Full proof of the existence of a degree 8 circulant graph of order $L(8, k)$ of arbitrary diameter $k$

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10th April 2014

This is the full proof of Theorem 3 in the paper “The degree-diameter problem for circulant graphs of degree 8 and 9” by the author [2]. To avoid the paper being unduly long it includes only the exceptions for the orthant of $v_1$ for diameter $k \equiv 0 \pmod{2}$ and for $k \equiv 1 \pmod{2}$. In the version below the exceptions for all eight orthants for diameter $k \equiv 0$ and $k \equiv 1 \pmod{2}$ are included in full. This proof closely follows the approach taken by Dougherty and Faber in their proof of the existence of the degree 6 graph of order $DF(6, k)$ for all diameters $k \geq 2$ [1].

**Theorem 3.** For all $k \geq 2$, there is an undirected Cayley graph on four generators of a cyclic group which has diameter $k$ and order $L(8, k)$, where

$$L(8, k) = \begin{cases} 
(k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{if } k \equiv 0 \pmod{2} \\
(k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{if } k \equiv 1 \pmod{2} 
\end{cases}$$

Moreover for $k \equiv 0 \pmod{2}$ a generator set is $\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k - 4k)/4\}$,  
and for $k \equiv 1 \pmod{2}$, $\{1, (k^3 + k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\}$.

**Proof.** We will show the existence of four-dimensional lattices $L_k \subseteq \mathbb{Z}^4$ such that $\mathbb{Z}^4/L_k$ is cyclic, $S_k + L_k = \mathbb{Z}^4$, where $S_k$ is the set of points in $\mathbb{Z}^4$ at a distance of at most $k$ from the origin under the $l^1$ (Manhattan) metric, and $|\mathbb{Z}^4 : L_k| = L(8, k)$ as specified in the theorem. Then, by Theorem 1 of [2], the resultant Cayley graph has diameter at most $k$.

Let $a = \begin{cases} k/2 & \text{for } k \equiv 0 \pmod{2} \\
(k + 1)/2 & \text{for } k \equiv 1 \pmod{2} \end{cases}$.

For $k \equiv 0 \pmod{2}$ let $L_k$ be defined by four generating vectors as follows:

$$\begin{align*}
v_1 &= (-a - 1, a + 1, a, -a + 1) \\
v_2 &= (a - 1, a + 1, a + 1, -a) \\
v_3 &= (-a - 1, -a + 1, a + 1, -a) \\
v_4 &= (-a, -a, a, a + 1)
\end{align*}$$
Hence we have 

\[-(a^2 + 2a + 1)v_1 + (2a^2 + a + 2)v_2 - (a + 2)v_3 + v_4 = (4a^3 + 4a^2 + 6a + 1, -1, 0, 0),\]

\[-(2a - 1)v_1 + (2a^2 - a^2 + 2a - 2)v_2 - (a^2 + a - 1)v_3 + (a - 1)v_4 = (4a^4 + 4a^2 - 4a, 0, -1, 0),\]

\[-2a^3v_1 + (2a^3 - a^2 + 2a - 1)v_2 - (a^2 + a - 1)v_3 + (a - 1)v_4 = (4a^4 + 4a^2 - 2a, 0, 0, -1)\]

Hence we have \(e_2 = (4a^3 + 4a^2 + 6a + 1)e_1, e_3 = (4a^4 + 4a^2 - 4a)e_1\) and \(e_4 = (4a^4 + 4a^2 - 2a)e_1\) in \(\mathbb{Z}/L_k\), and so \(e_1\) generates \(\mathbb{Z}/L_k\).

Also \(\det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 + 8a^3 + 12a^2 + 4a & 0 & 0 & 0 \\ 4a^3 + 4a^2 + 6a + 1 & -1 & 0 & 0 \\ 4a^4 + 4a^2 - 4a & 0 & -1 & 0 \\ 4a^4 + 4a^2 - 2a & 0 & 0 & -1 \end{pmatrix} = -(8a^4 + 8a^3 + 12a^2 + 4a) = -(k^4 + 2k^3 + 6k^2 + 4k)/2 = -L(8, k),\) as in the statement of the theorem.

Thus \(\mathbb{Z}/L_k\) is isomorphic to \(\mathbb{Z}_{L(8,k)}\) via an isomorphism taking \(e_1, e_2, e_3, e_4\) to \(1, 4a^3 + 4a^2 + 6a + 1, 4a^4 + 4a^2 - 4a, 4a^4 + 4a^2 - 2a\). As \(a = k/2\) this gives the first generator set specified in the theorem: \(\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k - 7)/4\} \).

Similarly for \(k \equiv 1 \pmod{2}\) let \(L_k\) be defined by four generating vectors as follows:

\[v_1 = (-a + 1, a + 1, -a + 1, a)\]
\[v_2 = (a + 1, a + 1, -a + 2, a - 1)\]
\[v_3 = (-a - 1, a - 1, a - 1, -a)\]
\[v_4 = (-a, a, a, -a)\]

In this case the following vectors are in \(L_k\):

\[-(2a^4 + a + 1)v_1 + (2a^2 + 2a + 1)v_2 - av_3 - v_4 = (4a^3 - 4a^2 + 6a - 1, -1, 0, 0),\]

\[-(2a^3 - a^2 - 2a - 2)v_1 + (2a^3 - 4a - 1)v_2 - (a^2 - a - 1)v_3 - (a - 1)v_4 = (4a^4 - 8a^3 + 8a^2 - 8a, 0, -1, 0),\]

\[-(2a^3 - a^2 - 2a - 1)v_1 + (2a^3 - 4a)v_2 - (a^2 - a - 1)v_3 - (a - 1)v_4 = (4a^4 - 8a^3 + 8a^2 - 6a, 0, 0, -1)\]

Hence we have \(e_2 = (4a^3 + 4a^2 + 6a - 1)e_1, e_3 = (4a^4 - 8a^3 + 8a^2 - 8a)e_1\) and \(e_4 = (4a^4 - 8a^3 + 8a^2 - 6a)e_1\) in \(\mathbb{Z}/L_k\), and so \(e_1\) generates \(\mathbb{Z}/L_k\).

Also \(\det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \det \begin{pmatrix} 8a^4 - 8a^3 + 12a^2 - 4a & 0 & 0 & 0 \\ 4a^3 - 4a^2 + 6a - 1 & -1 & 0 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 8a & 0 & -1 & 0 \\ 4a^4 - 8a^3 + 8a^2 - 6a & 0 & 0 & -1 \end{pmatrix} = -(8a^4 - 8a^3 + 12a^2 - 4a) = -(k^4 + 2k^3 + 6k^2 + 4k + 1)/2 = -L(8, k),\) as in the statement of the theorem.

Thus \(\mathbb{Z}/L_k\) is isomorphic to \(\mathbb{Z}_{L(8,k)}\) with generators \(1, 4a^3 - 4a^2 + 6a - 1, 4a^4 - 8a^3 + 8a^2 - 8a, 4a^4 - 8a^3 + 8a^2 - 6a\). As \(a = (k + 1)/2\) in this case, this gives the second generator set specified in the theorem: \(\{1, (k^3 + 2k^2 + 5k + 3)/2, (k^4 + 2k^2 - 8k - 11)/4, (k^4 + 2k^2 - 4k - 7)/4\} \).

It remains to show that \(S_k + L_k = \mathbb{Z}/4\). First we consider the case \(k \equiv 0 \pmod{2}\).

For \(k = 2\), it is straightforward to show directly that \(\mathbb{Z}/32\) with generators \(1, 4, 6, 15\) has
diameter 2. So we assume \( k \geq 4 \), so that \( a \geq 2 \). Now let

\[
\begin{align*}
v_5 &= v_1 - v_3 + v_4 = (-a, a, a - 1, a + 2) \\
v_6 &= v_1 - v_2 - v_4 = (-a, -a - 1, -a) \\
v_7 &= v_1 - v_2 - v_3 = (-a + 1, a, 1 - a - 2, a + 1) \\
v_8 &= v_2 - v_3 + v_4 = (a, a, a, a + 1)
\end{align*}
\]

with \( v_1, v_2, v_3, v_4 \) as defined for \( k \equiv 0 \pmod{2} \). Then the 16 vectors \( \pm v_i \) for \( i = 1, \ldots, 8 \) provide one element of \( L_k \) lying strictly within each of the 16 orthants of \( \mathbb{Z}^4 \). Most of the coordinates of these vectors have absolute value at most \( a + 1 \). Only \( \pm v_5 \) and \( \pm v_7 \) each have one coordinate with absolute value equal to \( a + 2 \).

Now we consider the case \( k \equiv 1 \pmod{2} \). For \( k = 3 \) it may be shown directly that \( \mathbb{Z}_{104} \) with generators 1, 16, 20, 27 has diameter 3. So we assume \( k \geq 5 \), so that \( a \geq 3 \), and let

\[
\begin{align*}
v_5 &= v_1 - v_2 - v_4 = (-a, -a, -a - 1, -a + 2) \\
v_6 &= v_2 + v_3 - v_4 = (a, a, -a + 1, -a) \\
v_7 &= v_1 + v_3 - v_4 = (-a, a, -a, -a + 1) \\
v_8 &= v_1 - v_2 - v_3 = (-a + 1, -a + 1, -a, a + 1)
\end{align*}
\]

with \( v_1, v_2, v_3, v_4 \) as defined for \( k \equiv 1 \pmod{2} \), so that the 16 vectors \( \pm v_i \) provide one element of \( L_k \) lying strictly within each of the orthants of \( \mathbb{Z}^4 \). In this case all the coordinates of these vectors have absolute value at most \( a + 1 \).

We must show that each \( x \in \mathbb{Z}^4 \) is in \( S_k + L_k \), which means that for any \( x \in \mathbb{Z}^4 \) we need to find a \( w \in L_k \) such that \( x - w \in S_k \). However \( x - w \in S_k \) if and only if \( \delta(x, w) \leq k \), where \( \delta \) is the \( l^1 \) metric on \( \mathbb{Z}^4 \). If \( x, y, z \in \mathbb{Z}^4 \) and each coordinate of \( y \) lies between the corresponding coordinate of \( x \) and \( z \) or is equal to one of them, then \( \delta(x, y) + \delta(y, z) = \delta(x, z) \). In such a case we say that “\( y \) lies between \( x \) and \( z \)”.

For any \( x \in \mathbb{Z}^4 \), we reduce \( x \) by adding appropriate elements of \( L_k \) until the resulting vector lies within \( l^1 \)-distance \( k \) of \( 0 \) or some other element of \( L_k \). The first stage is to reduce \( x \) to a vector whose coordinates all have absolute value at most \( a + 1 \). If \( x \) has a coordinate with absolute value above \( a + 1 \), then let \( v \) be one of the vectors \( \pm v_i (1 \leq i \leq 8) \) such that the coordinates of \( v \) have the same sign as the corresponding coordinates of \( x \). If a coordinate of \( x \) is 0 then either sign is allowed for \( v \) as long as the corresponding coordinate of \( v \) has absolute value \( \leq a + 1 \). So in the case \( k \equiv 0 \pmod{2} \) if the \( e_3 \) coordinate of \( x \) is 0 then we avoid \( v_7 \) and take \( v_5 \) instead. Also if the \( e_4 \) coordinate of \( x \) is 0 (or both \( e_3 \) and \( e_4 \) coordinates are 0) then instead of \( v_5 \) we take \( v_1 \).

