Level-spacing distributions of the Gaussian unitary random matrix ensemble

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Key words Random Matrix Theory, Gaussian Unitary Ensemble, Level Spacing Distribution, Series Expansion

PACS 05.45.-a, 05.45.Mt, 05.40.-a, 74.40.+k

Level-spacing distributions of the Gaussian Unitary Ensemble (GUE) of random matrix theory are expressed in terms of solutions of coupled differential equations. Series solutions up to order 50 in the level spacing are obtained, thus providing a very good description of the small-spacing part of the level-spacing distribution, which can be used to make comparisons with experimental or numerical data. The level-spacing distributions can be obtained by solving the system of differential equations numerically.

1 Introduction

Ever since the pioneering works of Wigner and Dyson [1, 2, 3, 4, 5, 6], random matrix theory has been a major tool in the investigation of complex systems in physics, see *e.g.* Refs. [7, 8]. In particular, the three "classical" random matrix ensembles, the Gaussian Orthogonal Ensemble (GOE), the Gaussian Unitary Ensemble (GUE) and the Gaussian Sympletic Ensemble (GSE) have been studied in much detail [9] and have proven of relevance in various applications, not alone in physics, see *e.g.* [10] for examples.

One of the characteristic features that is frequently used in investigations is the distribution of differences between eigenvalues. These level-spacing distributions are universal quantities, in the sense that they correctly describe the level-spacing distributions in many complex systems. In many applications, experimentally observed or numerically calculated discrete data, such as energy levels of large nuclei, are analyzed and their spacing distribution is compared to that of the appropriate random matrix ensemble, where the choice of the appropriate ensemble is dictated by the symmetry of the system. Besides the spacing between neighboring levels, also the spacing between levels separated by a fixed number of levels, *i.e.*, next-nearest neighbors, next-next-nearest neighbors, *etc.*, has been considered. A recent example of an astonishingly precise agreement between energy levels of a Hamiltonian and random matrix distributions was found in a simple tight-binding model of an electron moving on a planar quasiperiodic graph [11, 12, 13, 14], which is used to model electrons in quasicrystals [15, 16]. But the distribution functions of random matrix theory are even of relevance in pure mathematics; the paradigm of an application in number theory is the distribution of zeros of Riemann's zeta function on the critical line [17].

However, there exists no simple closed form for the level-spacing distributions of the above-mentioned ensembles of random matrix theory. Thus, in many cases, the empirical data are actually compared to a so-called Wigner surmise, which corresponds to the spacing distribution of neighboring levels obtained from two-by-two random matrices, rather than from the limiting distribution for matrices of infinite size. For example, in the GOE case, the difference between the Wigner surmise and the true GOE spacing distribution is rather small, but the example of the quasiperiodic tight-binding model [11, 12, 13, 14] shows that it may indeed be detected in physical applications. For the GUE case, the situation is somewhat better as the complete small spacing expansion for the nearest neighbor spacing distribution is known [7].

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Numerical values, series expansions and a Padé analysis for the nearest-neighbor spacing distributions can be found in [9, 7, 18]. However, level spacings beyond the nearest-neighbor distributions, the spacing distributions of levels separated by a number of other levels, have scarcely been considered. However, explicit power series expressions are available [19], although the coefficients in the series still involve determinants that need to be evaluated.

Here we employ the approach of Ref. [24], which is based on the relation to differential equations of Painlevé type discovered by Jimbo *at al.* [25], to obtain series expansions or numerical estimates of these functions, compare also [20, 21, 22, 23] for connections between random matrices and Painlevé transcendents. In this paper, we generalize this approach to calculate series expansions for small spacings up to order 50 for the higher level-spacing distributions of the GUE. The expansions provide a detailed account of the small-spacing behavior of these distributions, and thus can be applied to analyze data. The exact coefficients involve ratios of large integer numbers, which makes them somewhat difficult to present in a paper. Therefore, the analytic expressions of the expansion coefficients are not included; they are provided in form of MATHEMATICA [26] code on the author's homepage [27]. The other two classical ensembles, the GOE and the GSE, can be treated in a similar fashion, although the calculation is somewhat more involved [24]. The results for these two universality classes will be published separately.

