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Citation

Conder, Marston and Širáň, Jozef (2016). Orientably regular maps with given exponent group. *Bulletin of the London Mathematical Society*, 48(6) pp. 1013–1017.

URL

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Orientably regular maps with given exponent group

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ABSTRACT

We prove that for every integer $d \geq 3$ and every group U of units mod d , there exists an orientably regular map of valency d with exponent group U .

1. Introduction

An *orientably regular map* M is a 2-cell embedding of a connected graph in an orientable surface, such that the group of all orientation-preserving automorphisms Aut^+M of the embedding acts as regularly (sharply transitively) on the set of arcs of the graph. It follows that every vertex of M has the same valency, say d , and every face of M is bounded by a closed walk of the same length, say m .

If e is an arc at any vertex v of M , then regularity implies that Aut^+M contains an involution x acting like a 180-degree rotation of M about the centre of e , and an element y of order d acting like a d -fold rotation of M about v . Then by connectivity, the group Aut^+M is generated by x and y , and admits a presentation of the form $\text{Aut}^+M = \langle x, y \mid x^2 = y^d = (xy)^m = \dots = 1 \rangle$. The pair (d, m) is called the *type* of the map. Conversely, given any generating pair (x, y) for a group G with the above form, one may construct an orientably regular map M with $\text{Aut}^+M = G$ by taking edges, vertices and faces of M as the (right) cosets in G of the subgroups $\langle x \rangle$, $\langle y \rangle$ and $\langle xy \rangle$, respectively, and with incidence given by non-empty intersection of cosets. (Also the arcs may be taken as the elements of G .) Thus, orientably regular maps of valency d and face length m may be identified with 2-generator group presentations of the form $\langle x, y \mid x^2 = y^d = (xy)^m = \dots = 1 \rangle$.

Fundamentals of the theory of maps and orientably regular maps are explained in [8], some deep connections between such maps, Riemann surfaces and Galois groups are described in detail in [9], and a recent survey containing a large number of facts about regular maps is given in [11].

Next, let M and $G = \text{Aut}^+M = \langle x, y \rangle$ be as above. An integer j relatively prime to d is said to be an *exponent* of M if the assignment $(x, y) \mapsto (x, y^j)$ extends to an automorphism of G . Algebraically, this means that (x, y) and (x, y^j) satisfy the relations as each other, while from the point of view of maps, it means that if a new map M^j is constructed from M by replacing the clockwise local cyclic order π_v of arcs at each vertex v by π_v^j , then resulting map M^j is isomorphic to M . Orientably regular maps admitting the exponent -1 are isomorphic to their mirror image, and are therefore called *reflexible*.

The collection of all exponents of M forms a subgroup of the group of units \mathbb{Z}_d^* , and is called the *exponent group* of M . The notion of an exponent was introduced in [10], with applications in the classification of orientably regular maps with a given underlying graph. Previously, the

Received 25 February 2016; revised 7 July 2016.

2010 *Mathematics Subject Classification* 57M60 (primary), 05E18, 20B25, 20F05, 57M15 (secondary).

The first author was supported by New Zealand's Marsden Fund (grant no. UOA1323), and acknowledges the use of the MAGMA system [3] to discover and test the properties of various examples. The second author acknowledges support from the APVV Research Grants 0136/12 and 015-0220, and the VEGA Research Grants 1/0007/14 and 1/0026/16.

mapping $M \rightarrow M^j$ (even in the case when the two maps may not be isomorphic) was known as a *hole operator*, and studied by Wilson [14], but this mapping has also been attributed to Coxeter.

For the exponents of an orientably regular map of given valency d , there are two ‘extremes’: one where the exponent group is trivial, or consists only of 1 and -1 , and the other where the map admits the ‘full’ exponent group Z_d^* .

In [2], it was shown that for every $d \geq 3$ there are infinitely many finite orientably regular maps of valency d with trivial exponent group. This was done with the help of a method that allows one to forbid the creation of new automorphisms in lifted maps, but unfortunately the method offers no control over the face length. Also it was proved in [12] using residual finiteness of triangle groups that for every pair of positive integers d and m with $1/d + 1/m \leq 1/2$, there exist infinitely many finite orientably regular and reflexible maps of type (d, m) that admit no exponents other than 1 and -1 .

At the other end of the spectrum, it was shown in [13] that for every integer $d \geq 3$ there exist infinitely many finite orientably regular maps with exponent group Z_d^* . Again this was achieved using residual finiteness of triangle groups, but this time losing control over the face length of resulting maps. Such maps were called ‘kaleidoscopic’ in [1], where a covering construction was given for a kaleidoscopic d -valent regular map invariant also under duality and Petrie duality, for every even d . A different construction for such ‘super-symmetric’ d -valent maps was given for an infinite set of odd values of d in [6].

In this paper, we deal with the ‘intermediate’ cases, by considering arbitrary subgroups of the group of units modulo the valency d . We prove that for every $d \geq 3$ and every given subgroup U of Z_d^* , there exist infinitely many finite orientably regular maps of valency d with exponent group equal to (and not just isomorphic to) U .