Now consider \( x' = x - v \). If a coordinate of \( x \) has absolute value \( s, 1 \leq s \leq a + 1 \), then the corresponding coordinate of \( x' \) will have absolute value \( s' \leq a + 1 \) because of the sign matching and the fact that the coordinates of \( v \) have absolute value \( \leq a + 2 \). If a coordinate of \( x \) has absolute value \( s = 0 \), then as indicated above, the corresponding value of \( x' \) will have absolute value \( s' \leq a + 1 \) because \( v \) is chosen such that the corresponding coordinate has absolute value \( \leq a + 1 \). If a coordinate of \( x \) has absolute value \( s > a + 1 \), then the corresponding coordinate of \( x' \) will be strictly smaller in absolute value. Therefore repeating this procedure will result in a vector whose coordinates all have absolute value at most \( a + 1 \).
If the resulting vector $x'$ lies between $0$ and $v$, where $v = \pm v_i$ for some $i$, then we have $\delta(0, x') + \delta(x', v) = \delta(0, v)$. For $k \equiv 0 \pmod{2}$ all of the vectors $v$ satisfy $\delta(0, v) = 4a+1$, and for $k \equiv 1 \pmod{2}$ they all satisfy $\delta(0, v) = 4a-1$. So in either case we have $\delta(0, v) = 2k+1$. Since $\delta(0, x')$ and $\delta(x', v)$ are both non-negative integers, one of them must be at most $k$, so that $x' \in S_k + L_k$. Hence we also have $x \in S_k'' + L_k$ as required.

Now we are left with the case where the absolute value of each coordinate of the reduced $x$ is at most $a+1$, and $x$ is in the orthant of $v$, where $v = \pm v_i$ for some $i \leq 8$ but does not lie between $0$ and $v$. Since $L_k$ is centrosymmetric we only need to consider the eight orthants containing $v_1, ..., v_8$. For both cases $k \equiv 0$ and $k \equiv 1 \pmod{2}$ the exceptions need to be considered for each orthant in turn. We first consider all eight orthants for the case $k \equiv 0 \pmod{2}$ and then the same for $k \equiv 1 \pmod{2}$.

**Orthant of $v_1$, $k \equiv 0 \pmod{2}$**

Suppose that $k \equiv 0 \pmod{2}$ and $x$ lies within the orthant of $v_1$, but not between $0$ and $v_1$. Then as $v_1 = (-a-1, a+1, a, a^2+1)$, the third coordinate of $x$ is equal to $a+1$ or the fourth coordinate equals $-a$ or $a+1$. We distinguish three cases.

Case 1: $x = (-r, s, a+1, -u)$ where $0 \leq r, s \leq a+1$ and $a \leq u \leq a+1$. Let $x' = x - v_1 = (a+1-r, s-a-1, 1, 1-a-1-u)$, which lies between $0$ and $-v_7$ unless $r \leq 1$ or $s \leq 1$. Let $x'' = x' + v_7 = (2-r, s-2, -a-1, 2a-u)$. If $r \leq 1$ and $s \leq 1$ then $x''$ lies between $0$ and $-v_1$ unless $u = a$, in which case let $x''' = x'' + v_1 = (1-a-r, a-1+s, 1-a-1-u)$ which lies between $0$ and $v_7$. If $r \leq 1$ and $s \geq 2$ then $x''$ lies between $0$ and $-v_3$. If $r \geq 2$ and $s \leq 1$ then $x''$ lies between $0$ and $-v_2$.

Case 2: $x = (-r, s, a+1, -u)$ where $0 \leq r, s \leq a+1$ and $0 \leq u \leq a-1$. Let $x' = x - v_1 = (a+1-r, s-a-1, 1, 1-a-1-u)$, which lies between $0$ and $-v_6$ unless $r = 0$ or $s = 0$. Let $x'' = x' + v_6 = (1-r, s-1, -a, -u-1)$. If $r = 0$ and $s = 0$ then $x''$ lies between $0$ and $-v_5$. If $r = 0$ and $s \geq 1$ then $x''$ lies between $0$ and $-v_4$. If $r \geq 1$ and $s = 0$ then $x''$ lies between $0$ and $-v_8$.

Case 3: $x = (-r, s, t, -u)$ where $0 \leq r, s \leq a+1$ and $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_1 = (a+1-r, s-a-1, t-a, a-1-u)$, which lies between $0$ and $-v_5$ unless $r = 0$ or $s = 0$ or $t = 0$. If $r = 0$ and $s = 0$, then $x$ lies between $0$ and $-v_7$. Let $x'' = x' + v_5 = (1-r, s-1, t-1, 2a+1-u)$. If $r = 0, s \geq 1$ and $t \geq 1$ then $x''$ lies between $0$ and $v_8$. Let $x''' = x + v_4 = (-a-r, s-a, a+t, a+1-u)$. If $r = 0$ and $s \geq 1$ and $t = 0$, then $x'''$ lies between $0$ and $v_4$ unless $s = a+1$, in which case if $u = a$ then $x$ lies between $0$ and $v_2$, and if $u = a+1$ then $x'''$ lies between $0$ and $v_4$. Let $x''' = x - v_3 = (a+1-r, a-1+s, t-a-1, a-u)$. If $r \geq 1$, $s = 0$ and $t \geq 1$ then $x'''$ lies between $0$ and $-v_4$. If $r \geq 1, s = 0$ and $t = 0$ then $x'''$ lies between $0$ and $-v_3$ if $u = a$, and between $0$ and $v_6$ if $u = a+1$. If $r \geq 1, s \geq 1$ and $t = 0$ then $x'''$ lies between $0$ and $v_7$ unless $r = a+1$ or $s = a+1$. If $r = a+1$, $s \geq 1$ and $t = 0$ then $x$ lies between $0$ and $-v_8$. If $r \geq 1, s = a+1$ and $t = 0$ then $x'$ lies between $0$ and $-v_4$.

This completes the cases for the orthant of $v_1$ for $k \equiv 0 \pmod{2}$. 

Orhant of $v_2$, $k \equiv 0 \pmod{2}$

Now suppose that $x$ lies in the orthant of $v_2$ but not between $0$ and $v_2$. Then the first coordinate of $x$ is equal to $a$ or $a+1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $x = (r,s,t,-a-1)$ where $0 \leq r \leq a+1$ and $0 \leq s,t \leq a+1$. Let $x' = x - v_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $0$ and $-v_3$ unless $s = 0$ or $t \leq 1$, in which case let $x'' = x' + v_3 = (r-2a+1, s-1, t-2, a+1)$. If $s = 0$ and $t \leq 1$ then let $x''' = x + v_5 = (r-a, a, t+a-1, 1)$ which lies between $0$ and $v_8$. If $s = 0$ and $t \geq 2$ then let $x''' = x'' - v_8 = (r-2a, 0, t-1, -a)$ which lies between $0$ and $v_3$. If $s \geq 1$ and $t \leq 1$ then $x''$ lies between $0$ and $v_7$ unless $s = a + 1$, in which case let $x'' = x'' - v_7 = (r-a, 1, a+t, 0)$ which lies between $0$ and $v_2$.

Case 2: $x = (r,s,t,-u)$ where $a \leq r \leq a+1$, $0 \leq s, t \leq a+1$ and $0 \leq u \leq a$. Let $x' = x - v_2 = (r-a+1, s-a-1, t-a-1, a-u)$, which lies between $0$ and $-v_1$ unless $t = 0$ or $u = 0$. If $t = 0$ and $u = 0$ then $x$ lies between $0$ and $-v_3$ unless $a \leq s \leq a+1$. If $r = a+1$, $a \leq s \leq a+1$, $t = 0$ and $u = 0$ then let $x'' = x + v_3 = (r-a-1, s-a+1, t+a+1, -u-a)$ which lies between $0$ and $v_2$. If $a \leq r \leq a+1$, $s = a+1$, $t = 0$ and $u = 0$ then let $x'' = x - v_2 = (r-a+1, s-a-1, t-a-1, -u+a)$ which lies between $0$ and $-v_3$. If $r = a$, $s = a$, $t = 0$ and $u = 0$ then $x$ lies between $0$ and $v_8$. Now let $x''' = x' + v_1 = (r-2a, s, t-1, -u)$. If $t = 0$ and $1 \leq u \leq a$ then $x'''$ lies between $0$ and $v_6$ unless $s = a+1$, in which case let $x''' = x'' - v_6 = (r-a, s-a, t+a, a+1-u)$ which lies between $0$ and $v_8$. If $1 \leq t \leq a+1$ and $u = 0$ then $x'''$ lies between $0$ and $v_5$ unless $s = a+1$ or $t = a+1$, in which case let $x''' = x'' - v_5 = (r-a, s-a, t-a, -a-1-u)$. If $s = a+1$ and $t = a+1$ then $x'$ lies between $0$ and $v_8$. If $s = a+1$ and $1 \leq t \leq a$ then $x'''$ lies between $0$ and $-v_4$. If $0 \leq s \leq a$ and $t = a+1$ then $x'''$ lies between $0$ and $v_7$ unless $s = 0$, in which case $x''$ lies between $0$ and $v_4$.

Case 3: $x = (r,s,t,-a-1)$ where $0 \leq r \leq a-1$ and $0 \leq s,t \leq a+1$. Let $x' = x - v_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $0$ and $-v_8$ unless $s = 0$ or $t = 0$. If $s = 0$ then $x$ lies between $0$ and $-v_7$. If $t = 0$ then $x$ lies between $0$ and $-v_4$ unless $s = a+1$, in which case let $x'' = x + v_7 = (r-a, 1, a, 0)$ which lies between $0$ and $v_1$.

This completes the cases for the orthant of $v_2$.

Orhant of $v_3$, $k \equiv 0 \pmod{2}$

Now suppose that $x$ lies in the orthant of $v_3$ but not between $0$ and $v_3$. Then the second coordinate of $x$ is equal to $-a$ or $-a-1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $x = (-r, -s, t, -a-1)$ where $0 \leq r, t \leq a+1$ and $a \leq s \leq a+1$. Let $x' = x - v_3 = (a+1-r, a-1-s, t-a-1, -1)$, which lies between $0$ and $-v_5$ unless $r = 0$ or $t \leq 1$, in which case let $x'' = x + v_5 = (1-r, 2a-1-s, t-2, a+1)$. If $r = 0$ and $t \geq 2$ then $x''$ which lies between $0$ and $v_8$. If $r = 0$ and $t \leq 1$ then let $x''' = x + v_5 = (-a, a-s, a-1+t, 1)$ which lies between $0$ and $v_4$. If $r \geq 1$ and $t \leq 1$ then $x''$ lies between $0$ and $v_7$ unless $r = a+1$, in which case let $x'' = x'' - v_7 = (-1, a-s, a+t, 0)$ which lies between $0$ and $v_3$. 


Case 2: \( x = (-r, -s, t, -a -1) \) where \( 0 \leq r, t \leq a + 1 \) and \( 0 \leq s \leq a -1 \). Let \( x' = x - v_3 = (a +1-r, a-1-s, t-a-1, -1) \), which lies between \( 0 \) and \( -v_4 \) unless \( r = 0 \) or \( t = 0 \). Let \( x'' = x' + v_4 = (1-r, -1-s, t-1, a) \). If \( r = 0 \) and \( t \geq 1 \) then \( x'' \) lies between \( 0 \) and \(-v_6 \). If \( r \geq 1 \) and \( t = 0 \) then \( x'' \) lies between \( 0 \) and \(-v_2 \) unless \( r = a+1 \), in which case let \( x''' = x'' + v_2 = (-1, a-s, a, 0) \) which lies between \( 0 \) and \( v_1 \). If \( r = 0 \) and \( t = 0 \), then let \( x''' = x'' + v_1 = (-a, a-s, a-1, 1) \) which lies between \( 0 \) and \( v_5 \).

