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# The parabola theorem on continued fractions

Ian Short

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**Abstract** Using geometric methods borrowed from the theory of Kleinian groups, we interpret the parabola theorem on continued fractions in terms of sequences of Möbius transformations. This geometric approach allows us to relate the Stern–Stolz series, which features in the parabola theorem, to the dynamics of certain sequences of Möbius transformations acting on three-dimensional hyperbolic space. We also obtain a version of the parabola theorem in several dimensions.

**Keywords** Conical limit points · continued fractions · hyperbolic geometry · Möbius transformations · parabola theorem · Stern–Stolz series

**Mathematics Subject Classification (2000)** Primary: 40A15; Secondary: 30B70, 30F45.

## 1 Introduction

The object of this paper is to discuss the parabola theorem on continued fractions using the geometry of Möbius transformations. The parabola theorem is about *infinite complex continued fractions* (henceforth described more briefly as *continued fractions*) which are quantities

$$\mathbf{K}(a_n|b_n) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}},$$

where the coefficients  $a_i$  and  $b_j$  are complex numbers, and none of the coefficients  $a_i$  are 0. Let  $t_n(z) = a_n/(b_n + z)$  and  $T_n = t_1 \circ \cdots \circ t_n$  (in future we just write  $T_n = t_1 \cdots t_n$ ). The continued fraction  $\mathbf{K}(a_n|b_n)$  is said to *converge* if the sequence  $T_n(0)$  converges within the

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extended complex plane. Excepting Section 4, we will assume that  $b_j = 1$  for  $j = 1, 2, \dots$ , in which case  $t_n(z) = a_n/(1+z)$  and the continued fraction will be denoted by  $\mathbf{K}(a_n|1)$ .

The first version of the parabola theorem was proven by Scott and Wall in [19]. Shortly after, more general versions were found by Leighton and Thron [13] and Paydon and Wall [17]. The theorem was extended by some of these authors in a number of subsequent papers including [14, 21], and the statement of the theorem from [21] is recast in the books by Jones and Thron [11, Thm. 4.42] and Lorentzen and Waadeland [15, Thm. 3.43]. It is this version of the parabola theorem that we work with. Let  $P_\alpha$  be the region given by

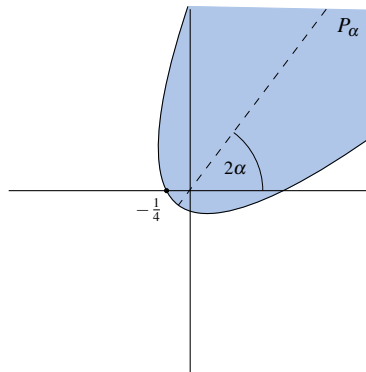
$$P_\alpha = \{z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq \frac{1}{2} \cos^2 \alpha\},$$

where  $-\pi/2 < \alpha < \pi/2$ , shown in Fig. 1.1, which is bounded by a parabola.

**The parabola theorem** *Suppose that  $a_n \in P_\alpha$  and  $a_n \neq 0$  for  $n = 1, 2, \dots$ . Then the odd and even sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  of the continued fraction  $\mathbf{K}(a_n|1)$  both converge. They converge to the same limit, and hence  $\mathbf{K}(a_n|1)$  converges, if and only if the series*

$$\left| \frac{1}{a_1} \right| + \left| \frac{a_1}{a_2} \right| + \left| \frac{a_2}{a_1 a_3} \right| + \left| \frac{a_1 a_3}{a_2 a_4} \right| + \left| \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \dots$$

*diverges.*



**Fig. 1.1** The parabolic region  $P_\alpha = \{z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq \frac{1}{2} \cos^2 \alpha\}$

Inspired by the philosophy set out by Beardon in [3], our aim is to understand in detail what the conditions of the parabola theorem say about the sequence of Möbius transformations  $T_n$  by using properties of Möbius transformations that are often employed in the theory of Kleinian groups. In particular, we wish to determine the behaviour of the sequence  $T_n$  not only at distinguished points such as 0 or  $\infty$ , but on the whole of the extended complex plane. To do this, we use the topological group structure of the group of Möbius transformations and the isometric action of this group on three-dimensional hyperbolic space. This approach gives insight into the parabola theorem and allows us to make precise statements about convergence. Another strength of the geometric approach is that it generalises easily to several dimensions, although for simplicity we state our results in the complex plane. Only at the very end of the paper do we discuss the parabola theorem in higher dimensions.

Our strategy is to split the parabola theorem in two, and deal with the condition  $a_n \in P_\alpha$ , and convergence of the series

$$\left| \frac{1}{a_1} \right| + \left| \frac{a_1}{a_2} \right| + \left| \frac{a_2}{a_1 a_3} \right| + \left| \frac{a_1 a_3}{a_2 a_4} \right| + \left| \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \cdots, \quad (1.1)$$

separately. To understand the significance of the condition  $a_n \in P_\alpha$ , we use a recent version of the Hillam–Thron theorem, a theorem that was originally proven in [7]. The type of argument we use for this part of the parabola theorem is well known to continued fraction theorists, so we describe it only briefly, in Section 3.

The main result of this paper (Theorem 1.1, below) is about the significance of the series (1.1), which is known as the *Stern–Stolz series*. It converges only if the sequence  $a_1, a_2, \dots$  grows sufficiently quickly. When  $a_n$  is large the map  $t_n(z) = a_n/(1+z)$  is close to the map  $s_n(z) = a_n/z$ , in a sense that will later be made precise. Thus it will be shown that if the Stern–Stolz series converges, then we can understand the behaviour of the maps  $T_n$  using the simpler maps  $S_n = s_1 \cdots s_n$ .

Theorem 1.1 (to follow) gives a host of equivalent conditions involving the maps  $S_n$  and  $T_n$ . Let us summarise the terminology used in that theorem. We denote the upper half-space model of three-dimensional hyperbolic space by  $\mathbb{H}^3$  (this is the upper half of  $\mathbb{R}^3$ ) and we denote the hyperbolic metric on  $\mathbb{H}^3$  by  $\rho$ . We identify the ideal boundary of  $\mathbb{H}^3$  with the extended complex plane  $\mathbb{C}_\infty$  in the usual way. The closure  $\overline{\mathbb{H}^3}$  of  $\mathbb{H}^3$  in  $\mathbb{R}^3 \cup \{\infty\}$  consists of  $\mathbb{H}^3$  together with its ideal boundary  $\mathbb{C}_\infty$ . The chordal metric  $\chi$  is a complete metric on  $\overline{\mathbb{H}^3}$ ; it is the metric inherited from the Euclidean metric on the unit ball by stereographic projection. The group of Möbius transformations acts on  $\mathbb{C}_\infty$ , and it can also act on  $\mathbb{H}^3$ . In fact, it is the full group of conformal isometries of  $\mathbb{H}^3$ . We can measure the distance between two Möbius transformations using the supremum metric  $\chi_0$ , which is given by

$$\chi_0(f, g) = \sup_{z \in \mathbb{C}_\infty} \chi(f(z), g(z)).$$

Let  $j = (0, 0, 1)$ . A point  $p$  in  $\mathbb{C}_\infty$  is said to be a *backward limit point* of a sequence  $F_1, F_2, \dots$  of Möbius transformations if there is a subsequence of  $F_1^{-1}(j), F_2^{-1}(j), \dots$  that converges to  $p$  in the chordal metric. A backward limit point  $p$  is a *conical limit point* of  $F_1, F_2, \dots$  if there is a geodesic  $\gamma$  in  $\mathbb{H}^3$  with one end-point at  $p$  and a subsequence of  $F_1^{-1}(j), F_2^{-1}(j), \dots$  that lies within a bounded hyperbolic distance of  $\gamma$  and converges to  $p$  in the chordal metric. The sequence  $F_1, F_2, \dots$  is said to be a *rapid escape sequence* if  $\sum_n \exp[-\rho(j, F_n(j))]$  converges. These are familiar concepts from Kleinian group theory. They will be explained in more detail later on.

