# Aggregation functions with given super-additive and sub-additive transformations

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**Abstract**

Aggregation functions and their transformations have found numerous applications in various kinds of systems as well as in economics and social science. Every aggregation function is known to be bounded above and below by its super-additive and sub-additive transformations. We are interested in the 'inverse' problem of whether or not every pair consisting of a super-additive function dominating a sub-additive function comes from some aggregation function in the above sense. Our main results provide a negative answer under mild extra conditions on the super- and sub-additive pair. We also show that our results are, in a sense, best possible.

*Keywords: aggregation function, sub-additive and super-additive transformation*

## 1 Introduction

Aggregation functions have found numerous applications in various kinds of systems and situations in which it is desirable to merge quantitative non-homogeneous information into a sole representative value. There is an abundance of such situations and examples range from compression of information by fusing inputs from several sources through decision making based on aggregating diverse scores up to applications in disciplines such as artificial intelligence, risk management, and so on. A variety of applications in economics, industry and social sciences prompted the study of a number of types of aggregation functions and within a large scale of related resources we will only refer to [2, 3] regarding basic facts.

In this article we will be dealing with aggregation functions defined on $[0,\infty[^n$. Throughout, an *aggregation function* is a mapping $A:\ [0,\infty[^n\to[0,\infty[$ increasing in every coordinate and such that $A(\mathbf{0})=A(0,\ldots,0)=0$.

In order to associate an aggregation function with 'more structured' objects one can mimic the way external and internal measure is introduced in measure theory. This was done in [4] where the authors introduced a pair of mutually dual transformations of aggregation functions defined as follows. If $A$ is as above, its *super-additive* and *sub-additive* transformations, $A^*$ and $A_*$, are functions $[0,\infty[^n\to[0,\infty]$ defined by

$$A^*(\mathbf{x})=\sup\ \{\sum_{j=1}^{k}A(\mathbf{x}^{(j)})\ ;\ \ \sum_{j=1}^{k}\mathbf{x}^{(j)}\le\mathbf{x}\}\ ,\quad\text{and}\tag{1}$$

$$A_*(\mathbf{x})=\inf\ \{\sum_{j=1}^{k}A(\mathbf{x}^{(j)})\ ;\ \ \sum_{j=1}^{k}\mathbf{x}^{(j)}\ge\mathbf{x}\}\ .\tag{2}$$

It is worth remarking that the transformations (1) and (2) of [4] were restricted to aggregation functions under the proviso that $A_*$ and $A^*$ do not attain the value $\infty$; see e.g. [9] for inclusion of this value. The adjectives in the names of the two transformations come from the basic fact proved in [4] that $A^*$ and $A_*$ are super-additive and sub-additive functions, respectively; that is,

$$A^*(\mathbf{u}+\mathbf{v})\ge A^*(\mathbf{u})+A^*(\mathbf{v})\ \ \text{and}\ \ A_*(\mathbf{u}+\mathbf{v})\le A_*(\mathbf{u})+A_*(\mathbf{v})\tag{3}$$

for every pair of points $\mathbf{u},\ \mathbf{v}\in[0,\infty[^n$, with the usual coordinate-wise addition.

Specification of aggregation functions $A$ for which $A=A^*$ or $A=A_*$ appears to be essential in certain economics applications, as illustrated e.g. in [4]. Following this resource, values of an aggregation function $A$ at some point $\mathbf{x}$ may represent production output subject to a vector $\mathbf{x}$ of manufacturing factors. In this situation, the optimal output for every vector $\overline{\mathbf{x}}$ of available resources would be equal to $A^*(\overline{\mathbf{x}})$. In a complete analogy, if $A(\mathbf{x})$ represents price for a collection of $n$-tuples of merchandise with quantities encoded in a vector $\mathbf{x}$, then an optimal purchase price of a preselected collection of goods represented by $\overline{\mathbf{x}}$ would be equal to $A_*(\overline{\mathbf{x}})$. In [4] one may found further examples of importance of distinguishing aggregation functions $A$ with $A=A^*$ and $A=A_*$ related to applications in economics.