Case 3: \( x = (-r, -s, t, -u) \) where \( 0 \leq r, t \leq a + 1 \), \( a \leq s \leq a +1 \) and \( 0 \leq u \leq a \). Let \( x' = x - v_3 = (a +1-r, a-1-s, t-a-1, a-u) \), which lies between \( 0 \) and \(-v_1 \) unless \( t = 0 \) or \( u = 0 \). Let \( x'' = x + v_8 = (a-r, a-s, a+t, a+1-u) \). If \( t = 0 \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( v_4 \) unless \( r \leq a -1 \), in which case let \( x''' = x + v_2 = (a-r, a+1-s, a+1+t, -a-u) \) which lies between \( 0 \) and \( v_2 \). If \( t = 0 \) and \( u \geq 1 \) then \( x'' \) lies between \( 0 \) and \(-v_5 \) unless \( r = a+1 \), in which case \( x'' \) lies between \( 0 \) and \( v_4 \). Let \( x''' = x - v_4 = (a-r, a-s, t-a, -a-1-u) \). If \( t \geq 1 \), \( u = 0 \) and \( r \leq a \) then \( x''' \) lies between \( 0 \) and \(-v_5 \) unless \( t = a+1 \). If \( t = a+1, u = 0 \) and \( r \leq a \) then \( x''' \) lies between \( 0 \) and \(-v_5 \) unless \( r = 0 \) in which case \( x''' \) lies between \( 0 \) and \( v_8 \). If \( t \geq 1, u = 0 \) and \( r = a+1 \), then \( x''' \) lies between \( 0 \) and \(-v_2 \).

This completes the cases for the orhant of \( v_3 \).

**Orhant of \( v_4, k \equiv 0 \pmod{2} \)**

Now suppose \( x \) lies in the orhant of \( v_4 \) but not between \( 0 \) and \( v_4 \). Then the first coordinate of \( x \) is equal to \(-a-1 \) or the second coordinate is equal to \(-a-1 \), or the third equals \( a+1 \). We distinguish seven cases.

Case 1: \( x = (-a-1, -a-1, a+1, u) \) where \( 0 \leq u \leq a+1 \). Let \( x' = x - v_4 = (-1, -1,1, u-a-1) \), which lies between \( 0 \) and \( v_4 \) if \( u = a+1 \) and between \( 0 \) and \( v_3 \) if \( u \leq a \).

Case 2: \( x = (-a-1, -a-1, t, u) \) where \( 0 \leq t \leq a \) and \( 0 \leq u \leq a+1 \). Let \( x' = x - v_4 = (-1, -1, t-a, u-a-1) \), which lies between \( 0 \) and \( -v_8 \).

Case 3: \( x = (-a-1, -s, a+1, u) \) where \( 0 \leq s \leq a \) and \( 0 \leq u \leq a+1 \). Let \( x' = x - v_4 = (-1, -a-s, 1, u-a-1) \), which lies between \( 0 \) and \( v_1 \) unless \( u \geq a \), in which case let \( x'' = x' - v_1 = (a-s-1, -a+1, u-2) \) which lies between \( 0 \) and \(-v_1 \).

Case 4: \( x = (-r, -a-1, a+1, u) \) where \( 0 \leq r \leq a \) and \( 0 \leq u \leq a+1 \). Let \( x' = x - v_4 = (a-r, -1,1, u-a-1) \), which lies between \( 0 \) and \(-v_7 \) unless \( r = 0 \), in which case let \( x'' = x' + v_7 = (1, a-2, -a-1, u) \) which lies between \( 0 \) and \(-v_3 \) unless \( u = a+1 \), in which case let \( x''' = x'' + v_3 = (-a, -1, 0, 1) \) which lies between \( 0 \) and \( v_4 \).

Case 5: \( x = (-r, -s, a+1, u) \) where \( 0 \leq r, s \leq a \) and \( 0 \leq u \leq a+1 \). Let \( x' = x - v_4 = (a-r, a-s, 1, u-a-1) \), which lies between \( 0 \) and \( v_2 \) unless \( r = 0 \) or \( u = 0 \) in which case let \( x'' = x' - v_2 = (1-r, -s-1, -a, u-1) \). If \( r \geq 1 \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \(-v_8 \) unless \( s = a \), in which case let \( x''' = x'' + v_8 = (a+1-r, -1,0, a) \) which lies between \( 0 \) and \(-v_6 \). If \( r = 0 \) then \( x'' \) lies between \( 0 \) and \(-v_1 \) unless \( u = 0 \) or \( u = a +1 \). If \( r = 0 \) and \( u = 0 \), then let \( x''' = x'' + v_5 = (-1, a-1-s, -1, a+1) \) which lies between \( 0 \) and \( v_7 \) unless \( s = a \), in which case let \( x''' = x''' + v_2 = (0, a, a, 1) \) which lies between \( 0 \) and \( v_8 \). If \( r = 0 \) and \( u = a +1 \), then let \( x''' = x''' + v_1 = (-a, a-s, 0, 1) \) which lies between \( 0 \) and \( v_5 \).
Case 6: $\mathbf{x} = (-r, -a - 1, t, u)$ where $0 \leq r, t \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a - r, -a - 1, t - a, u - a - 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (a - r, a - 1, -1, u + 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = a$ or $u = a + 1$. If $t = 0$ and $r = a$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (-1, 0, a + 1, u - a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u = a + 1$. If $t = 0$ and $u = a + 1$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (a - 1 - r, 0, a + 1, 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = a$. If $t = 0, r = a$ and $u = a + 1$, then $\mathbf{x}'' = (-1, 0, a + 1, 1)$. Let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_4 = (a - 1, a, 1, -a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 7: $\mathbf{x} = (-a - 1, -s, t, u)$ where $0 \leq s, t \leq a$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, a - s, t - a, u - a - 1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a - 1, -s, t + 1, -1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_7 = (0, -1, t - a - 1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

This completes the cases for the orthant of $\mathbf{v}_4$.

Orthant of $\mathbf{v}_3$, $k \equiv 0 \ (\text{mod} \ 2)$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_3$ but not between $\mathbf{0}$ and $\mathbf{v}_5$. Then the first coordinate of $\mathbf{x}$ is equal to $-a - 1$ or the second coordinate is equal to $a + 1$, or the third equals $a$ or $a + 1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a - 1, a + 1, t, u)$ where $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u \leq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, t - 2a + 1, u - 3)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a - 1, a + 1, t, u)$ where $0 \leq t \leq a - 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a - 1, -a + 1, t + 2, u - 2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 3: $\mathbf{x} = (-a - 1, a, t, u)$ where $0 \leq s \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, s - a, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = 0$ or $u \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, a - 1, t - 2a, u - 2)$. If $s = 0$ and $u \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, a - 1, -1, a + u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $s = 0$ and $u \geq 2$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, a, t - a, u - a - 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $s \geq 1$ and $u \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, a + 1, t, u)$ where $0 \leq r \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, 1, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = 0$ or $u \geq 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1 - r, -a, t - 2a, u - 2)$. If $r = 0$ and $u \geq 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $r = 0$ and $u \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1 - a, 0, -1, a + u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r \geq 1$ and $u \geq 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, t, u)$ where $0 \leq r, s \leq a$, $a \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a - r, s - a, t - a + 1, u - a - 2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $s = 0$ or $u = 0$. If $r = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $t = a + 1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (-a, s - a, 1, u - a - 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u = 0$. If $r = 0, t = a + 1$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_2$. If $r \geq 1$ and $s = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $t = a + 1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a - r, a, 1, u - a - 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $u = 0$. If $r \geq 1, s = 0$ and $u = 0$, then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1 - r, -1, -a, -1)$ which lies between $\mathbf{0}$
and \(-v_8\). If \(r \geq 1, s \geq 1\) and \(u = 0\), then \(x\) lies between \(0\) and \(v_1\) unless \(t = a + 1\), in which case let \(x' = x - v_1 = (a + 1 - r, s - a - 1, 1, a - 1)\) which lies between \(0\) and \(-v_7\).

Case 6: \(x = (-r, a + 1, t, u)\) where \(0 \leq r \leq a, 0 \leq t \leq a - 1\) and \(0 \leq u \leq a + 1\). Let \(x' = x - v_5 = (a - r, 1, t - a + 1, u - a - 2)\), which lies between \(0\) and \(-v_4\) unless \(u = 0\) in which case \(x\) lies between \(0\) and \(v_1\).

Case 7: \(x = (-a - 1, s, t, u)\) where \(0 \leq s \leq a, 0 \leq t \leq a - 1\) and \(0 \leq u \leq a + 1\). Let \(x' = x - v_5 = (-1, s - a, t - a + 1, u - a - 2)\), which lies between \(0\) and \(-v_8\) unless \(u = 0\) in which case \(x\) lies between \(0\) and \(v_1\).

This completes the cases for the orthant of \(v_5\).

**Orthant of** \(v_6, k = 0 \pmod{2}\)

Now suppose \(x\) lies in the orthant of \(v_6\) but not between \(0\) and \(v_6\). Then the first coordinate of \(x\) is equal to \(-a - 1\) or the second coordinate is equal to \(a + 1\), or the fourth equals \(-a - 1\). We distinguish seven cases.

Case 1: \(x = (-a - 1, a + 1, -t, -a - 1)\) where \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (-1, 1, a + 1 + t, a - u)\), which lies between \(0\) and \(v_5\) unless \(t \leq 1\), in which case let \(x'' = x' - v_5 = (a - 1, a - 2, -t, -u - 2)\) which lies between \(0\) and \(-v_7\) unless \(u = a\). If \(t = 1\) and \(u = a\) then \(x'\) lies between \(0\) and \(v_4\). If \(t = 0\) and \(u = a\) then let \(x'' = x' - v_1 = (a, -a, 1, a + 1)\) and \(x''' = x'' + v_6 = (0, 0, -a, 1)\) which lies between \(0\) and \(v_7\).

Case 2: \(x = (-a - 1, a + 1, -t, -u)\) where \(0 \leq t \leq a + 1\) and \(0 \leq u \leq a\). Let \(x' = x - v_6 = (-1, 1, a + 1 + t, a - u)\), which lies between \(0\) and \(v_5\) unless \(t \leq 1\), in which case let \(x'' = x' - v_5 = (a - 1, 1 - a, 2 - t, -u - 2)\) which lies between \(0\) and \(-v_7\) unless \(u = a\). If \(t = 1\) and \(u = a\) then \(x'\) lies between \(0\) and \(v_4\). If \(t = 0\) and \(u = a\) then let \(x'' = x' - v_1 = (a, -a, 1, a + 1)\) and \(x''' = x'' + v_6 = (0, 0, -a, 1)\) which lies between \(0\) and \(v_7\).

Case 3: \(x = (-a - 1, s, -t, -a - 1)\) where \(0 \leq s \leq a\) and \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (-1, s - a, a + 1 - t, -1)\), which lies between \(0\) and \(v_3\) unless \(s = a\) in which case let \(x'' = x' - v_3 = (a, -a, 1, -t, a - 1)\) which lies between \(0\) and \(-v_3\).

