SPECTRAL ANALYSIS OF A FAMILY OF BINARY INFLATION RULES

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Abstract. The family of primitive binary substitutions defined by $1 \mapsto 0 \mapsto 01^m$ with $m \in \mathbb{N}$ is investigated. The spectral type of the corresponding diffraction measure is analysed for its geometric realisation with prototiles (intervals) of natural length. Apart from the well-known Fibonacci inflation ($m = 1$), the inflation rules either have integer inflation factors, but non-constant length, or are of non-Pisot type. We show that all of them have singular diffraction, either of pure point type or essentially singular continuous.

1. Introduction

Due to the general interest in substitutions with a multiplier that is a Pisot–Vijayaraghavan (PV) number, and the renewed interest in substitutions of constant length, other cases and classes have been a bit neglected. In particular, the analysis of non-PV inflations is clearly incomplete, although they should provide valuable insight into systems with singular continuous spectrum. This was highlighted in a recent example [1], where the absence of absolutely continuous diffraction could be shown via estimates of certain Lyapunov exponents. The same method can also be used for substitutions of constant length [2, 20] to re-derive results that are known from [21, 9] in an independent way.

Here, we extend these methods to an entire family of binary inflation rules, namely those derived from the substitutions $1 \mapsto 0 \mapsto 01^m$ with $m \in \mathbb{N}$ by using prototiles of natural length. The inflations are not of constant length, and all have singular spectrum (either pure point or mainly singular continuous), as previously announced in [5]. More precisely, we prove the following result, the concepts and details of which are explained as we go along.

Theorem 1.1. *Consider the primitive, binary inflation rule $1 \mapsto 0 \mapsto 01^m$ with $m \in \mathbb{N}$, and let $\widehat{\gamma_u}$ be the diffraction measure of the corresponding Delone dynamical system that emerges from the left endpoints of the tilings with two intervals of natural length, where $u = (u_0, u_1)$ with $u_0 u_1 \neq 0$ are arbitrary complex weights for the two types of points. Then, one has the following three cases.*

(1) *For $m = 1$, this is the well-known Fibonacci chain, which has pure point diffraction and, equivalently, pure point dynamical spectrum.*
(2) *When $m = \ell(\ell{+}1)$ with $\ell \in \mathbb{N}$, the inflation multiplier is an integer, and the diffraction measure as well as the dynamical spectrum is once again pure point.*
(3) *In all remaining cases, the inflation tiling is of non-PV type, and the diffraction measure, apart from the trivial peak at 0, is purely singular continuous.*

The article is organised as follows. We begin with the introduction of our family of inflations and their properties in Section 2, where the cases (1) and (2) of Theorem 1.1 will already follow, and then discuss the displacement structure and its consequence on the pair correlations in the form of exact renormalisation relations in Section 3. This has strong implications on the autocorrelation and diffraction measures (Section 4), which are then further analysed via Lyapunov exponents in Section 5. The main result here is the absence of absolutely continuous diffraction for all members of our family of inflation systems. One ingredient is the logarithmic Mahler measure of a derived family of polynomials, which we analyse a little further in an Appendix.

2. Setting and general results

Consider the family of primitive substitution rules on the binary alphabet $\{0,1\}$ given by

$$\varrho_m : \quad 0 \mapsto 01^m\,,\ 1 \mapsto 0\,, \quad \text{with } m \in \mathbb{N}.$$

Its substitution matrix is $M_m = \left(\begin{smallmatrix} 1 & 1 \\ m & 0 \end{smallmatrix}\right)$ with eigenvalues $\lambda_m^{\pm} = \frac{1}{2}\bigl(1 \pm \sqrt{4m+1}\,\bigr)$, which are the roots of $x^2 - x - m = 0$. Whenever the context is clear, we will simply write λ instead of λ_m^{+}. Note that one has $\lambda_m^{-} = \frac{-m}{\lambda}$. For each $m \in \mathbb{N}$, there is a unique bi-infinite fixed point w of ϱ_m^2 with legal seed $0|0$ around the reference point (or origin), and the orbit closure of w under the shift action defines the discrete (or symbolic) hull $\mathbb{X}_m$. Then, $(\mathbb{X}_m, \mathbb{Z})$ is a topological dynamical system that is strictly ergodic by standard results; see [21, 4] and references therein for background and further details.

The Perron–Frobenius (PF) eigenvector of M_m, in frequency-normalised form, is

$$v_{\mathrm{PF}} \,=\, (\nu_0, \nu_1)^T \,=\, \tfrac{1}{\lambda}(1, \lambda{-}1)^T, \tag{2.1}$$

where the ν_i are the relative frequencies of the two letters in any element of the hull, $\mathbb{X}_m$. Next, $(\lambda, 1)$ is the corresponding left eigenvector, which gives the interval lengths for the corresponding geometric inflation rule. Up to scale, this is the unique choice to obtain a self-similar *inflation tiling* of the line from ϱ_m; see [4, Ch. 4] for background. This version, where 0 and 1 stand for intervals of length λ and 1, is convenient because $\mathbb{Z}[\lambda]$ is then the natural $\mathbb{Z}$-module to work with. The tiling hull $\mathbb{Y}_m$ emerges from the orbit closure of the tiling defined by w, now under the continuous translation action of $\mathbb{R}$. The topological dynamical system $(\mathbb{Y}_m, \mathbb{R})$ is again strictly ergodic, which can be proved by a suspension argument [14]. The unique invariant probability measure on $\mathbb{Y}_m$ is the well-known patch frequency measure of the inflation rule.

2.1. Cases with pure point spectrum.

Let us begin with the analysis of the case $m = 1$, which defines the Fibonacci chain. Here, the following result is standard [4, 21].

Fact 2.1. *For $m = 1$, our substitution defines the well-known Fibonacci chain or tiling system. Both dynamical systems, $(\mathbb{X}_1, \mathbb{Z})$ and $(\mathbb{Y}_1, \mathbb{R})$, are known to have pure point diffraction and dynamical spectrum.* □

Let us thus analyse the systems for $m > 1$, where we begin with an easy observation.

Fact 2.2. *The inflation multiplier $\lambda = \lambda_m^+$ is an integer if and only if $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, where $\lambda_m^+ = \ell + 1$ and $\lambda_m^- = -\ell$. In all remaining cases with $m > 1$, the inflation multiplier fails to be a PV number.* □

Let us take a closer look at the cases where λ is an integer, where we employ the concept of *mutual local derivability* (MLD) from [4]. This can be viewed as the natural extension of conjugacy via sliding block maps from symbolic dynamics to tiling dynamics.

Proposition 2.3. *When $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, the inflation tiling hull $\mathbb{Y}_m$ defined by ϱ_m is MLD with another inflation tiling hull that is generated by the binary constant length substitution $\tilde{\varrho}_m$, defined by $a \mapsto ab^\ell$, $b \mapsto a^{\ell+1}$, under the identifications $a \,\widehat{=}\, 0$ and $b \,\widehat{=}\, 1^{\ell+1}$.*

Consequently, for any such m, the dynamical system $(\mathbb{Y}_m, \mathbb{R})$ has pure point spectrum, both in the dynamical and in the diffraction sense.

Proof. The claim can be proved by comparing the two-sided fixed point w of ϱ_m^2, with seed $0|0$, with that of $\tilde{\varrho}_m^2$, with matching seed $a|a$, called u say, where we employ the tiling picture and assume that the letters a and b both stand for intervals of length $\lambda = \ell + 1$. Clearly, the local mapping defined by $a \mapsto 0$ and $b \mapsto 1^{\ell+1}$ sends u to w. For the other direction, each 0 is mapped to a, while the symbol 1 in w occurs in blocks of length $\ell(\ell+1)$, which are locally recognisable. Any such block is then replaced by b^ℓ, and this defines a local mapping that sends w to u. The transfer from the symbolic fixed points to the corresponding tilings is consistent, as the interval lengths match the geometric constraints. The extension to the entire hulls is standard.

The constant length substitution $\tilde{\varrho}_m$ has a coincidence in the first position, and thus defines a discrete dynamical system with pure point dynamical spectrum by Dekking's theorem [13]. Due to the constant length nature, $\tilde{\mathbb{Y}}_m$ emerges from $\tilde{\mathbb{X}}_m$ by a simple suspension with a constant roof function [14], so that the dynamical spectrum of $(\tilde{\mathbb{Y}}_m, \mathbb{R})$, and hence that of $(\mathbb{Y}_m, \mathbb{R})$ by conjugacy, is still pure point. By the equivalence theorem between dynamical and diffraction spectra [18, 7] in the pure point case, the last claim is clear. □

Let us mention in passing that all eigenfunctions are continuous for primitive inflation rules [21, 24]. For the systems considered in this paper, all eigenvalues are thus topological. So far, we have the following result.

