Self-similarity of solitary waves on inertia-dominated falling liquid films

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We propose the first consistent scaling of solitary waves on inertia-dominated falling liquid films, which accurately accounts for the driving physical mechanisms and leads to a self-similar characterization of solitary waves. Direct numerical simulations of the entire two-phase system are conducted using a state-of-the-art finite volume framework for interfacial flows in an open domain that was previously validated against experimental film-flow data with excellent agreement. We present a detailed analysis of the wave shape and the dispersion of solitary waves on 34 different water films with Reynolds numbers Re = 20 – 120 and surface tension coefficients σ = 0.0512 – 0.072 N m−1 on substrates with inclination angles β = 19° – 90°. Following a detailed analysis of these cases we formulate a consistent characterization of the shape and dispersion of solitary waves, based on a newly proposed scaling derived from the Nusselt flat film solution, that unveils a self-similarity as well as the driving mechanism of solitary waves on gravity-driven liquid films. Our results demonstrate that the shape of solitary waves, i.e. height and asymmetry of the wave, is predominantly influenced by the balance of inertia and surface tension. Furthermore, we find that the dispersion of solitary waves on the inertia-dominated falling liquid films considered in this study is governed by non-linear effects and only driven by inertia, with surface tension and gravity having a negligible influence.

I. INTRODUCTION

Falling liquid films have been an active research topic for several decades, starting with the pioneering experiments by the father-son team of the Kapitza family [1, 2], and extending to more recent work in both planar [3–6] and annular flow geometries [7, 8]. A falling liquid film is an open-flow hydrodynamic system that is convectively unstable to long-wave disturbances at small flow rates. It can be studied with the simplest experimental apparatus (e.g., [6]) while at the same time the theoretical analysis of a falling liquid film is facilitated by the substantial reduction of the complexity of the governing equations offered by the long-wave nature of the instability. The simplicity of the resulting model equations, single evolution equations or coupled averaged evolution equations, makes them useful prototypes for mathematical and numerical scrutiny. Yet, it yields a rich variety of spatial and temporal structures and a rich spectrum of wave forms and wave transitions that are generic to a large class of open-flow hydrodynamic and other non-linear systems: a unique and experimentally well-characterized sequence of non-linear secondary transitions that begins with a selected monochromatic disturbance and leads eventually to nonstationary and broad-banded (in both frequency and wavenumber) “turbulent” wave dynamics, a state of disorder/spatiotemporal chaos. At this stage, despite the apparent complexity one can still identify solitary waves in what appears to be a randomly disturbed surface. It is then essential that in order to understand the spatiotemporal evolution of the film, we fully understand the properties of individual solitary waves which in turn can help us understand the way they interact with each other. In fact, these coherent structures are truly elementary processes so that (by using techniques from nonlinear dynamics and dynamical systems theory) the dynamics of the film can be described by their superposition [9–14].

Applying a periodic forcing at the inlet with a sufficiently large amplitude, for instance by periodically changing the flow rate of the liquid film, interfacial waves form as a result of the long-wave instability mechanism [15, 16] and synchronize with the forcing frequency [17, 18]. Depending on the frequency of the forcing, two wave families can be distinguished [19, 20]: γ1-waves and γ2-waves. The γ1-waves have a speed smaller than the corresponding spatially-amplified infinitesimal wave at the same frequency and are observed close to the inlet at high forcing frequency, close but below the cut-off fre-
frequency above which the film is stable. At low forcing frequency, the exponential growth of the waves is followed by the formation of fast solitary waves of $\gamma_2$-type. These solitary waves have a dominant elevation with a long tail and steep front, typically with capillary ripples preceding the main wave hump. The $\gamma_1$ and $\gamma_2$ waves are also observed in falling films in the presence of various complexities, such as Marangoni effects due to heating, localized or uniform surfactants [22–24], chemical reactions [25–26], surface tension, viscous drag and inertial effects. In a previous study on the dynamics of solitary waves on inertia-dominated falling liquid film of Denner et al. [37], the results obtained with the DNS framework [17–19] used in this study showed excellent agreement with experimental measurements [6]. We propose a novel scaling for solitary waves, which we derive from the Nusselt flat film solution based on the physical mechanisms that underpin the growth and dispersion of solitary waves. Our results demonstrate that the proposed scaling leads to a (surprising) self-similar characterization of solitary waves on liquid films with respect to the proposed non-dimensional numbers and the inclination angle of the substrate, allowing an a priori prediction of the shape and dispersion of solitary waves for a given fluid. The presented results and observed correlations provide a detailed account of the acting physical mechanisms that drive and influence the dynamic behavior of solitary waves.

In Sec. III the governing equations are discussed briefly and in Sec. IV a new scaling for solitary waves is proposed. The applied numerical methodology is briefly outlined in Sec. V, Sec. VI describes the setup and parameter space of the conducted numerical experiments. A detailed analysis of the shape of solitary waves is presented in Sec. VII followed by an analysis of the non-linear dispersion of solitary waves in Sec. VIII. We close with a summary of our findings in Sec. IX.