Case 4: \(x = (-r, a + 1, -t, -a - 1)\) where \(0 \leq r \leq a\) and \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (a - r, 1, a + 1 - t, -1)\), which lies between \(0\) and \(v_2\) unless \(r = 0\) in which case let \(x'' = x' - v_2 = (1, -a, -t, -a - 1)\) which lies between \(0\) and \(-v_1\) unless \(t = a + 1\) in which case \(x' = (a, 1, 0, -1)\) which lies between \(0\) and \(-v_4\).

Case 5: \(x = (-r, s, -t, -a - 1)\) where \(0 \leq r, s \leq a\) and \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (a - r, s - a, a + 1 - t, -1)\), which lies between \(0\) and \(-v_7\) unless \(r = 0\) or \(s = 0\), in which case let \(x'' = x' + v_7 = (1 - r, s - 1, -t - 1, a)\). If \(r = 0\) and \(s \geq 1\) then \(x''\) lies between \(0\) and \(-v_3\) unless \(t = a + 1\), in which case let \(x''' = x'' + v_3 = (-a, s - a, -1, 0)\) which lies between \(0\) and \(v_6\). If \(r = 1\) and \(s = 0\) then \(x\) lies between \(0\) and \(-v_8\) unless \(t = a + 1\), in which case let \(x''' = x + v_8 = (a, a, -1, 0)\) which lies between \(0\) and \(-v_4\).

Case 6: \(x = (-r, a + 1, -t, -u)\) where \(0 \leq r, u \leq a\) and \(0 \leq t \leq a + 1\). Let \(x' = x - v_6 = (a - r, 1, a + 1 - t, a - u)\), which lies between \(0\) and \(v_8\) unless \(t = 0\) in which case let \(x = (-r, a + 1, 0, -u)\) which lies between \(0\) and \(v_1\) unless \(u = a\). If \(t = 0\) and \(u = a\) then let \(x'' = x - v_1 = (a + 1 - r, 0, -a, -1)\) which lies between \(0\) and \(-v_4\) unless \(r = 0\) in
which case $x = (0, a + 1, 0, -a)$ which lies between 0 and $v_1$.

Case 7: $x = (-a - 1, s, -t, -u)$ where $0 \leq s, u \leq a$ and $0 \leq t \leq a + 1$. Let $x' = x - v_6 = (-1, s - a, a + 1 - t, a - u)$, which lies between 0 and $v_4$ unless $t = 0$ in which case $x = (-a - 1, s, 0, -u)$ which lies between 0 and $v_1$ unless $u = a$. If $t = 0$ and $u = a$ then let $x'' = x - v_1 = (0, s - a - 1, -a, -1)$ which lies between 0 and $-v_8$ unless $s = 0$ in which case let $x''' = x'' + v_8 = (a, -1, 0, a)$ which lies between 0 and $-v_6$.
This completes the cases for the orthant of $v_6$.

**Orthant of $v_7$, $k \equiv 0 \pmod{2}$**

Now suppose $x$ lies in the orthant of $v_7$ but not between 0 and $v_7$. Then the first coordinate of $x$ is equal to $a$ or $-a + 1$ or the second equals $a$ or $a + 1$. We distinguish seven cases.

Case 1: $x = (-r, s, -t, u)$ where $a \leq r, s \leq a + 1$, $2 \leq t \leq a + 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_7 = (a - 1 - r, s - a, a + 2 - t, u - a - 1)$, which lies between 0 and $v_1$ unless $u \leq 1$, in which case let $x'' = x - v_6 = (a - r, s - a, a + 1 - t, a + u)$ which lies between 0 and $v_5$.

Case 2: $x = (-a, a, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a + 1$. If $u = 0$ then $x$ lies between 0 and $v_6$. If $u \geq 1$ then let $x' = x - v_7 - v_3 = (a, a, 1 - t, u - 1)$, which lies between 0 and $v_8$.

Case 3: $x = (-a - 1, a, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_1 = (0, 0, -a - t, a - 1 + u)$. If $u \leq 1$ then $x'$ lies between 0 and $-v_2$. If $u \geq 2$ then let $x'' = x' + v_2 = (a - 1, a, 1 - t, u - 1)$, which lies between 0 and $v_8$.

Case 4: $x = (-a, a + 1, -t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_1 = (1, 0, -a - t, a - 1 + u)$. If $u \leq 1$ then $x'$ lies between 0 and $-v_3$. If $u \geq 2$ then let $x'' = x' + v_3 = (-a, -a + 1, 1 - t, u - 1)$, which lies between 0 and $v_4$.

Case 5: $x = (-a - 1, a + 1, t, u)$ where $0 \leq t \leq 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_1 = (0, 0, -a - t, a - 1 + u)$. If $u \leq 1$ then $x'$ lies between 0 and $v_7$. If $u \geq 2$ then let $x'' = x' - v_7 = (a - 1, -a + 1, 2 - t, u - 2)$, which lies between 0 and $-v_6$.

Case 6: $x = (-r, s, -t, u)$ where $0 \leq r \leq a - 1$, $a \leq s \leq a + 1$ and $0 \leq t, u \leq a + 1$. Let $x' = x - v_7 = (a - 1 - r, s - a + 1, a + 2 - t, u - a - 1)$, which lies between 0 and $v_2$ unless $t = 0$ or $u = 0$, in which case let $x'' = x' - v_2 = (-r, s - 2a, 1 - t, u - 1)$. If $t = 0$ and $u \geq 1$ then $x''$ lies between 0 and $v_4$. If $t = 0$ and $u = 0$ then $x''$ lies between 0 and $v_3$ unless $s = a$, in which case let $x''' = x'' - v_3 = (a + 1 - r, -1, -a, a - 1)$ which lies between 0 and $-v_1$. If $t \geq 1$ and $u = 0$ then $x'' = (-r, s - 2a, 1 - t, -1)$ which lies between 0 and $-v_8$.

Case 7: $x = (-r, s, -t, u)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a - 1$ and $0 \leq t, u \leq a + 1$. Let $x' = x - v_7 = (a - 1 - r, s - a + 1, a + 2 - t, u - a - 1)$, which lies between 0 and $v_3$ unless $t = 0$ or $u = 0$, in which case let $x'' = x' - v_3 = (2a - r, s, 1 - t, u - 1)$. If $t = 0$ and $u = 0$ then $x$ lies between 0 and $v_1$. If $t = 0$ and $u \geq 1$ then $x''$ lies between 0 and $v_8$. If $t \geq 1$ and $u = 0$ then $x''$ lies between 0 and $-v_4$.
This completes the cases for the orthant of $v_7$. 

Orhant of $v_8$, $k \equiv 0 \pmod{2}$

Finally suppose $x$ lies in the orthant of $v_8$ but not between $0$ and $v_8$. Then at least one of the first three coordinate of $x$ is equal to $a + 1$. We distinguish seven cases.

Case 1: $x = (a+1, a+1, t, u)$ where $0 \leq u \leq a+1$ and $0 \leq r \leq a + 1$. Let $x' = x - v_8 = (1,1,1, u-a-1)$, which lies between $0$ and $v_2$ unless $u = 0$, in which case let $x'' = x' - v_2 = (-a + 2, -a, -a, -1)$ which lies between $0$ and $-v_8$.

Case 2: $x = (a + 1, a + 1, t, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (1,1,1, t-a, u-a-1)$, which lies between $0$ and $v_4$.

Case 3: $x = (a + 1, s, a + 1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (1, s-a, 1, u-a-1)$, which lies between $0$ and $v_7$ unless $s = 0$, in which case let $x'' = v' = (a, s-a, u-a)$, which lies between $0$ and $v_2$ unless $u = a + 1$. If $s = 0$ and $u = a + 1$ then $x' = (1, s-a, 0, u-a)$. If $t = 0$ and $u = a + 1$ then $x'' = (1, s-a, -a, 0)$ which lies between $0$ and $v_1$.

Case 4: $x = (r, a + 1, a + 1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (r-a, 1, 1, u-a-1)$, which lies between $0$ and $v_1$ unless $r = 0$, in which case let $x'' = v' = (a, r-a, u-a)$, which lies between $0$ and $v_7$ unless $u = a + 1$. If $r = 0$ and $u = a + 1$ then $x' = (1, s-a, -a, 0)$ which lies between $0$ and $v_1$.

Case 5: $x = (r, a + 1, t, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (r-a, t-a, u-a-1)$, which lies between $0$ and $v_5$ unless $t = 0$, in which case let $x'' = v' = (r-a, s-a, 0)$, which lies between $0$ and $v_2$ unless $u = a + 1$. If $s = 0$ and $u = a + 1$ then $x' = (1, s-a, -a, 0)$ which lies between $0$ and $v_1$.

Case 6: $x = (r, a + 1, t, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (r-a, t-a, u-a-1)$, which lies between $0$ and $v_6$ unless $u = 0$, in which case $x$ lies between $0$ and $v_2$ unless $r = a$. If $r = a$ and $u = 0$ then $x'' = v' = (r-a, s-a, 0)$, which lies between $0$ and $v_3$.

Case 7: $x = (r, s, a + 1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (r-a, s-a, 1, u-a-1)$, which lies between $0$ and $v_3$ unless $s = 0$ or $u = 0$, in which case let $x'' = v' = (r-a, s-a, u-a)$, which lies between $0$ and $v_1$ unless $u = a + 1$. If $s = 0$ and $u = a + 1$ then $x'' = v' = (r-a, s-a, 0)$, which lies between $0$ and $v_1$ unless $u = a + 1$. If $s = 0$ and $u = a + 1$ then $x'' = v' = (r-a, s-a, u-a)$, which lies between $0$ and $v_1$.

This completes the cases for the orthant of $v_8$.

This also completes the proof of the theorem for any $k \equiv 0 \pmod{2}$.

Now we consider the eight orthants $v_1, \ldots, v_8$ in turn for the case $k \equiv 1 \pmod{2}$.

Orhant of $v_1$, $k \equiv 1 \pmod{2}$

First suppose that $x$ lies within the orthant of $v_1$, but not between $0$ and $v_1$. Then the first coordinate of $x$ is equal to $-a$ or $-a - 1$, or the third coordinate equals $-a$ or $-a - 1$, or the fourth equals $a + 1$. We distinguish seven cases.

Case 1: $x = (-r, t, -a, a + 1)$ where $a \leq r, t \leq a + 1$ and $0 \leq s \leq a + 1$. Let $x' = x - v_1 = (a, a, a, a)$.
(a – 1 – r, s – a – 1, a – 1 – t, 1), which lies between 0 and v₈ unless s ≤ 1 in which case let \( x'' = x' - v₈ = (2a - 2 - r, s - 2, 2a - 1 - t, -a) \) which lies between 0 and \(-v₁\).

