Small partial latin squares that cannot be embedded in a Cayley table

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Dedicated to the memory of Dan Archdeacon

Abstract

We answer a question posed by Dénes and Keedwell that is equivalent to the following. For each order n what is the smallest size of a partial latin square that cannot be embedded into the Cayley table of any group of order n? We also solve some variants of this question and in each case classify the smallest examples that cannot be embedded. We close with a question about embedding of diagonal partial latin squares in Cayley tables.

1 Introduction

A *partial latin square* (PLS) is a matrix in which some cells may be empty and in which any two filled cells in the same row or column must contain distinct symbols. In this work we will insist that each row and column of a PLS must contain at least

one filled cell. The *size* of the PLS is the number of filled cells. The *order* of the PLS is the maximum of its number of rows, number of columns and the number of distinct symbols it contains. We say that a PLS P *embeds* in a group G if there is a copy of P in the Cayley table of G. Formally, this means there are injective maps I_1, I_2 and I_3 from, respectively, the rows, columns and symbols of P to G such that $I_1(r)I_2(c) = I_3(s)$ whenever a symbol s occurs in cell (r, c) of P.

Let $\psi(n)$ denote the largest number m such that for every PLS P of size m there is some group of order n in which P can be embedded. Rephrased in our terminology, Open Problem 3.8 in [5] (see also [8]) asks for the value of $\psi(n)$ for even n. In this note we solve this problem by showing:

Theorem 1.
$$\psi(n) = \begin{cases} 1 & \text{when } n = 1, 2, \\ 2 & \text{when } n = 3, \\ 3 & \text{when } n = 4, \text{ or when } n \text{ is odd and } n > 3, \\ 5 & \text{when } n = 6, \text{ or when } n \equiv 2, 4 \pmod 6 \text{ and } n > 4, \\ 6 & \text{when } n \equiv 0 \pmod 6 \text{ and } n > 6. \end{cases}$$

We also consider an abelian variant. Let $\psi_+(n)$ denote the largest number m such that for every PLS P of size m there is some *abelian* group of order n in which P can be embedded. We show that:

Theorem 2.
$$\psi_+(n) = \begin{cases} 1 & \text{when } n = 1, 2, \\ 2 & \text{when } n = 3, \\ 3 & \text{when } n = 4, \text{ or when } n \text{ is odd and } n > 3, \\ 5 & \text{when } n \text{ is even and } n > 4. \end{cases}$$

Narrowing the focus even further, let $\psi_\circ(n)$ be the largest number m such that every PLS P of size m embeds in the cyclic group $\mathbb{Z}_n$. We show that

Theorem 3. $\psi_\circ(n) = \psi_+(n)$ *for all positive integers* n.

Note that it is immediate from the definitions that $\psi_\circ(n) \leqslant \psi_+(n) \leqslant \psi(n)$ for all n.

In our investigations we will repeatedly use the observation that pre- and post-multiplication allow us, without loss of generality, to specify one row to be mapped to the identity by I_1 and one column to be mapped to the identity by I_2 (see Lemma 2 in [3]). We will also often find it convenient to consider the symbols in a PLS to be group elements (put another way, we may treat I_3 as the identity map).

We will use ε to denote the identity element of a group, except that when we know the group is abelian we will use additive notation with 0 as the identity.

2   Upper bounds

We first show that $\psi(n)$ never exceeds the values quoted in Theorem 1. To do this we construct, for each n, a PLS of size $\psi(n) + 1$ that cannot be embedded into any group of order n.

A *quasigroup* Q is a nonempty set with one binary operation such that for every a, $b \in Q$ there is a unique $x \in Q$ and a unique $y \in Q$ satisfying $ax = b = ya$. The definition of an embedding for a PLS converts without change from groups to quasigroups. The quasigroup analogue of finding $\psi(n)$ was the subject of a famous conjecture known as the Evans conjecture. It is now a theorem [1, 9].