Why are we interested in calculating these expansion to such high order in s? There are several reasons. First of all, the data suggest that the series have infinite radius of convergence, thus the level-spacing distributions appear to be analytic functions of s. This means that, the more coefficients we know, the larger the interval where the truncated series gives a good approximation of the spacing functions. Secondly, the higher level spacing distributions have leading terms of increasing power in s, so in order to at least obtain the leading terms of the higher spacing function we have to expand all function during the calculation up to that given order in s. In our case, expanding up to order s^{50} just suffices to gives us the leading terms of the spacing distribution of levels separated by five other levels. Thirdly, although it is, in principle, possible to integrate the system of coupled differential equations numerically and to obtain "numerically exact" results for the spacing functions, this does require the knowledge of at least the leading terms in the expansions as we cannot integrate directly from $s = 0$. Finally, it turns out to be very difficult to obtain a numerically exact distribution function in this way, because the distribution functions arise as differences between functions that grow exponentially with s, and thus requires high precision numerics. In fact, the author was not able to integrate numerically the set of equations for the spacing functions with four or five intermediate levels, even though the algebraic computer package MATHEMATICA [26] in principle allows for arbitrary precision arithmetics.

The paper is organized as follows. After this introduction, the method used to calculate the spacing distribution is briefly summarized, following closely the discussion of Ref. [24]. After that, the series expansion data are presented and their applicability is discussed. We end with some concluding remarks.

2 Level-spacing distributions

The basic objects that usually are considered are the probabilities $E_n(s)$ that an interval of length s contains exactly n eigenvalues. Here, s denotes the energy-level spacing in units of the mean level spacing. The gap probability $E_0(s)$ that an interval of length s contains no eigenvalue at all can be expressed as a Fredholm determinant of a certain integral operator K with a sine kernel, see [10] and references therein. Jimbo *et al.* [25] showed that this Fredholm determinant can be expressed as

$$D(s;\lambda) \;:=\; \det\left(I-\lambda K\right) \;=\; \exp\left(\int_0^{\pi s} dx\, \frac{\sigma(x;\lambda)}{x}\right) \tag{1}$$

where the function $\sigma(x;\lambda)$, considered as a function of the variable x whereas λ is regarded as a parameter, is a solution of a Painlevé V differential equation

$$\left(x\,\frac{d^2\sigma}{dx^2}\right)^2 \;+\; 4\Big(x\,\frac{d\sigma}{dx}-\sigma\Big)\left[x\,\frac{d\sigma}{dx}-\sigma+\Big(\frac{d\sigma}{dx}\Big)^2\right] \;=\; 0. \tag{2}$$

The parameter λ is introduced via the boundary condition

$$\sigma(x;\lambda) \;=\; -\frac{\lambda}{\pi}\,x \;+\; o(x) \tag{3}$$

for $x\rightarrow 0$. For the GUE, $D(s;\lambda)$ is the generating function of the probabilities $E_n(s)$, thus

$$E_n(s) = D_n(s), \tag{4}$$

where we defined

$$D_0(s) \;=\; D(s;1) \tag{5}$$

$$D_n(s) \;=\; \frac{(-1)^n}{n!}\,\frac{\partial^n D(s;\lambda)}{\partial\lambda^n}\bigg|_{\lambda=1}, \quad n>0. \tag{6}$$

The level-spacing distributions $P_n(s)$, which are the probability distributions to find two energy levels at a distance s with n levels lying in-between, are then obtained as

$$P_n(s) \;=\; \frac{d^2}{ds^2}\,\sum_{m=0}^{n}\,(n-m+1)\,E_m(s) \tag{7}$$

where $P_0(s)$ is the usual nearest-neighbour spacing distribution.

Now, it is rather straightforward to derive series expansions for the distributions $P_n(s)$ by making a power-series ansatz for the solution $\sigma(x;\lambda)$ of the differential equation (2) with the appropriate boundary condition (3). The distributions $P_n(s)$ were computed with the commercial algebraic computer program MATHEMATICA [26]. Whereas it is rather easy to calculate the first few series coefficients in this way, a straightforward implementation of the equations in an algebraic programming language like MATHEMATICA will generally not succeed to calculate higher orders. In order to achieve this, the calculation was split into smaller tasks and an iterative scheme was used, calculating the expansions coefficients one by one. In this way, it was possible to obtain the series expansions for the spacing functions $P_n(s)$ up to order s^{50}.