Case 2: \( x = (-r, s, -t, u) \) where \( a ≤ r, t ≤ a + 1 \) and \( 0 ≤ s ≤ a + 1 \) and \( 0 ≤ u ≤ a \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a) \), which lies between 0 and \( v₅ \) unless \( s = 0 \) or \( u ≤ 1 \), in which case let \( x'' = x' - v₅ = (2a - 1 - r, s - 1, 2a - t, u - 2) \). If \( s = 0 \) and \( u ≤ 1 \) then \( x'' \) lies between 0 and \(-v₁\), unless \( t = a \), in which case let \( x''' = x'' + v₁ = (a - r, a, u + a - 2) \) which lies between 0 and \( v₄ \). If \( s = 0 \) and \( u ≥ 2 \) then \( x'' \) lies between 0 and \(-v₇\). If \( s ≥ 1 \) and \( u ≤ 1 \) then \( x'' \) lies between 0 and \(-v₈\) unless \( s = a + 1 \), in which case let \( x''' = x'' + v₈ = (a - r, 1, a - t, u - 1) \) which lies between 0 and \( v₁ \).

Case 3: \( x = (-r, s, -t, a + 1) \) where \( a ≤ r ≤ a + 1 \), 0 ≤ \( s ≤ a + 1 \) and \( 0 ≤ t ≤ a - 1 \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1) \), which lies between 0 and \(-v₃\) unless \( s = 0 \), in which case let \( x'' = x' + v₆ = (2a - 1 - r, -1, -t, -a + 1) \) which lies between 0 and \(-v₄\).

Case 4: \( x = (-r, s, -t, a + 1) \) where \( 0 ≤ r ≤ a - 1 \), 0 ≤ \( s ≤ a + 1 \) and \( a ≤ t ≤ a + 1 \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1) \), which lies between 0 and \(-v₃\) unless \( s ≤ 1 \), in which case let \( x'' = x' + v₃ = (-2 - r, s - 2, 2a - 2 - t, -a + 1) \) which lies between 0 and \(-v₂\).

Case 5: \( x = (-r, s, -t, a + 1) \) where \( 0 ≤ r ≤ a - 1 \), 0 ≤ \( s ≤ a + 1 \) and \( 0 ≤ t ≤ a + 1 \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, 1) \), which lies between 0 and \(-v₃\) unless \( s = 0 \) in which case let \( x'' = x' + v₇ = (-r - 1, -1, -t - 1, -a + 2) \) which lies between 0 and \( v₅ \).

Case 6: \( x = (-r, s, -t, u) \) where \( 0 ≤ r ≤ a - 1 \), 0 ≤ \( s ≤ a + 1 \), \( a ≤ t ≤ a + 1 \) and \( 0 ≤ u ≤ a \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a) \), which lies between 0 and \(-v₄\) unless \( s = 0 \) or \( u = 0 \) in which case let \( x'' = x' + v₄ = (-r - 1, s - 1, 2a - 1 - t, u - 1) \). If \( s = 0 \) and \( u = 0 \) then let \( x''' = x'' + v₃ = (a - r, a, a + 1 - t, a - 2) \) which lies between 0 and \(-v₅\). If \( s = 0 \) and \( u ≥ 1 \) then \( x'' \) lies between 0 and \(-v₆\). If \( s ≥ 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( v₃ \) unless \( s = a + 1 \), in which case \( x' \) lies between 0 and \( v₆ \).

Case 7: \( x = (-r, s, -t, u) \) where \( a ≤ r ≤ a + 1 \), 0 ≤ \( s ≤ a + 1 \), \( 0 ≤ t ≤ a - 1 \) and \( 0 ≤ u ≤ a \). Let \( x' = x - v₁ = (a - 1 - r, s - a - 1, a - 1 - t, u - a) \), which lies between 0 and \(-v₂\) unless \( t = 0 \) or \( u = 0 \) in which case let \( x'' = x' + v₂ = (2a - r, s, -t + 1, u - 1) \). If \( t = 0 \) and \( u = 0 \) then let \( x''' = x'' + v₃ = (a + 1 - r, s - a + 1, -a + 1, a) \) which lies between 0 and \(-v₃\) unless \( a ≤ s ≤ a + 1 \), in which case let \( x''' = x - v₇ = (a - r, s - a, a, a - 1) \) which lies between 0 and \( v₄ \). If \( t = 0 \) and \( u ≥ 1 \) then \( x'' \) lies between 0 and \(-v₅\) unless \( s = a + 1 \) or \( u = u \) in which case let \( x'' = x'' + v₅ = (a - r, s - a, -a, -a + u + 1) \). If \( s = a + 1 \) then \( x' \) lies between 0 and \( v₃ \). If \( 1 ≤ s ≤ a \) and \( u = a \) then \( x'' \) lies between 0 and \(-v₈\). If \( t ≥ 1 \) and \( u = 0 \) then \( x'' \) lies between 0 and \( v₆ \) unless \( s = a + 1 \), in which case \( x' \) lies between 0 and \( v₃ \).

This completes the cases for the orthant of \( v₁ \).

Orthant of \( v₂ \), \( k ≡ 1 \pmod{2} \)

Now suppose that \( x \) lies in the orthant of \( v₂ \) but not between 0 and \( v₂ \). Then the third coordinate of \( x \) is equal to \(-a + 1\), \(-a\) or \(-a - 1\), or the fourth coordinate equals \( a \) or \( a + 1 \).
We distinguish three cases.

Case 1: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1 \), \( a - 1 \leq t \leq a + 1 \) and \( a \leq u \leq a + 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between \( 0 \) and \( v_8 \) unless \( r \leq 1 \) or \( s \leq 1 \), in which case let \( x'' = x' - v_8 = (r - 2, s - 2a - 2 - t, u - 2a) \). If \( r \leq 1 \) and \( s \geq 2 \) then \( x'' \) lies between \( 0 \) and \( v_3 \). If \( r \geq 2 \) and \( s \leq 1 \) then \( x'' \) lies between \( 0 \) and \( -v_1 \). If \( r \leq 1 \) and \( s \leq 1 \) then \( x'' \) lies between \( 0 \) and \( -v_2 \) unless \( t = a - 1 \) or \( u = a \). Let \( x''' = x'' + v_3 = (r + a - 1, s + a - 1, a - t, u - a - 1) \). If \( r \leq 1 \) and \( s \leq 1 \) and \( u = a \) then \( x''' \) lies between \( 0 \) and \( v_3 \). If \( r = 0 \) and \( s = 0 \) then \( x \) lies between \( 0 \) and \( v_1 \). If \( r \leq 1 \) and \( s \leq 1 \) and \( t = a - 1 \) and \( u = a + 1 \) then \( x''' \) lies between \( 0 \) and \( -v_5 \).

Case 2: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1 \), \( 0 \leq t \leq a - 2 \) and \( a \leq u \leq a + 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between \( 0 \) and \( -v_6 \) unless \( r = 0 \) or \( s = 0 \). Let \( x'' = x' + v_6 = (r - 1, s - 1, -t - 1, u - 2a + 1) \). If \( r = 0 \) and \( s = 0 \) then \( x'' \) lies between \( 0 \) and \( v_5 \) unless \( u = a \), in which case let \( x''' = x'' - v_5 = (r + a - 1, s + a - 1, a - t, u - a - 1) \) which lies between \( 0 \) and \( -v_8 \). If \( r = 0 \) and \( s \geq 1 \) then \( x''' \) lies between \( 0 \) and \( v_7 \). If \( r \geq 1 \) and \( s = 0 \) then \( x''' \) lies between \( 0 \) and \( -v_4 \).

Case 3: \( x = (r, s, -t, u) \) where \( 0 \leq r, s \leq a + 1 \), \( a - 1 \leq t \leq a + 1 \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_2 = (r - a - 1, s - a - 1, a - 2 - t, u - a + 1) \), which lies between \( 0 \) and \( v_5 \) unless \( r = 0 \) or \( s = 0 \) or \( u = 0 \), in which case let \( x'' = x' - v_5 = (r - 1, s - 1, 2a - 1 - t, u - 1) \). If \( r = 0 \) and \( s = 0 \) and \( u = 0 \) then \( x'' \) lies between \( 0 \) and \( v_3 \), in which case let \( x''' = x'' + v_3 = (r + a, s + a, a - t, u + a) \) which lies between \( 0 \) and \( -v_5 \). If \( r = 0 \) and \( s \geq 1 \) and \( u = 0 \) then \( x''' \) lies between \( 0 \) and \( v_3 \), in which case let \( x'' = x' - v_3 = (r + a, s + a, a - t, u + a) \) which lies between \( 0 \) and \( -v_5 \). If \( r = 0 \) and \( s \geq 1 \) and \( u = 0 \) then \( x''' \) lies between \( 0 \) and \( v_3 \), in which case let \( x''' = x'' + v_3 = (r + a, s + a, a - t, u + a) \) which lies between \( 0 \) and \( v_3 \). If \( r = a + 1 \) and \( t = a - 1 \) then \( x''' \) lies between \( 0 \) and \( -v_5 \).

This completes the cases for the orthant of \( v_2 \).

**Orthant of** \( v_3 \), \( k \equiv 1 \pmod{2} \)

Now suppose that \( x \) lies in the orthant of \( v_3 \) but not between \( 0 \) and \( v_3 \). Then the second coordinate of \( x \) is equal to \( a \) or \( a + 1 \), or the third coordinate equals \( a \) or \( a + 1 \), or the
fourth equals $-a - \frac{1}{2}$. We distinguish seven cases.

Case 1: $x = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $a \leq s, t \leq a + 1$. Let $x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $0$ and $-v_8$ unless $r \leq 1$, in which case let $x'' = x' + v_8 = (2 - r, s - 2a + 2, t - 2a + 1, a)$ which lies between $0$ and $-v_3$.

Case 2: $x = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$ and $a \leq s, t \leq a + 1$ and $0 \leq u \leq a$. Let $x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, u - a - u)$, which lies between $0$ and $-v_3$ unless $r = 0$ or $u \leq 1$, in which case let $x'' = x' + v_3 = (1 - r, s - 2a + 1, t - 2a, 2u - u)$.

If $r = 0$ and $u \leq 1$ then $x''$ lies between $0$ and $-v_3$, unless $r = 0$ and $u \geq 2$ then $x''$ lies between $0$ and $-v_3$. If $r \geq 1$ and $u \leq 1$ then $x''$ lies between $0$ and $v_7$.

Case 3: $x = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $0 \leq s, t \leq a - 1$. Let $x' = x - v_3 = (2 - r, s + 2, t - 2a + 2, a - 1)$ which lies between $0$ and $v_2$.

Case 5: $x = (-r, s, t, -a - 1)$ where $0 \leq r \leq a + 1$ and $0 \leq s, t \leq a - 1$. Let $x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, -1)$, which lies between $0$ and $v_6$ unless $r = 0$, in which case let $x'' = x' + v_6 = (1 - r, s - 2a + 1, t, a - 1)$ which lies between $0$ and $-v_7$.

Case 6: $x = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$, $0 \leq s \leq a - 1$, $a \leq t \leq a + 1$ and $0 \leq u \leq a$. Let $x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, u - a - u)$, which lies between $0$ and $-v_1$ unless $r = 0$ or $u = 0$, in which case let $x'' = x' + v_7 = (1 - r, s + 1, t - 2a + 1, 1 - u)$. If $r = 0$ and $u = 0$ then $x''$ lies between $0$ and $v_2$ unless $t = a$, in which case let $x'' = x' - v_2 = (1 - r, s - a, t - a - 1, 2 - a - u)$ which lies between $0$ and $v_5$. If $r = 0$ and $u \geq 1$ then $x''$ lies between $0$ and $v_6$. If $r \geq 1$ and $u = 0$ then $x''$ lies between $0$ and $v_1$ unless $r = a + 1$, in which case let $x'' = x'' - v_1 = (a - r, s - a, t - a, 1 - a - u)$ which lies between $0$ and $v_2$.