II. GOVERNING EQUATIONS

The dynamic behavior of isothermal, Newtonian fluids in the incompressible flow regime is governed by the momentum equations

\[ \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho g_i + f_{\sigma,i} \quad (1) \]

and the continuity equation

\[ \frac{\partial u_i}{\partial x_i} = 0 \quad (2) \]

where $\mathbf{x} \equiv (x, y, z)$ denotes a Cartesian coordinate system with $x$ in the streamwise direction, $y$ is cross-stream direction and $z$ the transverse direction (we adopt the Einstein notation), $t$ represents time, $\mathbf{u}$ is the velocity, $p$ is the pressure, $\rho$ is the density, $\mu$ is the dynamic viscosity, $g$ is the gravitational acceleration and $f_{\sigma,i}$ is the volumetric force due to surface tension acting at the gas-liquid interface. In falling liquid films in particular, the
The hydrodynamic balance of forces acting at the gas-liquid interface is given as \[ 50\]

\[
(p_g - p_l + \sigma \kappa) \dot{m}_i = \left[ \mu_g \left( \frac{\partial u_i}{\partial x_j} \right)_g + \frac{\partial u_j}{\partial x_i} \right] \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}, \tag{3}
\]

where subscript \( g \) denotes the gas phase, subscript \( l \) denotes the liquid phase, \( \sigma \) is the surface tension coefficient, \( \kappa \) is the curvature of the gas-liquid interface and \( \hat{m}_i \) is the outward (pointing into the liquid phase) unit normal vector of the gas-liquid interface.

### III. PARAMETRIZATION AND SCALING OF SOLITARY WAVES

Consider a laminar, falling liquid film flowing down a planar inclined substrate with angle \( \beta \) to the horizontal, schematically illustrated in Fig. 1. If this film is undisturbed, i.e. the film is flat and no external perturbations act on the film, a unique equilibrium solution exists, called the Nusselt flat film solution \[ 47, 48\]. The associated equilibrium/flat film height (Nusselt film height) is

\[
h_N = \sqrt{3 \mu \rho g \sin \beta}, \tag{4}
\]

and the corresponding average film velocity (Nusselt velocity) is

\[
u_N = \frac{g \sin \beta \rho_l h_N^2}{3 \mu}, \tag{5}
\]

where \( q \) is the flow rate per unit span, \( g \) is the magnitude of the gravitational acceleration and subscript \( l \) denotes properties of the liquid film. The Nusselt flat film solution typically serves as the basis for the characterization of laminar, gravity-driven film flows and the associated interfacial instabilities.

#### A. Physical mechanisms and pertinent dimensionless groups

Three physical mechanisms dominate the dynamic behavior of free-surface flows: inertia, viscous stresses and surface tension. Each of these effects can be quantified by their respective pressure scales, namely the dynamic pressure

\[
p_{dyn} = \frac{\rho_l u_N^2}{2}, \tag{6}
\]

the viscous pressure

\[
p_\mu = \frac{\mu_l u_N}{h_N}, \tag{7}
\]

and the pressure due to surface tension

\[
p_\sigma = \frac{\sigma}{h_N}. \tag{8}
\]

The relative importance of these three mechanisms can be unraveled by making use of the Reynolds number

\[
Re = \frac{2 \rho_{dyn}}{\rho_\mu} = \frac{\rho_l u_N h_N}{\mu_l} = \frac{\rho_l q}{\mu_l}, \tag{9}
\]

which compares inertia and viscous effects, the Weber number

\[
We = \frac{2 \rho_{dyn}}{p_\sigma} = \frac{\rho_l h_N u_N^2}{\sigma}, \tag{10}
\]

which compares the inertia of the film flow to the surface tension of the interface, and the capillary number

\[
Ca = \frac{p_\mu}{p_\sigma} = \frac{\mu_l u_N}{\sigma} = \frac{\rho_l g \sin \beta h_N}{3 \sigma}, \tag{11}
\]

which compares viscous effects and surface tension. All three non-dimensional numbers are related to each other as \( We = Re Ca \). In the context of falling liquid films the Reynolds number \( Re \) can also be regarded as the non-dimensional flow rate. Using the scalings introduced by Shkadov \[ 52\] for falling liquid films, the Weber number can be written as \( We = \delta \eta / 9 \), where \( \delta = 3 Re (3Ca)^{1/3} \) is the reduced Reynolds number and \( \eta = (3Ca)^{2/3} \) is the viscous dispersion number \[ 21\].