3 Series expansions

Small-s expansions

$$P_n(s) = \sum_{k=0}^{\infty} p_{n;k}s^k \tag{8}$$

of the level-spacing distributions $P_n(s)$ (7) can be obtained as follows. As a first step, we define functions $\sigma_n(x)$ by

$$\sigma_0(x) \;=\; \sigma(x;1), \tag{9}$$

$$\sigma_n(x) \;=\; \frac{\partial^n\sigma(x;\lambda)}{\partial\lambda^n}\bigg|_{\lambda=1}. \tag{10}$$

The function $\sigma_0(x)$ satisfies the differential equation (2) and behaves as

$$\sigma_0(x) \;=\; -\frac{1}{\pi}\,x \;+\; o(x) \tag{11}$$

for small x. This follows from Eq. (3), which also determines the small-x behavior of $\sigma_n(x)$ for $n > 0$,

$$\sigma_1(x) \;=\; -\frac{1}{\pi}\,x \;+\; o(x), \tag{12}$$
$$\sigma_n(x) \;=\; o(x) \qquad n > 1. \tag{13}$$

For any N, the functions $\sigma_n(x)$, $0 \leq n \leq N$ satisfy a coupled set of differential equations that are obtained by taking the derivative of the original Painlevé equation (2) with respect to λ and putting $\lambda = 1$. For instance, for $N = 2$ the set of differential equations becomes

$$0 \;=\; (x\sigma_0'')^2 + 4(x\sigma_0' - \sigma_0)(x\sigma_0' - \sigma_0 + {\sigma_0'}^2), \tag{14}$$
$$\begin{aligned} 0 \;=\;& x^2\sigma_1''\sigma_0'' + 2(x\sigma_1' - \sigma_1)(x\sigma_0' - \sigma_0 + {\sigma_0'}^2) \\ &+2(x\sigma_0' - \sigma_0)(x\sigma_1' - \sigma_1 + 2\sigma_1'\sigma_0'), \end{aligned} \tag{15}$$
$$\begin{aligned} 0 \;=\;& x^2\sigma_2''\sigma_0'' + (x\sigma_1'')^2 \\ &+2(x\sigma_2' - \sigma_2)(x\sigma_0' - \sigma_0 + {\sigma_0'}^2) \\ &+4(x\sigma_1' - \sigma_1)(x\sigma_1' - \sigma_1 + 2\sigma_1'\sigma_0') \\ &+2(x\sigma_0' - \sigma_0)(x\sigma_2' - \sigma_2 + 2\sigma_2'\sigma_0' + 2{\sigma_1'}^2). \end{aligned} \tag{16}$$

In order to obtain series expansions for the level-spacing distributions $P_n(s)$ (7) with $0 \leq n \leq N$, we first compute series expansions for the functions $\sigma_n(x)$ for $0 \leq n \leq N$. This is done by inserting polynomials

$$\sigma_n(x) = \sum_{k=0}^{K} c_{n;k} \left(\frac{x}{\pi}\right)^k \tag{17}$$

of order K into the set of differential equations obtained from (2), and implementing the initial conditions (11)–(13) by setting $c_{n;0} = 0$ for $0 \leq n \leq N$, $c_{0;1} = c_{1;1} = -1$ and $c_{n;1} = 0$ for $2 \leq n \leq N$. The initial conditions (11) and (12) motivate the convenient choice of the expansion variable x/π rather than x.

From the expansions of the functions $\sigma_n(x)$, $0 \leq n \leq N$, we can derive expansions for $D_n(s)$ with $0 \leq n \leq N$ by using Eqs. (1), (5), and (6). For example, we have

$$D_0(s) \;=\; \exp\left(\int_0^{\pi s} \frac{\sigma_0(x)\,dx}{x}\right) \tag{18}$$

$$D_1(s) \;=\; D_0(s)\,\int_0^{\pi s} \frac{\sigma_1(x)\,dx}{x}, \tag{19}$$

$$D_2(s) \;=\; D_0(s)\,\left[\int_0^{\pi s} \frac{\sigma_2(x)\,dx}{x} + \left(\int_0^{\pi s} \frac{\sigma_1(x)\,dx}{x}\right)^2\right], \tag{20}$$

Table 1 Numerical values of expansion coefficients (8) of the level-spacing distributions P_0, P_1, P_2, P_3, P_4, and P_5.