Theorem 2.4. *Consider the inflation tiling, with prototiles of natural length, defined by ϱ_m. For $m = 1$ and $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, the tiling has pure point diffraction, which can be calculated with the projection method.*[1] *The corresponding tiling dynamical system $(\mathbb{Y}_m, \mathbb{R})$ is strictly ergodic and has pure point dynamical spectrum.* □

[1]For $m \neq 1$, this works analogously to the case of the period doubling sequence; compare [4].

2.2. **Non-PV cases.** In all remaining cases, meaning those that are not covered by Theorem 2.4, the PF eigenvalue is irrational, but fails to be a PV number. None of the corresponding tilings can have non-trivial point spectrum [24, 1]. In particular, the only Bragg peak in the diffraction measure is the trivial one at $k = 0$. If we consider Dirac combs with point measures at the left endpoints of the intervals, which leads to the Dirac comb of Eq. (4.1) below, this Bragg peak has intensity

$$I_0 \,=\, \left|\nu_0 u_0 + \nu_1 u_1\right|^2, \tag{2.2}$$

where u_0, u_1 are the (possibly complex) weights for the two types of points, and ν_0, ν_1 are the frequencies from Eq. (2.1); compare [4, Prop. 9.2]. This gives the first part of the following result, the full proof of which will later follow from Lemma 4.4 and Proposition 5.5.

Theorem 2.5. *For all cases of our inflation family that remain after Theorem 2.4, the pure point part of the diffraction consists of the trivial Bragg peak at* 0*, with intensity* I_0 *according to Eq.* (2.2)*, while the remainder of the diffraction is purely singular continuous.*

The first example in our family with continuous spectral component is $m = 3$, where the eigenvalues are $\frac{1}{2}\bigl(1 \pm \sqrt{13}\,\bigr)$. This case was studied in detail in [1], where also general methods were developed that can be used for the entire family, as we shall demonstrate below.

2.3. **Some notation.** To continue, we need various standard results from the theory of unbounded (but translation bounded) measures on $\mathbb{R}$, for instance as summarised in [4, Ch. 8]. In particular, we use δ_x to denote the normalised Dirac measure at x and $\delta_S = \sum_{x \in S} \delta_x$ for the Dirac comb of a discrete point set S. A translation bounded measure μ is simultaneously considered as a regular Borel measure (then evaluated on bounded Borel sets) and as a Radon measure (hence as a linear functional on $C_{\mathsf{c}}(\mathbb{R})$, the space of continuous functions with compact support), which is justified by the general Riesz–Markov representation theorem; compare [25, Ch. 4].

For a continuous function g, the measure $g.\mu$ is defined by $\mu \circ g^{-1}$ as a Borel measure, while $g(.)\mu$ stands for the measure that is absolutely continuous relative to μ with Radon–Nikodym density g. The function $\widetilde{g}$ is specified by $\widetilde{g}(x) = \overline{g(-x)}$, which extends to Radon measures by $\widetilde{\mu}(g) = \overline{\mu(\widetilde{g})}$; for further details, we refer to [4].

3. Displacement structure and pair correlations

Let $m \in \mathbb{N}$ be arbitrary, but fixed, which will be suppressed in our notation from now on whenever reasonable. We will now review some properties of the inflation structure and how this can be used to get exact renormalisation relations for the pair correlation functions.

3.1. **Displacements and their algebraic structure.** First, we quantify the relative displacements of tiles in the inflation process by the set-valued matrix

$$T \,=\, \begin{pmatrix} \{0\} & \{0\} \\ S & \varnothing \end{pmatrix}, \quad \text{with } S := \{\lambda, \lambda+1, \dots, \lambda+m-1\},$$

where $T = (T_{ij})_{0 \leqslant i,j \leqslant 1}$ with T_{ij} being the set of relative positions of tiles (intervals) of type i in supertiles of type j. Here and below, the positions are always determined between the left endpoints of the tiles as markers.

From the measure matrix $\delta_T := \bigl(\delta_{T_{ij}}\bigr)_{0 \leqslant i,j \leqslant 1}$, with $\widehat{\cdot}$ denoting the standard Fourier transform as used in [4, Ch. 8], one obtains the Fourier matrix B of our inflation system as

$$B(k) \; := \; \overline{\widehat{\delta_T}(k)} \; = \; D_0 + p(k)\, D_\lambda \tag{3.1}$$

with the trigonometric polynomial

$$p(k) \; = \; z^{\lambda}(1 + z + \ldots + z^{m-1})\big|_{z = \mathrm{e}^{2\pi \mathrm{i} k}} \tag{3.2}$$

and digit matrices $D_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $D_\lambda = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, which are the same matrices for all $m \in \mathbb{N}$. The complex algebra $\mathcal{B}$ generated by them is the *inflation displacement algebra* (IDA) introduced in [2]. Invoking [1], one has the following result.

Fact 3.1. *For any $m \in \mathbb{N}$, the IDA $\mathcal{B}$ of the inflation defined by ϱ_m, with intervals of natural length as prototiles, is the full matrix algebra,* $\mathrm{Mat}(2, \mathbb{C})$*. This is also the IDA for all powers of the inflation.* □

The matrix function defined by $B(k)$ is analytic in k, and either 1-periodic (whenever λ is an integer) or quasiperiodic with incommensurate base frequencies 1 and λ. The case $m = 1$ is somewhat degenerate in this setting, as we shall explain in more detail later, in Section 5.2. In the genuinely quasiperiodic situation, via standard results from the theory of quasiperiodic functions, one has the representation

$$B(k) \; = \; \tilde{B}(x, y)\big|_{x = \lambda k, y = k} \tag{3.3}$$

with $\tilde{B}(x, y) = \left(\begin{smallmatrix} 1 & 1 \\ \tilde{p}(x,y) & 0 \end{smallmatrix}\right)$ and

$$\tilde{p}(x, y) \; = \; \mathrm{e}^{2\pi \mathrm{i} x}(1 + z + \ldots + z^{m-1})\big|_{z = \mathrm{e}^{2\pi \mathrm{i} y}}. \tag{3.4}$$

Here, both $\tilde{p}$ and $\tilde{B}$ are 1-periodic in *both* arguments. Our representation is chosen such that we have the correspondence

$$k \mapsto \lambda k \quad \longleftrightarrow \quad (x, y) \mapsto (x, y) M\,, \tag{3.5}$$

where $M = M_m$ is the substitution matrix of $\varrho = \varrho_m$. Each such M defines a toral endomorphism on the 2-torus, $\mathbb{T}^2$.

3.2. **Kronecker products.** Below, we also need the matrices $A(k) = B(k) \otimes \overline{B(k)}$ for $k \in \mathbb{R}$, which act on the space $W := \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^2$, and the structure of the $\mathbb{R}$-algebra $\mathcal{A}$ generated by them. While the $\mathbb{C}$-algebra $\mathcal{B}$ is irreducible by Fact 3.1, $\mathcal{A}$ is not, because each of its elements commutes with the $\mathbb{R}$-linear mapping $C : \; W \longrightarrow W$ defined by $x \otimes y \; \mapsto \; \overline{y} \otimes \overline{x}$, where W is considered as an $\mathbb{R}$-vector space of dimension 8. Now, $W = W_+ \oplus W_-$ with $W_\pm := \{ w \in W : C(w) = \pm w \}$, where $\dim_{\mathbb{R}}(W_+) = \dim_{\mathbb{R}}(W_-) = 4$ and $W_- = \mathrm{i} W_+$. These spaces are invariant under $\mathcal{A}$, and one has the following result.

Lemma 3.2. *The $\mathbb{R}$-algebra $\mathcal{A}$ satisfies $\dim_{\mathbb{R}}(\mathcal{A}) = 16$ and acts irreducibly on each of the four-dimensional invariant subspaces W_+ and W_- from above.*

Proof. The argument is analogous to that in the proof of [1, Lemma 5.3]. In particular, considering $\mathrm{Mat}(4,\mathbb{C})$ as an $\mathbb{R}$-algebra of dimension 32, the subalgebra $\mathcal{A}$ is conjugate to $\mathrm{Mat}(4,\mathbb{R}) \subset \mathrm{Mat}(4,\mathbb{C})$ via the conjugation $(.) \mapsto U(.)U^{-1}$ with the unitary matrix

$$U \, = \, \frac{1}{\sqrt{2}} \begin{pmatrix} 1-\mathrm{i} & 0 & 0 & 0 \\ 0 & 1 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 & 1-\mathrm{i} \end{pmatrix},$$

where $U(W_{\mp}) = \frac{1 \pm \mathrm{i}}{\sqrt{2}} \, \mathbb{R}^4$. □

One can check that $[U, A(0)] = 0$. More generally, one has

$$A_U(k) \, = \, U A(k) U^{-1} \, = \begin{pmatrix} 1 & 1 & 1 & 1 \\ c(k)+s(k) & s(k) & c(k) & 0 \\ c(k)-s(k) & c(k) & -s(k) & 0 \\ c(k)^2 + s(k)^2 & 0 & 0 & 0 \end{pmatrix}$$

with

$$c(k) \, = \sum_{\ell=0}^{m-1} \cos\bigl(2\pi(\lambda+\ell)k\bigr) \quad \text{and} \quad s(k) \, = \sum_{\ell=0}^{m-1} \sin\bigl(2\pi(\lambda+\ell)k\bigr).$$

This gives $c(k)^2 + s(k)^2 = \lvert p(k) \rvert^2 = \bigl(\sum_{\ell=0}^{m-1} \cos\bigl((2\ell+1-m)\pi k\bigr)\bigr)^2$ and $A_U(0) = A(0) = M \otimes M$ with the substitution matrix $M = M_m$.

