Full proof of the existence of a degree 8 circulant graph of order $L(8, k)$ of arbitrary diameter k

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This is the full proof of Theorem 3 in the paper "The degree-diameter problem for circulant graphs of degree 8 and 9" by the author [2]. To avoid the paper being unduly long it includes only the exceptions for the orthant of $\mathbf{v}_1$ for diameter $k \equiv 0 \pmod 2$ and for $k \equiv 1 \pmod 2$. In the version below the exceptions for all eight orthants for diameter $k \equiv 0$ and $k \equiv 1 \pmod 2$ are included in full. This proof closely follows the approach taken by Dougherty and Faber in their proof of the existence of the degree 6 graph of order $DF(6, k)$ for all diameters $k \geq 2$ [1].

Theorem 3. *For all $k \geq 2$, there is an undirected Cayley graph on four generators of a cyclic group which has diameter k and order $L(8, k)$, where*

$$L(8,k) = \begin{cases} (k^4 + 2k^3 + 6k^2 + 4k)/2 & \text{if } k \equiv 0 \pmod 2 \\ (k^4 + 2k^3 + 6k^2 + 6k + 1)/2 & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

Moreover for $k \equiv 0 \pmod 2$ a generator set is $\{1, (k^3 + 2k^2 + 6k + 2)/2, (k^4 + 4k^2 - 8k)/4, (k^4 + 4k^2 - 4k)/4\}$,
and for $k \equiv 1 \pmod 2$, $\{1, (k^3+k^2+5k+3)/2, (k^4+2k^2-8k-11)/4, (k^4+2k^2-4k-7)/4\}$.

Proof. We will show the existence of four-dimensional lattices $L_k \subseteq \mathbb{Z}^4$ such that $\mathbb{Z}^4/L_k$ is cyclic, $S_k + L_k = \mathbb{Z}^4$, where S_k is the set of points in $\mathbb{Z}^4$ at a distance of at most k from the origin under the l^1 (Manhattan) metric, and $|\mathbb{Z}^4 : L_k| = L(8, k)$ as specified in the theorem. Then, by Theorem 1 of [2], the resultant Cayley graph has diameter at most k.

Let $a = \begin{cases} k/2 & \text{for } k \equiv 0 \pmod 2 \\ (k+1)/2 & \text{for } k \equiv 1 \pmod 2. \end{cases}$

For $k \equiv 0 \pmod 2$ let L_k be defined by four generating vectors as follows:

$$\begin{aligned} \mathbf{v}_1 &= (-a-1, a+1, a, -a+1) \\ \mathbf{v}_2 &= (a-1, a+1, a+1, -a) \\ \mathbf{v}_3 &= (-a-1, -a+1, a+1, -a) \\ \mathbf{v}_4 &= (-a, -a, a, a+1) \end{aligned}$$

Then the following vectors are in L_k:

$$\begin{array}{l} -(2a^2+2a+1)\mathbf{v}_1+(2a^2+a+2)\mathbf{v}_2-(a+2)\mathbf{v}_3+\mathbf{v}_4=(4a^3+4a^2+6a+1,-1,0,0), \\ -(2a^3-1)\mathbf{v}_1+(2a^3-a^2+2a-2)\mathbf{v}_2-(a^2+a-1)\mathbf{v}_3+(a-1)\mathbf{v}_4=(4a^4+4a^2-4a,0,-1,0), \\ -2a^3\mathbf{v}_1+(2a^3-a^2+2a-1)\mathbf{v}_2-(a^2+a-1)\mathbf{v}_3+(a-1)\mathbf{v}_4=(4a^4+4a^2-2a,0,0,-1) \end{array}$$

Hence we have $\mathbf{e}_2=(4a^3+4a^2+6a+1)\mathbf{e}_1, \mathbf{e}_3=(4a^4+4a^2-4a)\mathbf{e}_1$ and $\mathbf{e}_4=(4a^4+4a^2-2a)\mathbf{e}_1$ in $\mathbb{Z}^4/L_k$, and so $\mathbf{e}_1$ generates $\mathbb{Z}^4/L_k$.

Also $\det\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\\ \mathbf{v}_4\end{pmatrix}=\det\begin{pmatrix} 8a^4+8a^3+12a^2+4a & 0 & 0 & 0\\ 4a^3+4a^2+6a+1 & -1 & 0 & 0\\ 4a^4+4a^2-4a & 0 & -1 & 0\\ 4a^4+4a^2-2a & 0 & 0 & -1\end{pmatrix}$
$=-(8a^4+8a^3+12a^2+4a)=-(k^4+2k^3+6k^2+4k)/2=-L(8,k)$, as in the statement of the theorem.

Thus $\mathbb{Z}^4/L_k$ is isomorphic to $\mathbb{Z}_{L(8,k)}$ via an isomorphism taking $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4$ to $1,\ 4a^3+4a^2+6a+1, 4a^4+4a^2-4a, 4a^4+4a^2-2a$. As $a=k/2$ this gives the first generator set specified in the theorem: $\{1,(k^3+2k^2+6k+2)/2,(k^4+4k^2-8k)/4,(k^4+4k^2-4k)/4\}$.

Similarly for $k\equiv 1 \pmod 2$ let L_k be defined by four generating vectors as follows:

$$\begin{array}{rcl} \mathbf{v}_1 &=& (-a+1,a+1,-a+1,a)\\ \mathbf{v}_2 &=& (a+1,a+1,-a+2,a-1)\\ \mathbf{v}_3 &=& (-a-1,a-1,a-1,-a)\\ \mathbf{v}_4 &=& (-a,a,a,a-1) \end{array}$$

In this case the following vectors are in L_k:
$-(2a^2+a+2)\mathbf{v}_1+(2a^2+2a+1)\mathbf{v}_2-a\mathbf{v}_3-\mathbf{v}_4=(4a^3-4a^2+6a-1,-1,0,0)$,
$-(2a^3-a^2-2a-2)\mathbf{v}_1+(2a^3-4a-1)\mathbf{v}_2-(a^2-a-1)\mathbf{v}_3-(a-1)\mathbf{v}_4=(4a^4-8a^3+8a^2-8a,0,-1,0)$,
$-(2a^3-a^2-2a-1)\mathbf{v}_1+(2a^3-4a)\mathbf{v}_2-(a^2-a-1)\mathbf{v}_3-(a-1)\mathbf{v}_4=(4a^4-8a^3+8a^2-6a,0,0,-1)$.

Hence we have $\mathbf{e}_2=(4a^3+4a^2+6a-1)\mathbf{e}_1$, $\mathbf{e}_3=(4a^4-8a^3+8a^2-8a)\mathbf{e}_1$ and $\mathbf{e}_4=(4a^4-8a^3+8a^2-6a)\mathbf{e}_1$, in $\mathbb{Z}^4/L_k$, and so $\mathbf{e}_1$ generates $\mathbb{Z}^4/L_k$.

Also $\det\begin{pmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \mathbf{v}_3\\ \mathbf{v}_4\end{pmatrix}=\det\begin{pmatrix} 8a^4-8a^3+12a^2-4a & 0 & 0 & 0\\ 4a^3-4a^2+6a-1 & -1 & 0 & 0\\ 4a^4-8a^3+8a^2-8a & 0 & -1 & 0\\ 4a^4-8a^3+8a^2-6a & 0 & 0 & -1\end{pmatrix}$
$=-(8a^4-8a^3+12a^2-4a)=-(k^4+2k^3+6k^2+6k+1)/2=-L(8,k)$, as in the statement of the theorem.