**Theorem 1.1** *The following are equivalent:*

- (i)  $\left| \frac{1}{a_1} \right| + \left| \frac{a_1}{a_2} \right| + \left| \frac{a_2}{a_1 a_3} \right| + \left| \frac{a_1 a_3}{a_2 a_4} \right| + \left| \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \cdots$  converges;
- (ii)  $\sum_n \chi_0(T_n S_n^{-1}, T_{n+1} S_{n+1}^{-1})$  converges;
- (iii)  $\sum_n \chi(T_n(0), T_{n+2}(0))$  converges, and the sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  converge to two distinct values;
- (iv)  $\sum_n \rho(T_n S_n^{-1}(j), T_{n+1} S_{n+1}^{-1}(j))$  converges;
- (v)  $S_n$  is a rapid escape sequence, and  $\infty$  is its only backward limit point;

(vi)  $T_n$  is a rapid escape sequence, and  $\infty$  is its only backward limit point and its only conical limit point.

Further, if (i)–(vi) hold, then there are distinct points  $p$  and  $q$  such that  $T_{2n-1} \rightarrow p$  and  $T_{2n} \rightarrow q$  locally uniformly on  $\mathbb{C}$ , and  $T_{2n-1}(\infty) \rightarrow q$  and  $T_{2n}(\infty) \rightarrow p$ .

The equivalence of (i) and (iii) has been proven already by Lane and Wall [12] in a direct algebraic fashion. Our shorter proof uses geometric properties of Möbius transformations.

Theorem 1.1 shows that if the Stern–Stolz series converges, then the sequences  $S_n$  and  $T_n$  have extremely strong convergence properties, even though the sequence  $T_n(0)$  itself does not converge.

As a consequence of Theorem 1.1, we obtain the following version of the parabola theorem.

**Theorem 1.2** *Let  $H$  be a half-plane that contains 0, but does not contain  $-1$ , even in its closure. Suppose that  $t_n(H) \subseteq H$  for  $n = 1, 2, \dots$ . If the Stern–Stolz series*

$$\left| \frac{1}{a_1} \right| + \left| \frac{a_1}{a_2} \right| + \left| \frac{a_2}{a_1 a_3} \right| + \left| \frac{a_1 a_3}{a_2 a_4} \right| + \left| \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \dots$$

*converges, then there are distinct points  $p$  and  $q$  such that  $T_{2n-1} \rightarrow p$  and  $T_{2n} \rightarrow q$  locally uniformly on  $\mathbb{C}$ , and  $T_{2n-1}(\infty) \rightarrow q$  and  $T_{2n}(\infty) \rightarrow p$ . If the Stern–Stolz series diverges then  $T_n$  converges locally uniformly on  $H$ , and almost everywhere on  $\mathbb{C}_\infty$ , to a single point.*

It is well known (and will be shown in the next section) that the condition  $t_n(H) \subseteq H$  in this theorem is equivalent to the condition that the coefficients  $a_n$  lie within a parabolic region such as that shown in Fig. 1.1. The more substantial difference between Theorem 1.2 and other statements of the parabola theorem, including the one given earlier, is that Theorem 1.2 contains detailed information about the behaviour of the sequence  $T_n$ . Some of the new features of this theorem could be extracted from existing accounts of the parabola theorem; however, other accounts tend to focus on the continued fraction  $\mathbf{K}(a_n|1)$  rather than describing in detail the dynamics of the sequence  $T_n$ .

In Section 6 we provide an example to show that when the Stern–Stolz series diverges, the sequence  $T_n$  may diverge on an uncountable, dense subset of  $\mathbb{C}_\infty \setminus H$ .

## 2 A geometric explanation of the parabolic region

Let  $H_\alpha$  be the half-plane given by

$$H_\alpha = \left\{ -\frac{1}{2} + e^{i\alpha} z : \operatorname{Re}[z] > 0 \right\},$$

and recall that  $P_\alpha$  denotes the parabolic region

$$\left\{ z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq \frac{1}{2} \cos^2 \alpha \right\}.$$

The following theorem, illustrated by Fig. 2.1, is well known (see, for example, [15, Thm. 3.43]).

**Theorem 2.1** *The Möbius transformation  $t_n(z) = a_n/(1+z)$  satisfies  $t_n(H_\alpha) \subseteq H_\alpha$  if and only if  $a_n \in P_\alpha$ .*

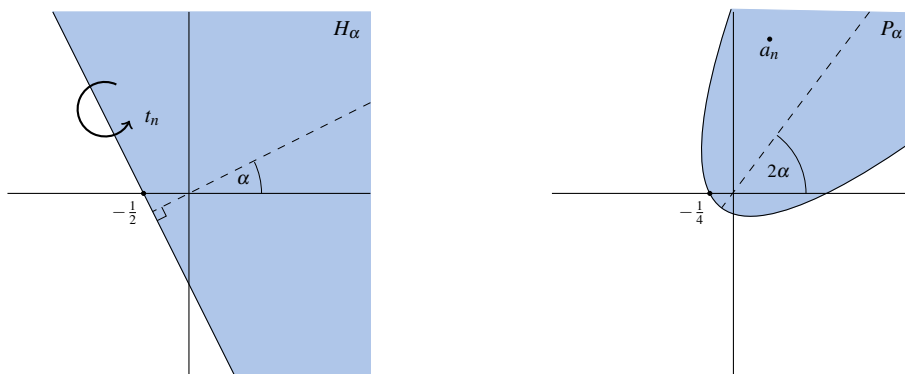


Fig. 2.1  $t_n(H_\alpha) \subseteq H_\alpha$  if and only if  $a_n \in P_\alpha$

Theorem 2.1 allows us to apply the Hillam–Thron theorem to help prove the parabola theorem in the usual manner, as we shall see in the next section. Here we give a new geometric proof of Theorem 2.1, which illuminates some of the features of the parabola theorem needed later, and, using only a few basic properties of Möbius transformations, it gets us off to an undemanding start.

Let  $H$  be an open Euclidean half-plane that contains 0, but does not contain  $-1$ , even in its closure. The boundary line  $\partial H$  must cut the real axis somewhere between  $-1$  and 0. If it cuts at  $-1/2$  then  $H$  is one of the half-planes  $H_\alpha$ , but there is no need for us to assume that. Let  $t(z) = a/(1+z)$ , where  $a \neq 0$ . Denote by  $u$  the inverse point of  $-1$  in the boundary line  $\partial H$ . This is the image of  $-1$  under reflection in  $\partial H$ . Inverse points of lines and circles are preserved by Möbius transformations, which implies that  $t(u)$  is the inverse point of  $t(-1)$  in  $t(\partial H)$ . Now, we know that  $t(-1) = \infty$ , which implies that  $t(\partial H)$  is a Euclidean circle with centre  $t(u)$ . We also know that  $t(\partial H)$  contains the point  $t(\infty) = 0$ .

The half-plane  $H$  and its image disc  $t(H)$  are shown in Fig. 2.2.

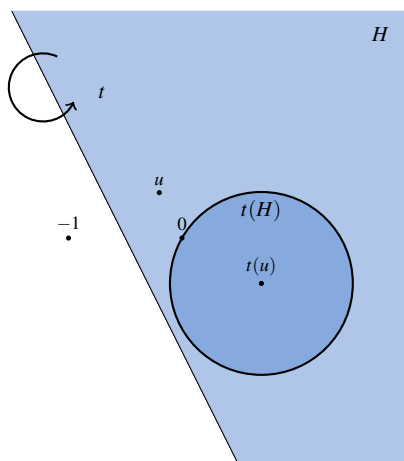


Fig. 2.2 The disc  $t(H)$  is contained within the half-plane  $H$

The radius of  $t(H)$  is  $|t(u) - 0| = |t(u)|$ . Since the centre of  $t(H)$  is  $t(u)$  it follows that  $t(H) \subseteq H$  if and only if  $t(u)$  lies in the set

$$\{z \in \mathbb{C} : |z| \leq d(z, \partial H)\},$$

where  $d$  denotes the Euclidean metric. This is a region bounded by a parabola with focus 0 and directrix  $\partial H$ . Since  $t(u) = a/(1+u)$ , we have proved the following theorem.

**Theorem 2.2** *The Möbius map  $t(z) = a/(1+z)$  satisfies  $t(H) \subseteq H$  if and only if the coefficient  $a$  lies in the parabolic region*

$$P = (1+u) \{z \in \mathbb{C} : |z| \leq d(z, \partial H)\}.$$

The parabolic region  $P$  has focus 0 and directrix  $(1+u)\partial H$ .