Observe that $A(0)^2$ is a strictly positive matrix, with determinant $\det(M)^4 > 0$. Now, consider $A_U^{(2)}(k) := A_U\bigl(\frac{k}{\lambda}\bigr) A_U(k)$, which defines a smooth, real-valued matrix function with $\lim_{k\to 0} A_U^{(2)}(k) = A(0)^2$. Consequently, there is some $\varepsilon = \varepsilon(m) > 0$ such that $A_U^{(2)}(k)$ is strictly positive, with positive determinant, for all $|k| \leqslant \varepsilon$. We can then state the following result, the proof of which is identical to that of [1, Prop. 5.4].

Proposition 3.3. *Let $k \in [0,\varepsilon]$ with the above choice of ε, and consider the iteration*

$$w_n \, := \, A_U^{(2)}\bigl(\tfrac{k}{\lambda^{2n-2}}\bigr) \cdots A_U^{(2)}\bigl(\tfrac{k}{\lambda^2}\bigr) A_U^{(2)}(k) \, w_0$$

for $n \geqslant 1$ and any non-negative starting vector $w_0 \neq 0$. Then, the vector w_n will be strictly positive for all $n \in \mathbb{N}$ and, as $n \to \infty$, it will diverge with asymptotic growth $c\lambda^{4n} w_{\mathrm{PF}}$. Here, c is a constant that depends on w_0 and k, while $w_{\mathrm{PF}} = v_{\mathrm{PF}} \otimes v_{\mathrm{PF}}$ is the statistically normalised PF eigenvector of $M \otimes M$, with eigenvalue λ^2 and v_{PF} as in Eq. (2.1). □

As we shall see later, this growth behaviour will collide with a local integrability condition, and then help to simplify our spectral problem by a dimensional reduction.

3.3. **Pair correlations.** To introduce the pair correlation functions, let $\mathbb{Y} = \mathbb{Y}_m$ be the tiling hull introduced earlier. Any $\mathcal{T} \in \mathbb{Y}$ is built from two prototiles (of length $\lambda > 1$ and 1, respectively). We now define the corresponding point set $\varLambda$ via the left endpoints of the tiles in $\mathcal{T}$, so $\varLambda = \varLambda^{(0)} \dot\cup \varLambda^{(1)}$ with $\varLambda^{(i)}$ denoting the left endpoints of type i. Clearly, $\mathcal{T}$ and $\varLambda$ are MLD, as are their hulls; see [4] for background. By slight abuse of notation, we use $\mathbb{Y}$ for both hulls, which means that we implicitly identify these two viewpoints.

Any two elements of $\mathbb{Y}$ are *locally indistinguishable* (LI), so $\mathbb{Y}$ consists of a single LI class, see [4, Thm. 4.1] or [21], which has the following important consequence.

Fact 3.4. *For any $i, j \in \{0,1\}$, the difference set $\varLambda^{(i)} - \varLambda^{(j)}$ is constant on the hull, which means that it does not depend on the choice of $\varLambda \in \mathbb{Y}$.* □

Given $\varLambda \in \mathbb{Y}$, let $\nu_{ij}(z)$ denote the (relative) frequency of occurrence of a point of type i (left) and one of type j (right) at distance z, where $\nu_{ij}(-z) = \nu_{ji}(z)$. By the strict ergodicity of our system, any such frequency exists (and uniformly so), and is the same for all $\varLambda \in \mathbb{Y}$. One can write the frequency as a limit,

$$\nu_{ij}(z) \, = \lim_{r\to\infty} \frac{\operatorname{card}\bigl(\varLambda_r^{(i)} \cap (\varLambda_r^{(j)} - z)\bigr)}{\operatorname{card}(\varLambda_r)} \, = \, \frac{1}{\operatorname{dens}(\varLambda)} \lim_{r\to\infty} \frac{\operatorname{card}\bigl(\varLambda_r^{(i)} \cap (\varLambda_r^{(j)} - z)\bigr)}{2r},$$

where $\varLambda \in \mathbb{Y}$, with $\varLambda = \varLambda^{(0)} \dot\cup \varLambda^{(1)}$ as above. The lower index r indicates the intersection of a set with the interval $[-r, r]$. Moreover, one has

$$\nu_{ij}(z) \, > \, 0 \quad \Longleftrightarrow \quad z \in \varDelta_{ij} := \varLambda^{(j)} - \varLambda^{(i)},$$

and $\varUpsilon_{ij} := \sum_{z\in\varDelta_{ij}} \nu_{ij}(z)\, \delta_z$ defines a positive pure point measure on $\mathbb{R}$ with locally finite support. Note that $\varUpsilon_{ii}$ is also positive definite. We call the $\nu_{ij}(z)$ the *pair correlation coefficients* and the $\varUpsilon_{ij}$ the corresponding *pair correlation measures* of $\mathbb{Y}$, where $\varUpsilon_{ij}(\{z\}) = \nu_{ij}(z)$. Our relative normalisation means that we have $\nu_{00}(0) + \nu_{11}(0) = 1$, so that

$$\nu_{00}(0) \, = \, \nu_0 \quad \text{and} \quad \nu_{11}(0) \, = \, \nu_1$$

are the relative tile (or letter) frequencies from Eq. (2.1).

Proposition 3.5. *The pair correlation coefficients of $\mathbb{Y}$ satisfy the exact renormalisation relations*

$$\nu_{ij}(z) \, = \, \frac{1}{\lambda} \sum_{k,\ell} \sum_{r\in T_{ik}} \sum_{s\in T_{j\ell}} \nu_{k\ell}\Bigl(\frac{z+r-s}{\lambda}\Bigr)$$

for any $i, j \in \{0,1\}$, subject to the condition that $\nu_{mn}(z) = 0$ whenever $z \notin \varDelta_{mn}$. In terms of the measures $\varUpsilon_{ij}$, this amounts to the convolution identity

$$\varUpsilon \, = \, \frac{1}{\lambda}\Bigl(\widetilde{\delta_T} \overset{*}{\otimes} \delta_T\Bigr) * (f.\varUpsilon),$$

where f is the dilation defined by $x \mapsto \lambda x$ and $\overset{*}{\otimes}$ denotes the Kronecker convolution product, while $\varUpsilon$ stands for the measure vector $(\varUpsilon_{00}, \varUpsilon_{01}, \varUpsilon_{10}, \varUpsilon_{11})^T$.

Sketch of proof. For $m = 1$, this was shown in [2], while the case $m = 3$ is treated in [1]. In general, the underlying observation is that, due the aperiodicity of ϱ and the ensuing local recognisability, each tile lies in a unique level-1 supertile, and the frequencies for the distance between tiles can uniquely be related to the frequencies of (generally different) supertile distances, after a change of scale. A simple computation then gives the first relation (a more general version of which will appear in [3]).

The second identity, in the form of measures, follows from the definition of the $\varUpsilon_{ij}$ by a straightforward calculation; see [1] for details in the case $m = 3$. $\square$

4. Autocorrelation and diffraction

As above, $m \in \mathbb{N}$ is arbitrary but fixed. For $\varLambda \in \mathbb{Y}$, we consider the weighted Dirac comb

$$\omega_u = \sum_{x \in \varLambda} u_x \, \delta_x = u_0 \, \delta_{\varLambda^{(0)}} + u_1 \, \delta_{\varLambda^{(1)}} , \tag{4.1}$$

where u_0 and u_1 are the (possibly complex-valued) weights of the two types of points. The corresponding *autocorrelation measure*, or autocorrelation for short, is defined by the volume-averaged (or Eberlein) convolution

$$\gamma_u = \omega_u \circledast \widetilde{\omega_u} = \operatorname{dens}(\varLambda) \sum_{ij} \overline{u_i} \, \varUpsilon_{ij} \, u_j ,$$

where we refer to [4, Chs. 8 and 9] for the general setting and to [1] for the detailed calculations in the case $m = 3$. Existence and uniqueness (with independence of $\varLambda$) are again a consequence of the strict (and hence in particular unique) ergodicity of our system.