Case 7: $x = (-r, s, t, -u)$ where $0 \leq r \leq a + 1$, $a \leq s \leq a + 1$, $0 \leq t \leq a - 1$ and $0 \leq u \leq a$. Let $x' = x - v_3 = (a + 1 - r, s - a + 1, t - a + 1, u - a - u)$, which lies between $0$ and $v_2$ unless $t = 0$ or $u = 0$. If $t = 0$ and $u = 0$ then $x''$ lies between $0$ and $v_6$ unless $r \leq 1$, in which case let $x'' = x' + v_8 = (a + 1 - r, s - a, t + a - 1, -a - u)$ which lies between $0$ and $v_1$. If $t = 0$ and $u \geq 1$ then $x''$ lies between $0$ and $v_8$ unless $r = 0$ or $u = a$, in which case let $x''' = x'' - v_5 = (a - r, s - a, t + a - 1, 1 - u)$. If $r = 0$, $t = 0$ and $u = a$ then let $x'' = x''' + v_8 = (1 - r, s - 2a + 1, t, 2a - u)$ which lies between $0$ and $v_3$. If $r = 0$, $t = 0$ and $1 \leq u \leq a - 1$ then $x'''$ lies between $0$ and $-v_5$. If $1 \leq r \leq a$, $t = 0$ and $u = a$ then $x'''$ lies between $0$ and $v_3$. If $r = a + 1$, $t = 0$ and $u = a$ then $x''$ lies between $0$ and $-v_6$.

If $t \geq 1$ and $u = 0$ then $x''$ lies between $0$ and $-v_6$.

This completes the cases for the orthant of $v_3$. 

R. R. Lewis

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Orthant of $v_4$, $k \equiv 1 \pmod{2}$

Now suppose $x$ lies in the orthant of $v_4$ but not between $0$ and $v_4$. Then the first coordinate of $x$ is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals $a+1$ or the fourth equals $a$ or $a+1$. We distinguish fifteen cases.

Case 1: $x = (-a-1, a+1, a+1, u)$ where $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, 1, 1, u-a+1)$, which lies between $0$ and $v_4$.

Case 2: $x = (-a-1, a+1, a+1, u)$ where $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, 1, 1, u-a+1)$, which lies between $0$ and $v_4$.

Case 3: $x = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, 1, t-a, u-a+1)$, which lies between $0$ and $v_7$.

Case 4: $x = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-s, a+1, u-a+1)$, which lies between $0$ and $-v_2$.

Case 5: $x = (-r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, 1, 1, u-a+1)$, which lies between $0$ and $-v_5$.

Case 6: $x = (-r, s, a+1, u)$ where $0 \leq r, s \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, s, a+1, u-a+1)$, which lies between $0$ and $-v_7$.

Case 7: $x = (-r, a+1, t, u)$ where $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, 1, t-a, u-a+1)$, which lies between $0$ and $v_2$ unless $t \leq 1$, in which case let $x'' = x' - v_2 = (-r-1, -a, t-2, u)$ and $x''' = x'' - v_8 = (a-r-2, -1, a+t-2, u-a-1)$.

Case 8: $x = (-a-1, s, t, u)$ where $0 \leq s, t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (-1, s-a, t-a, u-a+1)$, which lies between $0$ and $v_8$ unless $s = 0$, in which case let $x'' = x' - v_8 = (-2, s-1, t, u-2a)$ which lies between $0$ and $-v_1$ unless $t = a$. If $s = 0$ and $t = a$ then let $x''' = x'' + v_1 = (-1, s+a, t-a+1, u-a)$, which lies between $0$ and $v_4$.

Case 9: $x = (-r, a+1, a+1, u)$ where $0 \leq r \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (a-r, 1, 1, u-a+1)$, which lies between $0$ and $-v_8$ unless $r = 0$, in which case let $x'' = x' + v_8 = (1-r, 2-a, 1-a, u+2)$ which lies between $0$ and $v_8$.

Case 10: $x = (-a-1, s, a+1, u)$ where $0 \leq s \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, s-a, 1, u-a+1)$, which lies between $0$ and $-v_2$.

Case 11: $x = (-a-1, a+1, t, u)$ where $0 \leq t \leq a$ and $0 \leq u \leq a-1$. Let $x' = x - v_4 = (-1, 1, t-a, a-u+1)$ which lies between $0$ and $v_7$.

Case 12: $x = (-r, s, t, u)$ where $0 \leq r, s, t \leq a$ and $a \leq u \leq a+1$. Let $x' = x - v_4 = (a-r, s-a, t-a, u-a+1)$, which lies between $0$ and $-v_3$ unless $s = 0$ or $t = 0$. If $s = 0$ and $t = 0$ then let $x''' = x' + v_3 = (-1-r, s-1, t-1, u-2a+1)$ which lies between $0$ and $v_5$ unless $r = a$ or $u = a$ in which case let $x''' = x'' - v_5 = (a-1-r, s+a-1, t+a, u-a-1)$. If $r = a$ and $u = a$, then $x$ lies between $0$ and $-v_6$. If $r = a$ and $u = a+1$, then $x'''$
lies between 0 and v₄. If r ≤ a − 1 and u = a, then x‴ lies between 0 and −v₈. If s = 0 and 1 ≤ t ≤ a then let x‴ lies between 0 and −v₂ unless t = a, in which case let x‴ = x‴ + v₂ = (a − r, a + s, t − a + 1, u − a) which lies between 0 and −v₅. If 1 ≤ s ≤ a and t = 0 then x‴ lies between 0 and v₇ unless r = a, in which case let x‴ = x‴ − v₇ = (a − r, s − a − 1, t + a − 1, u − a) which lies between 0 and −v₆.

Case 13: x = (−r, s, a + 1, u) where 0 ≤ r, s ≤ a and 0 ≤ u ≤ a − 1. Let x′ = x − v₄ = (a − r, s − a, 1, u − a + 1) which lies between 0 and −v₁ unless r = 0, in which case let x″ = x′ + v₁ = (1 − r, s + 1, −a + 2, u + 1) which lies between 0 and v₂ unless u = a − 1. If r = 0 and u = a − 1 then let x‴ = x‴ − v₂ = (−a − r, s − a, 0, u − a + 2) which lies between 0 and v₅.

Case 14: x = (−r, a + 1, t, u) where 0 ≤ r, t ≤ a and 0 ≤ u ≤ a − 1. Let x′ = x − v₄ = (a − r, 1, t − a, u − a + 1), which lies between 0 and v₆ unless t = 0, in which case let x″ = x′ − v₆ = (−r − a, t − 1, u + 1) which lies between 0 and v₈ unless r = a. If r = a and t = 0 then let x‴ = x″ − v₈ = (a − 1 − r, 0, t + a − 1, u − a) which lies between 0 and v₃.

Case 15: x = (−a − 1, s, t, u) where 0 ≤ s, t ≤ a and 0 ≤ u ≤ a − 1. Let x′ = x − v₄ = (−1 − a, s, a + 1 − t, u − a + 1), which lies between 0 and v₅ unless t = 0 or u = 0 in which case let x″ = x′ − v₅ = (a − 1, s, t + 1, u − 1). If t = 0 and u = 0, then x‴ lies between 0 and −v₈ unless s = a, in which case let x‴ = x″ + v₈ = (0, s − a + 1, t − a + 1, u + 1) which lies between 0 and v₁. If t = 0 and 1 ≤ u ≤ a − 1, then x‴ lies between 0 and −v₅. If 1 ≤ t ≤ a − 1 and u = 0, then x‴ lies between 0 and −v₈ unless s = a, in which case x‴ lies between 0 and v₁. If t = a and u = 0 then x‴ lies between 0 and −v₆ unless s = a, in which case let x‴ = x‴ − v₆ = (a, s − 2a + 1, t − 2a + 1, u + 1) which lies between 0 and −v₃.

This completes the cases for the orthrant of v₄.

**Orphant of v₅, k ≡ 1 (mod 2)**

Now suppose x lies in the orthrant of v₅ but not between 0 and v₅. Then the first coordinate of x is equal to −a − 1 or the second coordinate is equal to −a − 1, or the fourth equals −a − 1, −a or −a − 1. We distinguish seven cases.

Case 1: x = (−a − 1, −a − 1, −t, −u) where 0 ≤ t ≤ a + 1 and a − 1 ≤ u ≤ a + 1. Let x′ = x − v₅ = (−1, −1, a + 1 − t, a − 2 − u), which lies between 0 and −v₂ unless t ≤ 2, in which case let x″ = x′ + v₂ = (a, a, 3 − t, 2a − 3 − u) which lies between 0 and −v₅.

Case 2: x = (−a − 1, −a − 1, −t, −u) where 0 ≤ t ≤ a + 1 and 0 ≤ u ≤ a − 2. Let x = x − v₅ = (−1, −1, a + 1 − t, a − 2 − u), which lies between 0 and −v₆ unless t ≤ 1, in which case let x″ = x′ + v₆ = (a + 1, a − 1, 2 − t, −2 − u) which lies between 0 and −v₈.

Case 3: x = (−a − 1, −s, −t, −u) where 0 ≤ s ≤ a, 0 ≤ t ≤ a + 1 and a − 1 ≤ u ≤ a + 1. Let x = x − v₅ = (−1, −s, a + 1 − t, a − 2 − u), which lies between 0 and v₃ unless s = 0 or t ≤ 1, in which case let x″ = x′ − v₃ = (a + 1 − s, 2 − t, 2a − 2 − u). If s = 0 and t ≤ 1 then x‴ lies between 0 and −v₅ unless u = a − 1, in which case let x‴ = x‴ + v₅ = (0, 1 − a − s, 1 − a − t, a − u) which lies between 0 and v₈. If s = 0 and t ≥ 2 then x‴ lies between 0 and v₂ unless t = a + 1, in which case let
\[ x''' = x'' - v_2 = (-1, -a - s, a - t, a - 1 - u) \] which lies between 0 and \( v_8 \). If \( 1 \leq s \leq a \) and \( t \leq 1 \) then \( x'' \) lies between 0 and \(-v_7\).

Case 4: \( x = (-r, -a - 1, -t, -u) \) where \( 0 \leq r \leq a, 0 \leq t \leq a + 1 \) and \( a - 1 \leq u \leq a + 1 \). Let \( x' = x - v_5 = (a - r, -a - s, a + 1 - t, a - 2 - u) \), which lies between 0 and \(-v_1\) unless \( r = 0 \) or \( t \leq 1 \), in which case let \( x'' = x' + v_1 = (1 - r, a, 2 - t, 2a - 2 - u) \). If \( r = 0 \) and \( t \leq 1 \) then \( x'' \) lies between 0 and \(-v_5\) unless \( u = a - 1 \), in which case let \( x''' = x'' + v_5 = (1 - a - r, 1 - a - t, a - u) \) which lies between 0 and \( v_8 \). If \( r = 0 \) and \( 2 \leq t \leq a + 1 \) then \( x'' \) lies between 0 and \( v_2 \) unless \( t = a + 1 \), in which case let \( x''' = x'' - v_2 = (-a - r, -1, -a - t, a - 1 - u) \) which lies between 0 and \( v_5 \). If \( r \geq 1 \) and \( t \leq 1 \) then \( x'' \) lies between 0 and \( v_4 \).