Theorem 4. *Each PLS of size at most $n - 1$ can be embedded into some quasigroup of order n. For each $n > 1$ there is a PLS of size n that cannot be embedded into any quasigroup of order n.*

The difficult part of Theorem 4 is the first statement. Examples that show the second statement are fairly obvious. For $1 \leqslant a < n$ a PLS $E_{n,a}$ of size n can be constructed as follows:

$$\begin{aligned} E_{n,a}(1, i) = i \text{ for } 1 \leqslant i \leqslant a, \\ E_{n,a}(i, n) = i \text{ for } a < i \leqslant n. \end{aligned} \tag{1}$$

Clearly, none of the symbols $1, 2, \ldots, n$ is available to fill the cell $(1, n)$, so $E_{n,a}$ cannot be embedded in a quasigroup of order n. By [1, 4], this is essentially the only way to build a PLS of size n that cannot be embedded in a quasigroup of order n. An immediate consequence of these examples is:

Corollary 5. $\psi(n) < n$ *for $n > 1$.*

There is another way in which Corollary 5 can be derived for certain values of n, which again connects to important prior work. A *complete mapping* for a group G is a permutation ϕ of the elements of G such that the map $x \mapsto x\phi(x)$ is also a permutation of the elements of G. See [5, 13] for context, including a proof that no group of order $n \equiv 2 \pmod 4$ has a complete mapping. This implies for $n \equiv 2 \pmod 4$ that $\psi(n) < n$, since no group can have an embedding of the PLS T_n of size n, where

$$T_n(i, i) = i \text{ for } i = 1, \ldots, n. \tag{2}$$

Later we need the following special case of a theorem by Brouwer et al. [2] and Woolbright [14].

Theorem 6. T_t *embeds in any group of order n, provided $t \leqslant \lceil n - \sqrt{n}\, \rceil$.*

Next we show:

Lemma 7. *For each $\ell \geqslant 2$ there exists a PLS of size 2ℓ that can only be embedded in groups whose order is divisible by ℓ.*

Proof. Consider the PLS

$$C_\ell = \begin{pmatrix} a_1 & a_2 & \cdots & a_{\ell-1} & a_\ell \\ a_2 & a_3 & \cdots & a_\ell & a_1 \end{pmatrix}$$

which is often known as a *row cycle* in the literature (see e.g. [10, 11]). Suppose that C_ℓ is embedded in rows indexed r_1 and r_2 of the Cayley table of a group G. From the regular representation of G as used in Cayley's theorem, it follows that $r_1^{-1} r_2$ has order ℓ in G. In particular ℓ divides the order of G. □

Finally, we exhibit a constant upper bound.

Lemma 8. $\psi(n) \leqslant 6$ *for all* n.

Proof. The following pair of PLS of size 7

$$\begin{pmatrix} a & b & \cdot \\ c & a & b \\ \cdot & c & d \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot \\ c & a & b \\ \cdot & d & a \end{pmatrix} \tag{3}$$

each fail the so-called quadrangle criterion [5, 8] and hence neither can be embedded into any group. □

Lemma 7 for $\ell \in \{2, 3\}$ combined with Lemma 8 and Corollary 5 gives the upper bounds on $\psi(n)$ that we set out to prove.

3   Proof of the main results

Let $\mathcal{S}_n$ denote the symmetric group of degree n. It can be convenient to view a PLS as a set of *triples* of the form (r, c, s) which record that symbol s occupies cell (r, c). When viewed in this way, there is a natural action, called *parastrophy*, of $\mathcal{S}_3$ on PLS where the triples are permuted uniformly. There is also a natural action of $\mathcal{S}_n \wr \mathcal{S}_3$ on the set of PLS of order n. Orbits under this action are known as *species* (the term *main classes* is also used). See [12] for full details.

To prove Theorem 1 it remains to show that every PLS of size no more than the claimed value of $\psi(n)$ can indeed be embedded in some group of order n. We also aim to identify every species of PLS of size $\psi(n) + 1$ which cannot be embedded into any group of order n. We refer to such a PLS as an *obstacle for* $\psi(n)$. Obstacles for $\psi_+(n)$ and $\psi_\circ(n)$ are defined similarly. By our work in the previous section we know that all obstacles have size at most 7. A catalogue of species representatives for all PLS of size at most 7 is simple to generate (it is much simpler task than the enumeration in [12], though some programs from that enumeration were reused in the present work). The number of species involved is shown in the following table:

size	1	2	3	4	5	6	7
#species	1	2	5	18	59	306	1861

$$\tag{4}$$

It suffices to consider one representative of each species because of the following Lemma. We omit the proof since it is identical to that of Lemma 1 in [3], and an easy consequence of Theorem 4.2.2 in [5].