Let us now show that this theorem implies Theorem 2.1. We can write  $H$  in the form  $s + e^{i\alpha}\mathbb{K}$ , where  $\mathbb{K}$  is the right half-plane, and  $-1 < s < 0$  and  $-\pi/2 < \alpha < \pi/2$ . Let  $\tau$  denote the reflection in  $\partial H$ , which is given by

$$\tau(z) = s + e^{2i\alpha}(s - \bar{z}).$$

Then  $u = \tau(-1)$ , and so

$$1+u = 1 + \tau(-1) = 2(s+1)e^{i\alpha} \cos \alpha.$$

Also, for  $z \in H$ ,

$$d(z, \partial H) = \frac{1}{2}|z - \tau(z)| = \operatorname{Re}[(z-s)e^{-i\alpha}] = \operatorname{Re}[ze^{-i\alpha}] - s \cos \alpha.$$

Therefore

$$\begin{aligned} P &= 2(s+1)e^{i\alpha} \cos \alpha \{z \in \mathbb{C} : |z| \leq \operatorname{Re}[ze^{-i\alpha}] - s \cos \alpha\} \\ &= \{z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq -2s(s+1) \cos^2 \alpha\}. \end{aligned}$$

When  $s = -1/2$ , so that  $H$  is the half-plane  $H_\alpha = -\frac{1}{2} + e^{i\alpha}\mathbb{K}$ , we find that

$$P = \{z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq \frac{1}{2} \cos^2 \alpha\},$$

which is the parabolic region  $P_\alpha$ . Therefore Theorem 2.2 does indeed imply Theorem 2.1.

Notice that the expression  $-2s(s+1)$  takes its maximum value, namely  $1/2$ , when  $s = -1/2$ . This shows that, for a given angle  $\alpha$ ,  $P_\alpha$  contains the parabolic region

$$\{z \in \mathbb{C} : |z| - \operatorname{Re}[ze^{-2i\alpha}] \leq -2s(s+1) \cos^2 \alpha\},$$

no matter the value of  $s$ . This explains why the original statement of the parabola theorem given near the beginning of the introduction cannot be improved by allowing  $s$  to take values other than  $-1/2$ .

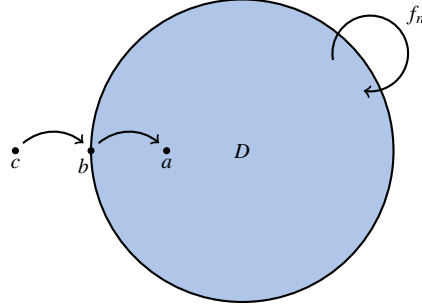
Let us finish this section with some remarks on related geometric constructions that arise in the theory of continued fractions. We have seen that a Möbius transformation  $t(z) = a/(1+z)$  satisfies  $t(H) \subseteq H$  if and only if  $a$  belongs to a parabolic region  $P$ . Suppose now that  $H$  is a Euclidean disc instead of a half-plane. It is only possible for  $t$  to map  $H$  within itself if  $-1 \notin \bar{H}$ . Given this condition, it can be shown that  $t(H) \subseteq H$  if and only if  $a$  belongs to a region bounded by a Cartesian oval, and there is a corresponding theorem in continued fraction theory called the oval theorem, which was first proven by Lorentzen (formerly known as Jacobsen) and Thron in [9].

Another possibility is that  $H$  is the complement of a Euclidean disc, which contains 0 in its interior. In this case,  $t(H) \subseteq H$  if and only if  $a$  belongs to another region bounded by a Cartesian oval. This case has received little if any attention.

### 3 The Hillam–Thron theorem

In this section we explore the ramifications of the condition  $a_n \in P_\alpha$  from the parabola theorem further. Theorem 2.1 tells us that  $a_n \in P_\alpha$  if and only if  $t_n(z) = a_n/(1+z)$  satisfies  $t_n(H_\alpha) \subseteq H_\alpha$ . We also know that  $t_n$  maps  $-1$  to  $\infty$  and  $\infty$  to  $0$ . The point  $-1$  lies outside the closure  $\overline{H_\alpha}$  of  $H_\alpha$ ,  $\infty$  lies on the boundary  $\partial H_\alpha$  of  $H_\alpha$ , and  $0$  lies inside  $H_\alpha$ .

More generally, consider a sequence  $f_n$  of Möbius transformations that satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$  for  $n = 1, 2, \dots$ , where  $D$  is an open disc in  $\mathbb{C}_\infty$ , and  $a, b$ , and  $c$  are three points with  $a \in D$ ,  $b \in \partial D$ , and  $c \notin \overline{D}$ . These properties are illustrated in Fig. 3.1. The sequence  $t_n$  is of this type, as we can see by choosing  $a = 0$ ,  $b = \infty$ ,  $c = -1$ , and  $D = H_\alpha$ .



**Fig. 3.1** The map  $f_n$  satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$

The next theorem shows that, given such a collection of maps  $f_n$ , the sequence  $F_n = f_1 \cdots f_n$  has strong convergence properties. This theorem uses the chordal metric  $\chi$  on  $\mathbb{C}_\infty$ , which is the metric inherited from the Euclidean metric on the unit sphere by stereographic projection. We could instead use the spherical metric on the unit sphere, which is additive along geodesics, but the chordal metric is simpler algebraically. It is given by

$$\chi(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad \chi(z, \infty) = \frac{2|z|}{\sqrt{1 + |z|^2}},$$

where  $z, w \neq \infty$ .

**Theorem 3.1** *Suppose that  $D$  is an open disc and  $a, b$ , and  $c$  are three points with  $a \in D$ ,  $b \in \partial D$ , and  $c \notin \overline{D}$ . Suppose also that  $f_1, f_2, \dots$  is a sequence of Möbius transformations that satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$  for  $n = 1, 2, \dots$ . Let  $F_n = f_1 \cdots f_n$ . Then  $\sum_n \chi(F_n(a), F_{n+2}(a))$  converges. Furthermore,  $F_{2n-1}$  converges locally uniformly on  $D$ , and almost everywhere on  $\mathbb{C}_\infty$ , to a point  $p$ , and  $F_{2n}$  converges locally uniformly on  $D$ , and almost everywhere on  $\mathbb{C}_\infty$ , to a point  $q$ .*

The first assertion of the parabola theorem, that the odd and even sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  both converge, follows immediately from Theorem 3.1.

Theorem 3.1 is a corollary of the following version of the Hillam–Thron theorem, which is [20, Thm. 4.1] paraphrased. This procedure of deducing a result such as Theorem 3.1 from a version of the Hillam–Thron theorem is standard within the continued fractions literature; the only original part of Theorem 3.1, which we need later, is the statement about almost everywhere convergence.



**Theorem 3.2** Suppose that  $D$  is an open disc,  $u$  is a point in  $D$ ,  $v$  is a point that lies outside  $\bar{D}$ , and  $g_1, g_2, \dots$  is a sequence of Möbius transformations with  $g_n(D) \subseteq D$  and  $g_n(v) = u$  for  $n = 1, 2, \dots$ . Then the sequence  $G_n = g_1 \cdots g_n$  satisfies

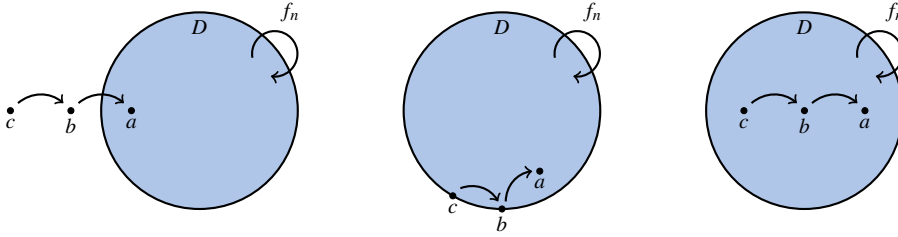
- (i)  $\sum_n \chi(G_n(u), G_{n+1}(u))$  converges;
- (ii)  $G_n$  converges locally uniformly on  $D$  to a point  $p$ ;
- (iii)  $G_n$  converges everywhere but on a set of Hausdorff dimension at most 1 to  $p$ .

We can now prove Theorem 3.1.