In the literature on falling liquid films, notably \[ 21, 52, 54\], the inverse of the capillary number \( Ca \) is referred to at times as a Weber number \( We_g = 1/3Ca \), where the subscript \( g \) indicates its association with viscous stresses. This presumably stems from the fact that most theoretical studies to date have focused on film flows with low inertia in which the balance of viscous stresses and surface tension plays a dominant role, a regime which can be adequately described using models based on the boundary-layer approximation, e.g. \[ 19, 21, 56, 52, 53-57\]. The Weber number \( We \) given in Eq. (10), on the
other hand, is widely used in the context of inertia-dominated two-phase flows, such as liquid jets, bubbles and drops, e.g. [55, 62]. It is important to note that some studies on film flows use a Weber number defined as the inverse of We as defined in Eq. (10), for instance [10, 38, 53].

Apart from the properties of the liquid and the inclination angle of the substrate, the Reynolds number, Weber number and capillary number depend on $h_N$, which in experiments can be modified by changing the flow rate $q = h_N N$. The Kapitza number $Ka$, on the other hand, is widely used in the context of inertia-dominated flows, e.g. [58–62]. It is important to note that experiments can be modified by changing the flow rate.

Consequently, since the driving Reynolds number (i.e. the driving Nusselt velocity) follow as

$$\text{Re}^* = \frac{2 \rho_N^* u_N^* h_N^*}{\mu} = \text{Re} \sin \beta$$  \hspace{1cm} (16)

$$\text{We}^* = \frac{2 \rho_N^* u_N^*}{\rho \sigma} = \text{We} (\sin \beta)^2$$ \hspace{1cm} (17)

$$\text{Ca}^* = \frac{\rho_N^* u_N^* h_N^*}{\rho \sigma} = \text{Ca} \sin \beta .$$ \hspace{1cm} (18)

All three driving non-dimensional numbers are related to each other as $\text{We}^* = \text{Re}^* \text{Ca}^*$.

IV. NUMERICAL METHODOLOGY

DNS of the full two-phase system, including the liquid film, the gas phase as well as the gas-liquid interface, are conducted by resolving all relevant length and time scales. The governing equations are solved numerically using a state-of-the-art finite-volume framework for interfacial flows, described in detail in Ref. [47]. The primitive variables are solved in a single linear system of equations using a coupled, implicit finite-volume framework with collocated variable arrangement. The momentum equations, Eq. (1), are discretized using a second-order backward Euler scheme for the transient term, while the convection, diffusion and pressure terms are discretized using a central differencing scheme. The continuity equation, Eq. (2), is discretized using a balanced-force implementation of the momentum-weighted interpolation method, proposed by Denner and van Wachem [17], which couples pressure and velocity. Each time-step consists of a finite number of non-linear iterations to account for the non-linearity of the governing equations, updating the deferred terms of the equation system based on the result of the previous non-linear iteration [66]. This iterative procedure continues until the non-linear problem has converged to a sufficiently small tolerance.

The Volume-of-Fluid (VOF) method [67] is adopted to capture the interface between the immiscible gas and liquid bulk phases. The local volume fraction of both liquid bulk phases. The local volume fraction of both immiscible gas and liquid bulk phases. The local volume fraction of both immiscible gas and liquid bulk phases. The local volume fraction of both immiscible gas and liquid bulk phases. The local volume fraction of both immiscible gas and liquid bulk phases. The local volume fraction of both immiscible gas and liquid bulk phases.

### B. Inclination-corrected scaling

An interfacial wave evolving from a long-wave perturbation is always unstable (for all Re) on a vertically falling liquid film ($\beta = 90^\circ$) and stable on a horizontal liquid film ($\beta = 0^\circ$) [21]. Consequently, since the horizontal component of velocity does not contribute to sustaining the solitary wave, a fully-developed solitary wave is only affected by the vertical component of the velocity, leading to the reference velocity

$$u_N^* = u_N \sin \beta = \frac{g (\sin \beta)^2 \rho h_N^2}{3 \mu} , \tag{13}$$

henceforth referred to as driving Nusselt velocity.

Based on the driving Nusselt velocity $u_N^*$, the effective dynamic pressure of the liquid film becomes

$$p_N^* = \frac{\rho_N^* u_N^* h_N^2}{2} = \frac{\rho h_N^2}{2} (\sin \beta)^2 , \tag{14}$$

and, similarly, the effective viscous pressure follows as

$$p_\mu^* = \frac{\mu_N^* u_N^*}{h_N} = \frac{\mu h_N}{h_N} \sin \beta . \tag{15}$$

The pressure due to surface tension $p_\sigma$ remains unchanged, as defined in Eq. (8), since the influence of surface tension does not depend on the velocity of the flow or the orientation of the gas-liquid interface. Consequently, the driving Reynolds number, the driving Weber number and the driving capillary number (i.e. the non-dimensional numbers based on the driving Nusselt velocity) follow as

$$\text{Re}^* = \frac{2 \rho_N^* u_N^* h_N^*}{\mu} = \text{Re} \sin \beta$$  \hspace{1cm} (16)

$$\text{We}^* = \frac{2 \rho_N^* u_N^*}{\rho \sigma} = \text{We} (\sin \beta)^2$$ \hspace{1cm} (17)

$$\text{Ca}^* = \frac{\rho_N^* u_N^* h_N^*}{\rho \sigma} = \text{Ca} \sin \beta .$$ \hspace{1cm} (18)

All three driving non-dimensional numbers are related to each other as $\text{We}^* = \text{Re}^* \text{Ca}^*$.
based on the underlying flow with velocity \( \mathbf{u} \). Equation [21] is discretized using a compressive VOF methodology [48], based on the CICSAM (Compressive Interface Capturing Scheme for Arbitrary Meshes) scheme [68].