Lemma 9. *Let G be an arbitrary group and P, P' two PLS from the same species. Then P embeds in G if and only if P' embeds in G.*

To find $\psi(n)$, $\psi_+(n)$ and $\psi_\circ(n)$ we wish to identify the smallest PLS that cannot be embedded into any group (resp. abelian group, cyclic group) of order n. The following two lemmas allow us to eliminate many candidates from consideration.

We use $\bullet$ to denote an arbitrary row, column or symbol (possibly a different one each time that $\bullet$ appears).

Lemma 10. *Let G be a group of order n and P a PLS of order at most n. If P contains a triple (r, c, s) with the following properties:*

(i) $P' := P \setminus \{(r, c, s)\}$ *has no triple* $(r, \bullet, \bullet)$,

(ii) *for each triple* (r', c', s') *in* P' *there is a triple* $(r', c, \bullet)$ *or* $(r', \bullet, s)$ *in* P',

then P can be embedded in G if P' can be embedded in G.

Proof. Consider an embedding (I_1, I_2, I_3) of P' in G. By condition (i) we know that $I_1(r)$ will not be defined. If $I_2(c)$ is undefined then P' must have fewer that n columns so we may simply define $I_2(c)$ to be any element of $G \setminus \{I_2(y) : (x, y, z) \in P'\}$. If $I_3(s)$ is undefined then we choose its value similarly. Then we define $I_1(r) = I_3(s)I_2(c)^{-1}$. By construction, both I_2 and I_3 are injective. So at this point, the only thing which could prevent us having an embedding of P in G would be if $I_1(r) = I_1(r')$ for some $(r', c', s') \in P'$. Suppose this is the case. Then by condition (ii) there is a triple $(r', c, \bullet)$ or $(r', \bullet, s)$ in P'. First suppose that (r', c, s'') is in P' for some symbol s''. Then $I_3(s'') = I_1(r')I_2(c) = I_1(r)I_2(c) = I_3(s)$ so $s'' = s$. But this means (r', c, s) and (r, c, s) are distinct triples in P, which is impossible. So it must be that (r', c'', s) is in P' for some c''. But this means that $I_2(c'') = I_1(r')^{-1}I_3(s) = I_1(r)^{-1}I_3(s) = I_2(c)$. Thus $c = c''$, which leads to the same contradiction as before. $\square$

Lemma 11. *Let G be a group of order n. Let P be a PLS with rows R, columns C and symbols S. Suppose that P contains triples $(r, c_1, s_1), \ldots, (r, c_\ell, s_\ell)$ with no other triples in row r. Let $P' = P \setminus \{(r, c_i, s_i) : 1 \leqslant i \leqslant \ell\}$ and let R', C' and S' be the rows, columns and symbols of P', respectively. Let $C_1 = \{1 \leqslant i \leqslant \ell : c_i \in C'\}$ and $S_1 = \{1 \leqslant i \leqslant \ell : s_i \in S'\}$. If*

(i) $C_1 \cap S_1 = \emptyset$,

(ii) $n \geqslant |R| + |C_1|(|S'| - 1) + |S_1|(|C'| - 1)$, *and*

(iii) $n \geqslant |C| + |S| - \ell$,

then P can be embedded in G if P' can be embedded in G.

Proof. Consider an embedding (I_1, I_2, I_3) of P' in G. By definition, P' contains at least one triple $(\bullet, c_i, \bullet)$ for each $i \in C_1$. Likewise, P' contains at least one triple $(\bullet, \bullet, s_i)$ for each $i \in S_1$. Hence the constraint (ii) ensures that we can choose a value for $I_1(r) \in G$ outside the set

$$\{I_1(r') : (r', c', s') \in P'\} \cup \{I_3(s)I_2(c_i)^{-1} : i \in C_1, s \in S'\}$$
$$\cup \{I_3(s_i)I_2(c)^{-1} : i \in S_1, c \in C'\}.$$

The injectivity of I_1 is immediate. By condition (i) we are free to define $I_2(c_i) = I_1(r)^{-1}I_3(s_i)$ for $i \in S_1$ and to define $I_3(s_i) = I_1(r)I_2(c_i)$ for $i \in C_1$.