Since all the measures $\varUpsilon_{ij} = \bigl(\delta_{-\varLambda^{(i)}} \circledast \delta_{\varLambda^{(j)}}\bigr) / \operatorname{dens}(\varLambda)$ are well-defined Eberlein convolutions of translation bounded measures and hence Fourier transformable by [2, Lemma 1], we also have the relation

$$\widehat{\gamma_u} = \operatorname{dens}(\varLambda) \sum_{ij} \overline{u_i} \, \widehat{\varUpsilon}_{ij} \, u_j \tag{4.2}$$

after Fourier transform, where $\widehat{\gamma_u}$ is a positive measure, for any $u \in \mathbb{C}^2$. Note that each $\widehat{\varUpsilon}_{ij}$ is a positive definite measure on $\mathbb{R}$, and also positive for $i = j$. Since $\widetilde{\varUpsilon}_{ij} = \widehat{\varUpsilon_{ij}}$ by definition, one has

$$\overline{\widehat{\varUpsilon_{ij}}} = \widehat{\widetilde{\varUpsilon_{ij}}} = \widehat{\varUpsilon_{ji}} .$$

This, in combination with Eq. (4.2), implies the following property.

Fact 4.1. *For any bounded Borel set $\mathcal{E} \subset \mathbb{R}$, the complex matrix $\bigl(\widehat{\varUpsilon}_{ij}(\mathcal{E})\bigr)_{0 \leqslant i,j \leqslant 1}$ is Hermitian and positive semi-definite.* $\square$

Note that, since $\widehat{\varUpsilon}_{00}$ and $\widehat{\varUpsilon}_{11}$ are positive measures on $\mathbb{R}$, the positive semi-definiteness of the matrix $\bigl(\widehat{\varUpsilon}_{ij}(\mathcal{E})\bigr)$ is simply equivalent to the determinant condition $\det\bigl(\widehat{\varUpsilon}_{ij}(\mathcal{E})\bigr) \geqslant 0$.

As a counterpart to Proposition 3.5, with $\widehat{\varUpsilon}$ denoting the vector of Fourier transforms of the pair correlation measures, we get the following result.

Proposition 4.2. *Under Fourier transform, the second identity of Proposition 3.5 turns into the relation*

$$\widehat{\varUpsilon} \, = \, \frac{1}{\lambda^2} \, A(.) \cdot \bigl(f^{-1}.\widehat{\varUpsilon}\bigr),$$

where $A(k) = B(k) \otimes \overline{B(k)}$ *with the Fourier matrix* $B(k)$ *from Eq.* (3.1) *and* $f(x) = \lambda x$. $\square$

4.1. **Pure point part.** By [4, Lemma 6.1], the identity from Proposition 4.2 must hold for each spectral component of $\widehat{\varUpsilon}$ separately. We write the pure point part as

$$\bigl(\widehat{\varUpsilon}\bigr)_{\mathsf{pp}} \, = \sum_{k \in K} \mathcal{I}(k) \, \delta_k$$

with the intensity vector $\mathcal{I}(k) = \widehat{\varUpsilon}(\{k\})$ and K the support of the pure point part, which is (at most) a countable set. Without loss of generality, we may assume that $\lambda K \subseteq K$, possibly after enlarging K appropriately. Inserting this into the above identity, one obtains

$$\mathcal{I}(k) \, = \, \frac{1}{\lambda^2} \, A(k) \, \mathcal{I}(\lambda k). \tag{4.3}$$

In particular, this gives $A(0) \, \mathcal{I}(0) = \lambda^2 \, \mathcal{I}(0)$, which means

$$\widehat{\varUpsilon}_{ij}(\{0\}) \, = \, \mathcal{I}(0) \, = \, \nu_i \, \nu_j \, = \, \frac{\mathrm{dens}(\varLambda^{(i)}) \, \mathrm{dens}(\varLambda^{(j)})}{\bigl(\mathrm{dens}(\varLambda)\bigr)^2} \tag{4.4}$$

with the relative frequencies ν_i from Eq. (2.1).

4.2. **Conditions on absolutely continuous part.** Likewise, if we represent $(\widehat{\varUpsilon})_{\mathsf{ac}}$ by the vector h of its Radon–Nikodym densities relative to Lebesgue measure, one obtains [1]

$$h(k) \, = \, \frac{1}{\lambda} \, A(k) \, h(\lambda k), \tag{4.5}$$

which holds for a.e. $k \in \mathbb{R}$. Here, the different exponent for the prefactor in comparison to Eq. (4.3) is the crucial point to observe and harvest. Since $\det(B(k)) = -p(k) = 0$ holds if and only if $k \in Z_m := \frac{1}{m}\mathbb{Z} \setminus \mathbb{Z}$, the matrix $A(k)$ is invertible for all $k \notin Z_m$, hence for a.e. $k \in \mathbb{R}$. For such k, we also have

$$h(\lambda k) \, = \, \lambda \, A^{-1}(k) \, h(k), \tag{4.6}$$

which is the outward-going counterpart to Eq. (4.5).

If we interpret the vector h as a matrix $(h_{ij})_{0 \leqslant i,j \leqslant 1}$, the iterations from Eqs. (4.5) and (4.6) can also be written as

$$\begin{aligned} \bigl(h_{ij}\bigl(\tfrac{k}{\lambda}\bigr)\bigr) \, &= \, \lambda^{-1} B\bigl(\tfrac{k}{\lambda}\bigr) \bigl(h_{ij}(k)\bigr) B^{\dagger}\bigl(\tfrac{k}{\lambda}\bigr) \quad \text{and} \\ \bigl(h_{ij}(\lambda k)\bigr) \, &= \, \lambda B^{-1}(k) \bigl(h_{ij}(k)\bigr) (B^{\dagger})^{-1}(k), \end{aligned} \tag{4.7}$$

where $B^{\dagger}$ denotes the Hermitian adjoint of B. This suggests a suitable decomposition of h.

Lemma 4.3. *For a.e.* $k \in \mathbb{R}$*, the Radon–Nikodym matrix* $\bigl(h_{ij}(k)\bigr)$ *is Hermitian and positive semi-definite. Moreover, it is of rank at most* 1.

Proof. From Fact 4.1, we know that, given any bounded Borel set $\mathcal{E}$, the complex matrix $\big(\widehat{\varUpsilon}_{ij}(\mathcal{E})\big)$ is Hermitian and positive semi-definite, which also holds for the absolutely continuous part of $\widehat{\varUpsilon}$. By standard arguments, this implies the first claim.

For a.e. $k \in \mathbb{R}$, the Radon–Nikodym matrix is then of the form $H = \left(\begin{smallmatrix} a & b+\mathrm{i}c \\ b-\mathrm{i}c & d \end{smallmatrix}\right)$ with $a,b,c,d \in \mathbb{R}$, $a,d \geqslant 0$ and $ad \geqslant (b^2+c^2) \geqslant 0$. When $\det(H)=0$, the rank of H is at most 1. On the other hand, when $\det(H)>0$, the rank of H is 2, with $ad>0$. In general, we have a unique decomposition as

$$H \, = \begin{pmatrix} a' & b+\mathrm{i}c \\ b-\mathrm{i}c & d \end{pmatrix} + a'' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.8}$$

with $a=a'+a''$ such that each matrix on the right-hand side is positive semi-definite (hence $a' \geqslant 0$ and $a'' \geqslant 0$) and of rank at most 1 (which means $a'd = b^2+c^2$).

It suffices to prove our second claim for a.e. $k \in \big[\frac{\varepsilon}{\lambda}, \varepsilon\big]$ for some $\varepsilon>0$, as the two iterations in Eq. (4.7) transport the property to all $k>0$, and then to $k<0$ via $h_{ij}(-k) = h_{ji}(k)$. Here, we choose ε as in Proposition 3.3. Whenever $h_{11}(k)=0$, we have $\det\big(h_{ij}(k)\big)=0$ and the Radon–Nikodym matrix has rank at most 1. Otherwise, we define $h''_{00}(k) = \det\big(h_{ij}(k)\big)/h_{11}(k)$ and $h'_{00}(k) = h_{00}(k) - h''_{00}(k)$, which are measurable functions and achieve the decomposition explained previously.