Thus $\mathbb{Z}^4/L_k$ is isomorphic to $\mathbb{Z}_{L(8,k)}$ with generators $1, 4a^3-4a^2+6a-1, 4a^4-8a^3+8a^2-8a, 4a^4-8a^3+8a^2-6a$. As $a=(k+1)/2$ in this case, this gives the second generator set specified in the theorem: $\{1,(k^3+k^2+5k+3)/2,(k^4+2k^2-8k-11)/4,(k^4+2k^2-4k-7)/4\}$.

It remains to show that $S_k+L_k=\mathbb{Z}^4$. First we consider the case $k\equiv 0 \pmod 2$.

For $k=2$, it is straightforward to show directly that $\mathbb{Z}_{32}$ with generators $1,4,6,15$ has diameter 2. So we assume $k \geq 4$, so that $a \geq 2$. Now let

$$\begin{array}{lll} \mathbf{v}_5 & = \mathbf{v}_1 - \mathbf{v}_3 + \mathbf{v}_4 = & (-a, a, a-1, a+2) \\ \mathbf{v}_6 & = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = & (-a, a, -a-1, -a) \\ \mathbf{v}_7 & = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = & (-a+1, a-1, -a-2, a+1) \\ \mathbf{v}_8 & = \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_4 = & (a, a, a, a+1) \end{array}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 0 \pmod 2$. Then the 16 vectors $\pm\mathbf{v}_i$ for $i = 1, ..., 8$ provide one element of L_k lying strictly within each of the 16 orthants of $\mathbb{Z}^4$. Most of the coordinates of these vectors have absolute value at most $a+1$. Only $\pm\mathbf{v}_5$ and $\pm\mathbf{v}_7$ each have one coordinate with absolute value equal to $a+2$.

Now we consider the case $k \equiv 1 \pmod 2$. For $k = 3$ it may be shown directly that $\mathbb{Z}_{104}$ with generators $1, 16, 20, 27$ has diameter 3. So we assume $k \geq 5$, so that $a \geq 3$, and let

$$\begin{array}{lll} \mathbf{v}_5 & = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_4 = & (-a, -a, -a-1, -a+2) \\ \mathbf{v}_6 & = \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = & (a, a, -a+1, -a) \\ \mathbf{v}_7 & = \mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_4 = & (-a, a, -a, -a+1) \\ \mathbf{v}_8 & = \mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = & (-a+1, -a+1, -a, a+1) \end{array}$$

with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as defined for $k \equiv 1 \pmod 2$, so that the 16 vectors $\pm\mathbf{v}_i$ provide one element of $\mathbf{L}_k$ lying strictly within each of the orthants of $\mathbb{Z}^4$. In this case all the coordinates of these vectors have absolute value at most $a+1$.

We must show that each $\mathbf{x} \in \mathbb{Z}^4$ is in $S_k + L_k$, which means that for any $\mathbf{x} \in \mathbb{Z}^4$ we need to find a $\mathbf{w} \in L_k$ such that $\mathbf{x} - \mathbf{w} \in S_k$. However $\mathbf{x} - \mathbf{w} \in S_k$ if and only if $\delta(\mathbf{x}, \mathbf{w}) \leq k$, where δ is the l^1 metric on $\mathbb{Z}^4$. If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^4$ and each coordinate of $\mathbf{y}$ lies between the corresponding coordinate of $\mathbf{x}$ and $\mathbf{z}$ or is equal to one of them, then $\delta(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{y}, \mathbf{z}) = \delta(\mathbf{x}, \mathbf{z})$. In such a case we say that "$\mathbf{y}$ lies between $\mathbf{x}$ and $\mathbf{z}$".

For any $\mathbf{x} \in \mathbb{Z}^4$, we reduce $\mathbf{x}$ by adding appropriate elements of L_k until the resulting vector lies within l^1-distance k of $\mathbf{0}$ or some other element of L_k. The first stage is to reduce $\mathbf{x}$ to a vector whose coordinates all have absolute value at most $a+1$. If $\mathbf{x}$ has a coordinate with absolute value above $a+1$, then let $\mathbf{v}$ be one of the vectors $\pm\mathbf{v}_i (1 \leq i \leq 8)$ such that the coordinates of $\mathbf{v}$ have the same sign as the corresponding coordinates of $\mathbf{x}$. If a coordinate of $\mathbf{x}$ is 0 then either sign is allowed for $\mathbf{v}$ as long as the corresponding coordinate of $\mathbf{v}$ has absolute value $\leq a+1$. So in the case $k \equiv 0 \pmod 2$ if the $\mathbf{e}_3$ coordinate of $\mathbf{x}$ is 0 then we avoid $\mathbf{v}_7$ and take $\mathbf{v}_5$ instead. Also if the $\mathbf{e}_4$ coordinate of $\mathbf{x}$ is 0 (or both $\mathbf{e}_3$ and $\mathbf{e}_4$ coordinates are 0) then instead of $\mathbf{v}_5$ we take $\mathbf{v}_1$.

Now consider $\mathbf{x}' = \mathbf{x} - \mathbf{v}$. If a coordinate of $\mathbf{x}$ has absolute value $s, 1 \leq s \leq a+1$, then the corresponding coordinate of $\mathbf{x}'$ will have absolute value $s' \leq a+1$ because of the sign matching and the fact that the coordinates of $\mathbf{v}$ have absolute value $\leq a+2$. If a coordinate of $\mathbf{x}$ has absolute value $s = 0$, then as indicated above, the corresponding value of $\mathbf{x}'$ will have absolute value $s' \leq a+1$ because $\mathbf{v}$ is chosen such that the corresponding coordinate has absolute value $\leq a+1$. If a coordinate of $\mathbf{x}$ has absolute value $s > a+1$, then the corresponding coordinate of $\mathbf{x}'$ will be strictly smaller in absolute value. Therefore repeating this procedure will result in a vector whose coordinates all have absolute value at most $a+1$.

If the resulting vector $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}$, where $\mathbf{v} = \pm\mathbf{v}_i$ for some i, then we have $\delta(\mathbf{0}, \mathbf{x}') + \delta(\mathbf{x}', \mathbf{v}) = \delta(\mathbf{0}, \mathbf{v})$. For $k \equiv 0 \pmod 2$ all of the vectors $\mathbf{v}$ satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a+1$, and for $k \equiv 1 \pmod 2$ they all satisfy $\delta(\mathbf{0}, \mathbf{v}) = 4a - 1$. So in either case we have $\delta(\mathbf{0}, \mathbf{v}) = 2k + 1$. Since $\delta(\mathbf{0}, \mathbf{x}')$ and $\delta(\mathbf{x}', \mathbf{v})$ are both non-negative integers, one of them must be at most k, so that $\mathbf{x}' \in S_k + L_k$. Hence we also have $\mathbf{x} \in S_k + L_k$ as required.

Now we are left with the case where the absolute value of each coordinate of the reduced $\mathbf{x}$ is at most $a + 1$, and $\mathbf{x}$ is in the orthant of $\mathbf{v}$, where $\mathbf{v} = \pm\mathbf{v}_i$ for some $i \leq 8$ but does not lie between $\mathbf{0}$ and $\mathbf{v}$. Since L_k is centrosymmetric we only need to consider the eight orthants containing $\mathbf{v}_1, ..., \mathbf{v}_8$. For both cases $k \equiv 0$ and $k \equiv 1 \pmod 2$ the exceptions need to be considered for each orthant in turn. We first consider all eight orthants for the case $k \equiv 0 \pmod 2$ and then the same for $k \equiv 1 \pmod 2$.