There are at least $n - (|C'| + |S'|) = n - (|C| - (\ell - |C_1|) + |S| - (\ell - |S_1|))$ elements of

$$G \setminus \left(\{I_2(c') : (r', c', s') \in P'\} \cup \{I_1(r)^{-1}I_3(s) : s \in S'\}\right) \tag{5}$$

So by condition (iii) there are at least $\ell - |C_1| - |S_1|$ elements of this set. This gives us the option, for $i \in \{1, \ldots, \ell\} \setminus (C_1 \cup S_1)$, to choose distinct values for $I_2(c_i)$ in (5) and put $I_3(s_i) = I_1(r)I_2(c_i)$. It is routine to check that this yields an embedding of P in G. $\qquad\square$

For ease of expression we have stated Lemmas 10 and 11 in a form that breaks the symmetry between rows, columns and symbols. This loses some generality, but we can get it back by applying the lemmas to each PLS in an orbit under parastrophy. We will do this without further comment when invoking these lemmas.

The fact that $\psi_\circ(1) = \psi_+(1) = \psi(1) = 1$ is a triviality. The fact that $\psi_+(n) = \psi(n) = n-1$ for $n \in \{2, 3, 4\}$ follows from Theorem 4 and the fact that for these orders every species of quasigroup contains an abelian group. Moreover, for $n \in \{2, 3, 4\}$ the obstacles for $\psi_+(n)$ and $\psi(n)$ are precisely those characterised in [1, 4]. Specifically, $E_{n,a}$ as given in (1) is an obstacle, and every obstacle belongs to the species of some $E_{n,a}$. However, $E_{n,a}$ is from the same species as $E_{n,n-a}$. Thus there are only $\lfloor n/2 \rfloor$ species described by (1). It is not hard to check that $\psi_\circ(n) = \psi_+(n)$ when $n \in \{2, 3, 4\}$. The obstacles for $\psi_\circ(n)$ are the same as for $\psi_+(n)$ except that there is one extra obstacle when $n = 4$, namely T_4 from (2). Henceforth, we may assume that $n \geqslant 5$.

Let G be any group of order $n \geqslant 5$ and let P be a PLS of size at most 3. Lemmas 10 and 11 together show that P cannot be the smallest PLS that does not embed in G. It then follows from Lemma 7 that $\psi_\circ(n) = \psi_+(n) = \psi(n) = 3$ for all odd $n \geqslant 5$. To confirm the uniqueness of C_2 as an obstacle we screened the PLS of size 4 with Lemmas 10 and 11. Most were eliminated immediately (for some, a quick manual check that they embed in $\mathbb{Z}_5$ was required because the lemmas only applied for $n \geqslant 7$). Apart from C_2, the only candidate left standing was:

$$\begin{pmatrix} a & b & \cdot \\ \cdot & a & b \end{pmatrix}. \tag{6}$$

However, this PLS can be embedded in any group which has an element of order more than 2, which is to say, any group other than an elementary abelian 2-group, as shown by:

$$
\begin{array}{c|ccc}
 & \varepsilon & b & b^2 \\
\hline
\varepsilon & \varepsilon & b & \cdot \\
b^{-1} & \cdot & \varepsilon & b
\end{array}
$$

Having completed the odd case, from now on we assume that n is even.

Let us next consider the case $n = 6$. Two independent computations confirm that, of the species of PLS of size at most 6, only six species do not embed into $\mathbb{Z}_6$. The six species all have size 6 and thus these six species are the obstacles for $\psi_\circ(6) = \psi_+(6) = 5$. They include the 3 species $E_{6,1}$, $E_{6,2}$, $E_{6,3}$ from (1) and the one species T_6 from (2). Representatives of the other two species are:

$$
\begin{pmatrix}
a & \cdot & \cdot & \cdot & \cdot & c \\
\cdot & a & \cdot & \cdot & b & \cdot \\
\cdot & \cdot & b & c & \cdot & \cdot
\end{pmatrix}
\tag{7}
$$

$$
\begin{pmatrix}
a & b & \cdot \\
c & \cdot & b \\
\cdot & c & d
\end{pmatrix}
\tag{8}
$$

These two species deserve individual scrutiny:

Lemma 12. *The PLS in* (7) *does not embed into any group of order* 6.