To continue, we switch to the vector notation from Eqs. (4.5) and (4.6), and observe that the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ corresponds to the vector $w_0 = (1,0,0,0)^T \in W_+$ in the notation of Section 3.2. Now, $U w_0 = \frac{1-\mathrm{i}}{\sqrt{2}}\, w_0$, and the iteration of w_0 under $A_U(k)$,

$$w_n \, = \, A_U\big(\tfrac{k}{\lambda^{n-1}}\big) \cdots A_U\big(\tfrac{k}{\lambda}\big)\, A_U(k)\, w_0 ,$$

grows asymptotically as $c\lambda^{2n} w_{\mathrm{PF}}$ by an application of Proposition 3.3, where $c>0$ depends on k. Observing that $U^{-1} w_{\mathrm{PF}} = \frac{1+\mathrm{i}}{\sqrt{2}}\, w_{\mathrm{PF}}$ and applying a standard argument on the basis of Lusin's theorem as in the proof of [1, Lemma 6.5], we see that $h''_{00}(k)>0$ would behave proportional to k^{-1} as $k \searrow 0$, which is impossible for a locally integrable function. Since also $h'_{00}(k) \geqslant 0$, there cannot be any cancellation with the other term of our decomposition under the inward iteration, and we must conclude that $h''_{00}(k)=0$ for a.e. $k \in \big[\frac{\varepsilon}{\lambda}, \varepsilon\big]$, and hence for a.e. $k \in \mathbb{R}$ as argued above. This implies our claim. $\square$

4.3. **Dimensional reduction and Lyapunov exponents.** If $H \in \mathrm{Mat}(2,\mathbb{C})$ is Hermitian, positive semi-definite and of rank at most 1, there are two complex numbers, v_0 and v_1 say, such that $H_{ij} = v_i \, \overline{v_j}$ holds for $i,j \in \{0,1\}$; compare [1, Fact 6.6] for more. The main consequence of Lemma 4.3 now is that it suffices to consider a vector $v(k) = \big(v_0(k), v_1(k)\big)^T$ of functions from $L^2_{\mathrm{loc}}(\mathbb{R})$ under the simpler iterations

$$v\big(\tfrac{k}{\lambda}\big) \, = \, \frac{1}{\sqrt{\lambda}}\, B\big(\tfrac{k}{\lambda}\big) v(k) \quad \text{and} \quad v(\lambda k) \, = \, \sqrt{\lambda}\, B^{-1}(k)\, v(k),$$

the latter for $k \notin Z_m$. In particular, one has

$$v(\lambda^n k) \, = \, \lambda^{n/2}\, B^{-1}(\lambda^{n-1}k) \cdots B^{-1}(k)\, v(k), \tag{4.9}$$

which holds for a.e. $k \in \mathbb{R} \setminus \bigcup_{\ell=0}^{n-1} \lambda^{-\ell} Z_m$, and thus still for a.e. $k \in \mathbb{R}$.

There are at most two Lyapunov exponents for this iteration, which agree with the extremal exponents [26] defined by

$$
\begin{aligned}
\chi_{\max}(k) \; &= \; \log \sqrt{\lambda} \, + \, \limsup_{n\to\infty} \frac{1}{n} \log \bigl\| B^{-1}(\lambda^{n-1}k) \cdots B^{-1}(\lambda k) B^{-1}(k) \bigr\| \quad \text{and} \\
\chi_{\min}(k) \; &= \; \log \sqrt{\lambda} \, - \, \limsup_{n\to\infty} \frac{1}{n} \log \bigl\| B(k) B(\lambda k) \cdots B(\lambda^{n-1}k) \bigr\| ,
\end{aligned}
\tag{4.10}
$$

where the term $\log \sqrt{\lambda}$ emerges from the prefactor on the right-hand side of Eq. (4.9). Our main concern will be the minimal exponent, for the following reason [1, 3].

Lemma 4.4. *If $\chi_{\min}(k) \geqslant c > 0$ holds for a.e. k in a small interval, the diffraction measure $\widehat{\gamma_u}$ is a singular measure, for any non-trivial choice of the weights u_0 and u_1, which means $u_1, u_2 \in \mathbb{C}$ with $u_1 u_2 \neq 0$.* □

5. Analysis via Lyapunov exponents

Let $m \in \mathbb{N}$ be fixed, and $\lambda = \lambda_m^+$ as before. Consider the matrix cocycle

$$
B^{(n)}(k) \, := \, B(k) B(\lambda k) \cdots B(\lambda^{n-1} k),
$$

which is motivated by Eq. (4.10). Note that $B^{(n)}(k)$ is invertible for $k \notin \bigcup_{\ell=0}^{n-1} \lambda^{-\ell} Z_m$. When $m = 1$, one has $Z_1 = \varnothing$ and $\bigl|\det\bigl(B(k)\bigr)\bigr| \equiv 1$, which makes this case considerably simpler. In general, one has the following result.

Proposition 5.1. *For a.e. $k \in \mathbb{R}$, one has $\lim_{n\to\infty} \frac{1}{n} \log \bigl|\det\bigl(B^{(n)}(k)\bigr)\bigr| = 0$.*

Proof. For $m = 1$, one has $\bigl|\det\bigl(B^{(n)}(k)\bigr)\bigr| \equiv 1$, and the claim trivially holds for all $k \in \mathbb{R}$. When $m \geqslant 2$, we invoke Sobol's theorem, as outlined in [1]; see [6] for a detailed exposition. Clearly, $\det\bigl(B(k)\bigr)$ is a Bohr almost periodic function, but it has zeros for $k \in Z_m$. Consequently, $\log\bigl|\det\bigl(B(k)\bigr)\bigr|$ cannot be Bohr almost periodic, because it has singularities at these points. Nevertheless, this function is locally Lebesgue-integrable on $\mathbb{R}$, and is continuous on $\mathbb{R} \setminus Z_m$, hence locally Riemann-integrable on the complement of $Z_m + (-\varepsilon, \varepsilon)$ for any $\varepsilon > 0$. Then, Sobol's theorem [23] (in the periodic case where λ is an integer) or its extension to almost periodic functions [6] can be applied as follows.

First, recall that the sequence $(\lambda^n k)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 for a.e. $k \in \mathbb{R}$. Next, one needs the property that Z_m is a Delone set and that, for any fixed $\varepsilon > 0$, and then for a.e. $k \in \mathbb{R}$, the inequality

$$
\operatorname{dist}(\lambda^{n-1} k, Z_m) \; \geqslant \; \frac{1}{n^{1+\varepsilon}}
$$

holds for almost all $n \in \mathbb{N}$ (meaning for all except at most finitely many); see [6, Lemma 6.2.6]. Then, again for any fixed $\varepsilon > 0$, it follows from [17, Thm. 5.13] that the discrepancy of

$(\lambda^n k)_{n\in\mathbb{N}}$, for a.e. $k \in \mathbb{R}$, is

$$\mathcal{D}_N \, = \, \mathcal{O}\left(\frac{\bigl(\log(N)\bigr)^{\frac{3}{2}+\varepsilon}}{\sqrt{N}}\right) \quad \text{as } N \to \infty .$$

Putting this together, we can apply [6, Thm. 6.4.8] which tells us that the Birkhoff-type averages of the function $\log\bigl|\det\bigl(B(.)\bigr)\bigr|$, for a.e. $k \in \mathbb{R}$, converge to the mean of this function (see Eq. (5.1) below for a definition), which gives

$$\begin{aligned} \frac{1}{n} \log\bigl|\det\bigl(B^{(n)}(k)\bigr)\bigr| \, &= \, \frac{1}{n} \sum_{\ell=0}^{n-1} \log\bigl|\det\bigl(B(\lambda^{\ell} k)\bigr)\bigr| \\ &\xrightarrow{\,n\to\infty\,} \int_0^1 \log\bigl|\det\bigl(B(t)\bigr)\bigr| \,\mathrm{d}t \, = \int_0^1 \log\bigl|1 + z + \ldots + z^{m-1}\bigr|_{z=\mathrm{e}^{2\pi\mathrm{i}t}} \,\mathrm{d}t \, = \, 0, \end{aligned}$$

where the last step follows via Jensen's formula from complex analysis (see [22, Prop. 16.1] for a formulation that fits our situation) because the polynomial $1 + z + \ldots + z^{m-1}$ either equals 1 (when $m = 1$) or has zeros only on the unit circle. □

Remark 5.2. When $m = \ell(\ell+1)$ with $\ell \in \mathbb{N}$, where $\lambda = \ell + 1$, the result of Proposition 5.1 easily follows from Birkhoff's ergodic theorem, because $\det\bigl(B(k)\bigr)$ is then a 1-periodic, locally Lebesgue-integrable function that gets averaged along orbits of the dynamical system defined by $x \mapsto \lambda x \bmod 1$ on the 1-torus, $\mathbb{T}$. This approach, however, does not extend to the other values of m with $m > 1$, because the sequence $(\lambda^n k)_{n\in\mathbb{N}}$, taken modulo 1, is then no longer an orbit of $x \mapsto \lambda x \bmod 1$ on $\mathbb{T}$; compare [6, Ex. 6.3.4]. ◊

We can now relate the two extremal exponents from Eq. (4.10) as follows.

Lemma 5.3. *For a.e.* $k \in \mathbb{R}$, *one has* $\chi_{\max}(k) + \chi_{\min}(k) = \log(\lambda)$.