Orthant of $\mathbf{v}_1$, $k \equiv 0 \pmod 2$

Suppose that $k \equiv 0 \pmod 2$ and $\mathbf{x}$ lies within the orthant of $\mathbf{v}_1$, but not between $\mathbf{0}$ and $\mathbf{v}_1$. Then as $\mathbf{v}_1 = (-a-1, a+1, a, -a+1)$, the third coordinate of $\mathbf{x}$ is equal to $a + 1$ or the fourth coordinate equals $-a$ or $-a - 1$. We distinguish three cases.

Case 1: $\mathbf{x} = (-r, s, a+1, -u)$ where $0 \leq r, s \leq a+1$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a+1-r, s-a-1, 1, a-1-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r \leq 1$ or $s \leq 1$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (2-r, s-2, -a-1, 2a-u)$. If $r \leq 1$ and $s \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (1-a-r, a-1+s, -1, a+1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r \leq 1$ and $s \geq 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r \geq 2$ and $s \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 2: $\mathbf{x} = (-r, s, a+1, -u)$ where $0 \leq r, s \leq a+1$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a+1-r, s-a-1, 1, a-1-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = 0$ or $s = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (1-r, s-1, -a, -u-1)$. If $r = 0$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $r = 0$ and $s \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-r, s, t, -u)$ where $0 \leq r, s \leq a+1$ and $0 \leq t \leq a$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a+1-r, s-a-1, t-a, a-1-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $s = 0$ or $t = 0$. If $r = 0$ and $s = 0$, then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1-r, s-1, t-1, 2a+1-u)$. If $r = 0, s \geq 1$ and $t \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. Let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_4 = (-a-r, s-a, a+t, a+1-u)$. If $r = 0$ and $s \geq 1$ and $t = 0$, then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $s = a+1$, in which case if $u = a$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_2$, and if $u = a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. Let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1+s, t-a-1, a-u)$. If $r \geq 1, s = 0$ and $t \geq 1$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \geq 1, s = 0$ and $t = 0$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ if $u = a$, and between $\mathbf{0}$ and $\mathbf{v}_6$ if $u = a+1$. If $r \geq 1, s \geq 1$ and $t = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = a+1$ or $s = a+1$. If $r = a+1$, $s \geq 1$ and $t = 0$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r \geq 1$, $s = a+1$ and $t = 0$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of $\mathbf{v}_1$ for $k \equiv 0 \pmod 2$.

Orthant of $\mathbf{v}_2$, $k \equiv 0 \pmod 2$

Now suppose that $\mathbf{x}$ lies in the orthant of $\mathbf{v}_2$ but not between $\mathbf{0}$ and $\mathbf{v}_2$. Then the first coordinate of $\mathbf{x}$ is equal to a or $a+1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $\mathbf{x} = (r, s, t, -a-1)$ where $a \le r \le a+1$ and $0 \le s, t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = 0$ or $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (r-2a+1, s-1, t-2, a+1)$. If $s = 0$ and $t \le 1$ then let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_5 = (r-a, a, t+a-1, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $s = 0$ and $t \ge 2$ then let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_8 = (r-2a, 0, t-1, -a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$. If $s \ge 1$ and $t \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $s = a+1$, in which case let $\mathbf{x}^v = \mathbf{x}'' - \mathbf{v}_7 = (r-a, 1, a+t, 0)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 2: $\mathbf{x} = (r, s, t, -u)$ where $a \le r \le a+1$, $0 \le s, t \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$. If $t = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $a \le s \le a+1$. If $r = a+1$, $a \le s \le a+1$, $t = 0$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_3 = (r-a-1, s-a+1, t+a+1, -u-a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$. If $a \le r \le a+1$, $s = a+1$, $t = 0$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -u+a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r = a$, $s = a$, $t = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. Now let $\mathbf{x}''' = \mathbf{x}' + \mathbf{v}_1 = (r-2a, s, t-1, 1-u)$. If $t = 0$ and $1 \le u \le a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $s = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_6 = (r-a, s-a, t+a, a+1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $1 \le t \le a+1$ and $u = 0$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $s = a+1$ or $t = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_5 = (r-a, s-a, t-a, -a-1-u)$. If $s = a+1$ and $t = a+1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $s = a+1$ and $1 \le t \le a$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $0 \le s \le a$ and $t = a+1$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 3: $\mathbf{x} = (r, s, t, -a-1)$ where $0 \le r \le a-1$ and $0 \le s, t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a+1, s-a-1, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = 0$ or $t = 0$. If $s = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $t = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = a+1$, in which case let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_4 = (r-a, 1, a, 0)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. This completes the cases for the orthant of $\mathbf{v}_2$.

Orthant of $\mathbf{v}_3$, $k \equiv 0 \pmod 2$

Now suppose that $\mathbf{x}$ lies in the orthant of $\mathbf{v}_3$ but not between $\mathbf{0}$ and $\mathbf{v}_3$. Then the second coordinate of $\mathbf{x}$ is equal to $-a$ or $-a-1$, or the fourth coordinate equals $-a-1$. We distinguish three cases.

Case 1: $\mathbf{x} = (-r, -s, t, -a-1)$ where $0 \le r, t \le a+1$ and $a \le s \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1-r, 2a-1-s, t-2, a+1)$. If $r = 0$ and $t \ge 2$ then $\mathbf{x}''$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $r = 0$ and $t \le 1$ then let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_5 = (-a, a-s, a-1+t, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $r \ge 1$ and $t \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_7 = (-1, a-s, a+t, 0)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 2: $\mathbf{x} = (-r, -s, t, -a-1)$ where $0 \le r, t \le a+1$ and $0 \le s \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$ or $t = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (1-r, -1-s, t-1, a)$. If $r = 0$ and $t \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r \ge 1$ and $t = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $r = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (-1, a-s, a, 0)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $r = 0$ and $t = 0$, then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-a, a-s, a-1, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 3: $\mathbf{x} = (-r, -s, t, -u)$ where $0 \le r, t \le a+1$, $a \le s \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, a-1-s, t-a-1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$. Let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_8 = (a-r, a-s, a+t, a+1-u)$. If $t = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $r \le a-1$, in which case let $\mathbf{x}''' = \mathbf{x} + \mathbf{v}_2 = (a-1-r, a+1-s, a+1+t, -a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$. If $t = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = a+1$, in which case $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. Let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_4 = (a-r, a-s, t-a, -a-1-u)$. If $t \ge 1, u = 0$ and $r \le a$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a+1$. If $t = a+1, u = 0$ and $r \le a$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ in which case $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $t \ge 1, u = 0$ and $r = a+1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$.
This completes the cases for the orthant of $\mathbf{v}_3$.

Orthant of $\mathbf{v}_4$, $k \equiv 0 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_4$ but not between $\mathbf{0}$ and $\mathbf{v}_4$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the second coordinate is equal to $-a-1$, or the third equals $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, -a-1, a+1, u)$ where $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -1, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$ if $u = a+1$ and between $\mathbf{0}$ and $\mathbf{v}_3$ if $u \le a$.

