

ARTICLE

On Hamilton decompositions of infinite circulant graphs

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Abstract

The natural infinite analog of a (finite) Hamilton cycle is a two-way-infinite Hamilton path (connected spanning 2-valent subgraph). Although it is known that every connected $2k$ -valent infinite circulant graph has a two-way-infinite Hamilton path, there exist many such graphs that do not have a decomposition into k edge-disjoint two-way-infinite Hamilton paths. This contrasts with the finite case where it is conjectured that every $2k$ -valent connected circulant graph has a decomposition into k edge-disjoint Hamilton cycles. We settle the problem of decomposing $2k$ -valent infinite circulant graphs into k edge-disjoint two-way-infinite Hamilton paths for $k = 2$, in many cases when $k = 3$, and in many other cases including where the connection set is $\pm\{1, 2, \dots, k\}$ or $\pm\{1, 2, \dots, k - 1, k + 1\}$.

KEYWORDS

circulant graph, hamilton decomposition, infinite graph

1 | INTRODUCTION AND NOTATION

Hamiltonicity refers to graph properties related to Hamilton cycles or Hamilton paths, and its study includes many classical problems in graph theory. One such problem is the Lovász Conjecture [20] that states that every finite connected vertex-transitive graph has a Hamilton path. There are only four nontrivial finite connected vertex-transitive graphs that are known to not admit a Hamilton cycle; the Petersen graph, the Coxeter graph, and the two graphs obtained from these by replacing each vertex with a triangle. None of these is a Cayley graph and a well-known conjecture states that every finite nontrivial

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connected Cayley graph has a Hamilton cycle, see [26]. Both the above-mentioned conjectures remain open.

A *decomposition* of a graph is a set of edge-disjoint subgraphs that collectively contain all the edges; a decomposition into Hamilton cycles is called a *Hamilton decomposition*, and a graph admitting a Hamilton decomposition is said to be *Hamilton-decomposable*. An obvious necessary condition for a Hamilton decomposition of a graph is that the graph be regular of even valency. Sometimes, a decomposition of a $(2k + 1)$ -valent graph into k Hamilton cycles and a perfect matching is also called a Hamilton decomposition, but here we do not consider these to be Hamilton decompositions.

In 1984, Alspach [1] asked whether every $2k$ -valent connected Cayley graph on a finite abelian group is Hamilton-decomposable. It is known that every connected Cayley graph on a finite abelian group has a Hamilton cycle [7], so it makes sense to consider the stronger property of Hamilton-decomposability. Alspach's question is now commonly referred to as Alspach's conjecture. It holds trivially when $k = 1$ and Bermond et al. proved that it holds for $k = 2$ [3]. The case $k = 3$ is still open, although many partial results exist, see [8,9,22–24]. There are also results for $k > 3$, see [2,12,17–19]. It was shown in [4] that there exist $2k$ -valent connected Cayley graphs on finite non-abelian groups that are not Hamilton-decomposable.

In this article, we study the natural extension of Alspach's question to the case of Cayley graphs on infinite abelian groups, specifically in the case of the infinite cyclic group \mathbb{Z} . We will not be considering any uncountably infinite graphs, so it should be assumed that the order of any graph in this article is countable. The natural infinite analog of a (finite) Hamilton cycle is a *two-way-infinite Hamilton path*, which is defined as a connected spanning 2-valent subgraph. This is, of course, an exact definition for a Hamilton cycle in the finite case, and accordingly we define a *Hamilton decomposition* of an infinite graph to be a decomposition into two-way-infinite Hamilton paths. A *one-way-infinite Hamilton path* is a connected spanning subgraph in which there is exactly one vertex of valency 1, and the remaining vertices have valency 2. For convenience, since we will not be dealing with one-way-infinite Hamilton paths, we refer to two-way-infinite Hamilton paths simply as *Hamilton paths*, or as *infinite Hamilton paths* if we wish to emphasize that the path is infinite.

Hamiltonicity of infinite circulant graphs, and of infinite graphs generally, has already been studied. In 1959, Nash-Williams [21] showed that every connected Cayley graph on a finitely generated infinite abelian group has a Hamilton path. It seems that Nash-Williams' article is largely unknown. For example, it is not cited in the 1984 survey by Witte and Gallian [26], and in 1995 Zhang and Huang [27] proved the above-mentioned result of Nash-Williams in the special case of infinite circulant graphs. Indeed, D. Jungreis' article [15] on Hamilton paths in infinite Cayley digraphs is one of the few articles to cite Nash-Williams' result. Other results on Hamilton paths in infinite Cayley digraphs can be found in [11,16].

Given the existence of Hamilton paths in Cayley graphs on finitely-generated infinite abelian groups, it makes sense to consider Hamilton-decomposability of these graphs. In this article, we investigate this problem in the special case of infinite circulant graphs. Witte [25] proved that an infinite graph with infinite valency has a Hamilton decomposition if and only if it has infinite edge-connectivity and has a Hamilton path. By combining this characterization with the result of Nash-Williams, we observe that if a connected Cayley graph on a finitely generated infinite abelian group has infinite valency, then it is Hamilton-decomposable, see Theorem 8.

In Lemma 2, we prove necessary conditions for an infinite circulant graph to be Hamilton-decomposable, thereby showing that not all connected infinite circulant graphs are Hamilton-decomposable. Since there are no elements of order 2 in \mathbb{Z} , any infinite circulant graph with finite connection set is regular of valency $2k$ and is $2k$ -edge-connected, for some nonnegative integer k . Thus, neither the valency nor the edge-connectivity is an immediate obstacle to Hamilton-decomposability.

We call infinite circulant graphs *admissible* if they satisfy the necessary conditions for Hamilton-decomposability given in Lemma 2 (see Definition 3). In Section 3, we prove that all admissible 4-valent infinite circulant graphs are Hamilton-decomposable. We also show, in Section 4, that several other infinite families of infinite circulant graphs are Hamilton-decomposable, including many 6-valent infinite circulant graphs, and several families with arbitrarily large finite valency.