Assuming a constant surface tension coefficient \( \sigma \) and neglecting mass transfer between the bulk phases, the surface force per unit volume is described by the Continuum Surface Force (CSF) model [69] as

\[
 f_{\sigma,i} = \sigma \kappa \frac{\partial \gamma}{\partial x_i},
\]

(22)

which translates the surface force resulting from surface tension into a volume force which can be discretized in a finite volume framework. To ensure a balanced-force implementation of the surface force, Eq. (22) is discretized on the same computational stencil as the pressure gradient [47]. In order to diminish aliasing errors in the evaluation of the interface normal vector \( \mathbf{m} = \nabla \gamma / | \nabla \gamma | \) and the interface curvature \( \kappa = - \nabla \cdot \mathbf{m} \), the volume fraction \( \gamma \) is convoluted by means of a cosine convolution kernel with a support of \( \varepsilon = 3 \Delta x \), where \( \Delta x \) is the mesh spacing [35, 41, 70]. Convolution to smooth the surface force is not needed [71, 72] and numerical artifacts that manifest in parasitic or spurious flow features near or at the interface are addressed by an interfacial shear stress correction [37].

Simulation results obtained with this numerical framework have been shown to be in excellent agreement with analytical solutions and experimental data for single-phase flows [73, 74], interfacial flows in general [47, 48, 71, 73] and solitary waves on falling liquid films [37] in particular.

V. SETUP OF THE NUMERICAL EXPERIMENTS

The shape and dispersion of solitary waves with constant frequency \( f = 20 \text{s}^{-1} \) on 34 different falling water films in contact with air are simulated and analyzed for different Reynolds numbers \( Re \), on substrates with different inclination angles \( \beta \) and for different surface tension coefficients \( \sigma \). The working fluid is taken to be water with a density of \( \rho_l = 998 \text{kg m}^{-3} \) and a viscosity of \( \mu_l = 8.967 \times 10^{-4} \text{Pas} \), air is taken to have a density of \( \rho_g = 1.17 \text{kg m}^{-3} \) and a viscosity of \( \mu_g = 1.836 \times 10^{-5} \text{Pa s} \). All relevant parameters of the different cases analyzed are given in Table I.

These cases can be divided into three groups of results. The first group consists of 23 cases with \( Re = 20 - 120 \) and a surface tension coefficient of \( \sigma = 0.072 \text{N m}^{-1} \), with inclination angles \( \beta = 45^\circ, \beta = 60^\circ, \beta = 75^\circ \) and \( \beta = 90^\circ \). The second group of results includes 15 cases with \( Re = 20 - 100 \) on a substrate with inclination angle \( \beta = 90^\circ \) and \( \sigma = 0.0512 - 0.0621 \text{N m}^{-1} \). The third group of results are 6 cases with \( Re = 100 \) and a surface tension coefficient \( \sigma = 0.072 \text{N m}^{-1} \), simulated on a substrate with inclination angles of \( \beta = 19^\circ - 90^\circ \). The chosen cases allow a detailed and comprehensive scrutiny of the influence of inertia, the inclination angle of the substrate as well as surface tension on the equilibrium shape and dispersion of solitary waves.

The applied three-dimensional computational domain, schematically illustrated in Fig. 2, has the dimensions \( 300 h_N \times 6 h_N \times 0.1 h_N \) (with \( 0.1 h_N \) being the transverse dimension) and is represented by a Cartesian mesh. We found that a domain height of \( L_y = 6 h_N \) has a negligible effect on the dynamic behaviour of the solitary waves. This is further supported by the study of Albert et al. [75] who concluded that a domain height of \( L_y = 4 h_N \) is sufficient. The region of the computational mesh in which the dynamically evolving gas-liquid interface is located has an equidistant resolution of 10 cells per \( h_N \), following the work of Albert et al. [75]. The mesh resolution gradually increases near the substrate, with the cell center of the mesh cells closest to the substrate located at a distance of \( 0.0035 h_N \) from the substrate, assuring a detailed spatial resolution of the flow separation and reversal [4, 38, 39] in the wave troughs observed in some of the studied cases. We note that the transverse dimension is resolved with only one mesh cell, so that the simulation can be regarded as being effectively two-dimensional. The numerical time-step applied in the simulations satisfies a Courant number of \( Co = \Delta t u / \Delta x \leq 0.25 \), as well as the capillary time-step constraint proposed by Denner and van Wachem [10].