2. The main result

THEOREM 1. *For every $d \geq 3$ and every subgroup U of Z_d^* , there are infinitely many finite orientably regular maps of degree d with exponent group equal to U .*

Proof. Let G be the free product $Z_2 * Z_d$ of the cyclic groups of order 2 and $d \geq 3$, with presentation $\langle X, Y \mid X^2 = Y^d = 1 \rangle$, and let $D = G'$ be the derived subgroup of G , of index $2d$ in G , with quotient $G/D \cong Z_2 \times Z_d$. By Reidemeister–Schreier theory [5], the group D is free of rank $d - 1$, generated by the commutators $W_j = [X, Y^j]$ for $j \in \{1, 2, \dots, d - 1\}$.

We will construct for any given subgroup U of Z_d^* an infinite family of quotients of G that give rise to orientably regular maps of degree d with exponent group U .

For any prime p , let $N_p = D'D^{(p)}$ be the subgroup of D generated by the commutators and p th powers of all elements of D . This subgroup is characteristic in D and hence normal in G , and the quotient D/N_p is isomorphic to the direct product Z_p^{d-1} of $d - 1$ copies of Z_p . Also G/N_p is an extension of $D/N_p \cong Z_p^{d-1}$ by $(G/N_p)/(D/N_p) \cong G/D \cong Z_2 \times Z_d$, and hence G/N_p has order $2dp^{d-1}$.

Next, for any $u \in Z_d^*$, let k_u be the automorphism of G that takes the generating pair (X, Y) to the generating pair (X, Y^u) . Note that this permutes the generators $W_j = [X, Y^j]$ of D among themselves, and therefore preserves D , and its characteristic subgroup N_p , and so induces an automorphism h_u of $G_p = G/N_p$, with $(Ng)^{h_u} = N(g^{k_u})$ for all $g \in G$.

Now, let U be any subgroup of Z_d^* . Then, $K_U = \{k_u : u \in U\}$ and $H_U = \{h_u : u \in U\}$ are groups of automorphisms of G and G_p (respectively), both isomorphic to U .

We will show that if the prime p is congruent to 1 mod d , then there exists a normal subgroup L_U of $G_p = G/N_p$ contained in D/N_p such that L_U is preserved by H_U , and furthermore, that L_U can be chosen so that it is not preserved by h_r for any $r \in Z_d^* \setminus U$. Under these

97 circumstances, the quotient G_p/L_U determines a finite orientably regular map M of valency d
 98 with exponent group containing U , and then finally, we will show that the exponent group of
 99 M is equal to U . We break this up into three steps below.

100 *Step 1.* Let x and y be the images of X and Y under the natural quotient homomorphism
 101 from G to $G/N_p = G_p$, and let $w_j = [x, y^j] = xy^{-j}xy^j$, which is the image of $W_j = [X, Y^j]$,
 102 for $j \in \{1, 2, \dots, d-1\}$. Then, these w_j are elements of the elementary abelian p -group $V_p =$
 103 $D/N_p \cong Z_p^{d-1}$, and so commute with each other. Moreover, it is easy to see that $xw_jx =$
 104 $y^{-j}xy^jx = w_j^{-1}$ and $y^{-1}w_jy = y^{-1}xy^{-j}xy^{j+1} = y^{-1}xyxy^{-(j+1)}xy^{j+1} = w_1^{-1}w_{j+1}$, for all $j \in$
 105 $\{1, 2, \dots, d-1\}$, if we define also $w_d = [x, y^d] = 1$.

106 Next, suppose $p \equiv 1 \pmod d$, and let t be any non-trivial d th root of 1 mod p , so that
 107 $1 + t + t^2 + \dots + t^{d-1} \equiv 0 \pmod p$. Define

$$108 \quad v_t = w_1^t w_2^{t^2} \cdots w_{d-2}^{t^{d-2}} w_{d-1}^{t^{d-1}},$$

109 which is an element of the abelian p -group $V_p = D/N$. Conjugation by x inverts v_t , while

$$\begin{aligned} 110 \quad y^{-1}v_t y &= (y^{-1}w_1 y)^t (y^{-1}w_2 y)^{t^2} \cdots (y^{-1}w_{d-2} y)^{t^{d-2}} (y^{-1}w_{d-1} y)^{t^{d-1}} \\ 111 &= (w_1^{-1}w_2)^t (w_1^{-1}w_3)^{t^2} \cdots (w_1^{-1}w_{d-1})^{t^{d-2}} (w_1^{-1})^{t^{d-1}} \\ 112 &= w_1^{-(t+t^2+\dots+t^{d-2}+t^{d-1})} w_2^t w_3^{t^2} \cdots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ 113 &= w_1 w_2^t w_3^{t^2} \cdots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ 114 &= (v_t)^{t^{-1}}. \end{aligned}$$

115 It follows that the cyclic subgroup L_t of $V_p = D/N_p$ generated by v_t is normal in G_p .

116 Now, take $L_U = \langle L_t^{h_u} : u \in U \rangle$. Since L_t is a normal subgroup of G_p , the image $L_t^{h_u}$ of L_t
 117 under each automorphism h_u is also a normal subgroup of G_p , and hence L_U is normal in G_p .
 118 Moreover, L_U is clearly preserved by H_U , as required.