Throughout the article we make use of the following notation and terminology. Let \mathcal{G} be a group with identity e and $S \subseteq \mathcal{G} - \{e\}$ that is inverse-closed, that is, $s^{-1} \in S$ if and only if $s \in S$. The *Cayley graph* on the group \mathcal{G} with *connection set* S , denoted $\text{Cay}(\mathcal{G}, S)$, is the undirected simple graph whose vertices are the elements of \mathcal{G} and whose edge set is $\{\{g, gs\} \mid g \in \mathcal{G}, s \in S\}$. When \mathcal{G} is an infinite group, we call $\text{Cay}(\mathcal{G}, S)$ an *infinite Cayley graph*. When \mathcal{G} is a cyclic group, a Cayley graph $\text{Cay}(\mathcal{G}, S)$ is called a *circulant graph*. Since we are interested in infinite circulant graphs, we will be considering graphs $\text{Cay}(\mathbb{Z}, S)$, where S is an inverse-closed set of distinct nonzero integers, which may be finite or infinite. We define $S^+ = \{a \in S \mid a > 0\}$. Observe that if $|S^+| = k$ then $\text{Cay}(\mathbb{Z}, S)$ is a $2k$ -valent graph.

If A is any subset of \mathbb{Z} and $t \in \mathbb{Z}$, then we write $A + t$ to represent the set $\{a + t \mid a \in A\}$. Furthermore, if G is any graph with $V(G) \subseteq \mathbb{Z}$ and $t \in \mathbb{Z}$, then $G + t$ is the graph with vertex set $\{x + t \mid x \in V(G)\}$ and edge set $\{\{x + t, y + t\} \mid \{x, y\} \in E(G)\}$. The *length* of any edge $\{u, v\}$, denoted $\ell(u, v)$, in a graph with vertex set \mathbb{Z} or \mathbb{Z}_n is the distance from u to v in $\text{Cay}(\mathbb{Z}, \{\pm 1\})$ or $\text{Cay}(\mathbb{Z}_n, \{\pm 1\})$ if the vertex set is \mathbb{Z} or \mathbb{Z}_n , respectively.

Next we discuss some notation for walks and paths in infinite circulant graphs that we will use throughout the remainder of the article. The finite path with vertex set $\{v_1, v_2, \dots, v_t\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{t-1}, v_t\}\}$ is denoted $[v_1, v_2, \dots, v_t]$. For $a \in \mathbb{Z}$ and $z_1, z_2, \dots, z_t \in S$, we define $\Omega_a(z_1, z_2, \dots, z_t)$ to be the walk in $\text{Cay}(\mathbb{Z}, S)$ where the sequence of vertices is

$$a, \quad a + z_1, \quad a + z_1 + z_2, \quad \dots \quad a + \sum_{i=1}^t z_i,$$

so the lengths of the edges in the walk are $|z_1|, |z_2|, \dots, |z_t|$. Whenever we write $\Omega_a(z_1, z_2, \dots, z_t)$ it will be the case that $[a, a + z_1, \dots, a + \sum_{i=1}^t z_i]$ is a path, and we will use the notation $\Omega_a(z_1, z_2, \dots, z_t)$ interchangeably for both the walk (with associated orientation, start, and end vertices) and the path $[a, a + z_1, \dots, a + \sum_{i=1}^t z_i]$ (which is a graph with no inherent orientation).

2 | NECESSARY CONDITIONS AND INFINITE VALENCY

The following two lemmas give a characterization of connected infinite circulant graphs, and necessary conditions for an infinite circulant graph to be Hamilton-decomposable. We remark that the main idea in the proof of Lemma 2 has been used in [5,13].

Lemma 1. [27] *If S is an inverse-closed set of distinct nonzero integers and $\gcd(S) = d$, then $\text{Cay}(\mathbb{Z}, S)$ has d connected components that are each isomorphic to $\text{Cay}(\mathbb{Z}, \{\frac{a}{d} \mid a \in S\})$. In particular, $\text{Cay}(\mathbb{Z}, S)$ is connected if and only if $\gcd(S) = 1$.*

Lemma 2. *If $\text{Cay}(\mathbb{Z}, S)$ is Hamilton-decomposable, then*

- (i) $S = \emptyset$ or $\gcd(S) = 1$; and
- (ii) if S is finite, then $\sum_{a \in S^+} a \equiv |S^+| \pmod{2}$.

Proof. Suppose $\text{Cay}(\mathbb{Z}, S)$ has Hamilton decomposition \mathcal{D} . A Hamilton-decomposable graph is clearly either empty or connected, and so (i) follows immediately from Lemma 1. If S is finite, then let $k = |S^+|$ and let $E = \{\{u, v\} \in E(\text{Cay}(\mathbb{Z}, S)) \mid u \leq 0, v \geq 1\}$. For each $a \in S^+$, there are exactly a edges of length a in E , and so we have $|E| = \sum_{a \in S^+} a$. However, it is clear that each of the k Hamilton paths in \mathcal{D} has an odd number of edges from E . This means that $|E| \equiv k \pmod{2}$, and (ii) holds. ■

It may seem plausible to use a Hamilton decomposition of a finite circulant graph to construct a Hamilton decomposition of an infinite circulant graph whose edges have the same lengths as those of the finite graph. However, this is not possible in general.

For example, $\text{Cay}(\mathbb{Z}_n, \pm\{1, 2\})$ is Hamilton-decomposable for every $n \geq 5$, yet $\text{Cay}(\mathbb{Z}, \pm\{1, 2\})$ is not Hamilton-decomposable by Lemma 2.

Definition 3. An infinite circulant graph $\text{Cay}(\mathbb{Z}, S)$ is **admissible** if it satisfies (i) and (ii) from Lemma 2.

There are infinitely many connected infinite circulant graphs that are not admissible (and thus not Hamilton-decomposable). For example, if S is finite and $\text{Cay}(\mathbb{Z}, S)$ is admissible, then for every even positive integer $s \notin S^+$, $\text{Cay}(\mathbb{Z}, S \cup \pm\{s\})$ is not admissible. We have found no admissible infinite circulant graphs that are not Hamilton-decomposable, and thus we pose the following problem.