Another type of connections of the transformations (1) and (2) with both theory and applications stem from [5] through the concepts of $(A,\mathcal{D})$-based sub-decomposition and super-decomposition integrals for an aggregation function $A$ and a so-called decomposition system $\mathcal{D}$ of its domain. In this context, the quantities $A^*(\mathbf{x})$ and $A_*(\mathbf{x})$, respectively, are the values of the $(A,\mathcal{D})$-based sub-decomposition and super-decomposition integrals at $\mathbf{x}$ if $\mathcal{D}$ is a decomposition system of the domain of $A$ made of singletons. A number of other applications of sub- and super-decomposition integrals in economics are also briefly discussed in [5] (e.g. in work distribution optimization and planning) that imply interest in determination of $A^*$ and $A_*$ for a given aggregation function $A$.

A natural question that arises in this connection is if, for every pair of functions $f, g : [0, \infty[^n \to [0, \infty]$ with $f(\mathbf{x}) \geq g(\mathbf{x})$ for every $\mathbf{x} \in [0, \infty[^n$, having zero value at the origin, and such that $f$ is super-additive and $g$ is sub-additive, there exists an aggregation function $A$ on $[0, \infty[^n$ such that $A^* = f$ and $A_* = g$.

The answer in general is in the negative, as it was demonstrated in [13]; cf. also [12] for a separate treatment in dimension one. To describe the corresponding results we need to introduce some terminology and notation. As usual, points in $[0, \infty[^n$ will be denoted $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, and so on; in particular, $\mathbf{0}$ and $\mathbf{1}$ will stand for the points $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. We will write $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{y} - \mathbf{x} \in [0, \infty[^n$, and $\mathbf{x} < \mathbf{y}$ if $\mathbf{x} \leq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$; if the inequality $\mathbf{x} < \mathbf{y}$ does *not* hold we write $\mathbf{x} \not< \mathbf{y}$.

A function $h : [0, \infty[^n \to [0, \infty[$ is said to be *strictly directionally convex* if $h(\mathbf{x}) + h(\mathbf{y}) < h(\mathbf{u}) + h(\mathbf{v})$ for every quadruple of points $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in [0, \infty[^n$ such that $\mathbf{u} < \mathbf{x}, \mathbf{y} < \mathbf{v}$ and $\mathbf{u} + \mathbf{v} = \mathbf{x} + \mathbf{y}$. Directional convexity (where the inequalities above are allowed to be non-strict) can also be defined with the help of increments [1] and is alternatively known as ultra-modularity [7]; we also note that directional convexity implies continuity and is equivalent to the conjunction of super-modularity and coordinate-wise convexity [1]. Further, we say that a function $f$ on $[0, \infty[^n$ *overruns* some super-additive function if there exists a super-additive function $h$ on $[0, \infty[^n$ such that $f/h$ is strictly increasing in every coordinate on $]0, \infty[^n$. The latter property is weaker than the former (and the two are not equivalent), as shown in [10, 11].

Returning to the negative answer to our question, in [13] it was proved that if $f$ and $g$ are functions $[0, \infty[^n \to [0, \infty[$ with zero value at the origin and $f \geq g$, such that $f$ is strictly directionally convex and $g$ is sub-additive but not linear, then there is no aggregation function $A$ on $[0, \infty[^n$ such that $A^* = f$ and $A_* = g$. This result was generalized in [10, 11] by showing that strict directional convexity of $f$ may be replaced by a weaker condition of $f$ overrunning some super-additive function. In both cases, 'dual' versions of these results were also proved (assuming that, for a non-linear $f$, the function $g$ is strictly directionally concave or underrunning some sub-additive function, with the obvious meaning of these concepts).

In this paper we show that the answer to the above question is negative if one assumes *continuity* and *strict super-additivity* of $f$, maintaining sub-additivity and non-linearity of $g$ (and, dually, *continuity* and *strict sub-additivity* of $g$, keeping super-additivity and non-linearity of $f$); the adjective *strict* here means strict inequalities for non-zero points in (3). We prove these results as corollaries of Theorems 1 and 2, in section 2. In the concluding section 3 we show that our results are, in a sense, best possible, as they fail to hold if the continuity or strict super- (or sub-) additivity assumptions are dropped, and we also discuss the relationship between our new conditions and the ones of [10, 11].

# 2 Results

Recall that a function $h: [0,\infty[^n \to [0,\infty[$ is *strictly super-additive* if $h(\mathbf{y})+h(\mathbf{z}) < h(\mathbf{y}+\mathbf{z})$ for every pair of points $\mathbf{y},\mathbf{z} \in [0,\infty[^n \backslash \mathbf{0}$; this terminology agrees e.g. with [8]. For example, the function $h_1(\mathbf{x}) = x_1^2 + \ldots + x_n^2$ is strictly super-additive while $h_2(\mathbf{x}) = x_1 + \ldots + x_n$ is super-additive but not strictly. To the best of our knowledge, strictly super-additive functions per se have not been studied in detail in the literature, although other variations than strict super-additivity have been considered, e.g., completely strong super-additivity [6]. We state and prove here an important property of strictly super-additive functions which we will use later.