k	$p_{0;k}$	$p_{1;k}$	$p_{2;k}$	$p_{3;k}$	$p_{4;k}$	$p_{5;k}$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	3.290	0	0	0	0	0
3	0	0	0	0	0	0
4	-4.329	0	0	0	0	0
5	0	0	0	0	0	0
6	3.052	0	0	0	0	0
7	-2.374×10^{-1}	2.374×10^{-1}	0	0	0	0
8	-1.339	0	0	0	0	0
9	2.104×10^{-1}	-2.104×10^{-1}	0	0	0	0
10	4.004×10^{-1}	0	0	0	0	0
11	-9.088×10^{-2}	9.088×10^{-2}	0	0	0	0
12	-8.685×10^{-2}	0	0	0	0	0
13	2.491×10^{-2}	-2.491×10^{-2}	0	0	0	0
14	1.446×10^{-2}	-3.450×10^{-4}	1.725×10^{-4}	0	0	0
15	-4.827×10^{-3}	4.827×10^{-3}	0	0	0	0
16	-1.974×10^{-3}	2.625×10^{-4}	-1.313×10^{-4}	0	0	0
17	7.038×10^{-4}	-7.038×10^{-4}	0	0	0	0
18	2.411×10^{-4}	-9.932×10^{-5}	4.966×10^{-5}	0	0	0
19	-8.042×10^{-5}	8.042×10^{-5}	0	0	0	0
20	-2.884×10^{-5}	2.496×10^{-5}	-1.248×10^{-5}	0	0	0
21	7.418×10^{-6}	-7.418×10^{-6}	0	0	0	0
22	3.527×10^{-6}	-4.714×10^{-6}	2.357×10^{-6}	0	0	0
23	-5.659×10^{-7}	5.678×10^{-7}	-2.805×10^{-9}	9.351×10^{-10}	0	0
24	-4.301×10^{-7}	7.180×10^{-7}	-3.590×10^{-7}	0	0	0
25	3.684×10^{-8}	-3.816×10^{-8}	1.983×10^{-9}	-6.610×10^{-10}	0	0
26	4.991×10^{-8}	-9.240×10^{-8}	4.620×10^{-8}	0	0	0
27	-2.210×10^{-9}	2.677×10^{-9}	-7.013×10^{-10}	2.338×10^{-10}	0	0
28	-5.357×10^{-9}	1.038×10^{-8}	-5.189×10^{-9}	0	0	0
29	1.484×10^{-10}	-2.587×10^{-10}	1.655×10^{-10}	-5.517×10^{-11}	0	0
30	5.265×10^{-10}	-1.040×10^{-9}	5.198×10^{-10}	0	0	0
31	-1.363×10^{-11}	3.320×10^{-11}	-2.936×10^{-11}	9.788×10^{-12}	0	0
32	-4.738×10^{-11}	9.430×10^{-11}	-4.715×10^{-11}	0	0	0
33	1.532×10^{-12}	-4.317×10^{-12}	4.178×10^{-12}	-1.393×10^{-12}	0	0
34	3.915×10^{-12}	-7.816×10^{-12}	3.908×10^{-12}	-1.072×10^{-16}	2.680×10^{-17}	0
35	-1.700×10^{-13}	5.011×10^{-13}	-4.966×10^{-13}	1.655×10^{-13}	0	0
36	-2.980×10^{-13}	5.956×10^{-13}	-2.979×10^{-13}	7.275×10^{-17}	-1.819×10^{-17}	0
37	1.703×10^{-14}	-5.084×10^{-14}	5.071×10^{-14}	-1.690×10^{-14}	0	0
38	2.095×10^{-14}	-4.191×10^{-14}	2.098×10^{-14}	-2.473×10^{-17}	6.184×10^{-18}	0
39	-1.516×10^{-15}	4.542×10^{-15}	-4.539×10^{-15}	1.513×10^{-15}	0	0
40	-1.365×10^{-15}	2.732×10^{-15}	-1.372×10^{-15}	5.620×10^{-18}	-1.405×10^{-18}	0
41	1.205×10^{-16}	-3.613×10^{-16}	3.612×10^{-16}	-1.204×10^{-16}	0	0
42	8.260×10^{-17}	-1.657×10^{-16}	8.379×10^{-17}	-9.603×10^{-19}	2.401×10^{-19}	0
43	-8.622×10^{-18}	2.586×10^{-17}	-2.586×10^{-17}	8.620×10^{-18}	0	0
44	-4.659×10^{-18}	9.384×10^{-18}	-4.824×10^{-18}	1.316×10^{-19}	-3.291×10^{-20}	0
45	5.600×10^{-19}	-1.680×10^{-18}	1.680×10^{-18}	-5.600×10^{-19}	0	0
46	2.459×10^{-19}	-4.994×10^{-19}	2.648×10^{-19}	-1.509×10^{-20}	3.772×10^{-21}	0
47	-3.324×10^{-20}	9.972×10^{-20}	-9.972×10^{-20}	3.324×10^{-20}	-1.483×10^{-26}	2.966×10^{-27}
48	-1.221×10^{-20}	2.516×10^{-20}	-1.407×10^{-20}	1.487×10^{-21}	-3.717×10^{-22}	0
49	1.814×10^{-21}	-5.441×10^{-21}	5.441×10^{-21}	-1.814×10^{-21}	9.814×10^{-27}	-1.963×10^{-27}
50	5.735×10^{-22}	-1.211×10^{-21}	7.343×10^{-22}	-1.287×10^{-22}	3.217×10^{-23}	0