At the substrate a no-slip condition is enforced and at the top (gas-side) boundary a free-slip/no-penetration boundary condition is applied. A monochromatic forcing is imposed at the domain inlet by periodically changing the flow rate with frequency \( f = 20 \text{s}^{-1} \) and amplitude \( A \) from the mean. The amplitude is chosen in order to obtain fully developed waves within the finite length of the computational domain. At the domain inlet a semi-parabolic velocity profile is prescribed for the liquid phase,

\[
 u(x,0,0 \leq y \leq h_N) = \frac{3}{2} [1 + A \sin(2\pi ft)]
\]

\[
 \left(\frac{2y}{h_N} - \frac{y^2}{h_N^2}\right) u_N,
\]

(23)

and a spatially-invariant velocity is prescribed for the gas
TABLE I: Non-dimensional numbers, inclination angle of the substrate $\beta$ and surface tension coefficient $\sigma$ of the analyzed falling liquid films.

<table>
<thead>
<tr>
<th>No.</th>
<th>Re</th>
<th>$\beta$</th>
<th>$\sigma$ [N m$^{-1}$]</th>
<th>We</th>
<th>Ca</th>
<th>Ka</th>
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<td>75°</td>
<td>7.20 \times 10^{-2}</td>
<td>3.80 \times 10^{-1}</td>
<td>3.80 \times 10^{-3}</td>
<td>3932</td>
</tr>
<tr>
<td>29</td>
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<td>90°</td>
<td>6.21 \times 10^{-2}</td>
<td>4.46 \times 10^{-1}</td>
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<td>3887</td>
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<td>5.21 \times 10^{-1}</td>
<td>4.34 \times 10^{-3}</td>
<td>3887</td>
</tr>
</tbody>
</table>


\[ u(x = 0, h_N < y \leq 6h_N) = \frac{3}{2} [1 + A \sin (2\pi ft)] u_N. \] (24)

The film height at the inlet is constant, $h(x = 0) = h_N$. The domain outlet is modelled as an open boundary, following Nosoko and Miyara [46] and Denner et al. [27], with

\[ \frac{\partial u_x}{\partial x} \bigg|_{\text{out}} = \frac{\partial p}{\partial x} \bigg|_{\text{out}} = 0, \] (25)

which assures that the flow can leave the domain with minimal reflections. Initially, at time $t = 0s$, the film is flat and the velocity field is fully developed.

VI. SHAPE OF SOLITARY WAVES

The instantaneous film height $h/h_N$ as function of the downstream distance $x/h_N$ for a representative selection of the considered cases is shown in Figs. 3 and 4. The maximum film height (i.e. height of the crest of the solitary wave) exhibits a clear dependency on the inclination angle $\beta$ and the surface tension coefficient $\sigma$, whereas the minimum film height (i.e. height of the trough preceding the solitary wave) is independent of $\beta$ and $\sigma$ for high Reynolds numbers. Similarly, the influence of the inclination angle and the surface tension coefficient on the number and amplitude of capillary ripples preceding the main solitary hump reduces for increasing Reynolds number, which is particularly relevant for the onset of flow separation [4, 17, 59] as well as the wave length and binary interactions of solitary waves [12, 13].
FIG. 3: Interface height $h$ of the falling water film as a function of downstream distance $x$, both normalized by the Nusselt film height $h_N$, with $x = 0$ located at the center of the front of the solitary wave, for selected cases of different Reynolds numbers $Re$ and inclination angles $\beta$.

A. Maximum film height

Figure 5 shows the maximum and minimum film height, normalized by the Nusselt film height $h_N$, as a function of the driving non-dimensional numbers defined in Sec. III B. As noted in the previous paragraph, the inclination angle has a strong influence on the maximum film height $h_{\text{max}}$, which is the result of gravity acting in the cross-stream direction, arresting the growth of the solitary wave at a smaller maximum film height for smaller inclination angles. Similarly, a higher surface tension coefficient means the growth of the solitary wave is arrested at a lower equilibrium film height, as surface tension acts to flatten the wave. The initial increase of the maximum film height for $Re^* \lessgtr 60$ correlates very well (i.e. the data collapse into a single curve) with the driving Reynolds number $Re^*$ and the driving Weber number $We^*$. The correlation with $We^*$ also holds if the surface tension coefficient changes, as seen in Fig. 6. For changing inclination angle $\beta$ but constant Reynolds number $Re$, the maximum film height is monotonically increasing, shown in Fig. 7. This suggests that, apart from the influence of the cross-stream component of gravity, the maximum film height is also affected by the balance of surface tension and the driving component of inertia.