**Proposition 1** *Let $h: [0,\infty[^n \to [0,\infty[$ be a continuous strictly super-additive function. Then, for every $\mathbf{x} \neq \mathbf{0}$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that $h(\mathbf{y})+h(\mathbf{z}) \leq h(\mathbf{x})-\delta$ for every $\mathbf{y},\mathbf{z} \in [0,\infty[^n$ such that $\mathbf{y}+\mathbf{z}=\mathbf{x}$ and $\mathbf{x}-\varepsilon\mathbf{1} \not< \mathbf{y},\mathbf{z}$.*

**Proof.** For $h$, $\mathbf{x}$ and $\varepsilon$ as above let $C(\mathbf{x},\varepsilon) = \{\mathbf{u} \in [0,\infty[^n;\ \mathbf{x}-\varepsilon\mathbf{1} \not< \mathbf{u} \leq \mathbf{x}\}$. Also, let $P(\mathbf{x},\varepsilon)$ be the set of all pairs $(\mathbf{y},\mathbf{z})$ of points in $C(\mathbf{x},\varepsilon)$ such that $\mathbf{y}+\mathbf{z}=\mathbf{x}$. We will assume that $\varepsilon > 0$ is sufficiently small so that $P(\mathbf{x},\varepsilon) \neq \emptyset$, for otherwise the statement is vacuously true. Let $\delta = \inf\{h(\mathbf{x})-h(\mathbf{y})-h(\mathbf{z});\ (\mathbf{y},\mathbf{z}) \in P(\mathbf{x},\varepsilon)\}$. By super-additivity of $h$ we have $\delta \geq 0$, and our aim is to show that $\delta > 0$.

Suppose the contrary and let $\delta = 0$. This means that for every positive integer $m$ there is a pair $(\mathbf{y}^{(m)},\mathbf{z}^{(m)}) \in P(\mathbf{x},\varepsilon)$ such that

$$h(\mathbf{x}) - m^{-1} < h(\mathbf{y}^{(m)}) + h(\mathbf{z}^{(m)}) < h(\mathbf{x})\ . \tag{4}$$

Since $C(\mathbf{x},\varepsilon)$ is a compact set, the sequence $(\mathbf{y}^{(m)})_{m=1}^{\infty}$ contains a subsequence $(\mathbf{y}^{(m_k)})_{k=1}^{\infty}$ converging to some point $\mathbf{y}^{(0)} \in C(\mathbf{x},\varepsilon)$. The subsequence $(\mathbf{z}^{(m_k)})_{k=1}^{\infty}$ given by $\mathbf{z}^{(m_k)} = \mathbf{x}-\mathbf{y}^{(m_k)}$ then obviously converges to the point $\mathbf{z}^{(0)} = \mathbf{x}-\mathbf{y}^{(0)} \in C(\mathbf{x},\varepsilon)$. Taking the limit in (4) for $m = m_k$ as $k \to \infty$ and using our assumption of that $h$ is continuous in $[0,\infty[^n$ gives $h(\mathbf{y}^{(0)}) + h(\mathbf{z}^{(0)}) = h(\mathbf{x})$. This, however, contradicts the strict super-additivity of $h$, since we clearly have $\mathbf{y}^{(0)},\mathbf{z}^{(0)} \neq \mathbf{0}$.

It follows that $\delta > 0$ and the way $\delta$ was defined implies that $h(\mathbf{y})+h(\mathbf{z}) \leq h(\mathbf{x})-\delta$ for every pair $(\mathbf{y},\mathbf{z}) \in P(\mathbf{x},\varepsilon)$, which was to be shown. □

For an aggregation function $f: [0,\infty[^n \to [0,\infty[$ we let $\nabla_f$ be the $n$-dimensional vector whose $i$-th component $(\nabla_f)_i$ is equal to $\liminf_{t\to 0^+} f(t\mathbf{e_i})/t$ for $i \in \{1,2,\ldots,n\}$, where $\mathbf{e_i}$ denotes the $i$-th unit vector. With this we are now ready to state and prove our first main result.