*Proof (of Theorem 3.1)* Recall that  $f_1, f_2, \dots$  is a sequence of Möbius transformations that satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$  for  $n = 1, 2, \dots$ . Let  $g_n = f_{2n-1}f_{2n}$ . Then  $g_n(D) \subseteq f_{2n-1}(D) \subseteq D$  and  $g_n(c) = f_{2n-1}(b) = a$ . It follows from Theorem 3.2 that the series  $\sum_n \chi(F_{2n}(a), F_{2n+2}(a))$  converges and  $F_{2n}$  converges locally uniformly on  $D$ , and everywhere on  $\mathbb{C}_\infty$  but on a set of Hausdorff dimension at most one, to a point  $p$ . A similar argument can be applied to  $F_{2n-1}$ . The result follows, because sets of Hausdorff dimension one have Lebesgue measure 0.  $\square$

As the proof indicates, each assertion about almost everywhere convergence in Theorem 3.1 can be replaced by a stronger assertion about convergence on a set whose complement has Hausdorff dimension at most one. The same can be said of Theorem 1.2. There is an example in Section 6 that shows that  $F_n$  may diverge on an uncountable, dense subset of  $\mathbb{C}_\infty \setminus D$ .

We conclude this section by discussing other theorems of a similar nature to Theorem 3.1. Suppose that we continue to assume that the sequence  $f_n$  satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$ , but adjust the configuration of  $a, b, c$ , and  $D$ . A few possibilities are shown in Fig. 3.2.



**Fig. 3.2** Each map  $f_n$  satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$

The left-hand configuration arises in the oval theorem, which was referred to near the end of the previous section. There are other theorems corresponding to the other configurations: some trivial, and some no less significant, from this perspective, than the parabola theorem.

#### 4 The Stern–Stolz series

For this section only we consider more general continued fractions of the form  $\mathbf{K}(a_n|b_n)$ , where the coefficients  $b_n$  need not necessarily equal 1. Our goal is to understand the geo-

metric significance of the *Stern–Stolz series*

$$\left| b_1 \frac{1}{a_1} \right| + \left| b_2 \frac{a_1}{a_2} \right| + \left| b_3 \frac{a_2}{a_1 a_3} \right| + \left| b_4 \frac{a_1 a_3}{a_2 a_4} \right| + \left| b_5 \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \cdots, \quad (4.1)$$

which features heavily in the continued fractions literature. Roughly speaking, we will show that the series measures how close  $T_n S_n^{-1}$  is to a Möbius transformation. Here, as usual,  $t_n(z) = a_n/(b_n + z)$  and  $T_n = t_1 \cdots t_n$ , and  $s_n(z) = a_n/z$  and  $S_n = s_1 \cdots s_n$ .

Let us first go over some of the theory of Möbius transformations, which can be found in [2, 18]. Using the Poincaré extension, the group  $\mathcal{M}$  of Möbius transformations acts on the upper half-space model  $\mathbb{H}^3$  of three-dimensional hyperbolic space. Each Möbius transformation is a conformal isometry of  $\mathbb{H}^3$ , and every conformal isometry of  $\mathbb{H}^3$  arises in this fashion. Let  $j = (0, 0, 1)$ . A useful formula for the hyperbolic metric  $\rho$  on  $\mathbb{H}^3$  is

$$\sinh^2 \frac{1}{2} \rho(u, v) = \frac{|u - v|^2}{4(u \cdot j)(v \cdot j)}, \quad (4.2)$$

where  $u \cdot j$  is the scalar product of  $u$  and  $j$  (that is, if  $u = (u_1, u_2, u_3)$  then  $u \cdot j = u_3$ ).

We also make use of the metric of uniform convergence  $\chi_0$  on  $\mathcal{M}$ , given by

$$\chi_0(f, g) = \sup_{z \in \mathbb{C}_\infty} \chi(f(z), g(z)).$$

The metric space  $(\mathcal{M}, \chi_0)$  is complete, and it is a topological group. The metric  $\chi_0$  is right-invariant. It is not left-invariant; however, given a Möbius transformation  $h$ , we can define  $M(h) = \exp[\rho(j, h(j))]$ , and then

$$\frac{1}{M(h)} \chi_0(f, g) \leq \chi_0(hf, hg) \leq M(h) \chi_0(f, g) \quad (4.3)$$

for all Möbius maps  $f$  and  $g$ .

The rest of this section is devoted to proving the following theorem, which itself will later be used to prove Theorem 1.1.

**Theorem 4.1** *The following are equivalent:*

- (i)  $\left| b_1 \frac{1}{a_1} \right| + \left| b_2 \frac{a_1}{a_2} \right| + \left| b_3 \frac{a_2}{a_1 a_3} \right| + \left| b_4 \frac{a_1 a_3}{a_2 a_4} \right| + \left| b_5 \frac{a_2 a_4}{a_1 a_3 a_5} \right| + \cdots$  converges;
- (ii)  $\sum_n \chi_0(T_n S_n^{-1}, T_{n+1} S_{n+1}^{-1})$  converges;
- (iii)  $\sum_n \chi(T_n(0), T_{n+2}(0))$  converges, and the sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  converge to two distinct values;
- (iv)  $\sum_n \rho(T_n S_n^{-1}(j), T_{n+1} S_{n+1}^{-1}(j))$  converges.

Further, if (i)–(iv) hold, then there is a Möbius transformation  $g$  such that  $\chi_0(T_n, g S_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

For continued fractions of the form  $\mathbf{K}(a_n | 1)$ , this theorem gives part of Theorem 1.1. For continued fractions of the form  $\mathbf{K}(1 | b_n)$ , this theorem gives [5, Thm. 1.7].

Define, for each positive integer  $n$ ,

$$\lambda_{2n-1} = \frac{a_2 a_4 \cdots a_{2n-2}}{a_1 a_3 \cdots a_{2n-1}}, \quad \lambda_{2n} = \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}}. \quad (4.4)$$

Then  $\sum_n |b_n \lambda_n|$  is the Stern–Stolz series. Observe that

$$S_{2n-1}(z) = \frac{1}{\lambda_{2n-1}z}, \quad S_{2n}(z) = \lambda_{2n}z, \quad S_{2n-1}^{-1}(z) = \frac{1}{\lambda_{2n-1}z}, \quad S_{2n}^{-1}(z) = \frac{z}{\lambda_{2n}}.$$

Let  $\tau_n(z) = z + b_n \lambda_n$  for  $n = 1, 2, \dots$ , and let  $\sigma(z) = 1/z$ .

**Lemma 4.1** *We have*

$$S_{n-1}t_n S_n^{-1} = \begin{cases} \sigma \tau_n \sigma & \text{if } n \text{ is odd,} \\ \tau_n & \text{if } n \text{ is even.} \end{cases}$$

*Proof* Let  $\beta_n(z) = z + b_n$ . Then  $t_n = s_n \beta_n$ . Therefore  $S_{n-1}t_n S_n^{-1} = S_n \beta_n S_n^{-1}$ . It is now straightforward to check the odd and even cases separately.  $\square$

We now give a pair of lemmas that will be used to handle statement (ii) of Theorem 4.1, which involves the metric  $\chi_0$ . We denote the identity Möbius transformation by  $I$ .

**Lemma 4.2** *Let  $\tau(z) = z + \mu$ . Then*

$$\chi_0(\tau, I) = \begin{cases} 2 & \text{if } |\mu| \geq 2, \\ \frac{8|\mu|}{4+|\mu|^2} & \text{if } |\mu| \leq 2. \end{cases}$$

*Proof* Observe that

$$\chi_0(\tau, I) = \sup_{z \in \mathbb{C}} \frac{2|\mu|}{\sqrt{1+|z|^2} \sqrt{1+|z+\mu|^2}}.$$

If  $|\mu| \geq 2$  then this supremum attains the value 2 (the largest possible value of  $\chi_0$ ) at  $z = -\frac{1}{2}\mu(1 + \sqrt{1-4/|\mu|^2})$ . If  $|\mu| \leq 2$  then the supremum can be obtained by finding the minimum of  $(1+|z|^2)(1+|z+\mu|^2)$  over  $\mathbb{C}$ . The minimum occurs at  $z = -\mu/2$ , and we omit the details.  $\square$

Recall that  $\tau_n(z) = z + b_n \lambda_n$  and  $\sigma(z) = 1/z$ . The map  $\sigma$  is a chordal isometry because, acting on the unit sphere, it is a rotation by  $\pi$  that interchanges the north and south poles. It follows that  $\chi_0(\sigma f, \sigma g) = \chi_0(f, g)$  for any Möbius maps  $f$  and  $g$ .