Case 5: \( x = (-r, -s, -t, -u) \) where \( 0 \leq r, s \leq a, 0 \leq t \leq a + 1 \) and \( a - 1 \leq u \leq a + 1 \). Let \( x' = x - v_5 = (a - r, -a - s, a + 1 - t, a - 2 - u) \), which lies between 0 and \(-v_8\) unless \( r = 0 \) or \( s = 0 \) or \( t = 0 \), in which case let \( x'' = x' + v_8 = (1 - r, 1 - s, 1 - t, 2a - 1 - u) \). If \( r = 0 \), \( s = 0 \) and \( t = 0 \) then \( x \) lies between 0 and \(-v_8\). If \( r = 0, s = 0 \) and \( 1 \leq t \leq a + 1 \) then \( x'' \) lies between 0 and \( v_2 \) unless \( u \leq a - 1 \), in which case let \( x''' = x'' - v_2 = (-a - r, -a - s, a - 1 - t, a - u) \). If \( a \leq t \leq a + 1 \) and \( a - 1 \leq u \leq a + 1 \) then \( x'' \) lies between 0 and \( v_5 \). If \( 1 \leq t \leq a - 1 \) and \( u = a - 1 \), then \( x'' \) lies between 0 and \(-v_5\). If \( a \leq t \leq a + 1 \) and \( u = a - 1 \), then \( x''' = x - v_7 = (a - r, -a - s, -a - t, a - 1 - u) \) which lies between 0 and \(-v_4\).

Case 6: \( x = (-r, -a - 1, -t, -u) \) where \( 0 \leq r \leq a, 0 \leq t \leq a + 1 \) and \( 0 \leq u \leq a - 2 \). Let \( x' = x - v_5 = (a - r, -a - s, a + 1 - t, a - 2 - u) \), which lies between 0 and \(-v_7\) unless \( t = 0 \), in which case let \( x'' = x + v_7 = (-r, a - 1 - t, -1, -u) \) which lies between 0 and \( v_3 \).

Case 7: \( x = (-a - 1, -s, -t, -u) \) where \( 0 \leq s \leq a, 0 \leq t \leq a + 1 \) and \( 0 \leq u \leq a - 2 \). Let \( x' = x - v_5 = (-1, a - s, a + 1 - t, a - 2 - u) \), which lies between 0 and \( v_4 \) unless \( t = 0 \), in which case let \( x'' = x - v_4 = (a - 1, -s, a - 1 - t, -1 - u) \) which lies between 0 and \(-v_1\).

This completes the cases for the orthant of \( v_5 \).

**Orthant of \( v_6 \), \( k \equiv 1 \pmod{2} \)**

Now suppose \( x \) lies in the orthant of \( v_6 \) but not between 0 and \( v_6 \). Then the first coordinate of \( x \) is equal to \( a + 1 \) or the second coordinate is equal to \( a + 1 \), or the third equals \(-a \) or \(-a - 1 \) or the fourth equals \(-a - 1 \). We distinguish fifteen cases.

Case 1: \( x = (a + 1, a + 1, -t, -a - 1) \) where \( a \leq t \leq a + 1 \). Let \( x' = x - v_6 = (1, 1, a - 1 - t, -1) \), which lies between 0 and \( v_6 \).

Case 2: \( x = (a + 1, a + 1, -t, -u) \) where \( a \leq t \leq a + 1 \) and \( 0 \leq u \leq a \). Let \( x' = x - v_6 = (1, 1, a - 1 - t, a - u) \), which lies between 0 and \( v_2 \) unless \( u = 0 \), in which case let \( x'' = x' - v_2 = (-a, a - 2a + 3 - t, 1 - u) \) which lies between 0 and \(-v_6\).

Case 3: \( x = (a + 1, a + 1, -t, -a - 1) \) where \( 0 \leq t \leq a - 1 \). Let \( x' = x - v_6 = (1, 1, a - 1 - t, -1) \), which lies between 0 and \(-v_8\).

Case 4: \( x = (a + 1, s, -t, -a - 1) \) where \( 0 \leq s \leq a \) and \( a \leq t \leq a + 1 \). Let \( x' = x - v_6 = (1, s - a, a - 1 - t, -1) \), which lies between 0 and \(-v_4\).

Case 5: \( x = (r, a + 1, -t, -a - 1) \) where \( 0 \leq r \leq a \) and \( a \leq t \leq a + 1 \). Let \( x' = x - v_6 =
(r - a, 1, a - 1 - t, -1), which lies between 0 and v7.

Case 6: x = (r, s, -t, -a - 1) where 0 \leq r, s \leq a and a \leq t \leq a + 1. Let x' = x - v6 = (r - a, s - a, a - 1 - t, -1), which lies between 0 and v5.

Case 7: x = (r, a + 1, -t, -a - 1) where 0 \leq r \leq a and 0 \leq t \leq a - 1. Let x' = x - v6 = (r - a, 1, a - 1 - t, -1), which lies between 0 and v3.

Case 8: x = (a + 1, s, -t, -a - 1) where 0 \leq s \leq a and 0 \leq t \leq a - 1. Let x' = x - v6 = (1, s - a, a - 1 - t, -1), which lies between 0 and -v1.

Case 9: x = (r, a, -t, -u) where 0 \leq r, u \leq a and a \leq t \leq a + 1. Let x' = x - v6 = (r - a, 1, a - 1 - t, a - u), which lies between 0 and v1 unless r = 0, in which case let x'' = x' - v1 = (r - 1, -a, 2a - 2 - t, -u) which lies between 0 and -v2 unless u = a in which case x' lies between 0 and v7.

Case 10: x = (a + 1, s, -t, -u) where 0 \leq s, u \leq a and a \leq t \leq a + 1. Let x' = x - v6 = (1, s - a, a - 1 - t, a - u), which lies between 0 and v3 unless s = 0, in which case let x'' = x' + v3 = (-a, s - 1, 2a - 2 - t, -u) which lies between 0 and -v2 unless u = a. If s = 0 and u = a then let x''' = x'' + v2 = (1, s + a, a - t, a - 1 - u) which lies between 0 and v2.

Case 11: x = (a + 1, a + 1, -t, -u) where 0 \leq t \leq a - 1 and 0 \leq u \leq a. Let x' = x - v6 = (1, 1, a - 1 - t, a - u), which lies between 0 and v5 unless a - 1 \leq u \leq a, in which case let x'' = x' + v5 = (1 - a, 1, a - 1 - t, 2 - u) which lies between 0 and v5.

Case 12: x = (r, s, -t, -a - 1) where 0 \leq r, s \leq a and 0 \leq t \leq a - 1. Let x' = x - v6 = (r - a, s - a, a - 1 - t, -1), which lies between 0 and -v2 unless t = 0, in which case let x'' = x' + v2 = (r + 1, s + 1, 1 - t, a - 2) and x''' = x'' + v5 = (r - a + 1, s - a + 1, a - 1) where x''' lies between 0 and v8.

Case 13: x = (r, s, -t, -a - 1) where 0 \leq r, s, u \leq a and a \leq t \leq a + 1. Let x' = x - v6 = (r - a, s - a, a - 1 - t, a - u), which lies between 0 and v8 unless r = 0 or s = 0, in which case let x'' = x' - v8 = (r - 1, s - 1, 2a - 1 - t, 1 - u). If r = 0 and s = 0 then x'' lies between 0 and -v2 unless t = a or a - 1 \leq u \leq a, in which case let x''' = x'' + v2 = (a, a, a + 1 - t, a - 2 - u) = 0. If t = a and 0 \leq a - 2 then x''' lies between 0 and v5. If a \leq t \leq a + 1 and a - 1 \leq u \leq a then let x'''' = x - v4 = (a, a, a, a - t, a - 1) which lies between 0 and v7. If r = 1 and 0 \leq s \leq a then let x'''' = x - v7 = (r + 1, s - a, a - 1 - u) which lies between 0 and v4. If 1 \leq r \leq a and s = 0 then let x'''' = x + v4 = (r - a, s + a, a - t, a - 1 - u) which lies between 0 and v1 unless u = a, in which case x''' lies between 0 and v7.

Case 14: x = (r, a + 1, -t, -u) where 0 \leq r, u \leq a and 0 \leq t \leq a - 1. Let x' = x - v6 = (r - a, 1, a - 1 - t, a - u), which lies between 0 and v4 unless u = 0, in which case let x'' = x' - v4 = (r - 1, 1, a, a - 1 - t, 1 - u) which lies between 0 and -v3 unless t = a - 1. If t = a - 1 and u = 0 then let x''' = x'' + v3 = (r - a - 1, 0, a - 2 - t, 1 - a - u) which lies between 0 and v7 unless r = 0, in which case x lies between 0 and v1.

Case 15: x = (a + 1, s, -t, -u) where 0 \leq s, u \leq a and 0 \leq t \leq a - 1. Let x' = x - v6 = (1, s - a, a - 1 - t, a - u), which lies between 0 and v7 unless u = 0, in which case let x'' = x' + v7 = (1 - a, s - 1, a - t, 1 - u) which lies between 0 and v1 unless t = a - 1. If t = a - 1 and u = 0 then let x''' = x'' - v1 = (0, s - a - 1, a - 2 - t, 1 - a - u) which lies
between \(0\) and \(-v_4\) unless \(s = 0\), in which case \(x\) lies between \(0\) and \(-v_3\).

This completes the cases for the orthant of \(v_6\).

**Orthant of \(v_7\), \(k \equiv 1 \, (\text{mod} \, 2)\)**

Now suppose \(x\) lies in the orthant of \(v_7\) but not between \(0\) and \(v_7\). Then the first coordinate of \(x\) is equal to \(-a - 1\) or the second is equal to \(a + 1\) or the third equals \(-a - 1\), or the fourth equals \(-a\) or \(-a - 1\). We distinguish fifteen cases.

**Case 1:** \(x = (-a - 1, a + 1, -a - 1, -u)\) where \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\), which lies between \(0\) and \(v_7\).

**Case 2:** \(x = (-a - 1, a + 1, -a - 1, -u)\) where \(0 \leq u \leq a - 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\), which lies between \(0\) and \(v_7\).

**Case 3:** \(x = (-a - 1, a + 1, -t, -u)\) where \(0 \leq t \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\), which lies between \(0\) and \(v_7\) unless \(t = 0\). Let \(x'' = x' - v_3 = (a, 2a - 1, t, 2a - 1 - u)\), which lies between \(0\) and \(-v_7\).

**Case 4:** \(x = (-a - 1, s, -a - 1, -u)\) where \(0 \leq s \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\), which lies between \(0\) and \(v_6\).

**Case 5:** \(x = (-r, a + 1, -a - 1, -u)\) where \(0 \leq r \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (a - r, 1, 1, a - 1 - u)\), which lies between \(0\) and \(-v_4\).

**Case 6:** \(x = (-r, s, -a - 1, -u)\) where \(0 \leq r, s \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (a - r, s, -a, 1 - u)\), which lies between \(0\) and \(-v_4\).

**Case 7:** \(x = (-r, a + 1, -t, -u)\) where \(0 \leq r, t \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (a - r, 1, a - t, a - 1 - u)\) which lies between \(0\) and \(-v_3\) unless \(t = 0\). Let \(x'' = x' + v_3 = (1 - r, 1, a - t, 2a - 1 - u)\) lies between \(0\) and \(-v_7\) unless \(t = a\). If \(r = 0\) and \(t = a\) then let \(x''' = x'' + v_3 = (a - r, 1, 1, a - 1 - t, a - u)\) which lies between \(0\) and \(-v_7\).