Case 2: $\mathbf{x} = (-a-1, -a-1, t, u)$ where $0 \le t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-a-1, -s, a+1, u)$ where $0 \le s \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, -a-s, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u \ge a$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -s-1, -a+1, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 4: $\mathbf{x} = (-r, -a-1, a+1, u)$ where $0 \le r \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, -1, 1, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1, a-2, -a-1, u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a, -1, 0, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 5: $\mathbf{x} = (-r, -s, a+1, u)$ where $0 \le r, s \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, a-s, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = 0$ or $u = 0$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -s-1, -a, u-1)$. If $r \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a+1-r, -1, 0, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = 0$ or $u = a+1$. If $r = 0$ and $u = 0$, then let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a, a-1-s, -1, a+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $s = a$, in which case let $\mathbf{x}^v = \mathbf{x}'''' + \mathbf{v}_2 = (0, a, a, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $r = 0$ and $u = a+1$, then let $\mathbf{x}^{vi} = \mathbf{x}'' + \mathbf{v}_1 = (-a, a-s, 0, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, -a-1, t, u)$ where $0 \le r, t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, -1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-r, a-1, -1, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = a$ or $u = a+1$. If $t = 0$ and $r = a$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (-1, 0, a+1, u-a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u = a+1$. If $t = 0$ and $u = a+1$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (a-1-r, 0, a+1, 1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = a$. If $t = 0, r = a$ and $u = a+1$, then $\mathbf{x}''' = (-1, 0, a+1, 1)$. Let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_4 = (a-1, a, 1, -a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 7: $\mathbf{x} = (-a-1, -s, t, u)$ where $0 \le s, t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, a-s, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a-1, -s, t+1, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_7 = (0, -1, t-a-1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.
This completes the cases for the orthant of $\mathbf{v}_4$.

Orthant of $\mathbf{v}_5$, $k \equiv 0 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_5$ but not between $\mathbf{0}$ and $\mathbf{v}_5$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals a or $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, a+1, t, u)$ where $a \le t \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u \le 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, t-2a+1, u-3)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \le t \le a-1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (a-1, -a+1, t+2, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 3: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \le s \le a$, $a \le t \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, s-a, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = 0$ or $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, s-1, t-2a, u-2)$. If $s = 0$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, a-1, -1, a+u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $s = 0$ and $u \ge 2$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, a, t-a, u-a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $s \ge 1$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \le r \le a$, $a \le t \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = 0$ or $u \ge 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -a, t-2a, u-2)$. If $r = 0$ and $u \ge 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $r = 0$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a, 0, -1, a+u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r \ge 1$ and $u \ge 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, t, u)$ where $0 \le r, s \le a$, $a \le t \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, s-a, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $s = 0$ or $u = 0$. If $r = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $t = a+1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (-a, s-a, 1, u-a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $u = 0$. If $r = 0, t = a+1$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_2$. If $r \ge 1$ and $s = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $t = a+1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, a, 1, u-a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $u = 0$. If $r \ge 1, s = 0$ and $u = 0$, then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1-r, -1, -a, -1)$ which lies between $\mathbf{0}$

and $-\mathbf{v}_8$. If $r \geq 1, s \geq 1$ and $u = 0$, then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $t = a+1$, in which case let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a+1-r, s-a-1, 1, a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 6: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \leq r \leq a$, $0 \leq t \leq a-1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, 1, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $u = 0$ in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$.

Case 7: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \leq s \leq a$, $0 \leq t \leq a-1$ and $0 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, s-a, t-a+1, u-a-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $u = 0$ in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$.
This completes the cases for the orthant of $\mathbf{v}_5$.

Orthant of $\mathbf{v}_6$, $k \equiv 0 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_6$ but not between $\mathbf{0}$ and $\mathbf{v}_6$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the fourth equals $-a-1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, a+1, -t, -a-1)$ where $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, 1, a+1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, 1, a-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 2: $\mathbf{x} = (-a-1, a+1, -t, -u)$ where $0 \leq t \leq a+1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, 1, a+1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (a-1, 1-a, 2-t, -u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = a$. If $t = 1$ and $u = a$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_1$ If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (a, -a, 1, a+1)$ and $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_6 = (0, 0, -a, 1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 3: $\mathbf{x} = (-a-1, s, -t, -a-1)$ where $0 \leq s \leq a$ and $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, s-a, a+1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = a$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, a-1, -t, a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 4: $\mathbf{x} = (-r, a+1, -t, -a-1)$ where $0 \leq r \leq a$ and $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a-r, 1, a+1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (1, -a, -t, -a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = a+1$ in which case $\mathbf{x}' = (a, 1, 0, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$

Case 5: $\mathbf{x} = (-r, s, -t, -a-1)$ where $0 \leq r, s \leq a$ and $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a-r, s-a, a+1-t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1-r, s-1, -t-1, a)$. If $r = 0$ and $s \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a, s-a, -1, 0)$ which lies between $\mathbf{0}$ and $\mathbf{v}_6$. If $r \geq 1$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a-r, a, -1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r = 0$ and $s = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $t = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x} + \mathbf{v}_8 = (a, a, -1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 6: $\mathbf{x} = (-r, a+1, -t, -u)$ where $0 \leq r, u \leq a$ and $0 \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (a-r, 1, a+1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $t = 0$ in which case let $\mathbf{x} = (-r, a+1, 0, -u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u = a$. If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_1 = (a+1-r, 0, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$ in which case $\mathbf{x} = (0, a+1, 0, -a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$.

Case 7: $\mathbf{x} = (-a-1, s, -t, -u)$ where $0 \le s, u \le a$ and $0 \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (-1, s-a, a+1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $t = 0$ in which case $\mathbf{x} = (-a-1, s, 0, -u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u = a$. If $t = 0$ and $u = a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_1 = (0, s-a-1, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = 0$ in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a, -1, 0, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.
This completes the cases for the orthant of $\mathbf{v}_6$.

Orthant of $\mathbf{v}_7$, $k \equiv 0 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_7$ but not between $\mathbf{0}$ and $\mathbf{v}_7$. Then the first coordinate of $\mathbf{x}$ is equal to $-a$ or $-a-1$ or the second equals a or $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, -t, u)$ where $a \le r, s \le a+1$, $2 \le t \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_6 = (a-r, s-a, a+1-t, a+u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a, a, -t, u)$ where $0 \le t \le 1$ and $0 \le u \le a+1$. If $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_6$. If $u \ge 1$ then let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 - \mathbf{v}_3 = (a, a, 1-t, u-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-a-1, a, -t, u)$ where $0 \le t \le 1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (0, -1, -a-t, a-1+u)$. If $u \le 1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $u \ge 2$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a-1, a, 1-t, u-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$.

Case 4: $\mathbf{x} = (-a, a+1, -t, u)$ where $0 \le t \le 1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (1, 0, -a-t, a-1+u)$. If $u \le 1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $u \ge 2$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-a, -a+1, 1-t, u-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 5: $\mathbf{x} = (-a-1, a+1, -t, u)$ where $0 \le t \le 1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (0, 0, -a-t, a-1+u)$. If $u \le 1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $u \ge 2$ then let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (a-1, -a+1, 2-t, u-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 6: $\mathbf{x} = (-r, s, -t, u)$ where $0 \le r \le a-1$, $a \le s \le a+1$ and $0 \le t, u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r, s-2a, 1-t, u-1)$. If $t = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $t = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_3 = (a+1-r, -1, -a, a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $t \ge 1$ and $u = 0$ then $\mathbf{x}'' = (-r, s-2a, 1-t, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 7: $\mathbf{x} = (-r, s, -t, u)$ where $a \le r \le a+1$, $0 \le s \le a-1$ and $0 \le t, u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-1-r, s-a+1, a+2-t, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (2a-r, s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $t = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $t \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.
This completes the cases for the orthant of $\mathbf{v}_7$.

Orthant of $\mathbf{v}_8$, $k \equiv 0 \pmod 2$

Finally suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_8$ but not between $\mathbf{0}$ and $\mathbf{v}_8$. Then at least one of the first three coordinate of $\mathbf{x}$ is equal to $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (a+1, a+1, a+1, u)$ where $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, 1, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-a+2, -a, -a, -1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 2: $\mathbf{x} = (a+1, a+1, t, u)$ where $0 \le t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, 1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 3: $\mathbf{x} = (a+1, s, a+1, u)$ where $0 \le s \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, s-a, 1, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (-a+2, -1, -a-1, u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a+1$. If $s = 0$ and $u = a+1$ then $\mathbf{x}' = (1, -a, 1, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 4: $\mathbf{x} = (r, a+1, a+1, u)$ where $0 \le r \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r-a, 1, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (r+1, -a, -a+-1, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = a$. If $r = a$ and $u \le 1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1, 0, 0, u+a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$.