**Theorem 1** *Let $A: [0,\infty[^n \to [0,\infty[$ be an aggregation function. If $A^*$ is continuous and strictly super-additive, then $A^*(\mathbf{x}) = A(\mathbf{x})$ and $A_*(\mathbf{x}) = \nabla_A{\cdot}\mathbf{x}$ for every $\mathbf{x} \in [0,\infty[^n$.*

**Proof.** We begin by showing that $A = A^*$. Assume the contrary and let $\overline{\mathbf{x}} \neq \mathbf{0}$ be such that $A(\bar{x}) < A^*(\bar{x})$. Recall that

$$A^*(\overline{\mathbf{x}}) = \sup \left\{ \sum_{j=1}^{k} A(\mathbf{x}^{(j)});\ \mathbf{0} \neq \mathbf{x}^{(j)} \in [0,\infty[^n\ (1 \leq j \leq k),\ \sum_{j=1}^{k} \mathbf{x}^{(j)} = \overline{\mathbf{x}} \right\}\ ; \tag{5}$$

assuming equality in the second summation means no loss of generality. If both sums consist of the single elements $A(\overline{\mathbf{x}})$ and $\overline{\mathbf{x}}$, then automatically $A(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$ by our assumption, and so it is sufficient to assume that $k \geq 2$ in our further arguments.

Before proceeding we need to introduce a few parameters. Let $\delta_1 = \frac{1}{2}(A^*(\overline{\mathbf{x}}) - A(\overline{\mathbf{x}}))$. Let $m$ be the smallest positive integer satisfying $m \geq A^*(\mathbf{1})/\delta_1$. We will also use the reciprocal value $\varepsilon = m^{-1}$, so that $0 < \varepsilon \leq \delta_1/A^*(\mathbf{1})$.

We will consider two cases, distinguished by (C1) and (C2) below, for $k$-tuples

$$\mathbf{x}^{(j)}\ (1 \leq j \leq k);\ \ \sum_{j=1}^{k} \mathbf{x}^{(j)} = \overline{\mathbf{x}} \tag{6}$$

appearing in (5).

(C1) *Assume that in* (6) *there exists some* $j \in \{1, 2, \ldots, k\}$ *such that* $\overline{\mathbf{x}} - \varepsilon\mathbf{1} < \mathbf{x}^{(j)}$. We may let $j = 1$, and then $\sum_{j=2}^{k} \mathbf{x}^{(j)} < \varepsilon\mathbf{1}$. To estimate $\sum_{j=1}^{k} A(\mathbf{x}^{(j)})$, observe first that $A(\mathbf{x}^{(1)}) \leq A(\overline{\mathbf{x}}) = A^*(\overline{\mathbf{x}}) - 2\delta_1$ by our way of choosing $\delta_1$. Super-additivity of $A^*$ applied to $\sum_{j=2}^{k} \mathbf{x}^{(j)} < \varepsilon\mathbf{1}$ further yields $\sum_{j=2}^{k} A(\mathbf{x}^{(j)}) \leq \sum_{j=2}^{k} A^*(\mathbf{x}^{(j)}) \leq A^*(\sum_{j=2}^{k} \mathbf{x}^{(j)}) \leq A^*(\varepsilon\mathbf{1})$. Recalling that $\varepsilon = m^{-1}$ and invoking super-additivity of $A^*$ one more time we obtain $mA^*(m^{-1}\mathbf{1}) \leq A^*(\mathbf{1})$, which means that $A^*(\varepsilon\mathbf{1}) \leq \varepsilon A^*(\mathbf{1})$. (In fact, by strict super-additivity of $A^*$ the last three inequalities are strict but we do not need this here.) By our choice of $m$ and hence $\varepsilon$ we have $\varepsilon A^*(\mathbf{1}) \leq \delta_1$, which in combination with the previous inequalities leads to our first partial conclusion: If the sum (6) satisfies (C1), then

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) = A(\mathbf{x}^{(1)}) + \sum_{j=2}^{k} A(\mathbf{x}^{(j)}) \leq A^*(\overline{\mathbf{x}}) - 2\delta_1 + A^*(\varepsilon\mathbf{1}) \leq A^*(\overline{\mathbf{x}}) - \delta_1\ . \tag{7}$$

(C2) *Assume that in the* $k$*-tuple* (6) *we have* $\overline{\mathbf{x}} - \varepsilon\mathbf{1} \not< \mathbf{x}^{(j)}$ *for every* $j \in \{1, 2, \ldots, k\}$. Then there exists a smallest $r \in \{1, 2, \ldots, k-1\}$ with the property that

$$\overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} \not< \sum_{j=1}^{r} \mathbf{x}^{(j)} \quad \text{but} \quad \overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} < \sum_{j=1}^{r+1} \mathbf{x}^{(j)}\ . \tag{8}$$