Problem 4. Is every admissible infinite circulant graph Hamilton-decomposable?

We now show that results from [25] and [21] combine to settle this problem for the case where S is infinite. An infinite graph G with infinite valency is ∞ -connected if $G \setminus U$ is connected for every finite subset $U \subset V(G)$ (that is, G has no finite cut-set). An infinite graph G with infinite valency has *infinite edge-connectivity* if $G \setminus A$ is connected for every finite subset $A \subset E(G)$ (that is, G has no finite edge-cut). It is easy to see that if G is ∞ -connected then G has infinite edge-connectivity, but the converse of this does not hold.

Theorem 5. [25] *Every vertex-transitive infinite graph G of infinite valency that has a Hamilton path is ∞ -connected.*

Theorem 6. [25] *A countably infinite graph of infinite valency has a Hamilton decomposition if and only if it has a Hamilton path and infinite edge-connectivity.*

Theorem 7. [21] *Every connected Cayley graph on a finitely-generated infinite abelian group has a Hamilton path.*

Theorem 8. *A Cayley graph of infinite valency on a finitely-generated infinite abelian group is Hamilton-decomposable if and only if it is connected.*

Proof. Let G be a Cayley graph of infinite valency on a finitely generated infinite abelian group. If G is Hamilton-decomposable, then clearly it is connected. For the converse, suppose G is connected. By Theorem 7, G has a Hamilton path. Since G is a Cayley graph, it is vertex-transitive. Thus, G is ∞ -connected by Theorem 5, and hence has infinite edge-connectivity. So G is Hamilton-decomposable by Theorem 6. ■

Since an infinite circulant graph with infinite connection set is admissible if and only if it is connected, Theorem 8 answers Problem 4 in the affirmative for the case of infinite connection sets. For the remainder of this article, we consider the case where the connection set is finite.

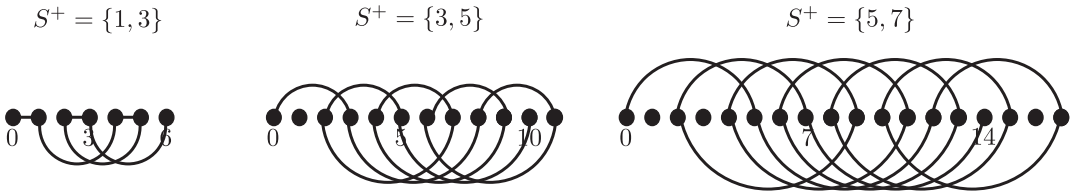


FIGURE 1 Example constructions from Theorem 9 when $b - a = 2$

3 | INFINITE 4-VALENT CIRCULANT GRAPHS

In this section, we prove that all admissible 4-valent infinite circulant graphs are Hamilton-decomposable, thus establishing the following theorem.

Theorem 9. *A 4-valent infinite circulant graph is Hamilton-decomposable if and only if it is admissible.*

Proof. Let $G = \text{Cay}(\mathbb{Z}, \pm\{a, b\})$ be a 4-valent admissible infinite circulant graph. Thus a and b are distinct nonzero integers, and we may assume that $1 \leq a < b$. Since G is admissible, $\text{gcd}(a, b) = 1$ and $a + b$ is even. Thus, a and b are both odd. Let $t = b - a$, and define m to be the integer in $\{0, 1, \dots, t - 1\}$ such that $m \equiv a \pmod{t}$. Note that t is even and $\text{gcd}(m, t) = 1$. For convenience, we use b and $a + t$ interchangeably.

For each $i \in \{0, 2, \dots, t - 2\}$, define α_i to be the integer in $\{0, 1, \dots, t - 1\}$ such that $\alpha_i \equiv im \pmod{t}$, and define $\alpha_t = t$. Let $F_v := \Omega_v(a, b)$ and $B_v := \Omega_v(a, -b)$. Note that F_v and B_v correspond to paths $[v, v + a, v + 2a + t]$ and $[v, v + a, v - t]$. Now define P as follows.

$$P := \left(\bigcup_{i \in \{0, 2, \dots, t - 2\}} F_{\alpha_i} \cup B_{2a + \alpha_i + t} \cup B_{2a + \alpha_i} \cup B_{2a + \alpha_i - t} \cup \dots \cup B_{t + \alpha_{(i+2)}} \right) \cup F_t,$$

see Figure 1. It is straightforward to check that P is a path with endpoints 0 and $2b$ having $2b$ edges. Since $\text{gcd}(a, b) = 1$ and the lengths of the edges of P alternate between a and b , it follows that P has exactly one vertex from each congruence class modulo $2b$ (except that the endpoints are both $0 \pmod{2b}$). Hence, $H_1 = \bigcup_{i \in \mathbb{Z}} (P + 2bi)$ is a Hamilton path in G .

The second Hamilton path is $H_2 = \bigcup_{i \in \mathbb{Z}} (Q + 2bi)$, where $Q = P + b$. To see that H_1 and H_2 are edge-disjoint, it suffices to show that $E(Q)$ is disjoint from both $E(P)$ and $E(P + 2b)$. Observe that the edge set of P is:

$$\begin{aligned} E(P) = & \{ \{x, x + a\} \mid x \text{ is even and } 0 \leq x \leq 2a + 2t - 2 \} \\ & \cup \{ \{x, x + b\} \mid (x \text{ is even and } 2 \leq x \leq 2a + t - 2) \text{ or} \\ & (x \text{ is odd and } a \leq x \leq a + t) \}. \end{aligned}$$

Since $E(Q) = \{ \{x + b, y + b\} \mid \{x, y\} \in E(P) \}$, where b is an odd integer, it follows that

$$\begin{aligned} E(Q) = & \{ \{x, x + a\} \mid x \text{ is odd and } a + t \leq x \leq 3a + 3t - 2 \} \\ & \cup \{ \{x, x + b\} \mid (x \text{ is odd and } a + t + 2 \leq x \leq 3a + 2t - 2) \text{ or} \\ & (x \text{ is even and } 2a + t \leq x \leq 2a + 2t) \}. \end{aligned}$$

Thus, $E(Q) \cap E(P) = \emptyset$. Similarly, $E(Q) \cap E(P + 2b) = \emptyset$ and hence H_1 and H_2 form a Hamilton decomposition of G . The result now follows by Lemma 2. ■

4 | INFINITE 2K-VALENT CIRCULANT GRAPHS

In this section, we consider $2k$ -valent infinite circulant graphs for $k \geq 3$, proving that there are many infinite families of such graphs that are Hamilton decomposable if and only if they are admissible.