Proof. We assume to the contrary that (I_1, I_2, I_3) embeds (7) into a group G of order 6, where I_3 is the identity. Let γ satisfy $I_1(1)I_2(\gamma) = b$. It is clear that $\gamma \in \{2, 4\}$. The two possible choices for γ are equivalent under the row-permutation $(2\,3)$, column-permutation $(1\,6)(2\,4)(3\,5)$ and symbol-permutation $(a\,c)$, so we may assume that $\gamma = 2$.

We may also assume that $I_1(1) = I_2(1) = a = \varepsilon$ from which it follows that $I_1(2) = b^{-1}$ and b must have order 2, 3 or 6. Clearly, b does not have order 2 as this would imply that $I_1(2)I_2(1) = b = I_1(2)I_2(5)$, violating the injectivity of I_2.

If b has order 3, then $b^2 = b^{-1}$ and we have:

$$
\begin{array}{c|cccccc}
 & \varepsilon & b & x & y & b^2 & c \\
\hline
\varepsilon & \varepsilon & b & \cdot & \cdot & \cdot & c \\
b^2 & b^2 & \varepsilon & \cdot & \cdot & b & \cdot \\
z & \cdot & \cdot & b & c & \cdot & \cdot
\end{array}
$$

where $G = \{\varepsilon, b, b^2, x, y, c\}$ and $z \in G \setminus \{\varepsilon, b^2, b, c\}$. By eliminating other possibilities within the second row we see that $b^2 x = c$ and hence $b^2 c = y$. However, this implies that $c = zy = zb^2 c$ and thus $z = b$, which is not possible. Therefore b cannot have order 3.

If b has order 6, then the group is abelian and we may assume that $b = 1$ giving:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & x & y & 2 & c \\
\hline
0 & 0 & 1 & \cdot & \cdot & \cdot & c \\
5 & \cdot & 0 & \cdot & \cdot & 1 & \cdot \\
z & \cdot & \cdot & 1 & c & \cdot & \cdot
\end{array}
$$

with $\{x, y, c\} = \{3, 4, 5\}$ and $z \in \{1, 2, 3, 4\}$. Now, $z + x = 1$ and $z + y = c$ so $c + x = y + 1$, which has no solution amongst the available values. Therefore b cannot have order 6. $\qquad\square$

Lemma 13. *The PLS in* (8) *does not embed into any abelian group.*

Proof. Assume to the contrary that (I_1, I_2, I_3) embeds this PLS into an abelian group. We may assume that I_1 maps the rows to 0, c and y, respectively, and I_2 maps the columns to 0, b and x, respectively.

$$
\begin{array}{c|ccc}
 & 0 & b & x \\
\hline
0 & 0 & b & \cdot \\
c & c & \cdot & b \\
y & \cdot & c & d
\end{array}
$$

We have $y + b = c$ and $c + x = b$ which imply that $b = y + b + x$. Hence, $d = x + y = 0$ which prevents I_3 from being injective. Thus no embedding into an abelian group is possible. $\qquad\square$

However, (8) does embed into each dihedral group of order at least 6. Using the presentation $D_{2k} = \langle r, s | r^k = s^2 = \varepsilon, sr = r^{-1}s \rangle$, we find this embedding:

$$
\begin{array}{c|ccc}
 & \varepsilon & rs & s \\
\hline
\varepsilon & \varepsilon & rs & \cdot \\
r & r & \cdot & rs \\
r^2 s & \cdot & r & r^2
\end{array}
$$

In particular, we have established that $\psi(6) = 5$ and there are precisely 5 species that are obstacles for $\psi(6)$, namely $E_{6,1}$, $E_{6,2}$, $E_{6,3}$, T_6 and (7).