and so forth. Finally, this translates into an expansion for the level-spacing distributions $P_n(s)$ via Eqs. (4) and (7).

As Eq. (7) involves two derivatives with respect to the variable s, we need to obtain the expansions of the functions $\sigma_n(x)$ to two additional orders in x; so in order to achieve a result for $P_n(s)$ that is correct to order 50 in s we have to compute the expansions of $\sigma_n(x)$ to order 52 in x/π. Furthermore, we need to expand the exponential function in $D_0(s)$ (18), whereas the integration on x is trivially performed on polynomials. As a result, the expansion coefficients $p_{n;k}$ (8) are expressions involving rather lengthy rational numbers and powers of π. The expansions for $P_n(s)$ with $0 \leq n \leq 5$ are given in Appendix A; for higher $n > 5$ the lowest order in the small-s expansion of $P_n(s)$ is larger than 50. The numerical values of the expansion coefficients are given in Table 1, compare also the numerical values of the coefficients $p_{0;k}$ for $k \leq 32$ given in Ref. [7, Table 4.1]. The modulus of the coefficients $p_{n;k}$ appears to decrease relatively rapidly with increasing order k, so we may expect the series to converge on a rather large domain around the origin.

4 Numerical integration

To give an account of the accuracy of the expansions, we compare them to the result of a numerical integration of the coupled set of differential equations (14)–(16) derived from the original Painlevé equation (2). These are also obtained using MATHEMATICA [26]. In practice, we use initial values for the functions $\sigma_n(x)$ for small, but non-zero x which are obtained from their expansions, and then integrate the system to larger values of x.

In fact, it turns out that this is not as easily done as stated. The reason is that some of the functions involved become very small or very large as s is increased, and the computation of $P_n(s)$ essentially involves the cancellation of these large terms. In particular, the functions $\sigma_n(x)$ behave asymptotically like [24]

$$\sigma_0(x) \sim -\frac{1}{4}x^2 \tag{21}$$

$$\sigma_n(x) \sim -\frac{n!}{(8\pi)^{n/2}}\frac{\exp(nx)}{x^{n/2-1}} \tag{22}$$

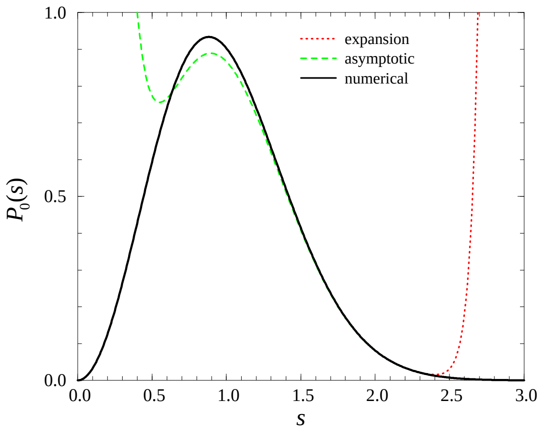

Fig. 1 The spacing distribution $P_0(s)$ as obtained by numerical integration (solid line), the corresponding small-s expansion (23) and the asymptotic form (30).