B. Minimum film height

As observed in Fig. 4, the cases with $Re^* \gtrsim 60$ have a common minimum film height which is independent of the inclination angle. Plotting the minimum film height for all cases as a function of the three proposed driving non-dimensional numbers in Figs. 5-7 reveals a global minimum of the minimum film height of $h_{\text{min}} = 0.37 h_N \pm 1.1\%$ for all considered cases, as indicated by the dotted lines in the figures. When the surface tension coefficient remains unaltered, the minimum film height is correlated with all three driving non-dimensional numbers, see Fig. 5, whereas comparing the cases with different surface tension coefficients the minimum film height is predominantly a function of the driving Weber number $We^*$, see Fig. 4. For $Re = 100$ and inclination angle $\beta = 19^\circ - 90^\circ$, shown in Fig. 4 the minimum film height remains approximately constant irrespective of the inclination angle.
The deepening of the trough preceding the solitary wave competes with the decelerating forces imposed by viscous stresses near the substrate (the crest and trough of a fully-developed solitary wave travel with the same speed \[ \frac{12}{70} \]), surface tension which acts to flatten the film as well as the narrowing cross-section of the film as a result of the wave trough approaching the substrate. The minimum film height, hence, represents the balance of inertia with viscous stresses and surface tension at the trough preceding the solitary wave, as shown by the correlation of the minimum film height with the driving Reynolds number \( Re^* \) and the driving Weber number \( We^* \) for all considered inclination angles and surface tension coefficients.

The observed initial decrease of the minimum film height for increasing Reynolds number when \( Re^* \lesssim 60 \), with a minimum at \( Re^* \approx 60 \), followed by an increase of the minimum film height when \( Re^* \gtrsim 60 \) corresponds well with the findings of Chakraborty et al. [44] concerning the onset of flow separation underneath the trough preceding the solitary wave. Chakraborty et al. [44] reported that for sufficiently high flow rates, inertia overcomes the adverse pressure gradient imposed by the convex shape of the interface at the trough preceding the solitary wave, since the low pressure region underneath the wave trough cannot extend into the low velocity region near the substrate, as shown by Denner et al. [57].
FIG. 6: Maximum film height \( h_{\text{max}} \) and minimum film height \( h_{\text{min}} \), normalized by the Nusselt height \( h_N \), as a function of driving Reynolds number \( \text{Re}^* \), driving Weber number \( \text{We}^* \) and driving capillary number \( \text{Ca}^* \), at an inclination angle \( \beta = 90^\circ \), for different surface tension coefficients \( \sigma \). The dotted line represents the global minimum film height of 0.37 \( h_N \).

FIG. 7: Maximum film height \( h_{\text{max}} \) and minimum film height \( h_{\text{min}} \), normalized by the Nusselt height \( h_N \), as a function of driving Reynolds number \( \text{Re}^* \), driving Weber number \( \text{We}^* \) and driving capillary number \( \text{Ca}^* \) for \( \text{Re} = 100 \) and inclination angles \( \beta = 19^\circ - 90^\circ \). The dotted line represents the global minimum film height of 0.37 \( h_N \).

C. Asymmetry of the solitary wave

We look now at the length \( d \) of the front of the solitary wave, which is defined as the distance between the crest of the solitary wave and the trough immediately downstream of the main wave hump, normalized by the wavelength \( \lambda \) of the solitary wave. With respect to solitary waves, the wavelength \( \lambda \) defines the length of the entire pulse, including the main wave hump and the preceding capillary ripples. The ratio \( d/\lambda \) can be regarded as measure of the asymmetry of the solitary wave, with \( d/\lambda = 0.5 \) representing a fully “symmetric wave” (as far as solitary waves on falling liquid films are concerned, symmetric waves are only observed at low Reynolds numbers and close to the domain inlet), as illustrated in Fig. 8.

An increasing inertia of the liquid film leads to a steepening of the wave front (i.e. smaller \( d/\lambda \)) and, thus, increasing asymmetry of the solitary wave. Figures 9-11 depict the length \( d \) of the solitary wave, where we can see that this asymmetry is correlated with the driving Weber number \( \text{We}^* \) and is, consequently, governed by
with wavenumber \( k \) propagates on a film of height \( h_N \) with phase velocity \( c_{\sigma-g} \),
\[
c_{\sigma-g} = \sqrt{\left(\frac{\sigma k}{\rho_l} + \frac{g \cos \beta}{k}\right) \tanh(k h_N)} ,
\]
whereas a pure capillary wave propagates with phase velocity
\[
c_{\sigma} = \sqrt{\frac{\sigma k}{\rho_l} \tanh(k h_N)} ,
\]
assuming \( \rho_l \gg \rho_g \) and the fluids being inviscid. Given the relatively long wavelength of the considered cases, viscous damping of the wave dispersion can be neglected and the fluids can be assumed inviscid for the application of linear wave theory. However, linear wave theory and, thus, Eqs. (27) and (28) are only valid in the limit of small amplitudes when \( (h_{\text{max}}-h_N)/h_N \ll 1 \).