We begin by considering graphs $\text{Cay}(\mathbb{Z}, S)$ where $|S^+| = k$ and $k \in S^+$ and no other element of S^+ is divisible by k . An example of such a graph is $\text{Cay}(\mathbb{Z}, \pm\{1, 5, 11, 12, 14\})$. The next lemma, Lemma 10, shows that in order to find a Hamilton decomposition of an admissible infinite circulant graph of this type, it suffices to find a Hamilton path in the complete graph with vertex set \mathbb{Z}_k with edges of appropriate lengths. In the following lemma and its proof, the notation $[x]$ denotes the congruence class of x modulo k .

Lemma 10. *Suppose $k \geq 3$ is odd and a_1, \dots, a_{k-1} are distinct positive integers that are not divisible by k such that $G = \text{Cay}(\mathbb{Z}, \pm\{a_1, \dots, a_{k-1}, k\})$ is an admissible infinite circulant graph. If there exists a Hamilton path Q in the complete graph with vertex set \mathbb{Z}_k where the multiset $\{\ell(a, b) \mid \{a, b\} \in E(Q)\}$ equals the multiset $\{\ell([0], [a_i]) \mid i = 1, 2, \dots, k - 1\}$, then G is Hamilton-decomposable.*

Proof. Let $Q = [v_0, v_1, \dots, v_{k-1}]$ be a Hamilton path in the complete graph with vertex set \mathbb{Z}_k such that the multiset $\{\ell(a, b) \mid \{a, b\} \in E(Q)\}$ equals the multiset $\{\ell([0], [a_i]) \mid i = 1, 2, \dots, k - 1\}$. Thus, there exist integers b_1, b_2, \dots, b_{k-1} such that $[b_i] = [v_i] - [v_{i-1}]$ for $i = 1, 2, \dots, k - 1$ and $\pm\{b_1, b_2, \dots, b_{k-1}\} = \pm\{a_1, a_2, \dots, a_{k-1}\}$. Define the path P_1 in G by $P_1 := \Omega_0(b_1, b_2, \dots, b_{k-1})$.

Let $\Sigma = \sum_{i=1}^k b_i$ and note that P_1 is a path in G with endpoints 0 and Σ , and that P_1 has exactly one vertex from each congruence class modulo k . Similarly, $P_1 + k$ is a path in G with endpoints k and $k + \Sigma$, $P_1 + k$ has exactly one vertex from each congruence class modulo k , and $P_1 + k$ is disjoint from P_1 .

Now define P to be the path in G with edge set

$$E(P_1) \cup E(P_1 + k) \cup \{\{\Sigma, \Sigma + k\}, \{k, 2k\}\}.$$

Thus, P is a path with endpoints 0 and $2k$ having $2k$ edges and having exactly one vertex from each congruence class modulo $2k$ (except that the the endpoints are both $0 \pmod{2k}$). Thus,

$$H_1 = \bigcup_{i \in \mathbb{Z}} (P + 2ki)$$

is a Hamilton path in G .

Recall that k is odd, so Σ is even by Lemma 2. The length k edges in P are $\{\Sigma, \Sigma + k\}$ and $\{k, 2k\}$, and the edges of length a_i in P are $\{x, x + a_i\}$ and $\{k + x, k + x + a_i\}$ for some integer x , for each $i = 1, \dots, k - 1$. Thus, for each $d \in \{a_1, a_2, \dots, a_{k-1}, k\}$, the set of length d edges in H_1 is $\{\{x, x + d\} \mid x \equiv x_1, x_2 \pmod{2k}\}$ where x_1 is odd and x_2 is even and hence the length d edges of each of $H_1, H_1 + 2, \dots, H_1 + 2k - 2$ are mutually disjoint. Thus $\{H_1, H_1 + 2, \dots, H_1 + 2k - 2\}$ is a Hamilton decomposition of G . ■

Buratti's Conjecture (see [14]) addresses the existence of paths with edges of specified lengths as required in Lemma 10. It states that if p is an odd prime and L is a multiset containing $p - 1$ elements from $\{1, 2, \dots, \frac{p-1}{2}\}$, then there exists a Hamilton path in the complete graph with vertex set \mathbb{Z}_p such that the lengths of the edges of the path comprise the multiset L . Buratti's Conjecture is open in general

and, as noted by Horak and Rosa [14], does not appear to be easy to solve. By Lemma 10, progress on Buratti's Conjecture can provide further constructions of Hamilton decompositions of admissible infinite circulant graphs. Since the conjecture has been verified for $p \leq 23$ by Mariusz Meszka (see [14] for example), we have the following result.

Theorem 11. *If p is an odd prime, where $p \leq 23$, and a_1, a_2, \dots, a_{p-1} are distinct positive integers, not divisible by p , then $\text{Cay}(\mathbb{Z}, \pm\{a_1, a_2, \dots, a_{p-1}, p\})$ is Hamilton-decomposable if and only if it is admissible.*

Several cases and generalizations of Buratti's Conjecture have been studied in [6,10,14]. If k is odd but not prime, it is not always possible to find a Hamilton path in K_k with specified edge lengths. For instance, there is no Hamilton path in K_9 where the multiset of edge lengths is one of the following: $\{1, 3, 3, 3, 3, 3, 3, 3\}$, $\{2, 3, 3, 3, 3, 3, 3, 3\}$, $\{3, 3, 3, 3, 3, 3, 3, 3\}$, and $\{3, 3, 3, 3, 3, 3, 3, 4\}$ [10].

