \in \{u\} \cup O(u) \cup \{u_1\} \cup N^{k-1}(u_1)$. By $k$-geodecity, $v_1 \neq u$ and by construction, $v_1 \neq u_1$. □

We now have enough information to identify one member of $O(u)$ and $O(v)$.

**Lemma 7.** $v_1 \in O(u)$ and $u_1 \in O(v)$.

**Proof.** We prove that $v_1 \in O(u)$. Suppose that neither $v_1$ nor $v$ lies in $O(u)$. Then by **Lemma 5** and **Corollary 4** we have $v, v_1 \in N^{k-1}(u_1)$. As $v_1$ is an out-neighbour of $v$, it follows that $v_1$ appears twice in $T_k(u_1)$, violating $k$-geodecity. Therefore $O(u) \cap \{v, v_1\} \neq \emptyset$.

Now assume that $v_1, v_3 \in T_k(u)$. Again by **Corollary 4**, $v_1 \in N^{k-1}(u_1)$. By $k$-geodecity we also have $v_3 \in T(u_1)$. However, $v_3 \in N^4(v_1)$, so $v_3$ appears twice in $T_k(u_1)$, which is impossible. Hence $O(u) \cap \{v_1, v_3\} \neq \emptyset$. Similarly, $O(u) \cap \{v_1, v_4\} \neq \emptyset$. In the terminology of the previous section, $G$ contains no bad pairs. Therefore, if $v_1 \notin O(u)$, then $\{v, v_3, v_4\} \subseteq O(u)$. Since these vertices are distinct, this is a contradiction and the result follows. □

**Lemma 7** allows us to conclude that for vertices sufficiently close to $v_1$ one of the potential situations mentioned in **Corollary 3** cannot occur.

**Lemma 8.** $T_{k-3}(v_1) \cap N^{k-1}(u_1) = T_{k-3}(u_1) \cap N^{k-1}(v_1) = \emptyset$.

**Proof.** Let $w \in T_{k-3}(v_1) \cap N^{k-1}(u_1)$. Consider the position of the vertices of $N^+(w)$ in $T_k(u) \cup O(u)$. As $v_1 \notin N^+(w)$, it follows from **Lemma 7** that at most one of the vertices of $N^+(w)$ can be an outlier of $u$, so let us write $w_1 \in N^+(w) \setminus O(u)$. By $k$-geodecity, $w_1 \notin T(u_1) \cup \{u\}$. Hence $w_1 \in T(u_2) = T(v_2)$. However, $w_1$ also lies in $T(v_1)$, so this violates $k$-geodecity. □

**Corollary 5.** There is at most one vertex in $T_{k-3}(v_1) \setminus \{v_1\}$ that does not lie in $T(u_1)$; for all other vertices $w \in T_{k-3}(v_1) \setminus \{v_1\}$, $d(u_1, w) = d(v_1, w)$. A similar result for $T_{k-3}(u_1) \setminus \{u_1\}$ also holds.

**Lemma 9.** For $k = 3$, $N^+(u_1) \cap N^2(v_1) = N^+(v_1) \cap N^2(u_1) = \emptyset$.

**Proof.** Suppose that $v_3 = u_2$. By the reasoning of **Lemma 8** we can set $u = v_2$ and $O(u) = \{v_1, v_8\}$, $v \notin O(u)$ and by $3$-geodecity $v \notin N^+(u_3)$, so we can assume that $u = u_9, u_3 \rightarrow v_3$ implies that $u_1 \notin T(v_1)$, so $O(v) = \{u_1, u_1\}$. We must have $\{u_4, u_8, u_{10}\} = \{v_4, v_3, v_{10}\}$. As $u_4 \rightarrow v$, it follows that $v_4 = u_8$ and hence $\{u_4, u_{10}\} = \{v_3, v_{10}\}$, which is impossible. □

As $u_3$ is an outlier of $v$, neither $v_3$ nor $v_4$ can be equal to $u_1$. It follows from **Corollary 5** and **Lemma 9** that either $N^+(u_1) = N^+(v_1)$ or $u_1$ and $v_1$ have a single common out-neighbour, with one vertex of $N^+(v_1)$ being an outlier of $u$.