We let $\mathbf{y} = \sum_{j=1}^{r} \mathbf{x}^{(j)}$ and $\mathbf{z} = \sum_{j=r+1}^{k} \mathbf{x}^{(j)}$. Obviously $\mathbf{y} + \mathbf{z} = \overline{\mathbf{x}}$ and $\overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} \not< \mathbf{y}$; we also show that $\overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} \not< \mathbf{z}$. Indeed, suppose that $\overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} < \mathbf{z}$. Then, $\mathbf{y} < (\varepsilon/2)\mathbf{1}$, but this together with the inequality $\overline{\mathbf{x}} - (\varepsilon/2)\mathbf{1} < \mathbf{y} + \mathbf{x}^{(r+1)}$ from (8) imply that $\overline{\mathbf{x}} - \varepsilon\mathbf{1} < \mathbf{x}^{(r+1)}$, contrary to the assumption (C2).

We are thus in a situation when the two points $\mathbf{y}, \mathbf{z}$ satisfy $\mathbf{y}+\mathbf{z} = \overline{\mathbf{x}}$ and $\overline{\mathbf{x}} - \varepsilon\mathbf{1} \not\leq \mathbf{y}, \mathbf{z}$. By Proposition 1 applied to our strictly super-additive function $A^*$, there exists a $\delta_2 > 0$ such that $A^*(\mathbf{y}) + A^*(\mathbf{z}) \leq A^*(\overline{\mathbf{x}}) - \delta_2$. Dominance of $A$ by $A^*$ and super-additivity of the latter now furnishes our second conclusion: If the sum (6) satisfies (C2), then

$$\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \leq \sum_{j=1}^{r} A^*(\mathbf{x}^{(j)}) + \sum_{j=r+1}^{k} A^*(\mathbf{x}^{(j)}) \leq A^*(\mathbf{y}) + A^*(\mathbf{z}) \leq A^*(\overline{\mathbf{x}}) - \delta_2 \ . \tag{9}$$

It is now easy to draw a conclusion regarding $A^*$. Observe that for every $k \geq 2$ a $k$-tuple as in (6) satisfies (C1) or (C2). Letting $\delta = \min\{\delta_1, \delta_2\} > 0$ it is clear that (7) and (9) imply the inequality $\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \leq A^*(\overline{\mathbf{x}}) - \delta$ whenever $k \geq 2$, and we know that $A(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$. By (5) we thus have $A^*(\overline{\mathbf{x}}) < A^*(\overline{\mathbf{x}})$, a contradiction. This proves that $A(\mathbf{x}) = A^*(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty[^n$.

To finish the proof it remains to show validity of the statement about $A_*$. Applying Theorem 1 of [9] to the function $x_i \mapsto A(x_i\mathbf{e}_i)$ of one variable $x_i \in [0,\infty[$, $1 \leq i \leq n$, we obtain the inequality $A_*(x_i\mathbf{e}_i) \leq (\nabla_A)_i x_i$ for every $x_i \in [0,\infty[$. We know by [4] that $A_*$ is sub-additive, which, for every $\mathbf{x} \in [0,\infty[^n$, implies that

$$A_*(\mathbf{x}) = A_*(\sum_{i=1}^{n} x_i\mathbf{e}_i) \leq \sum_{i=1}^{n} A_*(x_i\mathbf{e}_i) \leq \nabla_A\cdot\mathbf{x} \ . \tag{10}$$

To prove the reverse inequality we apply super-additivity of $A^* = A$ together with the inequality $A^*(x_i\mathbf{e}_i) \geq (\nabla_A)_i x_i$, which again follows from Theorem 1 of [9] when applied to the function $x_i \mapsto A(x_i\mathbf{e}_i)$. This results in the chain of inequalities

$$A(\mathbf{x}) = A^*(\mathbf{x}) = A^*(\sum_{i=1}^{n} x_i\mathbf{e}_i) \geq \sum_{i=1}^{n} A^*(x_i\mathbf{e}_i) \geq \sum_{i=1}^{n} (\nabla_A)_i x_i = \nabla_A\cdot\mathbf{x} \tag{11}$$

for every $\mathbf{x} \in [0,\infty[^n$. From (11) we deduce that $A_*(\mathbf{x}) \geq (\nabla_A\cdot\mathbf{x})_*$. Since $\nabla_A\cdot\mathbf{x}$ is a linear function, we have $(\nabla_A\cdot\mathbf{x})_* = \nabla_A\cdot\mathbf{x}$, and so $A_*(\mathbf{x}) \geq \nabla_A\cdot\mathbf{x}$ for every $\mathbf{x} \in [0,\infty[^n$. In conjunction with (10) this implies that $A_*(\mathbf{x}) = \nabla_A\cdot\mathbf{x}$ for every $\mathbf{x} \in [0,\infty[^n$. □