We note that when $k \geq 3$ is odd $Q = [0, 1, k-1, 2, k-2, 3, k-3, \dots, \frac{k-1}{2}, \frac{k+1}{2}]$ is a Hamilton path in the complete graph with vertex set \mathbb{Z}_k and the lengths of the edges of Q comprise the multiset $\{1, 1, 2, 2, \dots, \frac{k-1}{2}, \frac{k-1}{2}\}$. Thus, Lemma 10 immediately gives us the following result.

Theorem 12. *If $k \geq 3$ is odd and a_1, \dots, a_{k-1} are distinct positive integers such that, for each $i = 1, 2, \dots, k-1$, $a_i \equiv i \pmod{k}$, then $\text{Cay}(\mathbb{Z}, \pm\{a_1, \dots, a_{k-1}, k\})$ is Hamilton-decomposable if and only if it is admissible.*

We now address Hamilton-decomposability of $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 3, \dots, k\})$.

Theorem 13. *If k is any positive integer, then $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 3, \dots, k\})$ is Hamilton-decomposable if and only if it is admissible.*

Proof. Let $G = \text{Cay}(\mathbb{Z}, \pm\{1, 2, \dots, k\})$. Observe that G is admissible if and only if $k \equiv 0, 1 \pmod{4}$. If $k = 1$, then the result is trivial, and if $k \geq 5$ with $k \equiv 1 \pmod{4}$, then G is Hamilton-decomposable by Theorem 12. Thus, we can assume $k \equiv 0 \pmod{4}$. If $k = 4$, then with $P = [0, -1, 1, 5, 2, 3, 6, 4, 8]$, $H_1 = \cup_{i \in \mathbb{Z}} (P + 8i)$ is a Hamilton path in G and $\{H_1, H_1 + 2, H_1 + 4, H_1 + 6\}$ is a Hamilton decomposition of G .

Now suppose $k \geq 8$ with $k \equiv 0 \pmod{4}$ and let $u = \frac{k}{2}$ and $v = \frac{3k}{2}$. Define a path P with $2k$ edges as follows.

$$\begin{aligned} P := & \Omega_0(-1, 2) \\ & \cup \Omega_1(k-2, -(k-3), k-4, -(k-5), \dots, 4, -3) \\ & \cup \Omega_{u-1}(k, -(k-1), 1, k-1, -2) \\ & \cup \Omega_{v-2}(3, -4, 5, -6, \dots, k-3, -(k-2)) \\ & \cup \Omega_k(k), \end{aligned}$$

so that

$$\begin{aligned} E(P) = & \{\{x, x+1\}, | x = -1, u\} \\ & \cup \{\{x, x+2\} | x = -1, v-2\} \\ & \cup \{\{x, x+d\} | x = u - \lfloor d/2 \rfloor, v - \lfloor d/2 \rfloor - 1 \text{ where } d = 3, \dots, k-2\} \\ & \cup \{\{x, x+k-1\} | x = u, u+1\} \end{aligned}$$

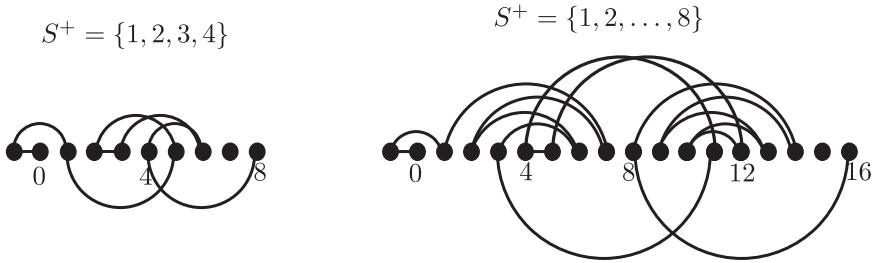


FIGURE 2 Construction of P from Theorem 13 when $k = 4$ and $k = 8$

$$\cup \{ \{x, x + k\}, | x = u - 1, k \}.$$

Figure 2 shows this construction for $k = 4$ and $k = 8$.

It is straightforward to check that $H_1 = \bigcup_{i \in \mathbb{Z}} (P + 2ki)$ is a Hamilton path in G . Observe that, for each $d \in \{1, 2, \dots, k\}$, the path P has two edges of the form $\{x, x + d\}$, one where x is even and one where x is odd (since u and v are even). Now it can be checked that $\{H_1, H_1 + 2, H_1 + 4, \dots, H_1 + 2k - 2\}$ is a Hamilton decomposition of G . ■

Although $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 3, \dots, k\})$ is not admissible when $k \equiv 2, 3 \pmod{4}$, the graph $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 3, \dots, k - 1, k + 1\})$ is admissible, and we next show that it is also Hamilton-decomposable.

Theorem 14. *If k is a positive integer, then $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 3, \dots, k - 1, k + 1\})$ is Hamilton-decomposable if and only if it is admissible.*

Proof 15. Let $G = \text{Cay}(\mathbb{Z}, \pm\{1, 2, \dots, k - 1, k + 1\})$ and observe that G is admissible if and only if $k \equiv 2, 3 \pmod{4}$. If $k = 2$, then the result follows by Theorem 9. If $k = 3$, define $P := [0, 1, -1, 3]$ (see Fig. 3) and let $H_1 = \bigcup_{i \in \mathbb{Z}} (P + 3i)$ so that $\{H_1, H_1 + 1, H_1 + 2\}$ is a Hamilton decomposition of G . Thus, we now assume $k \geq 6$.