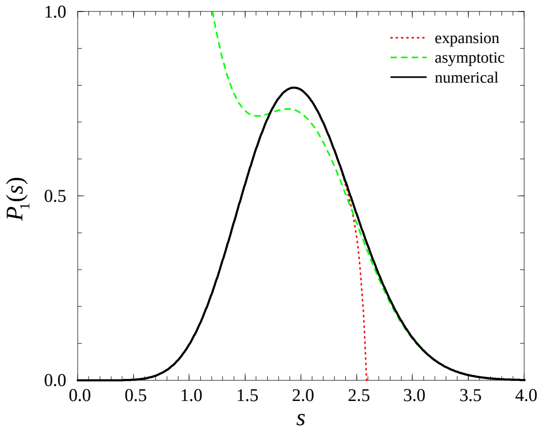

Fig. 2 The spacing distribution $P_1(s)$ as obtained by numerical integration (solid line), the corresponding small-s expansion (24) and the asymptotic form (31).

for large argument x. So, for $n > 0$, $\sigma_n(x)$ grows exponentially with x, whereas $P_n(s)$ is at most of order one, and eventually decreases as $\exp(-\pi^2 s^2/8)$ for large s, see Appendix B where the asymptotic behavior of the functions $P_n(s)$ for large spacings s is discussed.

Results for $n = 0$, $n = 1$ and $n = 2$ are shown in Figs. 1, 2 amd 3, respectively. Apparently, the numerical solutions agree well with the series expansions of Appendix A up to about $s \lesssim 2.5$. For $n = 0$, the distribution function is well reproduced by the asymptotic form of Eq. (30) down to $s \gtrsim 1.5$. Similarly, as shown in Figs. 2 and 3, the small-s expansions of Eqs. 24 and 25 describe the functions $P_1(s)$ and $P_2(s)$ for $s \lesssim 2.5$. The asymptotic forms of Eqs. (31)–(33) reproduce the functions $P_1(s)$ and $P_2(s)$ very well for $s \gtrsim 3$ and $s \gtrsim 4.5$, respectively. In principle, it should be possible to extend the s^{-1} expansion of Eq. (31) as well, at least for fixed values of n, and thus improve the situation considerably.

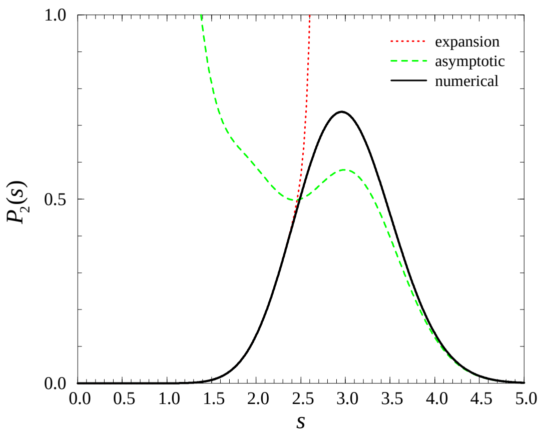

Fig. 3 The spacing distribution $P_2(s)$ as obtained by numerical integration (solid line), the corresponding small-s expansion (25) and the asymptotic form (31).

5 Concluding remarks

We presented series expansions for the spacing distributions $P_n(s)$ of the GUE. The expansion was calculated up to order 50 in the spacing s, and for all distributions $P_n(s)$. In practice, this means that for $0 \leq n \leq 5$ the leading terms up to order s^{50} were obtained, whereas for $n > 5$ the leading order is larger. The expansions agree with the complete series solutions obtained by Mehta [19], which were derived in a different way.

The explicit expansions give a very precise account of the small-spacing part of the spacing distributions, and might prove useful for comparisons with experimental or numerical data. The leading asymptotic behavior for large spacing was obtained in Ref. [24] by similar means.

Within the same framework, albeit slightly more involved, one can derive analogous series expansions for the spacing distributions of the GOE and the GSE.

It is conceivable that other forms of expansions of the level spacing distributions $P_n(s)$ for small s might improve the convergence. A candidate might be to start from a Wigner surmise type form as discussed in Ref. [28].

Acknowledgements The author is indebted to C. A. Tracy who pointed out Ref. [24] and provided him with the corresponding MATHEMATICA programs. These established a starting point and inspired the structure of the programs that were used to calculate the expansions presented in this paper. The author also acknowledges useful discussions with P. J. Forrester and R. A. Römer. He also thanks N. S. Witte for spotting some misprints in a couple of coeffi cients in a previous version of this manuscript, and for bringing Refs. [18] and [19] to his attention.