Case 5: $\mathbf{x} = (a+1, s, t, u)$ where $0 \le s, t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (1, s-a, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-a+1, s, -1, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $s = a$ or $u = a+1$. If $s = a$ and $t = 0$ then $\mathbf{x}' = (1, 0, -a, u-a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $t = 0$ and $u = a+1$ then $\mathbf{x}' = (1, s-a, -a, 0)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 6: $\mathbf{x} = (r, a+1, t, u)$ where $0 \le r, t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r-a, 1, t-a, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $u = 0$, in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = a$. If $r = a$ and $u = 0$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_2 = (1, 0, t-a-1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 7: $\mathbf{x} = (r, s, a+1, u)$ where $0 \le r, s \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (r-a, s-a, 1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (r+1, s-1, -a, u-1)$. If $s = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $s = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a+1$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $s \ge 1$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $r = a$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$.
This completes the cases for the orthant of $\mathbf{v}_8$.

This also completes the proof of the theorem for any $k \equiv 0 \pmod 2$.

Now we consider the eight orthants $\mathbf{v}_1, ..., \mathbf{v}_8$ in turn for the case $k \equiv 1 \pmod 2$.

Orthant of $\mathbf{v}_1$, $k \equiv 1 \pmod 2$

First suppose that $\mathbf{x}$ lies within the orthant of $\mathbf{v}_1$, but not between $\mathbf{0}$ and $\mathbf{v}_1$. Then the first coordinate of $\mathbf{x}$ is equal to $-a$ or $-a-1$, or the third coordinate equals $-a$ or $-a-1$, or the fourth equals $a+1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, -t, a+1)$ where $a \le r, t \le a+1$ and $0 \le s \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 =$

$(a-1-r, s-a-1, a-1-t, 1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $s \le 1$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (2a-2-r, s-2, 2a-1-t, -a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 2: $\mathbf{x} = (-r, s, -t, u)$ where $a \le r, t \le a+1$ and $0 \le s \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, u-a)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $s = 0$ or $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (2a-1-r, s-1, 2a-t, u-2)$. If $s = 0$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$, unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (a-r, a, 1, u+a-2)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $s = 0$ and $u \ge 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $s \ge 1$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_8 = (a-r, 1, a-t, a+u-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$.

Case 3: $\mathbf{x} = (-r, s, -t, a+1)$ where $a \le r \le a+1$, $0 \le s \le a+1$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (2a-1-r, -1, -t, -a+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 4: $\mathbf{x} = (-r, s, -t, a+1)$ where $0 \le r \le a-1$, $0 \le s \le a+1$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-2-r, s-2, 2a-2-t, -a+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, -t, a+1)$ where $0 \le r, t \le a-1$ and $0 \le s \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, 1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (-r-1, -1, -t-1, -a+2)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, s, -t, u)$ where $0 \le r \le a-1$, $0 \le s \le a+1$, $a \le t \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, u-a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-r-1, s-1, 2a-1-t, u-1)$. If $s = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a-r, a, a+1-t, a-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $s = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $s \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = a+1$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 7: $\mathbf{x} = (-r, s, -t, u)$ where $a \le r \le a+1$, $0 \le s \le a+1$, $0 \le t \le a-1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_1 = (a-1-r, s-a-1, a-1-t, u-a)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (2a-r, s, -t+1, u-1)$. If $t = 0$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a+1-r, s-a+1, -a+1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $a \le s \le a+1$, in which case let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_7 = (a-r, s-a, a, a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $t = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a+1$ or $u = a$ in which case let $\mathbf{x}^v = \mathbf{x}'' + \mathbf{v}_5 = (a-r, s-a, -a, -a+u+1)$. If $s = a+1$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_3$. If $1 \le s \le a$ and $u = a$ then $\mathbf{x}^v$ lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $s = 0$ and $u = a$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $t \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $s = a+1$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_3$.

This completes the cases for the orthant of $\mathbf{v}_1$.

Orthant of $\mathbf{v}_2$, $k \equiv 1 \pmod 2$

Now suppose that $\mathbf{x}$ lies in the orthant of $\mathbf{v}_2$ but not between $\mathbf{0}$ and $\mathbf{v}_2$. Then the third coordinate of $\mathbf{x}$ is equal to $-a+1, -a$ or $-a-1$, or the fourth coordinate equals a or $a+1$.

We distinguish three cases.

Case 1: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $a-1 \leq t \leq a+1$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r \leq 1$ or $s \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (r-2, s-2, 2a-2-t, u-2a)$. If $r \leq 1$ and $s \geq 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_3$. If $r \geq 2$ and $s \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $r \leq 1$ and $s \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a-1$ or $u = a$. Let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (r+a-1, s+a-1, a-t, u-a-1)$. If $r \leq 1$ and $s \leq 1$ and $u = a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $t = a-1$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_6 = (r-1, s-1, 2a-1-t, u-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$ if $s = 1$, and between $\mathbf{0}$ and $-\mathbf{v}_7$ if $r = 1$. If $r = 0$ and $s = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $r \leq 1$ and $s \leq 1$ and $t = a-1$ and $u = a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $0 \leq t \leq a-2$ and $a \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $r = 0$ or $s = 0$. Let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (r-1, s-1, -t-1, u-2a+1)$. If $r = 0$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_5 = (r+a-1, s+a-1, a-t, u-a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0$ and $s \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r \geq 1$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 3: $\mathbf{x} = (r, s, -t, u)$ where $0 \leq r, s \leq a+1$, $a-1 \leq t \leq a+1$ and $0 \leq u \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_2 = (r-a-1, s-a-1, a-2-t, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $r = 0$ or $s = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (r-1, s-1, 2a-1-t, u-1)$. If $r = 0$ and $s = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $a-1 \leq t \leq a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a+r, a+s, a+1-t, a-2+u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $r = 0$ and $s = 0$ and $u \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t = a-1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_6 = (r+a-1, s+a-1, a-t, u-a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0$ and $s \geq 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = a+1$ or $t = a-1$, in which case let $\mathbf{x}^v = \mathbf{x}'' - \mathbf{v}_3 = (r+a, s-a, a-t, u+a-1)$. If $s = a+1$ then $\mathbf{x}^v$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = a-1$. If $t = a-1$ then $\mathbf{x}^v$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $s = a+1$, in which case let $\mathbf{x}^{vi} = \mathbf{x}^v + \mathbf{v}_5 = (r, s-2a, -t-1, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $r = 0$ and $s \geq 1$ and $u \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $r \geq 1$ and $s = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = a+1$ or $t = a-1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (r-a, s+a, a-t, a+u-1)$. If $r = a+1$ and $t \geq a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$. If $r = a+1$ and $t = a-1$ then let $\mathbf{x}'''' = \mathbf{x}''' + \mathbf{v}_5 = (r-2a, s, -t-1, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $1 \leq r \leq a$ and $t = a-1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $r \geq 1$ and $s = 0$ and $u \geq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $r \geq 1$ and $s \geq 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = a+1$ or $s = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (r-a, s-a, a-1-t, u+a)$. If $r = a+1$ and $s = a+1$ then let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_2 = (r-2a-1, s-2a-1, 2a-3-t, u+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = a+1$ and $1 \leq s \leq a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $1 \leq r \leq a$ and $s = a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_1$.