Taking the ratio of the net phase velocity of the solitary wave, \( c_0 = c - u_N^* \) (i.e. the phase velocity at which the solitary wave propagates relative to the film flow), and the phase velocity of a capillary-gravity wave \( c_{\sigma-g} \) with the same wavenumber \( k \) correlates well with all three driving non-dimensional numbers, in particular the driving Weber number \( \text{We}^* \), as demonstrated in Fig. 13. By changing the surface tension coefficient, it becomes clear that the net phase velocity ratio is correlated with the driving Weber number \( \text{We}^* \), as shown in Fig. 14. Hence
\[
c_0 \sim c_{\sigma} \sqrt{\text{We}^*} .
\]

This correlation is also valid for a film with constant Reynolds number (Re = 100) on substrates with different inclination angles \( \beta \), shown in Fig. 15. Inserting the equations for the capillary phase velocity \( c_{\sigma} \), Eq. (28), and the driving Weber number \( \text{We}^* \), Eq. (17), the correlation given in Eq. (29) becomes
\[
c_0 \sim \sqrt{\frac{\sigma k}{\rho_l} \tanh(k h_N)} u_N^{*2}/\sigma ,
\]
with \( D = \sqrt{k h_N \tanh(k h_N)} \). Hence, the dispersion of the solitary waves represented by the net phase velocity \( c_0 \) is independent of the surface tension coefficient.

The phase velocity \( c \) of the solitary waves, shown in Figs. 16-18, is monotonically increasing with increasing inertia, with
\[
c \sim \sqrt{\text{Re}^*} .
\]
FIG. 9: Length $d$ of the front of the solitary wave normalized by the wavelength $\lambda$ as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$ for different inclination angles $\beta$. The dotted line represents the function $d/\lambda = A(We^*)^{-1/2} + B We^*$, with coefficients $A$ and $B$ fitted to the data of the cases with $\beta = 90^\circ$ and $\sigma = 0.072 \text{ N/m}^{-1}$.

FIG. 10: Length $d$ of the front of the solitary wave normalized by the wavelength $\lambda$ as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$, at an inclination angle $\beta = 90^\circ$, for different surface tension coefficients $\sigma$. The dotted line represents the function $d/\lambda = A(We^*)^{-1/2} + B We^*$, with coefficients $A$ and $B$ fitted to the data of the cases with $\beta = 90^\circ$ and $\sigma = 0.072 \text{ N/m}^{-1}$.

for all considered cases. This confirms that the phase velocity is in fact independent of surface tension, suggested in the previous paragraph based on the correlation given in Eq. (32), in particular since Eq. (33) also holds if the surface tension coefficient is changed, see Fig. 17. Since the train of solitary waves adopts the frequency $f$ of the periodically changing mass flow at the inlet ($f = 20 \text{ s}^{-1}$ in this study), the wavelength $\lambda = c/f$ exhibits the same relationship with respect to the driving Reynolds number, $\lambda \sim \sqrt{Re^*}$, as the phase velocity $c$.

The presented results suggest that the dispersion of solitary waves is governed by inertia, with an increasing inertia of the film flow resulting in a higher phase velocity. Both correlations proposed in Eqs. (29) and (33) further suggest a strong influence of the substrate and the associated viscous stresses on the dispersion of solitary waves, since all parts of a saturated solitary wave, including the wave trough preceding the main hump which is governed by viscous stresses near the substrate (see Sec. VI B), are propagating at the same speed. In particular, the absolute phase velocity of the analyzed solitary waves is a function of the ratio of inertia and viscous effects, see Eq. (33). Surface tension and gravity, on the other hand, have no discernible influence on the dispersion of solitary waves, contrary to the dispersion relation described by linear wave theory. Moreover, neither the net phase velocity ratio $c_0/c_\sigma$ nor the phase velocity $c$ are correlated with the driving capillary number $Ca^*$ in a consistent manner, which further supports the observation that surface tension has a negligible influence on the dispersion of solitary waves. This behavior can be attributed to the considerable amplitude of solitary waves.
FIG. 11: Length $d$ of the front of the solitary wave normalized by the wavelength $\lambda$ as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$ for $Re = 100$ and inclination angles $\beta = 19^\circ - 90^\circ$. The dotted line represents the function $d/\lambda = A(We^*)^{-1/2} + B We^*$, with coefficients $A$ and $B$ fitted to the data of the cases with $\beta = 90^\circ$ and $\sigma = 0.072 \text{ N m}^{-1}$.

FIG. 12: Ratio of the net phase velocity of the solitary wave $c_0 = c - u_N$ and the phase velocity of a capillary-gravity wave $c_{\sigma-g}$ with the same wavenumber $k$ as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$ for different inclination angles $\beta$.

on inertia-dominated falling liquid films, see Sec. VLA and the ensuing non-linearity of the hydrodynamic system. The observed self-similarity of the dispersion of solitary waves, however, is dependent on the inclination angle of the substrate, resulting in the observed correlation with the driving Weber number $We^*$, Eq. (24), and the driving Reynolds number $Re^*$, Eq. (33). Since the phase velocity of a pure capillary wave, given in Eq. (28), is based on linear wave theory, the ratio $c_0/c_{\sigma-g}$ can further be regarded as a measure of the non-linearity of the dispersion of solitary waves, which is proportional to the proposed driving Nusselt velocity $u_N^*$ of the falling liquid film.