**Lemma 10.** $N^2(u) \neq N^2(v)$

**Proof.** Let $N^2(u) = N^2(v)$, with $N^+(u_1) = N^+(v_1) = \{u_3, u_4\}$. Suppose that $v \notin O(u)$. By **Lemma 5**, $v \in N^{k-2}(u_3) \cup N^{k-2}(u_4)$. But then there is a $k$-cycle through $v$. It follows that $O(u) = \{v_1, v_1\}$, $O(v) = \{u_1, u_1\}$. By **Lemma 6**, $u_2 \in O(u_1) \cap O(v_1)$. Therefore by **Lemma 8** $\{u_2, v_1\}$, $O(v_1) = \{u_2, u_1\}$.

Consider the in-neighbour $u'$ of $u_1$ that is distinct from $u$. We have either $|N^+(u') \cap N^+(u)| = 1$ or $|N^+(u') \cap N^+(u)| = 2$.

In the first case, it follows from **Lemma 7** that $u_2 \in O(u)$. Every vertex of $G$ is an outlier of exactly two vertices, so $u' = u_1$ or $u_1$. In either case, we have a contradiction. Therefore $N^+(u') = N^+(u)$. It now follows from **Lemma 8** that $u' \in O(u) = \{v, v_1\}$, which is impossible. □

Noticing that $u_1$ and $v_1$ also have a unique common out-neighbour, we have the following corollary.

**Corollary 6.** Without loss of generality, $u_3 = v_3$, $u_9 = v_9$, $O(u) = \{v_1, v_4\}$, $O(v) = \{u_1, u_4\}$, $O(u_1) = \{v_4, v_{10}\}$ and $O(v_1) = \{u_4, u_{10}\}$.

We are now in a position to complete the proof by deriving a contradiction.

**Theorem 2.** There are no digereular $(2, k, +2)$-digraphs for $k \geq 3$.

**Proof.** $u, v \notin \{u_1, u_3, v_1, v_3\}$, so by **Lemma 5** $d(u, v) = d(v, u) = k$. In fact, $u_3 = v_3$ implies that $v \in N^{k-2}(u_3)$ and $u \in N^{k-2}(v_3)$. Let $k \geq 4$. Then $u, v \notin \{u_9, u_{10}\}$, so $u, v \in T_k(u_1) \cap T_k(v_1)$. If $u \in T_k(u_2) = T(v_2)$, then $u$ would appear twice in $T_k(u_1)$, so $u \in N^{k-1}(u_1)$. However, as $u$ and $v$ have a common out-neighbour, this violates $k$-geodecity.

Finally, suppose that $k = 3$. The above analysis will hold unless $u = u_{10}$ and $v = u_{10}$. Let $N^-(u_1) = \{u, u', N^-(v_1) = \{v, v'\}$. It is evident that $u' \notin \{v_1, v_4\}$, so that $v' \in T_3(u)$. As $v \in N^+(u_4)$, we must have $v' \in N^{2}(u_2)$. Similarly $u' \in N^{2}(u_2)$. Since $u_1$ and $v_1$ have a common out-neighbour, we can assume that $u' \in N^+(u_3)$ and $v' \in N^+(u_6)$. $u_3$ and $v_3$ can be the outlier of only two vertices, namely $u$ and $u_1$, so $v_4 \in N^3(u_2)$ and likewise $u_4 \in N^3(u_2)$. By $3$-geodecity $v_4 \in N^{2}(u_3)$ and $u_4 \in N^{2}(u_6)$. It follows that $u, v \notin N^3(u)$, so $u \notin T_3(u_1) \cup T_3(u_2)$. Hence $O(u) = N^-(u) = \{v_1, v_4\}$, which again is impossible. □

It is interesting to note that a similar argument can be used to provide an alternative proof of the result of [6].
References