This completes the cases for the orthant of $\mathbf{v}_2$.

Orthant of $\mathbf{v}_3$, $k \equiv 1 \pmod 2$

Now suppose that $\mathbf{x}$ lies in the orthant of $\mathbf{v}_3$ but not between $\mathbf{0}$ and $\mathbf{v}_3$. Then the second coordinate of $\mathbf{x}$ is equal to a or $a+1$, or the third coordinate equals a or $a+1$, or the fourth equals $-a-1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, s, t, -a-1)$ where $0 \le r \le a+1$ and $a \le s, t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (2-r, s-2a+2, t-2a+1, a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

Case 2: $\mathbf{x} = (-r, s, t, -u)$ where $0 \le r \le a+1$ and $a \le s, t \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $r = 0$ or $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1-r, s-2a+1, t-2a, 2-u)$. If $r = 0$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a-r, s-a, t-a-1, 2-a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r = 0$ and $u \ge 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $r \ge 1$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_8 = (a-r, s-a, t-a, 1-a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 3: $\mathbf{x} = (-r, s, t, -a-1)$ where $0 \le r \le a+1$, $a \le s \le a+1$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (1-r, s-2a+1, t, a-1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-r, s, t, -a-1)$ where $0 \le r \le a+1$ and $0 \le s \le a-1$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (2-r, s+2, t-2a+2, a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, s, t, -a-1)$ where $0 \le r \le a+1$ and $0 \le s, t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (1-r, s+1, t+1, a-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, s, t, -u)$ where $0 \le r \le a+1$, $0 \le s \le a-1$, $a \le t \le a+1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $r = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1-r, s+1, t-2a+1, 1-u)$. If $r = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a-r, s-a, t-a-1, 2-a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$. If $r = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_6$. If $r \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $r = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_1 = (a-r, s-a, t-a, 1-a-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 7: $\mathbf{x} = (-r, s, t, -u)$ where $0 \le r \le a+1$, $a \le s \le a+1$, $0 \le t \le a-1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_3 = (a+1-r, s-a+1, t-a+1, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r, s-2a, t-1, 1-u)$. If $t = 0$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r \le 1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-1-r, s-a-1, t+a-1, -a-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$. If $t = 0$ and $u \ge 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $r = 0$ or $u = a$, in which case let $\mathbf{x}'''' = \mathbf{x}'' - \mathbf{v}_5 = (a-r, s-a, a+t, a-u-1)$. If $r = 0$, $t = 0$ and $u = a$ then let $\mathbf{x}^v = \mathbf{x}'''' + \mathbf{v}_8 = (1-r, s-2a+1, t, 2a-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $r = 0$, $t = 0$ and $1 \le u \le a-1$ then $\mathbf{x}''''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \le r \le a$, $t = 0$ and $u = a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = a+1$, $t = 0$ and $u = a$ then $\mathbf{x}^v$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $t \ge 1$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

This completes the cases for the orthant of $\mathbf{v}_3$.

Orthant of $\mathbf{v}_4$, $k \equiv 1 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_4$ but not between $\mathbf{0}$ and $\mathbf{v}_4$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the second coordinate is equal to $a+1$, or the third equals $a+1$ or the fourth equals a or $a+1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (-a-1, a+1, a+1, u)$ where $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, 1, u - a + 1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 2: $\mathbf{x} = (-a-1, a+1, a+1, u)$ where $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 3: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \le t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-a-1, s, a+1, u)$ where $0 \le s \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 5: $\mathbf{x} = (-r, a+1, a+1, u)$ where $0 \le r \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, s, a+1, u)$ where $0 \le r, s \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 7: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \le r, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-r-1, -a, t-2, u)$ and $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-r-2, -1, a+t-2, u-a-1)$. $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $a-1 \le r \le a$, in which case let $\mathbf{x}'''' = \mathbf{x}''' + \mathbf{v}_2 = (2a-r-1, a, t, u-2)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a+1$. If $a-1 \le r \le a$, $t \le 1$ and $u = a+1$, then let $\mathbf{x}^{v} = \mathbf{x}'''' + \mathbf{v}_5 = (a-r-1, 0, t-a-1, u-a)$ and $\mathbf{x}^{vi} = \mathbf{x}^{v} - \mathbf{v}_8 = (2a-r-2, a-1, t-1, u-2a-1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 8: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \le s, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (a-2, s-1, t, u-2a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = a$. If $s = 0$ and $t = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, s+a, t-a+1, u-a)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 9: $\mathbf{x} = (-r, a+1, a+1, u)$ where $0 \le r \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1-r, 2-a, 1-a, u+2)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$.

Case 10: $\mathbf{x} = (-a-1, s, a+1, u)$ where $0 \le s \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$.

Case 11: $\mathbf{x} = (-a-1, a+1, t, u)$ where $0 \le t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 12: $\mathbf{x} = (-r, s, t, u)$ where $0 \le r, s, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, s-a, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$ or $t = 0$. If $s = 0$ and $t = 0$ then let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-1-r, s-1, t-1, u-2a+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $r = a$ or $u = a$ in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_5 = (a-1-r, s+a-1, t+a, u-a-1)$. If $r = a$ and $u = a$, then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $r = a$ and $u = a+1$, then $\mathbf{x}'''$

lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $r \le a-1$ and $u = a$, then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $s = 0$ and $1 \le t \le a$ then let $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a-r, a+s, t-a+1, u-a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \le s \le a$ and $t = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_7 = (a-1-r, s-a-1, t+a-1, u-a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 13: $\mathbf{x} = (-r, s, a+1, u)$ where $0 \le r, s \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, s-a, 1, u-a+1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1-r, s+1, -a+2, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $u = a-1$. If $r = 0$ and $u = a-1$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a-r, s-a, 0, u-a+2)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 14: $\mathbf{x} = (-r, a+1, t, u)$ where $0 \le r, t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (a-r, 1, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_6 = (-r, 1-a, t-1, u+1)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r = a$. If $r = a$ and $t = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-1-r, 0, t+a-1, u-a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 15: $\mathbf{x} = (-a-1, s, t, u)$ where $0 \le s, t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_4 = (-1, s-a, t-a, u-a+1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $t = 0$ or $u = 0$ in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (a-1, s, t+1, u-1)$. If $t = 0$ and $u = 0$, then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (0, s-a+1, t-a+1, u+a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $t = 0$ and $1 \le u \le a-1$, then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $1 \le t \le a-1$ and $u = 0$, then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$, in which case $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $t = a$ and $u = 0$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $s = a$, in which case let $\mathbf{x}'''' = \mathbf{x}''' - \mathbf{v}_4 = (a, s-2a+1, t-2a+1, u+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

This completes the cases for the orthant of $\mathbf{v}_4$.