Proof. Recall that, for any invertible matrix B, one has $B^{-1} = \frac{1}{\det(B)} B^{\mathsf{ad}}$, where B^{ad} is the (classical) adjoint of B. The adjoint satisfies $(AB)^{\mathsf{ad}} = B^{\mathsf{ad}} A^{\mathsf{ad}}$.

Now, in the formula for the extremal exponents, we are free to choose any matrix norm, as this does not affect the limit. For 2×2-matrices, one has $\|B^{\mathsf{ad}}\|_{\mathrm{F}} = \|B\|_{\mathrm{F}}$, where $\|.\|_{\mathrm{F}}$ denotes the Frobenius norm. Our claim now follows from Proposition 5.1 after a simple calculation. □

In view of Lemma 5.3, we define

$$\chi^B(k) \, := \, \limsup_{n\to\infty} \frac{1}{n} \log\bigl\|B(k)B(\lambda k)\cdots B(\lambda^{n-1}k)\bigr\|,$$

so that $\chi_{\max}(k) = \log\sqrt{\lambda} + \chi^B(k)$ and $\chi_{\min}(k) = \log\sqrt{\lambda} - \chi^B(k)$ holds for a.e. $k \in \mathbb{R}$, together with $\chi_{\max}(k) \geqslant \chi_{\min}(k)$. We can thus simply analyse χ^B from now on, which clearly is a non-negative function.

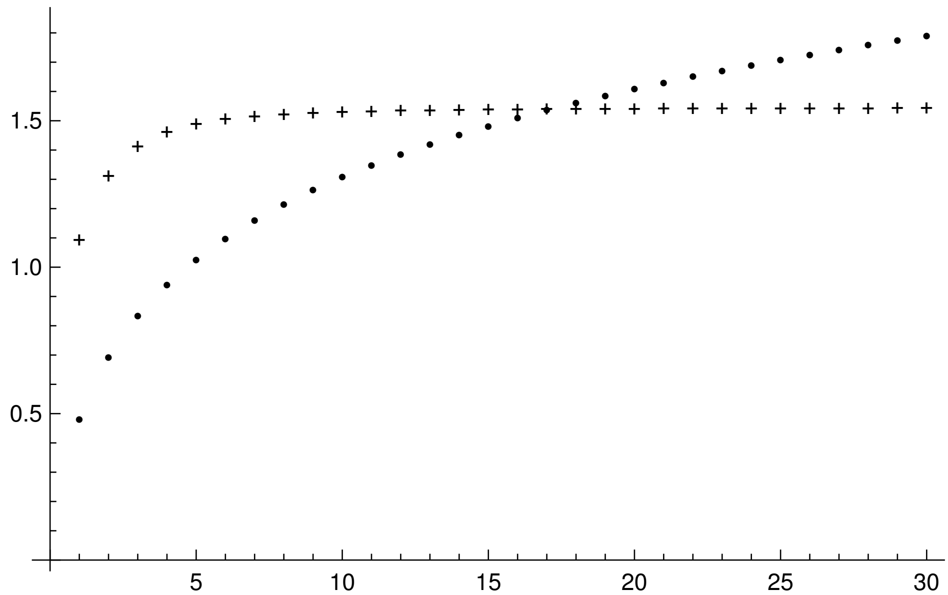

FIGURE 1. The values of $\log(\lambda)$ (dots) and $\mathfrak{m}(q)$ (crosses) for $1 \leqslant m \leqslant 30$.

5.1. **Arguments in common.** Below, we need the *mean* of a function. If f is a Bohr almost periodic function on $\mathbb{R}$ (and thus in particular uniformly continuous and bounded), its mean, $\mathbb{M}(f)$, is defined by

$$\mathbb{M}(f) \, = \lim_{T\to\infty} \frac{1}{T} \int_{x}^{x+T} f(t) \, \mathrm{d}t, \tag{5.1}$$

where $x \in \mathbb{R}$ is arbitrary. By standard results, the mean of such an f exists for all $x \in \mathbb{R}$, is independent of x, and the convergence is actually uniform in x. This is also true when f is almost periodic in the sense of Stepanov, which in particular covers some of our later situations; see [12, 6] for details. When f is a periodic function, with fundamental period T, the mean is simply given by $\mathbb{M}(f) = \frac{1}{T} \int_{0}^{T} f(t) \, \mathrm{d}t$.

Observing that $|p(.)|^2$ with p from Eq. (3.2) is 1-periodic (while p itself need not be), the simplest sufficient criterion for the positivity of all Lyapunov exponents is given by

$$\log(\lambda) \, > \, \mathbb{M}\bigl(\log \|B(.)\|_{\mathrm{F}}^{2}\bigr) \, = \int_{0}^{1} \log\bigl(2 + |p(t)|^2\bigr) \, \mathrm{d}t \, = \int_{0}^{1} \log \bigl|q(z)\bigr|_{z=\mathrm{e}^{2\pi\mathrm{i}t}} \, \mathrm{d}t \, = \, \mathfrak{m}(q) \tag{5.2}$$

where q is the polynomial

$$q(z) \, = \, 2z^{m-1} + \bigl(1 + z + \ldots + z^{m-1}\bigr)^2 \tag{5.3}$$

and the validity of $\overline{z} = z^{-1}$ on the unit circle was used. Here, $\mathfrak{m}(q)$ denotes the *logarithmic Mahler measure* of q; see [15, 22] for background. The integral can now once again be calculated by means of Jensen's formula; see the Appendix for some details. The comparison between $\log(\lambda)$ and $\mathfrak{m}(q)$ is illustrated in Figure 1.

More generally, one has the following result.

TABLE 1. Some relevant values for the quantities in the inequality of Eq. (5.4). The numerical error is less than 10^{-3} in all cases listed.

m	1	2	3	4	5	6	7	8	9	10
$\log(\lambda)$	0.481	0.693	0.834	0.941	1.027	1.099	1.161	1.216	1.265	1.309
$N = N(m)$	6	4	4	3	3	3	2	2	2	2
$\frac{1}{N}\mathbb{M}\bigl(\log\lVert B^{(N)}(.)\rVert_{\mathrm{F}}^{2}\bigr)$	0.439	0.677	0.770	0.924	0.949	0.964	1.144	1.152	1.157	1.161
m	11	12	13	14	15	16	17	18	19	20
$\log(\lambda)$	1.349	1.386	1.421	1.453	1.483	1.511	1.538	1.563	1.587	1.609
$N = N(m)$	2	2	2	2	2	2	2	1	1	1
$\frac{1}{N}\mathbb{M}\bigl(\log\lVert B^{(N)}(.)\rVert_{\mathrm{F}}^{2}\bigr)$	1.164	1.166	1.168	1.169	1.170	1.171	1.172	1.546	1.547	1.547

Lemma 5.4. *For any $m \geqslant 18$ and then a.e. $k \in \mathbb{R}$, all Lyapunov exponents of the outward iteration* (4.9) *are strictly positive and bounded away from* 0.

Proof. Since $\chi_{\max}(k) \geqslant \chi_{\min}(k)$, we need to show that $\log\sqrt{\lambda} - c \geqslant \chi^{B}(k)$ holds for some $c > 0$ and a.e. $k \in \mathbb{R}$. A sufficient criterion for this is the inequality from Eq. (5.2). Since $\mathfrak{m}(q)$ is bounded, see Lemma 5.9 from the Appendix, it is clear that this inequality holds for all sufficiently large $m \in \mathbb{N}$. By Lemma 5.9, this is so for all $m \geqslant 40$, and the slightly better estimate from Remark 5.10 improves this to all $m \geqslant 23$.

In any case, a (precise) numerical investigation of the remaining cases shows that that our claim is indeed true for all $m \geqslant 18$; compare Figure 1. □

In order to establish our goal for the remaining values of m, we need to determine a suitable $N = N(m)$ such that

$$\log(\lambda) > \frac{1}{N}\,\mathbb{M}\bigl(\log\lVert B^{(N)}(.)\rVert_{\mathrm{F}}^{2}\bigr) = \begin{cases} N^{-1}\int_0^1 \log\bigl\lVert B^{(N)}(k)\bigr\rVert_{\mathrm{F}}^{2}\,\mathrm{d}k, & \text{if } \lambda \in \mathbb{Z}, \\ N^{-1}\int_{[0,1]^2} \log\bigl\lVert \tilde{B}^{(N)}(x,y)\bigr\rVert_{\mathrm{F}}^{2}\,\mathrm{d}x\,\mathrm{d}y, & \text{otherwise.} \end{cases} \tag{5.4}$$

When λ is not an integer, $\lVert B^{(N)}(.)\rVert_{\mathrm{F}}^{2}$ is generally not a periodic, but a quasiperiodic function. In this case, we use the representation as a section through a doubly 1-periodic function according to Eq. (3.3), which permits the simple expression for the mean in (5.4). The latter can now be calculated numerically with good precision, and without ambiguity. Note that the choice of the Frobenius norm $\lVert . \rVert_{\mathrm{F}}$ does not give the best bounds, but is rather convenient otherwise. The result is given in Table 5.1, with minimal values for $N(m)$. Consequently, we can sharpen Lemma 5.4 and complete the proof of Theorem 2.5 as follows.