A Leading terms of series expansions

Here we list the leading terms of the small-s expansions for $P_n(s)$. The complete expansions can be downloaded as Mathematica code from [27]. The numerical values of the coefficients up to order s^{50} are given in Table 1.

$$P_0(s) = \frac{\pi^2 s^2}{3} - \frac{2\pi^4 s^4}{45} + \frac{\pi^6 s^6}{315} - \frac{\pi^6 s^7}{4050} - \frac{2\pi^8 s^8}{14175} + \frac{11\pi^8 s^9}{496125} + \frac{2\pi^{10} s^{10}}{467775} - \ldots \quad (23)$$

$$P_1(s) = \frac{\pi^6 s^7}{4050} - \frac{11\pi^8 s^9}{496125} + \frac{13\pi^{10} s^{11}}{13395375} - \frac{4586\pi^{12} s^{13}}{170188239375} - \frac{\pi^{12} s^{14}}{2679075000} + \ldots \quad (24)$$

$$P_2(s) = \frac{\pi^{12} s^{14}}{5358150000} - \frac{17\pi^{14} s^{16}}{1181472075000} + \frac{1577\pi^{16} s^{18}}{2859162421500000} - \ldots \quad (25)$$

$$P_3(s) = \frac{\pi^{20} s^{23}}{937852533135000000} - \frac{13\pi^{22} s^{25}}{170222023476400250000000} + \ldots \quad (26)$$

$$P_4(s) = \frac{\pi^{30} s^{34}}{306454025102648638446000000000000} - \ldots \quad (27)$$

$$P_5(s) = \frac{\pi^{42} s^{47}}{2559635896086661747545004101009723000000000000000} - \ldots \quad (28)$$

For the level-spacing distributions $P_n(s)$ with $n > 5$ the leading terms are of higher order than s^{50}.

B Asymptotic behavior

The large-s asymptotics of the level-spacing distributions $P_n(s)$ are also known. For $n = 0$, Dyson's asymptotic result [29, 7] reads

$$E_0^{(\mathrm{a})}(s) = \left(\frac{2}{\pi s}\right)^{1/4} \exp\left(\frac{\ln 2}{12} + 3\zeta'(-1) - \frac{\pi^2 s^2}{8}\right) \quad (29)$$

and thus

$$P_0^{(a)}(s) = \frac{d^2}{ds^2} E_0^{(a)}(s) = \frac{\pi^4}{16}\left(s^2 - \frac{2}{\pi^2} + \frac{5}{\pi^4 s^2}\right) E_0^{(a)}(s). \quad (30)$$

Here, $\zeta'(-1) \approx -0.165421$ denotes the derivative of Riemann's ζ-function evaluated at -1.

For $n > 0$, the asymptotics are given in Ref. [24]. For the GUE, this gives

$$\frac{E_n^{(a)}(s)}{E_0^{(a)}(s)} = \frac{B_n \exp(n\pi s)}{s^{n^2/2}} \Big(1 + \frac{(2n^2+7)n}{8\,\pi\, s} + \frac{(4n^4+48n^2+229)n^2}{128\,(\pi\, s)^2} + O[(\pi\, s)^{-3}]\Big) \quad (31)$$

where

$$B_n = 2^{-\frac{n^2+2n}{2}}\, \pi^{-\frac{n^2+n}{2}} \prod_{m=1}^{n-1} m!\,. \quad (32)$$

From this, we can easily calculate the corresponding asymptotic behavior $P_n^{(a)}(s)$ as

$$P_n^{(a)}(s) = \frac{d^2}{ds^2} \sum_{m=0}^{n} (n-m+1)\, E_m^{(a)}(s), \quad (33)$$

which is just Eq. (7) applied to the asymptotic expressions. The resulting terms are rather involved; however, it is apparent from Eqs. (29), (31), and (33) that the leading behavior in all cases is $\exp(-\pi^2 s^2/8)$.