**Lemma 4.3** *We have  $\chi_0(S_n T_n^{-1}, S_{n-1} T_{n-1}^{-1}) = \chi_0(\tau_n, I)$ .*

*Proof* Using right-invariance we obtain

$$\chi_0(S_n T_n^{-1}, S_{n-1} T_{n-1}^{-1}) = \chi_0(S_n, S_{n-1} t_n) = \chi_0(I, S_{n-1} t_n S_n^{-1}).$$

When  $n$  is even the result follows immediately from Lemma 4.1. When  $n$  is odd, Lemma 4.1 tells us that  $\chi_0(I, S_{n-1} t_n S_n^{-1}) = \chi_0(I, \sigma \tau_n \sigma)$ . Since  $\chi_0$  is right-invariant, and  $\sigma$  is a chordal isometry, it again follows that  $\chi_0(I, S_{n-1} t_n S_n^{-1}) = \chi_0(I, \tau_n)$ .  $\square$

Next we give a pair of lemmas that will be used to handle statement (iv) of Theorem 4.1, which involves the hyperbolic metric  $\rho$ . Recall that  $j = (0, 0, 1)$ .

**Lemma 4.4** *Let  $\tau(z) = z + \mu$ . Then  $2 \sinh \frac{1}{2} \rho(\tau(j), j) = |\mu|$ .*

*Proof* This follows immediately from the hyperbolic metric formula (4.2).  $\square$

On  $\mathbb{H}^3$ , the map  $\sigma(z) = 1/z$  acts as an inversion in the unit sphere followed by a reflection in the plane  $x_2 = 0$ . In particular,  $\sigma$  fixes the point  $j$ .

**Lemma 4.5** We have  $\rho(T_n S_n^{-1}(j), T_{n+1} S_{n+1}^{-1}(j)) = \rho(\tau_n(j), j)$ .

*Proof* Using left-invariance we obtain

$$\rho(T_n S_n^{-1}(j), T_{n-1} S_{n-1}^{-1}(j)) = \rho(t_n S_n^{-1}(j), S_{n-1}^{-1}(j)) = \rho(S_{n-1} t_n S_n^{-1}(j), j).$$

When  $n$  is even the result follows immediately from Lemma 4.1. When  $n$  is odd, Lemma 4.1 tells us that  $\rho(S_{n-1} t_n S_n^{-1}(j), j) = \rho(\sigma \tau_n \sigma(j), j)$ . Since  $\rho$  is left-invariant, and  $\sigma(j) = j$ , it again follows that  $\rho(S_{n-1} t_n S_n^{-1}(j), j) = \rho(\tau_n(j), j)$ .  $\square$

Next, we develop some preparatory results for dealing with statement (iii) of Theorem 4.1. We need the *cross ratio*  $[a, b, c, d]$  of four points  $a, b, c$ , and  $d$  in  $\mathbb{C}_\infty$ . It is given by

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-c)(b-d)},$$

where the usual conventions are adopted if one of the points  $a, b, c$ , or  $d$  is  $\infty$ . It is well known that if  $g$  is a Möbius transformation then

$$[g(a), g(b), g(c), g(d)] = [a, b, c, d].$$

Remember that  $t_n(z) = a_n/(b_n + z)$  and  $T_n = t_1 \cdots t_n$ .

**Lemma 4.6** We have

$$\frac{T_n(0) - T_{n-2}(0)}{T_n(0) - T_{n-1}(0)} = \frac{b_n}{a_n} T_{n-1}^{-1}(\infty).$$

*Proof* The result follows by expanding the identity

$$[t_n(0), \infty, 0, T_{n-1}^{-1}(\infty)] = [T_n(0), T_{n-1}(\infty), T_{n-1}(0), \infty]$$

and observing that  $T_{n-1}(\infty) = T_{n-2}(0)$ .  $\square$

Recall the sequence  $\lambda_n$  defined by (4.4).

**Corollary 4.1** We have

$$\frac{T_n(0) - T_{n-2}(0)}{T_n(0) - T_{n-1}(0)} = -\lambda_{n-1} \lambda_n T_{n-1}^{-1}(\infty) T_n^{-1}(\infty) + 1.$$

*Proof* Notice that

$$T_n^{-1}(\infty) = t_n^{-1}(T_{n-1}^{-1}(\infty)) = -b_n + \frac{a_n}{T_{n-1}^{-1}(\infty)}.$$

Therefore

$$\frac{b_n}{a_n} T_{n-1}^{-1}(\infty) = -\frac{1}{a_n} T_{n-1}^{-1}(\infty) T_n^{-1}(\infty) + 1.$$

The result follows, because  $\lambda_{n-1} \lambda_n = 1/a_n$ .  $\square$

We need one final lemma, on the convergence of series.

**Lemma 4.7** Suppose that  $z_n$  is a sequence of complex numbers such that  $\sum_n |z_n z_{n+1} - 1|$  converges. Then  $z_{2n-1}$  converges to a non-zero limit  $z$ , and  $z_{2n}$  converges to  $1/z$ .

*Proof* As the sum  $\sum_n |z_{2n}z_{2n+1} - 1|$  converges, we see that the product  $z_2z_3 \cdots z_{2n+1}$  converges to a non-zero value, and as the sum  $\sum_n |z_{2n-1}z_{2n} - 1|$  converges, we see that the product  $z_1z_2 \cdots z_{2n}$  converges to a non-zero value. Thus  $z_{2n+1}$  converges to a non-zero number  $z$ , and  $z_{2n} \rightarrow 1/z$  because  $z_n z_{n+1} \rightarrow 1$ .  $\square$

Finally we can prove Theorem 4.1.

*Proof (of Theorem 4.1)* We begin by proving the final statement of Theorem 4.1. That statement follows from (ii), because (ii) implies that the sequence  $T_n S_n^{-1}$  is a Cauchy sequence in  $(\mathcal{M}, \chi_0)$ , and hence there is a Möbius transformation  $g$  such that  $\chi_0(T_n S_n^{-1}, g) \rightarrow 0$ . Then right-invariance of  $\chi_0$  implies that  $\chi_0(T_n, g S_n) \rightarrow 0$ .

Next we show that the series  $\sum_n \chi_0(T_n S_n^{-1}, T_{n+1} S_{n+1}^{-1})$  converges if and only if the series  $\sum_n \chi_0(S_n T_n^{-1}, S_{n+1} T_{n+1}^{-1})$  converges. Suppose that  $\sum_n \chi_0(T_n S_n^{-1}, T_{n+1} S_{n+1}^{-1})$  converges. Using (4.3) we have

$$\begin{aligned} \chi_0(S_n T_n^{-1}, S_{n+1} T_{n+1}^{-1}) &= \chi_0(S_n T_n^{-1} T_{n+1} S_{n+1}^{-1}, S_n T_n^{-1} T_n S_n^{-1}) \\ &\leq M(S_n T_n^{-1}) \chi_0(T_{n+1} S_{n+1}^{-1}, T_n S_n^{-1}). \end{aligned}$$

We have seen that  $T_n S_n^{-1}$  converges to a Möbius map  $g$ . Since  $(\mathcal{M}, \chi_0)$  is a topological group it follows that  $S_n T_n^{-1}$  converges to  $g^{-1}$ . Therefore the sequence  $M(S_n T_n^{-1})$  is bounded above. Hence  $\sum_n \chi_0(S_n T_n^{-1}, S_{n+1} T_{n+1}^{-1})$  converges. This argument can be run in reverse, so  $\sum_n \chi_0(T_n S_n^{-1}, T_{n+1} S_{n+1}^{-1})$  converges if and only if  $\sum_n \chi_0(S_n T_n^{-1}, S_{n+1} T_{n+1}^{-1})$  converges.