**Example.** We illustrate Theorem 1 on three instances of aggregation functions $A : [0,\infty[^n \to [0,\infty[$. If $A(\mathbf{x}) = x_1^2+\ldots+x_n^2$, then by strict super-additivity of $A$ we have $A^* = A$ and $A_* = 0$ since in this case $\nabla_A$ is the zero vector, confirming the assertion of Theorem 1. On the other hand, taking the sub-additive function $A(\mathbf{x}) = \sum_{i=1}^{n}(\sqrt{1+x_i} - 1)$ one has $A(\mathbf{x}) = A_*(\mathbf{x}) \neq \nabla_A \cdot \mathbf{x}$ while $A^*(\mathbf{x}) = x_1 + \ldots + x_n$, which is not a *strictly* super-additive function. Also, note that the conditions listed in Theorem 1 are sufficient but not necessary, as illustrated by the function $A(\mathbf{x}) = \min_i(x_i)$, for which we have $A_*(\mathbf{x}) = 0 = \nabla_A \cdot \mathbf{x}$ and $A^* = A$ but again with $A^*$ not *strictly* super-additive.

In an entirely similar way one can prove a dual statement regarding aggregation functions $A$ for which $A_* = A$. We say that a function $h : [0,\infty[^n \to [0,\infty[$ is *strictly sub-additive* if $h(\mathbf{y})+h(\mathbf{z}) > h(\mathbf{y}+\mathbf{z})$ for every non-zero points $\mathbf{y}, \mathbf{z} \in [0,\infty[^n$. A straightforward modification of the proof of Proposition 1 gives:

**Proposition 2** *Let* $h: [0,\infty[^n \to [0,\infty[$ *be a continuous strictly sub-additive function. Then, for every* $\mathbf{x} \neq \mathbf{0}$ *and each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $h(\mathbf{y}) + h(\mathbf{z}) \geq h(\mathbf{x}) + \delta$ *for every* $\mathbf{y}, \mathbf{z} \in [0,\infty[^n$ *such that* $\mathbf{y} + \mathbf{z} = \mathbf{x}$ *and* $\mathbf{x} - \varepsilon\mathbf{1} \not< \mathbf{y}, \mathbf{z}$. □

Further, for an aggregation function $g: [0,\infty[^n \to [0,\infty[$ let $\nabla^g$ be the $n$-dimensional vector with $i$-th component $(\nabla^g)_i$ equal to $\limsup_{t\to 0^+} g(t\mathbf{e_i})/t$ for $i \in \{1,2,\ldots,n\}$. By reversing the chains of inequalities appropriately, the proof of Theorem 1 can obviously be turned into a justification of the following result.

**Theorem 2** *Let* $A: [0,\infty[^n \to [0,\infty[$ *be an aggregation function. If* $A_*$ *is continuous and strictly sub-additive, then* $A_*(\mathbf{x}) = A(\mathbf{x})$ *and* $A^*(\mathbf{x}) = \nabla^A \cdot \mathbf{x}$ *for every* $\mathbf{x} \in [0,\infty[^n$. □

Recall that for any aggregation function $A$ the transformed functions $A^*$ and $A_*$ are super-additive and sub-additive, respectively, the first one dominating the second. This leads to the interesting question whether or not for any given super-additive and sub-additive functions $f$ and $g$, respectively, with $f \geq g$, there exists an aggregation function $A$ such that $A^* = f$ and $A_* = g$. The following consequence of both Theorems 1 and 2 provides an interesting sufficient condition under which the answer is in the negative.

**Theorem 3** *Let* $f, g: [0,\infty[^n \to [0,\infty[$ *be continuous functions such that* $f(0) = g(0) = 0$ *and* $f(\mathbf{x}) \geq g(\mathbf{x})$ *for every* $\mathbf{x} \in [0,\infty[^n$. *If*

(a) $f$ *is strictly super-additive and* $g$ *is sub-additive but not linear, or*

(b) $g$ *is strictly sub-additive and* $f$ *is super-additive but not linear,*

*then there is no aggregation function* $A: [0,\infty[^n \to [0,\infty[$ *with* $A^* = f$ *and* $A_* = g$.