Case 1 Suppose $k \equiv 2 \pmod{4}$ and $k \geq 6$. If $k = 6$, then with $P = [0, 2, -3, -2, -5, -1, 6]$, $H_1 = \bigcup_{i \in \mathbb{Z}} (P + 6i)$ is a Hamilton path in G and $\{H_1, H_1 + 1, H_1 + 2, H_1 + 3, H_1 + 4, H_1 + 5\}$ is a Hamilton decomposition of G . Now suppose $k \geq 10$ and let $u = \frac{k}{2}$. Define a path with k edges as follows:

$$\begin{aligned} P &:= \Omega_0(u - 1, -(k - 1)) \\ &\cup \Omega_{-u}(-2, 3, -4, 5, \dots, -(u - 3), u - 2) \\ &\cup \Omega_{-(u+3)/2}(1) \\ &\cup \Omega_{-(u+1)/2}(-u, u + 1, -(u + 2), u + 3 \dots, -(k - 3), k - 2) \end{aligned}$$

$$S^+ = \{1, 2, 4\}$$



FIGURE 3 Construction of P from Theorem 14 when $k = 3$

$$S^+ = \{1, 2, 3, 4, 5, 7\}$$

$$S^+ = \{1, 2, \dots, 7, 8, 9, 11\}$$

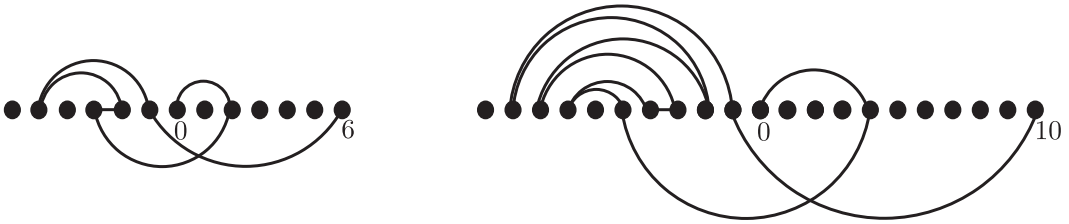


FIGURE 4 Construction of P from Theorem 14 when $k = 6$ and $k = 10$

$$S^+ = \{1, 2, 3, 4, 5, 6, 8\}$$

$$S^+ = \{1, 2, \dots, 9, 10, 12\}$$

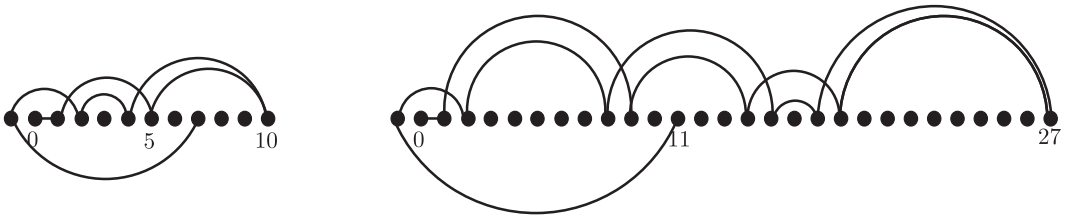


FIGURE 5 Construction of P from Theorem 14 when $k = 7$ and $k = 11$

$$\cup \Omega_{-1}(k + 1).$$

Figure 4 shows this construction for $k = 6$ and $k = 10$.

It is straightforward to check that $H_1 = \bigcup_{i \in \mathbb{Z}} (P + ki)$ is a Hamilton path in G and $\{H_1, H_1 + 1, H_1 + 2, \dots, H_1 + k - 1\}$ is a Hamilton decomposition of G .

Case 2 Suppose $k \equiv 3 \pmod{4}$ and $k \geq 7$. Let P be the path with k edges defined as follows:

$$\begin{aligned} P := & \Omega_0(1) \cup \Omega_1(k - 3, 5, k - 7, 9, k - 11, 13, \dots, 4, k - 2) \\ & \cup \Omega_v(-(k - 1)) \\ & \cup \Omega_{v-k+1}(-2, -(k - 4), -6, -(k - 8), \dots, -(k - 5), -3) \\ & \cup \Omega_{-1}(k + 1), \end{aligned}$$

where $v = (1 + 5 + 9 + \dots + k - 2) + (4 + 8 + 12 + \dots + k - 3) = \frac{k+1}{4}(k - 2)$. Figure 5 shows this construction for $k = 7$ and $k = 11$.

Note that, modulo k , the vertices of P are congruent to:

$$\begin{aligned} & 0, 1, k - 2, 3, k - 4, 5, k - 6, 7, \dots, \frac{k+3}{2}, \frac{k-1}{2}, \\ & \frac{k+1}{2}, \frac{k-3}{2}, \frac{k+5}{2}, \frac{k-7}{2}, \dots, k - 3, 2, k - 1, 0. \end{aligned}$$

Thus, P is a path with endpoints 0 and k having exactly one vertex from each congruence class modulo k (except that the endpoints are both $0 \pmod k$). Hence $H_1 = \bigcup_{i \in \mathbb{Z}} (P + ki)$ is a Hamilton path in G . For each $d \in \{1, 2, \dots, k - 1, k + 1\}$ the path P uses exactly one edge of the form $\{x, x + d\}$. Thus $\{H_1, H_1 + 1, H_1 + 2, \dots, H_1 + k - 1\}$ is a Hamilton decomposition of G . ■

Next, we consider graphs $\text{Cay}(\mathbb{Z}, S)$ where S^+ consists of consecutive even integers together with 1.

Theorem 15. *If t is a positive integer, then $\text{Cay}(\mathbb{Z}, \pm\{1, 2, 4, 6, 8, \dots, 2t\})$ is Hamilton-decomposable if and only if it is admissible.*

Proof. Suppose $G = \text{Cay}(\mathbb{Z}, \pm\{1, 2, 4, 6, 8, \dots, 2t\})$ is admissible and let $k = |S^+| = t + 1$. Since $1 + 2 + 4 + \dots + 2t$ is odd, it follows that k is odd and hence t is even. If $t = 2$, then the result follows by Theorem 14. Thus, we can assume $t \geq 4$.