Orthant of $\mathbf{v}_5$, $k \equiv 1 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_5$ but not between $\mathbf{0}$ and $\mathbf{v}_5$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the second coordinate is equal to $-a-1$, or the fourth equals $-a+1$, $-a$ or $-a-1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-a-1, -a-1, -t, -u)$ where $0 \le t \le a+1$ and $a-1 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, -1, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \le 2$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a, a, 3-t, 2a-3-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 2: $\mathbf{x} = (-a-1, -a-1, -t, -u)$ where $0 \le t \le a+1$ and $0 \le u \le a-2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, -1, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (a-1, a-1, 2-t, -2-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 3: $\mathbf{x} = (-a-1, -s, -t, -u)$ where $0 \le s \le a$, $0 \le t \le a+1$ and $a-1 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, a-s, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $s = 0$ or $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, 1-s, 2-t, 2a-2-u)$. If $s = 0$ and $t \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a-1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, 1-a-s, 1-a-t, a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $s = 0$ and $t \ge 2$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = a+1$, in which case let

$\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-1, -a-s, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $1 \leq s \leq a$ and $t \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-r, -a-1, -t, -u)$ where $0 \leq r \leq a$, $0 \leq t \leq a+1$ and $a-1 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, -1, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$ or $t \leq 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1-r, a, 2-t, 2a-2-u)$. If $r = 0$ and $t \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a-1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a-r, 0, 1-a-t, a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $r = 0$ and $2 \leq t \leq a+1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a-r, -1, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$. If $r \geq 1$ and $t \leq 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 5: $\mathbf{x} = (-r, -s, -t, -u)$ where $0 \leq r, s \leq a$, $0 \leq t \leq a+1$ and $a-1 \leq u \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, a-s, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$ or $s = 0$ or $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1-r, 1-s, 1-t, 2a-1-u)$. If $r = 0$, $s = 0$ and $t = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$. If $r = 0, s = 0$ and $1 \leq t \leq a+1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $a \leq t \leq a+1$ or $u = a-1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_2 = (-a-r, -a-s, a-1-t, a-u)$. If $a \leq t \leq a+1$ and $a \leq u \leq a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_5$. If $1 \leq t \leq a-1$ and $u = a-1$, then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $a \leq t \leq a+1$ and $u = a-1$, then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_7 = (a-r, -a-s, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 6: $\mathbf{x} = (-r, -a-1, -t, -u)$ where $0 \leq r \leq a$, $0 \leq t \leq a+1$ and $0 \leq u \leq a-2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (a-r, -1, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_7 = (-r, a-1, 1-t, -1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$

Case 7: $\mathbf{x} = (-a-1, -s, -t, -u)$ where $0 \leq s \leq a$, $0 \leq t \leq a+1$ and $0 \leq u \leq a-2$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_5 = (-1, a-s, a+1-t, a-2-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_4 = (a-1, -s, 1-t, -1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

This completes the cases for the orthant of $\mathbf{v}_5$.

Orthant of $\mathbf{v}_6$, $k \equiv 1 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_6$ but not between $\mathbf{0}$ and $\mathbf{v}_6$. Then the first coordinate of $\mathbf{x}$ is equal to $a+1$ or the second coordinate is equal to $a+1$, or the third equals $-a$ or $-a-1$ or the fourth equals $-a-1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (a+1, a+1, -t, -a-1)$ where $a \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a-1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 2: $\mathbf{x} = (a+1, a+1, -t, -u)$ where $a \leq t \leq a+1$ and $0 \leq u \leq a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_2 = (-a, -a, 2a-3-t, 1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$.

Case 3: $\mathbf{x} = (a+1, a+1, -t, -a-1)$ where $0 \leq t \leq a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a-1-t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 4: $\mathbf{x} = (a+1, s, -t, -a-1)$ where $0 \leq s \leq a$ and $a \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s-a, a-1-t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 5: $\mathbf{x} = (r, a+1, -t, -a-1)$ where $0 \leq r \leq a$ and $a \leq t \leq a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 =$

$(r-a, 1, a-1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 6: $\mathbf{x} = (r, s, -t, -a-1)$ where $0 \le r, s \le a$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, s-a, a-1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 7: $\mathbf{x} = (r, a+1, -t, -a-1)$ where $0 \le r \le a$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, 1, a-1-t, -1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 8: $\mathbf{x} = (a+1, s, -t, -a-1)$ where $0 \le s \le a$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s-a, a-1-t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 9: $\mathbf{x} = (r, a+1, -t, -u)$ where $0 \le r, u \le a$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, 1, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_1 = (r-1, -a, 2a-2-t, -u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a$ in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 10: $\mathbf{x} = (a+1, s, -t, -u)$ where $0 \le s, u \le a$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s-a, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-a, s-1, 2a-2-t, -u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a$. If $s = 0$ and $u = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (1, s+a, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 11: $\mathbf{x} = (a+1, a+1, -t, -u)$ where $0 \le t \le a-1$ and $0 \le u \le a$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, 1, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $a-1 \le u \le a$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (1-a, 1-a, -2-t, 2-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 12: $\mathbf{x} = (r, s, -t, -a-1)$ where $0 \le r, s \le a$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, s-a, a-1-t, -1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (r+1, s+1, 1-t, a-2)$ and $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (r-a+1, s-a+1, -a-t, 0)$. Then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_8$.

Case 13: $\mathbf{x} = (r, s, -t, -u)$ where $0 \le r, s, u \le a$ and $a \le t \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, s-a, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r = 0$ or $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (r-1, s-1, 2a-1-t, -1-u)$. If $r = 0$ and $s = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t = a$ or $a-1 \le u \le a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a, a, a+1-t, a-2-u)$. If $t = a$ and $0 \le u \le a-2$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$. If $a \le t \le a+1$ and $a-1 \le u \le a$ then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_4 = (a, -a, t-a, u-a+1)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_7$. If $r = 0$ and $1 \le s \le a$ then let $\mathbf{x}'' = \mathbf{x} - \mathbf{v}_7 = (r+a, s-a, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $u = a$, in which case $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $1 \le r \le a$ and $s = 0$ then let $\mathbf{x}'' = \mathbf{x} + \mathbf{v}_4 = (r-a, s+a, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $u = a$, in which case $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 14: $\mathbf{x} = (r, a+1, -t, -u)$ where $0 \le r, u \le a$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (r-a, 1, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_4 = (r, 1-a, -1-t, 1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a-1$. If $t = a-1$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (r-a-1, 0, a-2-t, 1-a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $r = 0$, in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_1$.

Case 15: $\mathbf{x} = (a+1, s, -t, -u)$ where $0 \le s, u \le a$ and $0 \le t \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_6 = (1, s-a, a-1-t, a-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_7 = (1-a, s, -1-t, 1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$ unless $t = a-1$. If $t = a-1$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_1 = (0, s-a-1, a-2-t, 1-a-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $s = 0$, in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$.

This completes the cases for the orthant of $\mathbf{v}_6$.

Orthant of $\mathbf{v}_7$, $k \equiv 1 \pmod 2$

Now suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_7$ but not between $\mathbf{0}$ and $\mathbf{v}_7$. Then the first coordinate of $\mathbf{x}$ is equal to $-a-1$ or the seond is equal to $a+1$ or the third equals $-a-1$, or the fourth equals $-a$ or $-a-1$. We distinguish fifteen cases.

Case 1: $\mathbf{x} = (-a-1, a+1, -a-1, -u)$ where $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, -1, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 2: $\mathbf{x} = (-a-1, a+1, -a-1, -u)$ where $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, -1, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_1$.

Case 3: $\mathbf{x} = (-a-1, a+1, -t, -u)$ where $0 \le t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, a-t, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (a, 2-a, 1-t, 2a-1-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_7$.

Case 4: $\mathbf{x} = (-a-1, s, -a-1, -u)$ where $0 \le s \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s-a, -1, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 5: $\mathbf{x} = (-r, a+1, -a-1, -u)$ where $0 \le r \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, 1, -1, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 6: $\mathbf{x} = (-r, s, -a-1, -u)$ where $0 \le r, s \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, s-a, -1, a-1-u)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

Case 7: $\mathbf{x} = (-r, a+1, -t, -u)$ where $0 \le r, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, 1, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_8 = (1-r, 2-a, -t, 2a-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $t = a$. If $r = 0$ and $t = a$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_3 = (-a-r, 1, a-1-t, a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 8: $\mathbf{x} = (-a-1, s, -t, -u)$ where $0 \le s, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s-a, a-t, a-1-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (a, s+1, 2-t, 2a-2-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a$. If $s = a$ and $t \le 1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (0, s-a+1, -a+1-t, a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 9: $\mathbf{x} = (-r, a+1, -a-1, -u)$ where $0 \le r \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, 1, -1, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_2$.