119 *Step 2.* Suppose further that t is a primitive d th root of 1 mod p , and for each $j \in Z_d^*$, define
 120 the element $v_t^{(j)}$ of V_p by

$$121 \quad v_t^{(j)} = h_{j-1}(v_t) = \prod_{i \in Z_d^*} h_{j-1}(w_i^{t^i}) = \prod_{i \in Z_d^*} (w_{j-1-i})^{t^i} = \prod_{\ell \in Z_d^*} w_\ell^{(t^j)^\ell}.$$

122 We claim that these $\phi(d) = |Z_d^*|$ elements $v_t^{(j)}$ generate a subgroup of order $p^{\phi(d)}$ in V_p , or
 123 equivalently, that they are linearly independent over Z_p when V_p is considered as a vector space
 124 over Z_p of dimension $d-1$. To see this, if we take the set $\{w_1, w_2, \dots, w_{d-1}\}$ as a basis for
 125 V_p , and write any element $w_1^{a_1} w_2^{a_2} \cdots w_{d-1}^{a_{d-1}}$ of V_p as a $(d-1)$ -tuple $(a_1, a_2, \dots, a_{d-1})$, then
 126 by its definition above, $v_t^{(j)}$ can be written as the $(d-1)$ -tuple $(t^j, t^{2j}, \dots, t^{(d-1)j})$. Hence,
 127 the set $\{v_t^{(j)} : j \in Z_d^*\}$ can be represented by a $\phi(d) \times (d-1)$ sub-matrix of the Vandermonde
 128 matrix

$$129 \quad \begin{pmatrix} t & t^2 & t^3 & \cdots & t^{d-1} \\ t^2 & t^4 & t^6 & \cdots & t^{2(d-1)} \\ t^3 & t^6 & t^9 & \cdots & t^{3(d-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{d-1} & t^{2(d-1)} & t^{3(d-1)} & \cdots & t^{(d-1)(d-1)} \end{pmatrix}.$$

130 This matrix has determinant $\prod_{1 \leq i < j \leq d-1} (t^j - t^i)$, which is non-zero in Z_p since t is a primitive
 131 d th root of 1 mod p , and it follows that for any subset S of Z_d^* , the rows with first entry t^j with
 132 $j \in S$ are linearly independent over Z_p . In particular, taking $S = Z_d^*$, we see the above claim is
 133 true.
 134

145 But also this shows that $h_r(L_U) \neq L_U$ for any $r \in \mathbb{Z}_d^* \setminus U$, because if $h_r(L_U) = L_U$, then
 146 $L_U = h_{r^{-1}}(L_U)$ and so the vector corresponding to $v_t^{(r)} = h_{r^{-1}}(v_t)$ is a linear combination of
 147 the vectors corresponding to the elements $v_t^{(u)}$ for $u \in U$, which is impossible.

148 *Step 3.* It remains to show that the exponent group of the orientable-regular map M arising
 149 from the quotient G_p/L_U of G is equal to U . By Step 1, we know that this exponent group
 150 contains U . To prove the reverse inclusion, suppose that j is any exponent of this map M .
 151 Then also j^{-1} is an exponent of M , and hence there exists an automorphism θ of G_p/L_U that
 152 fixes the element xL_U and takes yL_U to $y^{j^{-1}}L_U$. But now $v_t \in L_t \subseteq L_U$, so the coset v_tL_U
 153 is trivial in G_p/L_U , and it follows that the coset containing $v_t^{(j)} = h_{j^{-1}}(v_t)$ is trivial as well.
 154 Thus $v_t^{(j)}$ lies in L_U , and by Step 2, we deduce that $j \in U$.

155 This completes the proof.

156 3. Concluding remarks

159 The method we have used does not enable control over the face length of the resulting maps.
 160 This is no accident, as it is *not* true that there exist orientably regular maps of given type
 161 (d, m) with $1/d + 1/m \leq \frac{1}{2}$ and having a given exponent group. For example, in the case of
 162 triangulations (with $m = 3$), it was shown in [13] that an orientably regular map of type $(d, 3)$
 163 with valency $d \equiv \pm 1 \pmod{6}$ cannot have more than $\phi(d)/2$ exponents, and that if d is a prime
 164 such that $d \equiv -1 \pmod{8}$ and $(d-1)/2$ is also prime, then such a triangulation cannot have
 165 exponents other than ± 1 .

166 Finally, for completeness, we mention some interesting connections with the case where the
 167 exponent group U does not contain -1 . Orientably regular maps with this property are known
 168 as *chiral*. In [4], it was shown by a direct permutation construction that for every pair (d, m)
 169 such that $1/d + 1/m \leq \frac{1}{2}$, there exist infinitely many finite orientably regular but chiral maps
 170 of type (d, m) . The same thing was proved in [7] by a different method, with the help of
 171 holomorphic differentials.

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