Lemmas 4.2 and 4.3 tell us that  $\sum_n \chi_0(S_n T_n^{-1}, S_{n+1} T_{n+1}^{-1})$  converges if and only if the Stern–Stolz series  $\sum_n |b_n \lambda_n|$  converges. The equivalence of (i) and (ii) follows immediately. Lemmas 4.4 and 4.5 tell us that the series  $\sum_n \rho(T_n S_n^{-1}(j), T_{n+1} S_{n+1}^{-1}(j))$  converges if and only if  $\sum_n |b_n \lambda_n|$  converges, which implies that (i) and (iv) are equivalent.

Next we show that (ii) implies (iii). Observe first that

$$\chi(T_n(0), T_{n+2}(0)) = \chi(T_{n+1}(\infty), T_{n+2}(0)).$$

If  $n$  is odd then  $S_{n+1}^{-1}(\infty) = \infty$  and  $S_{n+2}^{-1}(\infty) = 0$ , and if  $n$  is even then  $S_{n+1}^{-1}(0) = \infty$  and  $S_{n+2}^{-1}(0) = 0$ . Therefore  $\chi(T_{n+1}(\infty), T_{n+2}(0))$  is less than

$$\chi(T_{n+1} S_{n+1}^{-1}(0), T_{n+2} S_{n+2}^{-1}(0)) + \chi(T_{n+1} S_{n+1}^{-1}(\infty), T_{n+2} S_{n+2}^{-1}(\infty)).$$

Since both terms in this expression do not exceed  $\chi_0(T_{n+1} S_{n+1}^{-1}, T_{n+2} S_{n+2}^{-1})$  we see that

$$\chi(T_n(0), T_{n+2}(0)) \leq 2\chi_0(T_{n+1} S_{n+1}^{-1}, T_{n+2} S_{n+2}^{-1}).$$

Therefore  $\sum_n \chi(T_n(0), T_{n+2}(0))$  converges. Recall from the start of this proof that  $T_n S_n^{-1}$  converges to a Möbius map  $g$ . It follows that  $T_{2n-1}(0) \rightarrow g(\infty)$  and  $T_{2n}(0) \rightarrow g(0)$ , and these two limits are distinct.

It remains to show that (iii) implies (i). We can assume, by adjusting  $b_1$  if necessary, that the two distinct limits  $p$  and  $q$  of the sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  are finite. Since Euclidean and chordal metrics are locally equivalent, and  $\sum_n \chi(T_n(0), T_{n-2}(0))$  converges, we deduce that  $\sum_n |T_n(0) - T_{n-2}(0)|$  converges. Furthermore, because  $|T_n(0) - T_{n-1}(0)| \rightarrow |p - q|$  we see that

$$\sum_n \left| \frac{T_n(0) - T_{n-2}(0)}{T_n(0) - T_{n-1}(0)} \right|$$

converges. Let  $z_n = \lambda_n T_n^{-1}(\infty)$ . Then, from Corollary 4.1, we find that  $\sum_n |z_n z_{n+1} - 1|$  converges. Hence, by Lemma 4.7, the sequence  $|z_n|$  is bounded below by a positive constant  $M$ . Next, from Lemma 4.6 we have that  $\sum_n \left| \frac{b_n}{a_n} T_{n-1}^{-1}(\infty) \right|$  converges. Now

$$|b_n \lambda_n| = \left| \frac{b_n}{\lambda_{n-1} a_n} \right| = \left| \frac{1}{z_{n-1}} \right| \left| \frac{b_n}{a_n} T_{n-1}^{-1}(\infty) \right| \leq \frac{1}{M} \left| \frac{b_n}{a_n} T_{n-1}^{-1}(\infty) \right|.$$

Therefore the Stern–Stolz series  $\sum_n |b_n \lambda_n|$  converges.  $\square$

## 5 Proof of Theorem 1.1

Before we prove Theorem 1.1 let us revise a few concepts from the theory of Kleinian groups that apply to continued fractions. These ideas, within the context of Kleinian groups, can be found in [16].

Recall that  $j = (0, 0, 1)$ . A sequence  $F_1, F_2, \dots$  of Möbius transformations is *restrained* if the sequence  $F_n(j)$  accumulates only on the ideal boundary  $\mathbb{C}_\infty$  of  $\mathbb{H}^3$ . This terminology was introduced by Lorentzen (formerly Jacobsen) and Thron in [10]. Our definition is taken from [1] and [3, Sec. 7], and it differs from (but is equivalent to) Lorentzen’s original definition. The sequence  $F_n$  is restrained if and only if  $\rho(F_n(j), j) \rightarrow \infty$  as  $n \rightarrow \infty$ . Since  $\rho(F_n(j), j) = \rho(j, F_n^{-1}(j))$  it follows that  $F_n^{-1}$  is also restrained. We are often interested in sequences  $F_n$  for which  $F_n(j)$  converges to a point on the ideal boundary  $\mathbb{C}_\infty$  (in the chordal metric). Such sequences are said to be *generally convergent*; they were first studied in the context of continued fractions by Lorentzen in [8].

A *backward limit point* of a restrained sequence  $F_n$  is an accumulation point of the backward orbit  $F_n^{-1}(j)$ . That is, a point  $p$  in  $\mathbb{C}_\infty$  is a backward limit point of  $F_n$  if there is a subsequence of  $F_1^{-1}(j), F_2^{-1}(j), \dots$  that converges to  $p$  in the chordal metric. The *backward limit set*  $\Lambda(F_n)$  of  $F_n$  is the collection of all backward limit points. A *forward limit point* of  $F_n$  is an accumulation point of the forward orbit  $F_1(j), F_2(j), \dots$ , although we have little use for this definition. If  $F_1, F_2, \dots$  are all the elements of a Kleinian group then, because groups are closed under taking inverses, backward limit points and forward limit points are the same. This is why there is no distinction between backward and forward limit points in the theory of Kleinian groups.

The point  $j$  has no special significance in the definitions so far, and we can replace it with any other point in  $\mathbb{H}^3$  without consequence. In particular, the set of accumulation points of  $F_1^{-1}(w), F_2^{-1}(w), \dots$  is  $\Lambda(F_n)$  for any point  $w$  in  $\mathbb{H}^3$ . In fact, providing a point  $p$  in  $\mathbb{C}_\infty$  is not a forward limit point of  $F_n$ , the set of accumulation points of  $F_1^{-1}(p), F_2^{-1}(p), \dots$  is also  $\Lambda(F_n)$  (see, for example, [1, Thm. 3.5]).

A point  $p$  in  $\mathbb{C}_\infty$  is a *conical limit point* of the restrained sequence  $F_n$  if there is a hyperbolic geodesic  $\gamma$  with one end-point at  $p$  and a subsequence of  $F_1^{-1}(j), F_2^{-1}(j), \dots$  that lies within a bounded hyperbolic distance of  $\gamma$  and converges to  $p$  in the chordal metric. Elementary hyperbolic geometry can be used to show that this definition is independent of the choice of geodesic  $\gamma$ , and  $j$  can be replaced by any other point in  $\mathbb{H}^3$ . The *conical limit set*  $\Lambda_c(F_n)$  of  $F_n$  is the set of all conical limit points of  $F_n$ . This set is contained in the backward limit set of  $F_n$ . The conical limit set is important in continued fraction theory because of the following theorem due to Aebischer [1, Thm. 5.2].

**Theorem 5.1** *Let  $F_n$  be a restrained sequence of Möbius transformations and let  $p \in \mathbb{C}_\infty$ . Then  $\chi(F_n(j), F_n(p)) \rightarrow 0$  if and only if  $p \notin \Lambda_c(F_n)$ .*

We also record a corollary of this theorem [1, Prop. 5.3]. Remember that a sequence  $F_n$  is generally convergent if  $F_n(j)$  converges in the chordal metric to a point on  $\mathbb{C}_\infty$ .

**Corollary 5.1** *If  $F_n$  is a generally convergent sequence of Möbius transformations then, providing it has more than one backward limit point,  $F_n$  diverges everywhere on its conical limit set.*

A sequence  $F_1, F_2, \dots$  of Möbius transformations is said to be a *rapid escape sequence* if, for a point  $w$  in  $\mathbb{H}^3$ , the sum  $\sum_n \exp[-\rho(w, F_n(w))]$  converges. Once again, this definition is independent of the particular point  $w$  chosen. We use the phrase “rapid escape” because the forward orbit of a rapid escape sequence at a point  $w$  approaches the ideal boundary particularly quickly. Clearly, rapid escape sequences are restrained. The Hausdorff dimension of the conical limit set of a rapid escape sequence does not exceed one (see [16, Cor. 9.3.2]). That is the reason why we obtain convergence everywhere but on a set of Hausdorff dimension at most one in the proof of Theorem 3.1. This issue is covered in more detail in [20]. Rapid escape sequences, and the related concept of the *critical exponent*, play an important role in Kleinian group theory (see [16]).