**Proof.** Suppose for a contradiction that, assuming (a), there was an aggregation function $A: [0,\infty[^n \to [0,\infty[$ such that $A^* = f$ and $A_* = g$. Since $f$ is assumed to be strictly super-additive, so is $A^*$. Applying Theorem 1 we conclude that $A^* = A$ and $A_*$ is linear, contradicting the assumption of non-linearity of $g$ in the part (a). The argument for the part (b) is entirely similar and we omit the details. □

# 3 Discussion

Our main results in Theorems 1 – 3 are related to the latest results of [10, 11] that were based on the extra assumption of overrunning and underrunning some super-additive and sub-additive function, respectively. Namely, every function that overruns (underruns) a super-additive (sub-additive) function is automatically strictly super-additive (strictly sub-additive), as it was noted in [10] in dimension one and in [11] in the multi-dimensional case. Despite of the fact that we do not know whether or not the reverse implication holds, our conditions in Theorems 1 – 3 are conceptually simpler and allow for simpler proofs.

We conclude by showing that Theorems 1 – 3 are best possible in the sense that the assumptions of *continuity*, or *strictness* in super- and sub-additivity, cannot be dropped in general. (Of course, one *has to* assume super-and sub-additivity, which are basic properties of $A^*$ and $A_*$.) Indeed, Example 1 of [13] shows that, keeping continuity, the *strictness* assumption in super- sub-additivity in the results of section 2 cannot be omitted. In the second case, that is, when strictness is maintained, we will show that the continuity assumption cannot be dropped, not even in one-dimensional case, and not even if discontinuity arises at just one single point. This will be illustrated by a construction of a counterexample to part (a) of Theorem 3.

**Example 1** *Let $\varepsilon$ be such that $0 < \varepsilon < \frac{1}{5}$ and let $\delta = (1+\varepsilon)/(1-\varepsilon)$; note that $1 < \delta < \frac{3}{2}$. Let $A:\ [0,\infty[\to [0,\infty[$ be given by*

$$A(x) = \begin{cases} x+\varepsilon x^2 & \textit{if } x \in [0,1]; \\ 1+\varepsilon(x+1) & \textit{if } x \in \left]1, \frac{3}{2}\right]; \\ (\frac{2}{3}+\varepsilon)x^2 & \textit{if } x \in \left]\frac{3}{2}, \infty\right[. \end{cases}$$

*Clearly, $A$ is a (strictly increasing) aggregation function. Further, we introduce functions $f:\ [0,\infty[\ \to [0,\infty[$ and $g:\ [0,\frac{3}{2}] \to [0, 1+\frac{5}{2}\varepsilon]$ by letting*

$$f(x) = \begin{cases} A(x) & \textit{if } x \in [0,1]\cup\left]\frac{3}{2},\infty\right[; \\ 1+2\varepsilon+A(x-1) & \textit{if } x \in \left]1,\frac{3}{2}\right]; \end{cases} \qquad g(x) = \begin{cases} x & \textit{if } x \in [0,\delta]; \\ 1+\varepsilon(x+1) & \textit{if } x \in [\delta, \frac{3}{2}]. \end{cases}$$

*The situation with the three functions restricted to the interval $[0,2]$ for $A$ and $f$, and to the interval $[0,\frac{3}{2}]$ for $g$, is depicted in Fig. 1. It is a matter of routine to check that $f$ is strictly super-additive and $g$ is sub-additive. We will show that $A^* = f$ on $[0,\infty[$ and $A_* = g$ on $[0,\frac{3}{2}]$. (It would be, of course, possible to determine $A_*$ completely but the resulting formula depends on $\varepsilon$ and does not have a particularly nice form; more importantly, this is not needed for our purposes.)*