Case 10: $\mathbf{x} = (-a-1, s, -a-1, -u)$ where $0 \le s \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s-a, -1, a-1-u)$ lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_8 = (a-2, s-1, a-1, -2-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $u = a-1$. If $s = 0$ and $u = a-1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_1 = (-1, s+a, 0, a-2-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 11: $\mathbf{x} = (-a-1, a+1, -t, -u)$ where $0 \le t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, 1, a-t, a-1-u)$, which lies between $\mathbf{0}$ and $\mathbf{v}_4$.

Case 12: $\mathbf{x} = (-r, s, -t, -u)$ where $0 \le r, s, t \le a$ and $a \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, s-a, a-t, a-1-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $r = 0$ or $t = 0$, in which case let

$\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (1-r, s+1, 1-t, 2a-1-u)$. If $r = 0$ and $t = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $s = a$ or $u = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_5 = (1-a-r, s-a+1, -a-t, a+1-u)$. If $u = a$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_6$. If $s = a$ and $u = a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$. If $r = 0$ and $1 \le t \le a$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = a$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$. If $1 \le r \le a$ and $t = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $s = a$, in which case $\mathbf{x}'$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$.

Case 13: $\mathbf{x} = (-r, s, -a-1, -u)$ where $0 \le r, s \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, s-a, -1, a-1-u)$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$ unless $s = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_3 = (-1-r, s-1, a-2, -1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $u = a-1$. If $s = 0$ and $u = a-1$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_2 = (a-r, a+s, 0, a-2-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 14: $\mathbf{x} = (-r, a+1, -t, -u)$ where $0 \le r, t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (a-r, 1, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_5 = (-r, -a+1, -t-1, 1-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$ unless $r = a$ or $t = a$. If $r = a$ and $u = 0$ then $\mathbf{x}'$ lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $0 \le r \le a-1$, $t = a$ and $u = 0$ then let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_8 = (a-1-r, 0, a-1-t, -a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 15: $\mathbf{x} = (-a-1, s, -t, -u)$ where $0 \le s, t \le a$ and $0 \le u \le a-1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_7 = (-1, s-a, a-t, a-1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_6$ unless $t = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_6 = (a-1, s, 1-t, -1-u)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $s = a$. If $s = a$ and $t = 0$ then let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (0, s-a+1, 1-a-t, a-u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$.

This completes the cases for the orthant of $\mathbf{v}_7$.

Orthant of $\mathbf{v}_8$, $k \equiv 1 \pmod 2$

Finally suppose $\mathbf{x}$ lies in the orthant of $\mathbf{v}_8$ but not between $\mathbf{0}$ and $\mathbf{v}_8$. Then the first coordinate of $\mathbf{x}$ is equal to $-a$ or $-a-1$, or the second is equal to $-a$ or $-a-1$, or the third is equal to $-a-1$. We distinguish seven cases.

Case 1: $\mathbf{x} = (-r, -s, -a-1, u)$ where $a \le r, s \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_5$ unless $u \le 2$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_5 = (2a-1-r, 2a-1-s, a, u-3)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_8$.

Case 2: $\mathbf{x} = (-r, -s, -t, u)$ where $a \le r, s \le a+1$, $0 \le t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_2$ unless $t \le 1$ or $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_2 = (2a-r, 2a-s, 2-t, u-2)$. If $t \le 1$ and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_8$ unless $r = a$ or $s = a$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_8 = (a+1-r, a+1-s, 2-a-t, u+a-1)$. If $t \le 1$, $u \le 1$, $r = a$ and $s = a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_2$ unless $t = 1$ or $u = 1$. If $r = a$, $s = a$, $t = 1$ and $u \le 1$ then let $\mathbf{x}'''' = \mathbf{x} - \mathbf{v}_5 = (a-r, a-s, a+1-t, a-2+u)$ which lies between $\mathbf{0}$ and $\mathbf{v}_4$. If $r = a$, $s = a$, $t = 0$ and $u = 1$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_6$. If $t \le 1$, $u \le 1$, $r = a$ and $s = a+1$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $-\mathbf{v}_3$. If $t \le 1$, $u \le 1$, $r = a+1$ and $s = a$ then $\mathbf{x}'''$ lies between $\mathbf{0}$ and $\mathbf{v}_1$. If $t \le 1$ and $2 \le u \le a+1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_5$ unless $u = a+1$, in which case let $\mathbf{x}'''' = \mathbf{x}'' + \mathbf{v}_5 = (a-r, a-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_8$. If $2 \le t \le a+1$, and $u \le 1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_6$.

Case 3: $\mathbf{x} = (-r, -s, -a-1, u)$ where $a \le r \le a+1$, $0 \le s \le a-1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_7$ unless $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_7 = (2a-1-r, -1-s, a-1, u-2)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_1$.

Case 4: $\mathbf{x} = (-r, -s, -a-1, u)$ where $0 \le r \le a-1$, $a \le s \le a+1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_4$ unless $u \le 1$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_4 = (-1-r, 2a-1-s, a-1, u-2)$ which lies between $\mathbf{0}$ and $\mathbf{v}_3$.

Case 5: $\mathbf{x} = (-r, -s, -a-1, u)$ where $0 \le r, s \le a-1$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, -1, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_6$ unless $u = 0$, in which case $\mathbf{x}$ lies between $\mathbf{0}$ and $\mathbf{v}_5$.

Case 6: $\mathbf{x} = (-r, -s, -t, u)$ where $0 \le r \le a-1$, $a \le s \le a+1$, $0 \le t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and $-\mathbf{v}_1$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' + \mathbf{v}_1 = (-r, 2a-2-s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $t = 0$ and $1 \le u \le a+1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_4$ unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' - \mathbf{v}_4 = (a-r, a-2-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and $-\mathbf{v}_3$. . If $1 \le t \le a$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $\mathbf{v}_7$.

Case 7: $\mathbf{x} = (-r, -s, -t, u)$ where $a \le r \le a+1$, $0 \le s \le a-1$, $0 \le t \le a$ and $0 \le u \le a+1$. Let $\mathbf{x}' = \mathbf{x} - \mathbf{v}_8 = (a-1-r, a-1-s, a-t, u-a-1)$, which lies between $\mathbf{0}$ and $\mathbf{v}_3$ unless $t = 0$ or $u = 0$, in which case let $\mathbf{x}'' = \mathbf{x}' - \mathbf{v}_3 = (2a-r, -s, 1-t, u-1)$. If $t = 0$ and $u = 0$ then $\mathbf{x}$ lies between $\mathbf{0}$ and $-\mathbf{v}_2$. If $t = 0$ and $1 \le u \le a+1$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_7$ unless $u = a+1$, in which case let $\mathbf{x}''' = \mathbf{x}'' + \mathbf{v}_7 = (a-r, a-s, 1-a-t, u-a)$ which lies between $\mathbf{0}$ and $\mathbf{v}_1$. . If $1 \le t \le a$ and $u = 0$ then $\mathbf{x}''$ lies between $\mathbf{0}$ and $-\mathbf{v}_4$.

This completes the cases for the orthant of $\mathbf{v}_8$.

This also completes the proof of the theorem for any $k \equiv 1 \pmod 2$, and therefore for all $k \ge 2$. □

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