**Case 8:** \(x = (-a - 1, s, -a - 1, -u)\) where \(0 \leq s, t \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\) lies between \(0\) and \(-v_2\) unless \(t \leq 1\). Let \(x'' = x' + v_2 = (a, s + 1, 2 - t, 2a - 2 - u)\) lies between \(0\) and \(-v_5\) unless \(s = a\). If \(s = a\) and \(t \leq 1\) then let \(x''' = x'' + v_5 = (0, s - a + 1, -a + 1, t, a - u)\) which lies between \(0\) and \(-v_7\).

**Case 9:** \(x = (-r, a + 1, -a - 1, -u)\) where \(0 \leq r \leq a\) and \(0 \leq u \leq a - 1\). Let \(x' = x - v_7 = (a - r, 1, a - 1 - u)\), which lies between \(0\) and \(v_2\).

**Case 10:** \(x = (-a - 1, s, -a - 1, -u)\) where \(0 \leq s \leq a\) and \(0 \leq u \leq a - 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\) lies between \(0\) and \(-v_3\) unless \(s = 0\). Let \(x'' = x' - v_3 = (-a - 2, s - 1, 1, a - 1 - u)\) lies between \(0\) and \(-v_1\) unless \(u = a - 1\). If \(s = 0\) and \(u = a - 1\) then let \(x''' = x'' + v_1 = (-a - 2, s + a, 0, a - 2 - u)\) which lies between \(0\) and \(-v_7\).

**Case 11:** \(x = (-a - 1, a + 1, -t, -u)\) where \(0 \leq t \leq a\) and \(0 \leq u \leq a - 1\). Let \(x' = x - v_7 = (-1, 1, 1, -1, a - 1 - u)\), which lies between \(0\) and \(v_4\).

**Case 12:** \(x = (-r, s, -t, -u)\) where \(0 \leq r, s, t \leq a\) and \(a \leq u \leq a + 1\). Let \(x' = x - v_7 = (a - r, s - a, 1 - t, a - 1 - u)\) lies between \(0\) and \(-v_1\) unless \(r = 0\) or \(t = 0\), in which case let
\[ x'' = x' + v_1 = (1 - r, s + 1, 1 - t, 2a - 1 - u) \]. If \( r = 0 \) and \( t = 0 \) then \( x'' \) lies between 0 and \(-v_5\) unless \( s = a \) or \( u = a \), in which case let \( x''' = x'' + v_5 = (1 - a - r, s - a + 1, a - t, a + 1 - u) \).

If \( u = a \) then \( x \) lies between 0 and \( v_6 \). If \( s = a \) and \( u = a + 1 \) then \( x'' \) lies between 0 and \( v_7 \). If \( r = 0 \) and \( 1 \leq t \leq a \) then \( x'' \) lies between 0 and \( v_2 \) unless \( t = a \), in which case \( x' \) lies between 0 and \(-v_4\). If \( 1 \leq r \leq a \) and \( t = 0 \) then \( x'' \) lies between 0 and \( v_4 \) unless \( s = a \), in which case \( x' \) lies between 0 and \(-v_5\).

Case 13: \( x = (-r, s, -a - 1, -u) \) where \( 0 \leq r, s \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (a - r, s - a, -1, a - 1 - u) \) lies between 0 and \(-v_3\) unless \( s = 0 \), in which case let \( x'' = x' + v_3 = (-r, s - 1, a - 2, 1 - u) \) which lies between 0 and \(-v_2\) unless \( u = a - 1 \). If \( s = 0 \) and \( u = a - 1 \) then let \( x''' = x'' + v_2 = (a - r, a + s, 0, a - 2 - u) \) which lies between 0 and \( v_6 \).

Case 14: \( x = (-r, a + 1, -1, -u) \) where \( 0 \leq r, t \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (a - r, 1, a - t, a - 1 - u) \) which lies between 0 and \(-v_5\) unless \( u = 0 \), in which case let \( x'' = x' + v_5 = (-r, a + 1 - t, 1 - 1 - u) \) which lies between 0 and \( v_8 \) unless \( r = a \) or \( t = a \). If \( r = a \) and \( u = 0 \) then \( x' \) lies between 0 and \( v_4 \). If \( 0 \leq r \leq a - 1 \), \( t = a \) and \( u = 0 \) then let \( x'' = x'' - v_8 = (a - 1 - r, 0, a - 1 - t, -a - u) \) which lies between 0 and \( v_6 \).

Case 15: \( x = (-a - 1, s, -1, -u) \) where \( 0 \leq s, t \leq a \) and \( 0 \leq u \leq a - 1 \). Let \( x' = x - v_7 = (-1, s - a, a - t, a - 1 - u) \) which lies between 0 and \(-v_5\) unless \( t = 0 \), in which case let \( x'' = x' + v_5 = (a - 1 - s, 1 - t, 1 - u) \) which lies between 0 and \(-v_8\) unless \( s = a \). If \( s = a \) and \( t = 0 \) then let \( x''' = x'' + v_8 = (0, s - a + 1, 1 - a - t, a - u) \) which lies between 0 and \( v_1 \).

This completes the cases for the orthant of \( v_7 \).

**Orthant of \( v_8 \), \( k \equiv 1 \) (mod 2)**

Finally suppose \( x \) lies in the orthant of \( v_8 \) but not between 0 and \( v_8 \). Then the first coordinate of \( x \) is equal to \(-a\) or \(-a - 1\), or the second is equal to \(-a\) or \(-a - 1\), or the third is equal to \(-a - 1\). We distinguish seven cases.

Case 1: \( x = (-r, -s, -a - 1, u) \) where \( a \leq r, s \leq a + 1 \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, 1 - u, a - u - 1) \) which lies between 0 and \(-v_5\) unless \( u \leq 2 \), in which case let \( x'' = x' + v_5 = (2a - 1 - r, 2a - 1 - s, 1 - u, a - u - 3) \) which lies between 0 and \(-v_8\).

Case 2: \( x = (-r, -s, -t, u) \) where \( a \leq r, s \leq a + 1 \) and \( 0 \leq t \leq a \) and \( 0 \leq u \leq a + 1 \). Let \( x' = x - v_8 = (a - 1 - r, a - 1 - s, 1 - t, u - a - 1) \), which lies between 0 and \(-v_2\) unless \( t \leq 1 \) or \( u \leq 1 \), in which case let \( x'' = x' + v_2 = (2a - r, 2a - s, 2 - t, u - 2) \). If \( t \leq 1 \) and \( u \leq 1 \) then \( x'' \) lies between 0 and \(-v_8\) unless \( r = a \) or \( s = a \) in which case let \( x''' = x'' + v_8 = (a + 1 - r, a + 1 - s, 2 - a - t, u + a - 1) \). If \( t \leq 1 \), \( u \leq 1 \), \( r = a \) and \( s = a \) then \( x''' \) lies between 0 and \( v_2 \) unless \( t = 1 \) or \( u = 1 \). If \( r = a \) and \( s = a \) and \( t = 1 \) and \( u \leq 1 \) then let \( x''' = x - v_5 = (a - r, a - s, a + 1 - t, a + 2 + u) \) which lies between 0 and \( v_4 \). If \( r = a \), \( s = a \), \( t = 0 \) and \( u = 1 \) then \( x \) lies between 0 and \(-v_6\). If \( t \leq 1 \), \( u \leq 1 \), \( r = a \) and \( s = a + 1 \) then \( x'' \) lies between 0 and \(-v_3\). If \( t \leq 1 \), \( u \leq 1 \), \( r = a + 1 \) and \( s = a \) then \( x''' \) lies between 0 and \( v_1 \). If \( t \leq 1 \) and \( 2 \leq u \leq a + 1 \) then \( x'' \) lies between 0 and \(-v_5\) unless \( u = a + 1 \), in which case let \( x''' = x'' + v_5 = (a - r, a - s, 1 - a - t, u - a) \) which lies between 0 and \( v_8 \). If \( 2 \leq t \leq a + 1 \), and \( u \leq 1 \) then \( x'' \) lies between 0 and \( v_6 \).
Case 3: $x = (-r, -s, -a - 1, u)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a - 1$ and $0 \leq u \leq a + 1$.
Let $x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1)$, which lies between $0$ and $v_7$ unless $u \leq 1$, in which case let $x'' = x' - v_7 = (2a - 1 - r, -1 - s, a - 1, u - 2)$ which lies between $0$ and $-v_1$.

Case 4: $x = (-r, -s, -a - 1, u)$ where $0 \leq r \leq a - 1$, $a \leq s \leq a + 1$ and $0 \leq u \leq a + 1$.
Let $x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1)$, which lies between $0$ and $-v_4$ unless $u \leq 1$, in which case let $x'' = x' + v_4 = (-1 - r, 2a - 1 - s, a - 1, u - 2)$ which lies between $0$ and $v_3$.

Case 5: $x = (-r, -s, -a - 1, u)$ where $0 \leq r, s \leq a - 1$ and $0 \leq u \leq a + 1$. Let $x' = x - v_8 = (a - 1 - r, a - 1 - s, -1, u - a - 1)$, which lies between $0$ and $v_6$ unless $u = 0$, in which case $x$ lies between $0$ and $v_5$.

Case 6: $x = (-r, -s, -t, u)$ where $0 \leq r \leq a - 1$, $a \leq s \leq a + 1$, $0 \leq t \leq a$ and $0 \leq u \leq a + 1$.
Let $x' = x - v_8 = (a - 1 - r, a - 1 - s, a - t, u - a - 1)$, which lies between $0$ and $-v_1$ unless $t = 0$ or $u = 0$, in which case let $x'' = x' + v_1 = (-r, 2a - 2 - s, 1 - t, u - 1)$. If $t = 0$ and $u = 0$ then $x$ lies between $0$ and $-v_2$. If $t = 0$ and $1 \leq u \leq a + 1$ then $x''$ lies between $0$ and $v_4$ unless $u = a + 1$, in which case let $x'' = x'' - v_4 = (a - r, a - 2 - s, 1 - a - t, u - a)$ which lies between $0$ and $-v_3$. If $1 \leq t \leq a$ and $u = 0$ then $x''$ lies between $0$ and $v_7$.

Case 7: $x = (-r, -s, -t, u)$ where $a \leq r \leq a + 1$, $0 \leq s \leq a - 1$, $0 \leq t \leq a$ and $0 \leq u \leq a + 1$.
Let $x' = x - v_8 = (a - 1 - r, a - 1 - s, a - t, u - a - 1)$, which lies between $0$ and $v_3$ unless $t = 0$ or $u = 0$, in which case let $x'' = x' + v_3 = (2a - r, -s, 1 - t, u - 1)$. If $t = 0$ and $u = 0$ then $x$ lies between $0$ and $-v_2$. If $t = 0$ and $1 \leq u \leq a + 1$ then $x''$ lies between $0$ and $-v_7$ unless $u = a + 1$, in which case let $x'' = x'' + v_7 = (a - r, a - s, 1 - a - t, u - a)$ which lies between $0$ and $v_1$. If $1 \leq t \leq a$ and $u = 0$ then $x''$ lies between $0$ and $-v_4$.

This completes the cases for the orthant of $v_8$.

This also completes the proof of the theorem for any $k \equiv 1 \pmod{2}$, and therefore for all $k \geq 2$. \hfill \square

References