VIII. CONCLUSIONS

We have scrutinized the solitary-wave characteristics, namely maximum-minimum amplitude, phase velocity and dispersion, as a function of the pertinent dimensionless parameters in inertia-dominated falling liquid films. We have proposed the first consistent scaling, derived from the Nusselt flat film solution based on the appropriate driving physical mechanisms, which leads to an unexpected self-similar characterization of solitary waves and allows an a priori description of the wave shape and dispersion.

Following a detailed analysis of the driving physical mechanisms and correlations with the proposed scaling of the main dimensionless numbers, our DNS results show a previously unknown self-similarity of the shape and dispersion of solitary waves. The correlations derived as
part of the presented self-similar characterization of solitary waves follow directly from rational corrections to the relevant parameters and do not depend on any tuneable coefficients.

Our results reveal that solitary waves on inertia-dominated falling liquid films are governed by a complex interplay of inertia, inclination angle of the substrate, viscous stresses in the vicinity of the substrate as well as surface tension. Both the maximum and minimum film height exhibit strong correlations with the driving Reynolds number and the driving Weber number. The maximum film height (i.e. the height at the crest of the solitary wave) has been shown to depend on the inclination angle as well as surface tension, while viscosity has no discernible influence on the maximum film height. The minimum film height (i.e. the height of the trough preceding the main wave hump) has been shown to be governed by a balance of the inertia of the liquid film with viscous stresses in the vicinity of the substrate and the surface tension acting at the gas-liquid interface in the trough preceding the main wave hump. A global minimum of the minimum film height of $h_{\text{min}} \approx 0.37 h_N$ is observed. Furthermore, the asymmetry of solitary waves has been shown to depend on the balance of inertia and surface tension, with an increase in inertia resulting in a steeper wave front and, therefore, more asymmetric solitary wave.

The dispersion of solitary waves has been found to be
FIG. 15: Ratio of the net phase velocity of the solitary wave $c_0 = c - u_N$ and the phase velocity of a capillary wave $c_\sigma$ with the same wavenumber $k$ as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$ for $Re = 100$ and inclination angles $\beta = 19^\circ - 90^\circ$. The dotted line represents the function $c_0/c_\sigma = A \sqrt{We^*} + B We^*$, with coefficients $A$ and $B$ fitted to the data of the cases with $\beta = 90^\circ$ and $\sigma = 0.072 \text{ N m}^{-1}$.

FIG. 16: Phase velocity $c$ of the solitary wave as a function of driving Reynolds number $Re^*$, driving Weber number $We^*$ and driving capillary number $Ca^*$ for different inclination angles $\beta$. The dotted line represents the function $c = A \sqrt{Re^*} + B Re^*$, with coefficients $A$ and $B$ fitted to the data of the cases with $\beta = 90^\circ$ and $\sigma = 0.072 \text{ N m}^{-1}$.

governed by a balance of inertia and viscous stresses. The absolute phase velocity is correlated with the driving Reynolds number, whereas the net phase velocity ratio (ratio between net phase velocity of the solitary wave and a capillary wave with the same wavenumber but infinitesimal amplitude) is correlated with the driving Weber number. Further analysis showed that the net phase velocity of the solitary waves is proportional to the driving Nusselt velocity of the falling liquid film. These correlations persist irrespective of the inclination angle of the substrate or the surface tension coefficient of the gas-liquid interface, suggesting that the dispersion of the solitary waves in the inertia-dominated flow regime is dominated by inertia and that dispersion according to linear wave theory (i.e. frequency dispersion) has no significant influence.

The proposed scaling of the main dimensionless numbers allows, for the first time, to predict the shape and phase velocity of a solitary wave on a falling liquid film of a given fluid with any flow rate (provided that the solitary wave does not break) and on a substrate with an arbitrary inclination angle, $0 \leq \beta \leq 90^\circ$, based on results at only two different flow rates or two different inclination angles.

Our computations also revealed that the (driving) capillary number is not suitable to describe, categorize and predict the shape and dispersion of solitary waves on inertia-dominated falling liquid films. In our view this is a significant finding because many studies, including previously proposed low-dimensional models for interfacial instabilities on falling liquid films, have used the capillary number or a similar formulation (e.g. inverse of the cap-
Even more important, our results indicate that despite their apparent complexity, film flows seem to be inherently simple nonlinear systems in that global quantities of interest can be described and understood in terms of simple relations. And self-similarity is at the heart of this “simplicity”. It is also precisely the reason why in physics multiscale phenomena are amenable to a theoretical description in the first place. To put it simply, this means that phenomena can be appropriately reduced so that there are only a few dominant scales, and the “complexity” then is all in the transformation from one set of original scales to another. And in that respect, the falling film is rather similar to many other seemingly unrelated physical problems where self-similarity is prevalent.

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