With the aid of Lemma 11 we know that these 5 obstacles for $\psi(6)$ can be embedded into every group of order at least 11. It follows that $\psi_\circ(n) = \psi_+(n) = 5$ and $\psi(n) = 6$ whenever $n \equiv 0 \pmod{6}$ and $n > 6$. In this case, the species of the PLS in (8) is the unique obstacle for $\psi_\circ(n)$ and $\psi_+(n)$. Characterising the obstacles for $\psi(n)$ requires more work. Using Lemmas 10 and 11 and Theorem 6 we immediately eliminate all but 50 of the PLS of size 7. Of these 50 PLS, 42 embed in $\mathbb{Z}_6$ and hence are not obstacles. Let Ω be the set of the remaining 8 PLS. There are 6 PLS in Ω that contain a PLS from the species represented by (8), which explains why they do not embed in $\mathbb{Z}_6$. Two of these 6 are the obstacles that we know from (3), and the

other 4 all embed in D_6 and hence are not obstacles. That leaves just two PLS in Ω that we have not discussed. One of them is

$$\begin{pmatrix} a & b & c \\ b & a & \cdot \\ c & \cdot & a \end{pmatrix} \tag{9}$$

which can embed in any group that has more than one element of order 2. In particular, it embeds in D_{2k} for any $k \geqslant 2$ which means it is not an obstacle for $\psi(2k)$. The final PLS in Ω is

$$\begin{pmatrix} a & b & c \\ b & c & \cdot \\ c & \cdot & a \end{pmatrix}. \tag{10}$$

Suppose (I_1, I_2, I_3) is an embedding of this PLS in a group G. We may assume that I_1 maps the rows to ε, b, c respectively, and I_2 maps the columns to ε, b, c respectively. It then follows that b is an element of order 4. We conclude that G cannot have order 2 (mod 4). Conversely, it is clear that (10) embeds in $\mathbb{Z}_4$ and hence into $\mathbb{Z}_n$ for any n divisible by 4. In conclusion, we know that for n divisible by 12 the only obstacles for $\psi(n)$ are the two species given in (3). When $n = 12k + 6$ for an integer $k > 1$, there are 3 obstacles as given in (3) and (10).

It remains to consider orders $n \geqslant 8$ which are divisible by 2 but not by 3. For such orders, Lemma 7 shows that $\psi_\circ(n) \leqslant \psi_+(n) \leqslant \psi(n) \leqslant 5$, Let G be a group of order n. Screening the PLS of size up to 5 using Lemmas 10 and 11, we found only three candidates for the smallest PLS that does not embed in G. The first was C_2, which can embed in G by Sylow's Theorem. The second was T_5 from (2), which can embed in G by Theorem 6. The third was the PLS (6), which can be embedded in any cyclic group of order greater than 2, so $\psi_\circ(n) = \psi_+(n) = \psi(n) = 5$.

To find the obstacles for $\psi(n)$, $\psi_+(n)$ and $\psi_\circ(n)$ for $n \equiv 2, 4 \pmod 6$ we proceed as before. Using Lemmas 10 and 11 and Theorem 6 we eliminated all but 11 of the PLS of size 6 (for 11 others we needed to find an embedding in $\mathbb{Z}_8$, whilst the lemmas took care of all larger groups). The 11 remaining candidates for obstacles included C_3 which we know is an obstacle for $\psi(n)$, $\psi_+(n)$ and $\psi_\circ(n)$ and (8), which we know is an obstacle for $\psi_+(n)$ and $\psi_\circ(n)$ but not for $\psi(n)$.

The remaining 9 PLS can be embedded into cyclic groups of any order at least 6 and hence are not obstacles for $\psi(n)$, $\psi_+(n)$ or $\psi_\circ(n)$. The claimed embeddings for these PLS are shown in the following, with each PLS embedding as per the bordered

table to its right:

$$\begin{pmatrix} a & b & \cdot & \cdot \\ \cdot & a & \cdot & \cdot \\ \cdot & \cdot & a & b \\ \cdot & \cdot & b & \cdot \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot & \cdot \\ \cdot & a & c & \cdot \\ \cdot & \cdot & a & b \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot \\ \cdot & a & c \\ d & \cdot & a \end{pmatrix}$$

	0	−1	−3	−4
0	0	−1	·	·
1	·	0	−2	·
3	3	·	0	−1
2	·	·	−1	·

$$\begin{pmatrix} a & c & \cdot & \cdot \\ \cdot & a & \cdot & b \\ \cdot & \cdot & b & c \end{pmatrix} \qquad \begin{pmatrix} a & \cdot & \cdot & c \\ \cdot & a & \cdot & b \\ c & \cdot & b & \cdot \end{pmatrix}$$