**Lemma 5.1** *Let  $F_n$  and  $G_n$  be two sequences of Möbius transformations such that  $F_n$  is restrained and  $G_n F_n^{-1}$  converges uniformly to another Möbius transformation  $g$ . Then  $G_n$  is also restrained and*

- (i)  *$p$  is a backward limit point of  $F_n$  if and only if  $p$  is a backward limit point of  $G_n$ ;*
- (ii)  *$p$  is a conical limit point of  $F_n$  if and only if  $p$  is a conical limit point of  $G_n$ ;*
- (iii)  *$F_n$  is a rapid escape sequence if and only if  $G_n$  is a rapid escape sequence.*

*Proof* Since  $G_n F_n^{-1}(j) \rightarrow g(j)$  it follows that the sequence  $\rho(G_n F_n^{-1}(j), j)$  is bounded above. But  $\rho(F_n^{-1}(j), G_n^{-1}(j)) = \rho(G_n F_n^{-1}(j), j)$ , so the sequence  $\rho(F_n^{-1}(j), G_n^{-1}(j))$  is bounded above. It is now immediate that  $G_n$  is restrained and (i), (ii), and (iii) hold.  $\square$

Recall that  $s_n(z) = a_n/z$  and  $S_n = s_1 \cdots s_n$ . Remember also the definition of  $\lambda_n$  given in (4.4), and the formulas for  $S_n^{-1}$  that follow that definition.

**Lemma 5.2** *If  $S_n^{-1}(j)$  has modulus greater than 1 then*

$$\exp[-\rho(S_n^{-1}(j), j)] = |\lambda_n|.$$

*Proof* If  $\mu$  is a complex number with modulus greater than 1 then  $\rho(\mu j, j) = \log |\mu|$ . Thus  $\exp[-\rho(\mu j, j)] = 1/|\mu|$ . Since  $S_n^{-1}(j) = j/|\lambda_n|$ , we see that  $\exp[-\rho(S_n^{-1}(j), j)] = |\lambda_n|$ .  $\square$

The sequence  $S_n^{-1}(j)$  is confined to the vertical geodesic from 0 to  $\infty$ . It follows that the only possible backward limit points of  $S_n$  are 0 and  $\infty$ , and each of these points is a backward limit point if and only if it is a conical limit point.

We need one final lemma on hyperbolic geometry (see [2, Thm. 7.9.1]).

**Lemma 5.3** *Let  $\gamma$  be a geodesic in  $\mathbb{H}^3$  that lands at points  $a$  and  $b$  in  $\mathbb{C}_\infty$ . Then*

$$\cosh \rho(j, \gamma) = \frac{2}{\chi(a, b)}.$$

*In particular,*

$$\exp[-\rho(j, \gamma)] \geq \frac{\chi(a, b)}{4}.$$

We can now prove Theorem 1.1.

*Proof (of Theorem 1.1)* The equivalence of (i)–(iv) follows from Theorem 4.1.

Let us show that (i) implies (v). If the Stern–Stolz series  $\sum_n |\lambda_n|$  converges then  $\lambda_n \rightarrow 0$ , so eventually  $|\lambda_n| < 1$ . Since  $|S_n^{-1}(j)| = 1/|\lambda_n|$  it follows that eventually  $|S_n^{-1}(j)| > 1$ , and then Lemma 5.2 tells us that the sum  $\sum_n \exp[-\rho(S_n^{-1}(j), j)]$  converges. Also, because  $S_n^{-1}(j) \rightarrow \infty$ , we see that  $\infty$  is the only backward limit point of  $S_n$ .

Now we show that (v) implies (i). Because  $S_n$  is a rapid escape sequence it follows that  $\rho(S_n^{-1}(j), j) \rightarrow \infty$  (that is,  $S_n$  is restrained). Since  $\infty$  is the only backward limit point of  $S_n$  we deduce that  $S_n^{-1}(j) \rightarrow \infty$  as  $n \rightarrow \infty$ . Therefore  $|S_n^{-1}(j)| > 1$  for sufficiently large  $n$ , and again we appeal to Lemma 5.2, this time to see that convergence of  $\sum_n \exp[-\rho(S_n^{-1}(j), j)]$  implies convergence of the Stern–Stolz series  $\sum_n |\lambda_n|$ .

Next we show that (ii) and (v) imply (vi). We know that  $S_n$  is restrained, by (v), because it is a rapid escape sequence. Furthermore, because  $S_n^{-1}(j)$  is constrained to lie on the vertical geodesic between 0 and  $\infty$ ,  $S_n$  has only a single backward limit point, and only a single conical limit point,  $\infty$ . From Theorem 4.1, and using (ii), we know that  $T_n S_n^{-1}$  converges to another Möbius transformation  $g$ . We can now apply Lemma 5.1 to deduce that  $T_n$ , like  $S_n$ , is a rapid escape sequence with only a single backward limit point, and only a single conical limit point,  $\infty$ .

Last we show that (vi) implies (iii). Let  $\gamma$  denote the hyperbolic geodesic between  $-1$  and 0. Let  $w_n$  denote the point on  $\gamma$  such that  $\rho(\gamma, T_n^{-1}(j)) = \rho(w_n, T_n^{-1}(j))$ . Since  $T_n$  is restrained, and it has only a single backward limit point  $\infty$ , we see that  $T_n^{-1}(j) \rightarrow \infty$  as  $n \rightarrow \infty$ . It follows that  $w_n$  converges to the highest point on  $\gamma$ , namely  $(-1 + j)/2$ . In particular, there is a positive constant  $K$  such that  $\rho(w_n, j) < K$  for all positive integers  $n$ . Applying the triangle inequality gives

$$\rho(j, T_n^{-1}(j)) \leq \rho(j, w_n) + \rho(w_n, T_n^{-1}(j)) < K + \rho(\gamma, T_n^{-1}(j)).$$

Therefore, as the series  $\sum_n \exp[-\rho(j, T_n^{-1}(j))]$  converges, the series  $\sum_n \exp[-\rho(\gamma, T_n^{-1}(j))]$  also converges. Hence  $\sum_n \exp[-\rho(T_n(\gamma), j)]$  converges. It follows from Lemma 5.3 that the series  $\sum_n \chi(T_n(-1), T_n(0))$  converges. Since  $T_n(-1) = T_{n-2}(0)$  we see that the series  $\sum_n \chi(T_{n-2}(0), T_n(0))$  converges too. This implies that the sequences  $T_{2n-1}(0)$  and  $T_{2n}(0)$  each converge. Suppose they converge to the same limit  $p$ . Then  $T_n(0) \rightarrow p$  and  $T_n(\infty) \rightarrow p$ . Moreover, because  $j$  lies on the geodesic between 0 and  $\infty$  it follows that  $T_n(j) \rightarrow p$ . However, we know from Theorem 5.1 that  $\chi(T_n(j), T_n(\infty)) \rightarrow 0$  as  $n \rightarrow \infty$  because  $\infty$  is a conical limit point. This contradiction shows that the limits of  $T_{2n-1}(0)$  and  $T_{2n}(0)$  are distinct.

It remains to prove the final assertion in Theorem 1.1. Recall that  $S_{2n-1}(z) = 1/(\lambda_{2n-1}z)$  and  $S_{2n}(z) = \lambda_{2n}z$ , where  $|\lambda_n|$  is the  $n$ th term in the Stern–Stolz series. When  $\sum_n |\lambda_n|$  converges (statement (i)), and hence  $\lambda_n \rightarrow 0$ , it follows that  $S_{2n-1} \rightarrow \infty$  and  $S_{2n} \rightarrow 0$  locally uniformly on  $\mathbb{C}$ , and  $S_{2n-1}(\infty) = 0$  and  $S_{2n}(\infty) = \infty$ . Now, we know from the final assertion in Theorem 4.1 that there is a Möbius map  $g$  such that  $\chi_0(T_n, gS_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $T_{2n-1} \rightarrow g(\infty)$  and  $T_{2n} \rightarrow g(0)$  locally uniformly on  $\mathbb{C}$ , and  $T_{2n-1}(\infty) \rightarrow g(0)$  and  $T_{2n}(\infty) \rightarrow g(\infty)$ .  $\square$

## 6 Proof of Theorem 1.2

Theorem 1.2 can easily be deduced from Theorems 1.1 and 3.1.