Proposition 5.5. *For any $m \in \mathbb{N}$ and then a.e. $k \in \mathbb{R}$, all Lyapunov exponents of the outward iteration* (4.9) *are strictly positive and bounded away from* 0. □

5.2. **The Fibonacci case.** Here, the leading eigenvalue is $\lambda = \tau$, the golden ratio, which is a PV number. Essentially as a consequence of [16, Thm. 2.9 and Prop. 3.8], which need some modification and extension to be applicable here, the extremal Lyapunov exponents exist as limits, for a.e. $k \in \mathbb{R}$. Let us look into this in more detail, in a slightly different way that provides an independent derivation of this property. Here, we have

$$B(k) \; = \; \begin{pmatrix} 1 & 1 \\ \mathrm{e}^{2\pi\mathrm{i}\tau k} & 0 \end{pmatrix},$$

which is τ^{-1}-periodic. However, this observation does not help because already

$$B^{(2)}(k) \; = \; B(k)\, B(\tau k) \; = \; \begin{pmatrix} 1 + \mathrm{e}^{2\pi\mathrm{i}(\tau+1)k} & 1 \\ \mathrm{e}^{2\pi i\tau k} & \mathrm{e}^{2\pi i\tau k} \end{pmatrix}$$

is genuinely quasiperiodic, with fundamental frequencies τ and 1. In line with our general approach from Eqs. (3.3) and (3.4), we now define $\tilde{B}^{(n+1)}(x,y) = \tilde{B}(x,y)\, \tilde{B}^{(n)}\big((x,y)M\big)$ with

$$\tilde{B}^{(1)}(x,y) \; = \; \tilde{B}(x,y) \; = \; \begin{pmatrix} 1 & 1 \\ \mathrm{e}^{2\pi\mathrm{i}x} & 0 \end{pmatrix} \quad \text{and} \quad M \; = \; \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, $\tilde{B}^{(n)}(x,y)$ defines a matrix cocycle over the dynamical system defined on $\mathbb{T}^2$ by the toral automorphism $(x,y) \mapsto (x,y)M \bmod 1$. By Oseledec's theorem, see [26], the Lyapunov exponents for $\tilde{B}^{(n)}$ exist as limits, for a.e. $(x,y) \in \mathbb{T}^2$, and are constant.

However, what we really need is the existence of the Lyapunov exponents for

$$B^{(n)}(k) \; = \; \tilde{B}^{(n)}(x,y)|_{x=\tau k,\, y=k}$$

for a.e. $k \in \mathbb{R}$, which is a statement along the line $\mathbb{R}(\tau,1)$, respectively its wrap-up on $\mathbb{T}^2$. This is the subspace defined by the left PF eigenvector of M. The problem here is that this defines a null set for Lebesgue measure on $\mathbb{T}^2$, so that the previous argument does not immediately imply what we need. However, for Lebesgue-a.e. starting point on the line $\mathbb{R}(\tau,1)$, the iteration sequence on this line, taken modulo 1, is also equidistributed in $\mathbb{T}^2$, by standard arguments around Weyl's lemma. This allows for a relation between the result on the line and that on $\mathbb{T}^2$ as follows.

Any initial condition for the cocycle $\tilde{B}^{(n)}$ is following an orbit of the toral automorphism that converges, exponentially fast, towards an orbit on this special subspace. This is a consequence of the PV property of τ and the fact that the second eigenvalue of M is $1-\tau \approx -0.618$, the algebraic conjugate of τ. Assume that the Lyapunov exponents for $B^{(n)}$ fail to exist as limits for a subset of $\mathbb{R}(\tau,1)$ of positive measure. Then, this must also be true of the exponents for $\tilde{B}^{(n)}$ for all initial conditions that lead to orbits which approach the failing orbits on $\mathbb{R}(\tau,1)$. By standard arguments, these initial conditions would constitute a set of positive measure, now with respect to Lebesgue measure on $\mathbb{T}^2$, in contradiction to our previous finding. We thus have the following result.

Fact 5.6. *For $m = 1$ and a.e. $k \in \mathbb{R}$, the Lyapunov exponents from Eq.* (4.10) *exist as limits, and are constant.* □

One can check numerically that $\chi^B(k) \approx 0.16(3)$ in this case, and some further analysis with a Furstenberg-type representation should result in a more reliable value.

5.3. **Integer inflation multipliers.** In these cases, we know the absence of any continuous spectral components already from Proposition 2.3. Moreover, in view of Lemma 4.4, our treatment in Section 5.1 also confirms the absence of absolutely continuous diffraction via the Lyapunov exponents. Here, the exponents also exist as limits for a.e. $k \in \mathbb{R}$, by an application of Oseledec's theorem to the matrix cocycle, viewed over the dynamical system defined on $\mathbb{T}$ by $x \mapsto \lambda x \bmod 1$ with $\lambda = \ell + 1$ according to Fact 2.2.

Fact 5.7. *When $m = \ell(\ell+1)$ for $\ell \in \mathbb{N}$, hence $\lambda = \ell + 1$, the Lyapunov exponents from Eq.* (4.10) *exist as limits, for a.e. $k \in \mathbb{R}$, and are constant.* □

As another way to look at the problem, let us add a quick analysis of the constant length substitution

$$\tilde{\varrho}_m : \begin{cases} a \mapsto ab^{\ell} \\ b \mapsto a^{\ell+1} \end{cases} \tag{5.5}$$

with $\ell \in \mathbb{N}$, which defines a hull that is MLD to the one defined via ϱ_m for $m = \ell(\ell+1)$ by Proposition 2.3, so the spectral type of both systems must be the same. The displacement matrix is

$$T = \begin{pmatrix} 0 & \{0, 1, 2, \dots, \ell\} \\ \{1, 2, \dots, \ell\} & \varnothing \end{pmatrix},$$

which results in the Fourier matrix

$$B(k) = \begin{pmatrix} 1 & \psi_{\ell}(z) \\ z\, \psi_{\ell-1}(z) & 0 \end{pmatrix}_{z = \mathrm{e}^{2\pi \mathrm{i} k}}$$

with $\psi_{\ell}(z) := 1 + z + \ldots + z^{\ell}$. One gets an analogue to Eq. (5.2) in the form

$$\mathbb{M}\bigl(\log \|B(.)\|_{\mathrm{F}}^2\bigr) = \int_0^1 \log \left| \frac{s(z)}{(z-1)^2} \right|_{z = \mathrm{e}^{2\pi \mathrm{i} t}} \mathrm{d}t = \mathfrak{m}(s),$$

with $s(z) = z^{2\ell+2} + z^{2\ell+1} + z^{\ell+2} - 6z^{\ell+1} + z^{\ell} + z + 1$. As we explain in more detail in the Appendix, we used $\mathfrak{m}\bigl((z-1)^2\bigr) = 0$ in an intermediate step.

One could now repeat the general analysis of Section 5.1 in this case, with an outcome of a similar kind. However, there is a more efficient way as follows. First, observe that we now have $\det(B(k)) = -z\, \psi_{\ell-1}(z)\, \psi_{\ell}(z)$ with $z = \mathrm{e}^{2\pi i k}$, which is a product of a monic polynomial (in the variable z) with two cyclotomic ones. Consequently, the corresponding logarithmic Mahler measures vanish, and we once again get

$$\lim_{n\to\infty} \frac{1}{n} \log \bigl|\det(B^{(n)}(k))\bigr| = \int_0^1 \log \bigl|\det(B(t))\bigr| \,\mathrm{d}t = \mathfrak{m}\bigl(z\, \psi_{\ell-1}(z)\, \psi_{\ell}(z)\bigr) = 0, \tag{5.6}$$

for a.e. $k \in \mathbb{R}$, as in Proposition 5.1. Next, observe that $v = (1,1)$ is a common left eigenvector of $B(k)$ for all $k \in \mathbb{R}$, with eigenvalue $\psi_\ell(z)$ for $z = \mathrm{e}^{2\pi \mathrm{i} k}$. This gives

$$\lim_{n\to\infty} \frac{1}{n} \log \bigl\| v B^{(n)}(k) \bigr\| \; = \; \mathfrak{m}(\psi_\ell) \; = \; 0$$

for a.e. $k \in \mathbb{R}$. In view of Eq. (5.6), this implies that both exponents of the cocycle $B^{(n)}$ vanish in this case.

Now, we still have $\chi_{\min} + \chi_{\max} = \log(\lambda)$ as in Lemma 5.3, where $\lambda = \ell + 1$, despite the fact that we now consider the substitution from Eq. (5.5). With this derivation, we have actually shown the following result.