References

[1] E. P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, Proc. Camb. Philos. Soc. **47**, 790–798 (1951).
[2] F. J. Dyson, Statistical theory of energy levels of complex systems, I, J. Math. Phys. **3**, 140–156 (1962).
[3] F. J. Dyson, Statistical theory of energy levels of complex systems, II, J. Math. Phys. **3**, 157–165 (1962).
[4] F. J. Dyson, Statistical theory of energy levels of complex systems, III, J. Math. Phys. **3**, 166–175 (1962).
[5] F. J. Dyson, Brownian motion model for the eigenvalues of a random matrix, J. Math. Phys. **3**, 1199–1215 (1962).
[6] F. J. Dyson and M. L. Mehta, Statistical theory of energy levels of complex systems, IV, J. Math. Phys. **4**, 701–712 (1963).
[7] F. Haake, Quantum Signatures of Chaos (Springer, Berlin,1991).
[8] T. Guhr, A. Müller-Groeling and H. A. Weidenmüller", Random-matrix theories in quantum physics: common concepts, Phys. Rep. **299**, 189–425 (1999).
[9] M. L. Mehta, Random Matrices, second ed. (Academic Press, Boston, 1991).
[10] C. A. Tracy and H. Widom, Universality of the distribution functions of random matrix theory, in: Statistical Physics on the Eve of the 21st Century: In Honour of J. B. McGuire on the Occasion of His 65th Birthday, edited by M. T. Batchelor and L. T. Wille (World Scientifi c, Singapore, 1999)
[11] J.-X. Zhong, U. Grimm, R. A. Römer and M. Schreiber, Level-spacing distributions of planar quasiperiodic tight-binding models, Phys. Rev. Lett. **80**, 3996–3999 (1998); cond-mat/9710006.
[12] M. Schreiber, U. Grimm, R. A. Römer and J.-X. Zhong, Application of random matrix theory to quasiperiodic systems, Physica A **266**, 477–480 (1999); cond-mat/9809370.
[13] M. Schreiber, U. Grimm, R. A. Römer and J.-X. Zhong, Energy levels of quasiperiodic Hamiltonians, spectral unfolding, and random matrix theory, Comp. Phys. Commun. **121–122**, 499–501 (1999); cond-mat/9811359.
[14] U. Grimm, R. A. Römer, M. Schreiber and J.-X. Zhong, Universal level-spacing statistics in quasiperiodic tight-binding models, Mat. Sci. Eng. A **294–296**, 564–567 (2000); cond-mat/9908063.

[15] U. Grimm and M. Schreiber, Aperiodic Tilings on the Computer, in: Quasicrystals: An Introduction to Structure, Physical Properties and Applications, edited by J.-B. Suck, M. Schreiber and P. Häussler, Springer Series in Materials Science, vol. 55 (Springer, Berlin, 2002) pp. 49–66; cond-mat/9903010.

[16] U. Grimm and M. Schreiber, Energy spectra and eigenstates of quasiperiodic tight-binding Hamiltonians, in: Quasicrystals: Structure and Physical Properties, edited by H.-R. Trebin, (Wiley-VCH, Berlin, 2003) pp. 210–235; cond-mat/0212140.

[17] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, Math. Comp. **48**, 273–308 (1987).

[18] B. Dietz and F. Haake, Taylor and Pade analysis of the level spacing distributions of random-matrix ensembles, Z. Phys. B **80**, 153–158 (1990).

[19] M. L. Mehta, Power series for level spacing functions of random matrix ensembles, Z. Phys. B **86**, 285–290 (1992).

[20] N. S. Witte, P. J. Forrester and C. M. Cosgrove, Integrability, random matrices and Painlevé transcendents, Preprint math-ph/0008033.

[21] P. J. Forrester and N. S. Witte, Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE, Preprint math-ph/0103025.

[22] P. J. Forrester and N. S. Witte, Discrete Painlevé equations and random matrix averages, Preprint math-ph/0304020 (2003).

[23] P. J. Forrester, Growth models, random matrices and Painlevé transcendents, Nonlinearity **16**, R27–R49 (2003); math-ph/0309058.

[24] E. L. Basor, C. A. Tracy and H. Widom, Asymptotics of level-spacing distributions for random matrices, Phys. Rev. Lett. **69**, 5–8 (1992).

[25] M. Jimbo, T. Miwa, Y. Môri and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica D **1**, 80–158 (1980).

[26] S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, second ed. (Addison-Wesley, Reading MA, 1991).

[27] U. Grimm, LevelSpacingGUE.m, `http://mcs.open.ac.uk/ugg2/RMT/LevelSpacingGUE.m`

[28] P. J. Forrester and N. S. Witte, Exact Wigner surmise type evaluation of the spacing distribution in the bulk of the scaled random matrix ensembles, Lett. Math. Phys. **53**, 195–200 (2000); math-ph/0009023.

[29] F. J. Dyson, Fredholm determinants and inverse scattering problems, Commun. Math. Phys. **47**, 171–183 (1976).