Observe that the elements of S^+ , namely $1, 2, 4, \dots, t - 2, t, t + 2, t + 4, \dots, 2t$, are congruent to $1, 2, 4, \dots, t - 2, t, -t, -(t - 2), \dots, -4, -2$ modulo k , respectively. For each odd $k \geq 5$ we define a path P with k edges as follows.

$$P := \begin{cases} [0, 1, 3, 7, -1, 5] & \text{if } k = 5 \\ [0, 1, 5, 3, -3, 9, -1, 7] & \text{if } k = 7 \\ [0, 1, 7, 3, 5, 13, -3, 11, -1, 9] & \text{if } k = 9 \\ [0, 1, 9, 3, 7, 5, -5, 15, -3, 13, -1, 11] & \text{if } k = 11. \end{cases}$$

If $k \geq 13$ and $k \equiv 1 \pmod 4$ then

$$\begin{aligned} P := & \Omega_0(1) \\ & \cup \Omega_1(t - 2, -(t - 4), t - 6, -(t - 8), \dots, 6, -4) \\ & \cup \Omega_{(t-2)/2}(2, t) \\ & \cup \Omega_{(3t+2)/2}(-2t, 2t - 2, -(2t - 4), 2t - 6, \dots, -(t + 4), t + 2); \end{aligned}$$

and if $k \geq 15$ and $k \equiv 3 \pmod 4$ then

$$\begin{aligned} P := & \Omega_0(1) \\ & \cup \Omega_1(t - 2, -(t - 4), t - 6, -(t - 8), \dots, 4, -2) \\ & \cup \Omega_{t/2}(-t, 2t) \\ & \cup \Omega_{3t/2}(-(2t - 2), 2t - 4, -(2t - 6), 2t - 8, \dots, -(t + 4), t + 2). \end{aligned}$$

Let $u = \frac{k-1}{2}$ and note that, modulo k , the vertices of P are congruent to

$$\begin{aligned} & 0, 1, k - 2, 3, \dots, u - 1, u + 1, u, u + 2, u - 2, \dots, 2, k - 1, 0 \quad \text{if } k \equiv 1 \pmod 4 \\ & 0, 1, k - 2, 3, \dots, u + 2, u, u + 1, u - 1, u + 3, \dots, 2, k - 1, 0 \quad \text{if } k \equiv 3 \pmod 4. \end{aligned}$$

In either case, P is a path with endpoints 0 and k having exactly one vertex from each congruence class modulo k (except that the endpoints are both $0 \pmod k$). Thus $H_1 = \bigcup_{i \in \mathbb{Z}} (P + ki)$ is a Hamil-

ton path in G . It is straightforward to check that $\{H_1, H_1 + 1, H_1 + 2 \dots, H_1 + k - 1\}$ is a Hamilton decomposition of G . ■

We conclude this section on $2k$ -valent infinite circulant graphs with a discussion of the case $k = 3$. By Theorem 11, if a and b are distinct positive integers, not divisible by 3, then $G = \text{Cay}(\mathbb{Z}, \pm\{3, a, b\})$ is Hamilton-decomposable if and only if G is admissible. Note that graphs of the form $\text{Cay}(\mathbb{Z}, \pm\{a, 3t, 3\})$, where $a \not\equiv 0 \pmod{3}$ are admissible but not covered by Theorem 11, however we have verified by computer that several small admissible cases of this form are Hamilton-decomposable.

A straightforward corollary of Theorem 9 is that if $a, b \in \mathbb{Z}^+$ are odd and relatively prime then $\text{Cay}(\mathbb{Z}, \pm\{1, a, b\})$ is Hamilton-decomposable. It remains an open problem to determine whether $\text{Cay}(\mathbb{Z}, \pm\{1, a, b\})$ is Hamilton-decomposable when a, b are both even and when a, b are both odd but not relatively prime. The next result answers this question when $a = 2$.

Theorem 16. *If $c \geq 3$ is an integer, then $\text{Cay}(\mathbb{Z}, \pm\{1, 2, c\})$ is Hamilton-decomposable if and only if it is admissible.*

Proof 18. Suppose $G = \text{Cay}(\mathbb{Z}, \pm\{1, 2, c\})$ is admissible. Then c is even and we may assume that $c = 2t$ for some integer $t \geq 2$. If $t = 2$, then G is Hamilton-decomposable by Theorem 14. Thus, we can assume that $t \geq 3$.

We divide the proof into three cases, t odd, $t \equiv 0 \pmod{4}$, and $t \equiv 2 \pmod{4}$ and provide a construction for a Hamilton decomposition in each case. The constructions are based on a starter path P that has $3t$ edges, using t edges of each length 1, 2, and $2t$. In each case it can be checked that $H_1 = \bigcup_{i \in \mathbb{Z}} (P + 3ti)$ is a Hamilton path and that $\{H_1, H_1 + t, H_1 + 2t\}$ is a Hamilton decomposition of G .

In the cases below, we use the following notation:

$$\begin{aligned} A_v &:= \Omega_v(1, 2t, 1, -2t) \\ B_v &:= \Omega_v(2t, -2, -2t, -2) \\ C_v &:= \Omega_v(2t, 2, -2t, 2). \end{aligned}$$

Case 1: t odd

Define

$$P := \bigcup_{i=0}^{(t-3)/2} A_{2i} \cup [t - 1, t + 1, t + 3, \dots, 2t - 2, 2t, 2t - 1, 2t - 3, 2t - 5, \dots, t + 2, t, 3t].$$

Thus, P is of the form

$$\bigcup_{i=0}^{(t-3)/2} A_{2i} \cup \Omega_{t-1}(2, 2, \dots, 2) \cup \Omega_{2t}(-1, -2, -2, -2, \dots, -2) \cup \Omega_t(2t),$$

and

$$\begin{aligned} E(P) = & \{ \{x, x + 1\} \mid x \in \{0, 2, \dots, t - 3, 2t - 1, 2t + 1, 2t + 3, \dots, 3t - 2\} \} \cup \\ & \{ \{x, x + 2\} \mid x \in \{t - 1, t, t + 1, \dots, 2t - 2\} \} \cup \end{aligned}$$