Corollary 5.8. *The extremal Lyapunov exponents for the outward iteration defined by the constant-length substitution* (5.5) *are equal, and given by* $\chi_{\min} = \chi_{\max} = \log \sqrt{\ell + 1}$. □

Also this approach implies the diffraction spectrum to be singular. However, as before, this is only a consistency check because Dekking's criterion (see the proof of Proposition 2.3) already gives a stronger result, namely the pure point nature of the spectrum.

Appendix

Here, we consider some logarithmic Mahler measures, in particular $\mathfrak{m}(q)$ for the polynomial q from Eq. (5.3) with $m \in \mathbb{N}$. The polynomials q seem to be irreducible over $\mathbb{Z}$, though we have no general proof for this observation. As follows from a simple calculation, $\mathfrak{m}(q)$ takes the values $\log(3)$ for $m = 1$ and $\log\bigl(2 + \sqrt{3}\,\bigr)$ for $m = 2$. It is known that one must have $\mathfrak{m}(q) = \log(\xi)$ where ξ is a Perron number. A little experimentation shows that ξ is a Salem number for $m = 3$, namely the largest root of $z^4 - 3z^3 - 4z^2 - 3z + 1$, and a Pisot number for $m = 4$, this time the largest root of $z^4 - 4z^3 - 2z^2 + 2z + 1$. For $m = 5$, one finds that ξ is the largest root of $z^8 - 6z^7 + 7z^6 - 3z^4 + 7z^2 - 6z + 1$, which is genuinely Perron, as the second largest root of this irreducible polynomial, with approximate value $1.354 > 1$, lies outside the unit circle. It would be interesting to know more about the numbers that show up here.

More generally, expressing $|p(t)|^2$ in Eq. (5.2) as $\bigl(\sin(m\pi t)/\sin(\pi t)\bigr)^2$, one has

$$\mathfrak{m}(q) \; = \int_0^1 \log \left(2 + \left(\frac{\sin(m\pi t)}{\sin(\pi t)} \right)^2 \right) \mathrm{d}t. \tag{5.7}$$

Since $\sin(m\pi t)^2 \leqslant 1$, one gets a simple upper bound as

$$\begin{aligned} \mathfrak{m}(q) \; &\leqslant \int_0^1 \log \frac{1 + 2\sin(\pi t)^2}{\sin(\pi t)^2} \,\mathrm{d}t \; = \; \log(2) + \int_0^1 \log \frac{2 - \cos(2\pi t)}{1 - \cos(2\pi t)} \,\mathrm{d}t \\ &= \; \log(2) + \mathfrak{m}\bigl(z^2 - 4z + 1\bigr) - \mathfrak{m}\bigl((z-1)^2\bigr) \; = \; \log\bigl(4 + 2\sqrt{3}\,\bigr) \; \approx \; 2.010, \end{aligned}$$

where the logarithmic Mahler measures of the quadratic polynomials were evaluated via Jensen's formula again. This shows that $\mathfrak{m}(q)$ is bounded for our family of polynomials. A slightly better bound can be obtained as follows.

Lemma 5.9. *For any $m \in \mathbb{N}$, the logarithmic Mahler measure of the polynomial q from Eq. (5.3) satisfies the inequality $\mathfrak{m}(q) < \log \sqrt{46} \approx 1.914321$.*

Proof. Here, we employ an argument from [11, 10] that was also used, in a similar context, in [20]. By a simple geometric series calculation, one finds that $q(z) = \frac{r(z)}{(z-1)^2}$ with

$$r(z) \; = \; z^{2m} + 2z^{m+1} - 6z^m + 2z^{m-1} + 1 \; = \sum_{\ell=0}^{2m} c_\ell \, z^\ell . \tag{5.8}$$

Consequently, we have $\mathfrak{m}(q) = \mathfrak{m}(r) - \mathfrak{m}\big((z-1)^2\big) = \mathfrak{m}(r)$.

Let $\mathfrak{M}(r) = \exp(\mathfrak{m}(r))$ be the (ordinary) Mahler measure of r; compare [15, Sec. 1.2]. By the strict convexity of the exponential function and Jensen's inequality, see [19, Ch. 2.2] for a suitable formulation, one finds

$$\mathfrak{M}(r) \; < \; \int_0^1 \big| r(z) \big|_{z=\mathrm{e}^{2\pi\mathrm{i}t}} \, \mathrm{d}t \; = \; \| r \|_1 \; \leqslant \; \| r \|_2 ,$$

where $r = r(t)$ is considered as a trigonometric polynomial on $\mathbb{T}$ (with the usual 1-periodic extension to $\mathbb{R}$). In fact, since r is not a monomial, we also have $\|r\|_1 < \|r\|_2$.

Assume that $m \geqslant 2$, so that the exponents of $r(z)$ in Eq. (5.8) are distinct. Consequently, by Parseval's equation, we may conclude that

$$\| r \|_2^2 \; = \sum_{\ell=0}^{2m} |c_\ell|^2 \; = \; 46 ,$$

so that $\mathfrak{M}(r) < \sqrt{46}$, independently of m. This inequality trivially also holds for $m = 1$, and we get $\mathfrak{m}(q) = \mathfrak{m}(r) < \log \sqrt{46}$ for all $m \in \mathbb{N}$ as claimed. □

With this bound, one has $\log(\lambda) > \mathfrak{m}(q)$ for all $m \geqslant 40$, where $\lambda = \lambda_m^+$ as before.

Remark 5.10. An even better bound can be obtained from Eq. (5.7) by observing that, as t varies a little, $\sin(m\pi t)^2$ oscillates quickly when m is large (with mean $\frac{1}{2}$), while $\big(\sin(\pi t)\big)^2$ remains roughly constant. Under the integral, one can then replace $\big(\sin(m\pi t)/\sin(\pi t)\big)^2$ by $\frac{1}{2}\big(\sin(\pi t)\big)^{-2}$, which still gives an upper bound for $\mathfrak{m}(q)$ because $\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(t) < 0$ on $\mathbb{R}_+$. Now,

$$\mathfrak{m}(q) \; \leqslant \; \int_0^1 \log \frac{3 - 2\cos(2\pi t)}{1 - \cos(2\pi t)} \, \mathrm{d}t \; = \; \mathfrak{m}\big(z^2 - 3z + 1\big) + \log(2) \; = \; \log\big(3 + \sqrt{5}\,\big) \; \approx \; 1.655571,$$

which is smaller than $\log(\lambda)$, where $\lambda = \lambda_m^+$ as above, for all $m \geqslant 23$. ◊

The values $\mathfrak{m}(q)$, as a function of $m \in \mathbb{N}$, seem to be increasing, so that $\lim_{m\to\infty} \mathfrak{m}(q)$ would be the optimal upper bound. The limit exists because $\mathfrak{m}(q) = \mathfrak{m}(r)$, and the polynomial r satisfies $r(z) = \tilde{r}(z, z^m)$ with

$$\tilde{r}(z,w) \; = \; -w \left(6 - 2\big(z + z^{-1}\big) - \big(w + w^{-1}\big) \right) .$$

By a classic approximation theorem for two-dimensional Mahler measures, see [15, Thm. 3.21], one has $\lim_{m\to\infty} \mathfrak{m}\big(\tilde{r}(z,z^m)\big) = \mathfrak{m}\big(\tilde{r}(z,w)\big)$, where

$$\begin{aligned}\mathfrak{m}(\tilde{r}) \;&=\; \int_{\mathbb{T}^2} \log\big(6 - 2\cos(2\pi t_1) - 4\cos(2\pi t_2)\big)\, \mathrm{d}t_1 \, \mathrm{d}t_2 \\ &=\; 2\int_0^1 \operatorname{arsinh}\big(\sqrt{2}\,\sin(\pi t_2)\big)\, \mathrm{d}t_2 \;\approx\; 1.550675.\end{aligned}$$

So, when $\mathfrak{m}(q)$ is an increasing function (which we did not prove), we immediately get the estimate $\log(\lambda) > \mathfrak{m}(q)$ for all $m \geqslant 18$.

Remark 5.11. The polynomial s from Section 5.3 can be analysed in a completely analogous way. Here, one has $s(z) \;=\; -z^{\ell+1}\left(6 - \left(z+z^{-1}\right) - \left(w+w^{-1}\right) - \left(zw + (zw)^{-1}\right)\right)$, and the approximation theorem results in

$$\lim_{\ell\to\infty} \mathfrak{m}(s) \;=\; \int_{\mathbb{T}^2} \log\big(6 - 2\cos(2\pi t_1) - 2\cos(2\pi t_2) - 2\cos(2\pi(t_1+t_2))\big)\, \mathrm{d}t_1\, \mathrm{d}t_2 \;\approx\; 1.615.$$

Moreover, various other properties are similar to those of the polynomial q from above. $\Diamond$

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