	0	2	−2	1
0	0	2	·	1
−2	·	0	·	−1
1	1	·	−1	2

$$\begin{pmatrix} a & b & \cdot & \cdot \\ \cdot & a & b & \cdot \\ \cdot & \cdot & a & b \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot \\ \cdot & a & b \\ c & \cdot & a \end{pmatrix}$$

	0	−1	−2	−3
0	0	−1	·	·
1	·	0	−1	·
2	2	·	0	−1

$$\begin{pmatrix} a & b & \cdot \\ \cdot & a & c \\ c & \cdot & b \end{pmatrix}$$

	0	−2	−3
0	0	−2	·
2	·	0	1
1	1	·	−2

$$\begin{pmatrix} a & \cdot & c \\ \cdot & a & b \\ b & c & \cdot \end{pmatrix}$$

	0	−1	1
0	0	·	1
1	·	0	2
2	2	1	·

This completes the proofs of Theorem 1, Theorem 2 and Theorem 3 and the characterisation of all obstacles.

4   Concluding remarks

In the process of answering Problem 3.8 from [5, 8] we have solved three related problems. Namely, we have found, for each order n, the smallest partial latin square that cannot be embedded into (i) any group of order n, (ii) any abelian group of order n and (iii) the cyclic group of order n. We have also identified the unique species of the smallest PLS that cannot be embedded into an abelian group of any order, namely (8). And we found the two species of PLS which share the honour of being the smallest that cannot embed into any group, namely (3). As a byproduct of our investigations, we can also be sure that (9) represents the unique species of smallest PLS that can embed into some abelian group but not into any cyclic group. Similar questions had previously been answered for an important restricted class of PLS known as a *separated, connected latin trades*. Let χ denote the set of such PLS. In [3] it was found that the smallest PLS in χ to not embed in any group has size 11, the smallest PLS in χ to embed in some group but not into any abelian group has size 14, while the smallest PLS in χ to embed in some abelian group but not into any cyclic group has size 10.

We close with a discussion of what seems to be an interesting special case of embedding PLS in groups. By a *diagonal PLS* we will mean a PLS in which cells off the main diagonal are empty. We have already seen an example in (2) and, modulo parastrophy, $E_{n,1}$ from (1) is another example. The question of which groups have an embedding for T_n has proved a particularly deep and fruitful line of enquiry. It seems that asking the same question for other diagonal PLS might yield some interesting results. Equivalently, we may ask the following question for each given group G of order n and partition Π of n. Is it possible to find a permutation π of G such that the multiplicities of the elements of G in the multiset $\{g \cdot \pi(g) : g \in G\}$ form the partition Π? A theorem of Hall [7] characterises the possible multisets $\{g \cdot \pi(g) : g \in G\}$ when G is abelian. We offer the following extra result as a "teaser".

Theorem 14. *Let Δ be the diagonal PLS of size n with $\Delta(i,i) = a$ for $i \leqslant 3$ and $\Delta(i,i) = b$ for $4 \leqslant i \leqslant n$. Then Δ has an embedding into a group G of order n if and only if n is divisible by 3.*

Proof. First suppose that 3 divides n so that G has an element u of order 3. By [6, Lem. 3.1], there is an embedding (I_1, I_2, I_3) of Δ in G with $I_3(a) = u$ and $I_3(b) = \varepsilon$.

Next suppose that Δ has an embedding (I_1, I_2, I_3) in G. By post-multiplying $I_3(a)$, $I_3(b)$ and $I_2(i)$ for $1 \leqslant i \leqslant n$ by $I_3(b)^{-1}$, we may assume that $I_3(b) = \varepsilon$. It then follows from [6, Lem. 3.1] that G contains an element of order 3. Hence G has order divisible by 3, as required. $\square$

The paper [6] looked at not just whether a diagonal PLS can be embedded in a given quasigroup, but how many different ways each such PLS can be embedded. It was shown that there are examples of quasigroups $Q_1 = (Q, \star)$ and $Q_2 = (Q, \otimes)$ from different species that cannot be distinguished by this information. That is, for each diagonal PLS D and each injection I_3 from the symbols of D to Q, there are the same number of embeddings (I_1, I_2, I_3) of D in Q_1 as there are in Q_2. A question was posed in [6] whether this is possible when Q_1 is a group.

Acknowledgements

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