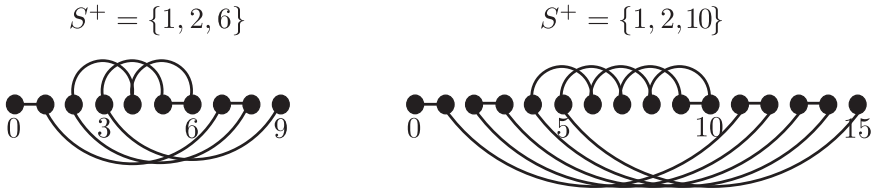


FIGURE 6 Construction of P from Theorem 16 when $t = 3$ and $t = 5$

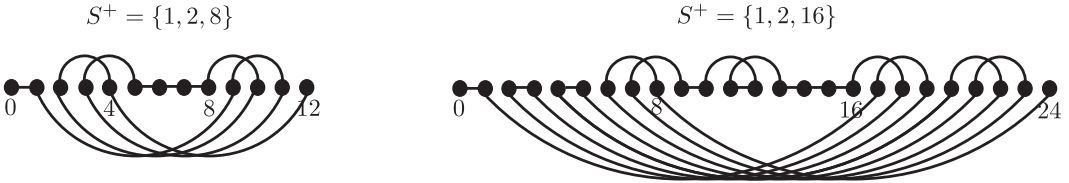


FIGURE 7 Construction of P from Theorem 16 when $t = 4$ and $t = 8$

$$\{\{x, x + 2t\} \mid x \in \{1, 2, \dots, t\}\}.$$

Figure 6 shows this construction for $t = 3$ and $t = 5$.

Case 2: $t \equiv 0 \pmod{4}$

If $t = 4$, define $P := [0, 1, 9, 11, 3, 5, 6, 7, 8, 10, 2, 4, 12]$. For $t \geq 8$, we define

$$P := [0, 1, 2t + 1, 2t + 3, 3, 2, 2t + 2, 2t, 2t - 1, 2t - 2, \dots, t + 5, t + 3, t + 4, t + 2, t + 1, t - 1]$$

$$\bigcup_{i=0}^{(t-12)/4} B_{t-1-4i} \cup [7, 2t + 7, 2t + 5, 5, 4] \bigcup_{i=1}^{(t-4)/4} C_{4i} \cup [t, 3t].$$

We remark that $\bigcup_{i=0}^{(t-12)/4} B_{t-1-4i}$ is empty in the case $t = 8$ as the upper index is negative. Thus, P is of the form

$$\Omega_0(1, 2t, 2, -2t, -1, 2t, -2) \cup \Omega_{2t}(-1, -1, \dots, -1) \cup \Omega_{t+5}(-2, 1, -2, -1, -2)$$

$$\bigcup_{i=0}^{(t-12)/4} B_{t-1-4i} \cup \Omega_7(2t, -2, -2t, -1) \bigcup_{i=1}^{(t-4)/4} C_{4i} \cup \Omega_t(2t),$$

and

$$\begin{aligned} E(P) = & \{\{x, x + 1\} \mid x \in \{0, 2, 4, t + 1, t + 3\} \text{ or } t + 5 \leq x \leq 2t - 1\} \\ & \cup \{\{x, x + 2\} \mid 6 \leq x \leq t + 3 \text{ where } x \equiv 2, 3 \pmod{4} \text{ or} \\ & 2t \leq x \leq 3t - 3 \text{ where } x \equiv 0, 1 \pmod{4}\} \\ & \cup \{\{x, x + 2t\} \mid 1 \leq x \leq t\}. \end{aligned}$$

Figure 7 shows the construction for $t = 4$ and $t = 8$.

Case 3: $t \equiv 2 \pmod{4}$

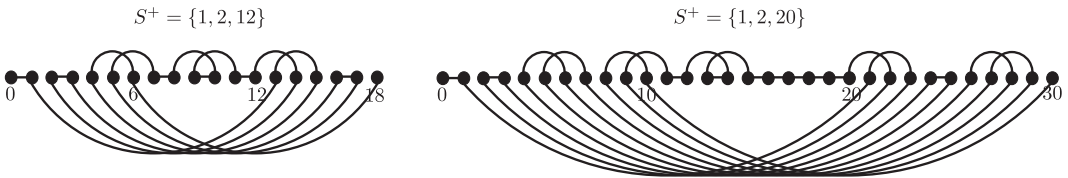


FIGURE 8 Construction of P from Theorem 16 when $t = 6$ and $t = 10$

We define

$$P := [0, 1, 2t + 1, 2t + 3, 3, 2, 2t + 2, 2t, 2t - 1, 2t - 2, \dots, t + 5, t + 3, t + 4, t + 2, t + 1, t - 1]$$

$$\bigcup_{i=0}^{(t-10)/4} B_{t-1-4i} \cup [5, 2t + 5, 2t + 4, 4, 6] \bigcup_{i=1}^{(t-6)/4} C_{4i+2} \cup [t, 3t].$$

We remark that $\bigcup_{i=0}^{(t-10)/4} B_{t-1-4i}$ is empty in the case $t = 6$ as the upper index is negative. Thus, P is of the form

$$\Omega_0(1, 2t, 2, -2t, -1, 2t, -2) \cup \Omega_{2t}(-1, -1, \dots, -1) \cup \Omega_{t+5}(-2, 1, -2, -1, -2) \\ \bigcup_{i=0}^{(t-10)/4} B_{t-1-4i} \cup \Omega_5(2t, -1, -2t, 2) \bigcup_{i=1}^{(t-6)/4} C_{4i+2} \cup \Omega_t(2t),$$

and

$$E(P) = \{ \{x, x + 1\} \mid x \in \{0, 2, t + 1, t + 3, 2t + 4\} \text{ or } t + 5 \leq x \leq 2t - 1 \} \\ \cup \{ \{x, x + 2\} \mid x \in \{2t, 2t + 1\} \text{ or } \\ 4 \leq x \leq t + 3 \text{ where } x \equiv 0, 1 \pmod{4} \text{ or } \\ 2t + 6 \leq x \leq 3t - 3 \text{ where } x \equiv 2, 3 \pmod{4} \} \\ \cup \{ \{x, x + 2t\} \mid 1 \leq x \leq t \}.$$

Figure 8 shows this construction for $t = 6$ and $t = 10$. ■

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