*To prove that $A^* = f$ we begin by observing that super-additivity of $f$, dominance of $A$ by $f$ on $[0,\infty[$ and the fact that $f = A$ for every $x \in [0,1]\cup]\frac{3}{2},\infty[$ imply that $A = A^* = f$ on $[0,1]\cup]\frac{3}{2},\infty[$ and $A^* \le f$ on $]1,\frac{3}{2}]$. It remains to prove that $A^* \ge f$ on $]1,\frac{3}{2}]$. Let us write an arbitrary $x \in ]1,\frac{3}{2}]$ in the form $x = y+z$, where $y > 1$ and $z \in [0,1[$. The definition of $A^*$ and the way $A$ has been introduced imply that $A^*(x) \ge A(y)+A(z) = 1+\varepsilon(y+1)+z+\varepsilon z^2$. Taking the limit of the right-hand side as $y \to 1^+$ gives $z \to x-1$ and yields the inequality $A^*(x) \ge 1+2\varepsilon+A(x-1) = f(x)$. This shows that $A^* \ge f$ on $]1,\frac{3}{2}]$, and hence $A^* = f$ on $[0,\infty[$.*

*Showing that $A_* = g$ on $[0,\frac{3}{2}]$ is even easier. First, note that sub-additivity of $g$ on this interval gives $A_*(x) \ge g(x)$ for every $x \in [0,\frac{3}{2}]$. Since $g(x) = A(x)$ for every $x \in [\delta,\frac{3}{2}]$, it follows that $A = A_* = g$ on $[\delta,\frac{3}{2}]$, and it remains to prove that $A_*(x) \le g(x) = x$ for every $x \in [0,\delta]$. This, however, follows from Theorem 1 of [9] because $lim_{x\to 0^+} A(x)/x = 1$. Therefore, $g = A_*$ on $[0,\frac{3}{2}]$.* □

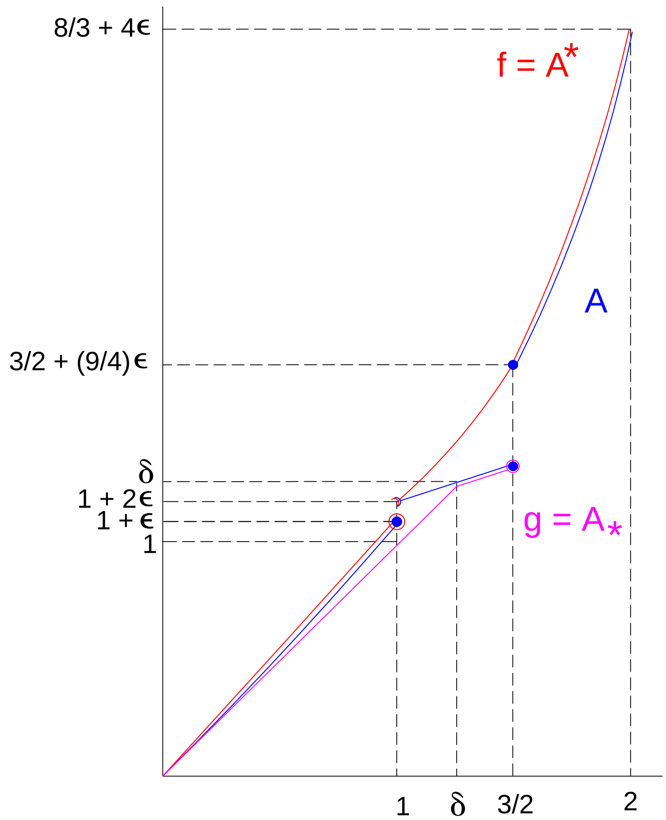


Figure 1: A function $A$ with non-linear $A_*$ and discontinuous strictly super-additive $A^*$.

Summing up, by Example 1 we have a strictly super-additive function $f$ discontinuous just at a single point and dominating a non-constant sub-additive function $g$ (explicitly given just on $[0, \frac{3}{2}]$), and yet $f = A^*$ and $g = A_*$ for some aggregation function $A$ (the function $g$ extends to $]\frac{3}{2}, \infty[$ by simply letting $g$ be equal to $A_*$ on this interval). This shows that the continuity assumption on $f$ in part (a) of Theorem 3 cannot be omitted.

Multi-dimensional examples of such an ilk can be obtained by replacing each function $h \in \{A, f, g\}$ from Example 1 by $\mathbf{x} \mapsto h(\sum_{i=1}^{n} x_i)$ for every $\mathbf{x} = (x_1, \ldots, x_n) \in [0, \infty[^n$, with discontinuity limited to a section of a single hyperplane. Analogously one can construct multi-dimensional examples of discontinuous strictly sub-additive functions $g$ for which part (b) of Theorem 3 fails to hold.

The interesting problem of a complete characterization of functions $f$ and $g$ for which there exists an aggregation function $A$ with the property that $A^* = f$ and $A_* = g$ still remains open.

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