*Proof (of Theorem 1.2)* Suppose first that the Stern–Stolz series converges. Then the final assertion of Theorem 1.1 tells us that there are distinct points  $p$  and  $q$  such that  $T_{2n-1} \rightarrow p$  and  $T_{2n} \rightarrow q$  locally uniformly on  $\mathbb{C}$ , and  $T_{2n-1}(\infty) \rightarrow q$  and  $T_{2n}(\infty) \rightarrow p$ .

Suppose now that the Stern–Stolz series diverges. Theorem 3.1 tells us that  $T_{2n-1}$  and  $T_{2n}$  converge locally uniformly on  $H$ , and almost everywhere on  $\mathbb{C}_\infty$ , to points  $p$  and  $q$ . That theorem also says that  $\sum_n \chi(T_n(0), T_{n+2}(0))$  converges. By comparing statements (i) and (iii) of Theorem 1.1 we see that  $p = q$ . Therefore  $T_n$  converges locally uniformly on  $H$ , and almost everywhere on  $\mathbb{C}_\infty$ , to a single point.  $\square$

It is interesting that many well-known concepts from continued fraction theory can be interpreted using geometric properties of Möbius transformations. The geometric approach often provides insight into continued fractions that is difficult to obtain from algebra alone. We provide another example of this here, by constructing a convergent continued fraction  $\mathbf{K}(a_n|1)$  such that the sequence  $t_1, t_2, \dots$  satisfies  $t_n(H) \subseteq H$  for  $n = 1, 2, \dots$  and  $T_n$  diverges on an uncountable, dense subset of the complement of  $H$ . This shows that the assertion about almost everywhere convergence in Theorem 1.2 cannot be strengthened significantly.

For simplicity, let us choose  $H$  to be the half-plane given by  $\operatorname{Re}[z] > -1/2$ , although a similar construction works for other half-planes. Suppose that  $\mathbf{K}(a_n|1)$  is a convergent continued fraction such that  $t_n(H) \subseteq H$  for  $n = 1, 2, \dots$ . The value of  $\mathbf{K}(a_n|1)$  – that is, the limit of the sequence  $T_n(0)$  – is necessarily contained in  $\overline{H}$  (and it cannot be 0 or  $\infty$ ). In fact, it is known that every element in  $\overline{H} \setminus \{0, \infty\}$  is the value of some such continued fraction (see, for example, [15, Thm. 3.47]). This observation, which is implicit in our arguments below, is key to our construction. We shortcut the usual proofs of the observation by considering the dynamics of anticonformal Möbius transformations.

Choose any point  $q$  of  $H$ , and define  $t$  to be the anticonformal Möbius transformation  $t(z) = a/(1 + \bar{z})$ , where  $a = q + |q|^2$ . This has exactly two fixed points, namely  $q$  and  $\tau(q)$ , where  $\tau$  is the reflection in  $\partial H$ , given by  $\tau(z) = -1 - \bar{z}$ . Each anticonformal Möbius map with exactly two fixed points is conjugate to a map of the form  $z \mapsto \lambda \bar{z}$ , where  $\lambda > 1$ . One can check that  $q$  is the attracting fixed point of  $t$  and  $\tau(q)$  is the repelling fixed point. Therefore the sequence  $t^n$  converges to  $q$  locally uniformly on the complement of  $\tau(q)$ . It follows, in particular, that  $t(H) \subseteq H$ .

**Lemma 6.1** *Given points  $p$  and  $q$  in  $H$ , and  $\varepsilon > 0$ , there are Möbius maps  $t_i(z) = a_i/(1 + \bar{z})$ ,  $i = 1, \dots, n$ , that satisfy  $t_i(H) \subseteq H$  and are such that  $\chi(t_1 \cdots t_n(p), q) < \varepsilon$ .*

*Proof* Choose  $n$  to be suitably large that  $\chi(t^n(\bar{p}), q) < \varepsilon$  and  $\chi(t^n(p), q) < \varepsilon$ , where  $t(z) = a/(1 + \bar{z})$  and  $a = q + |q|^2$ . Let  $a_i$  equal  $a$  if  $i$  is odd, and  $\bar{a}$  if  $i$  is even. Then  $t_1 \cdots t_n(p)$  is equal to either  $t^n(\bar{p})$  if  $n$  is odd, or  $t^n(p)$  if  $n$  is even, so the result follows.  $\square$

Using Lemma 6.1 we can choose an infinite sequence  $t_1, t_2, \dots$  of Möbius maps, where  $t_n(z) = a_n/(1 + \bar{z})$  and  $t_n(H) \subseteq H$ , such that the orbit  $t_n \cdots t_1(0)$ ,  $n = 1, 2, \dots$ , is dense in  $H$ . Notice that we work with the sequence  $t_n \cdots t_1(0)$  rather than the usual sequence  $t_1 \cdots t_n(0)$ . Let  $\alpha$  be the involution given by  $\alpha(z) = -1 - \bar{z}$ , and let  $L$  denote the half-plane given by  $\operatorname{Re}[z] < -1/2$ . The map  $\alpha$  interchanges  $L$  and  $H$ . Since  $\alpha(-1) = 0$ , it follows that the orbit  $\alpha t_n \cdots t_1 \alpha(-1)$  is dense in  $L$ . Observe that  $\alpha t_i \alpha = t_i^{-1}$  for each positive integer  $i$ , and so

$$\alpha t_n \cdots t_1 \alpha = t_n^{-1} \cdots t_1^{-1} = T_n^{-1}.$$

Therefore we have shown that the orbit  $T_n^{-1}(-1)$  is dense in  $L$ . Now,  $-1$  is not a forward limit point of  $T_n$ , or in other words the sequence  $T_n(j)$  does not accumulate at  $-1$ , because this sequence is confined to the hyperbolic half-space with ideal boundary  $H$ . It follows that the backward limit set  $\Lambda(T_n)$  of  $T_n$ , which is the set of accumulation points of

$T_1^{-1}(j), T_2^{-1}(j), \dots$ , is equal to the set of accumulation points of  $T_1^{-1}(-1), T_2^{-1}(-1), \dots$ . Therefore  $\Lambda(T_n) = \bar{L}$ .

Since the backward limit set of  $T_n$  contains points other than  $\infty$ , we see from the equivalence of (i) and (vi) in Theorem 1.1 that the Stern–Stolz series diverges. Therefore  $T_n(0)$  and  $T_n(\infty)$  converge to a point  $p$  (in the chordal metric). Since  $T_n(j)$  lies on the geodesic joining  $T_n(0)$  and  $T_n(\infty)$ , we see that  $T_n(j)$  converges to  $p$  also. That is,  $T_n$  is generally convergent, with limit  $p$ .

It is known that if the backward limit set of a restrained sequence of Möbius transformations contains an open disc  $E$ , then the conical limit set is uncountable and its closure contains  $E$  (see [6, Lem. 5.12]). As  $\Lambda(T_n) = \bar{L}$ , it follows that  $\Lambda_c(T_n)$  is uncountable and  $\overline{\Lambda_c(T_n)} \supseteq L$ . Therefore  $\Lambda_c(T_n)$  is dense in  $L$ . This completes our construction because, by Corollary 5.1,  $T_n$  diverges everywhere on  $\Lambda_c(T_n)$ .

## 7 The geometry of the parabola theorem

As we saw in Section 2, the significance of the parabolic region in the parabola theorem is that it gives rise to the inclusions  $t_n(H) \subseteq H$ , where  $H$  is a half-plane. This leads to a nested sequence of discs

$$H \supseteq T_1(H) \supseteq T_2(H) \supseteq T_3(H) \supseteq \dots$$

Associated to this sequence of discs is a sequence of points  $T_1(\infty), T_2(\infty), \dots$ . Since  $\infty \in \partial H$ , it follows that  $T_n(\infty) \in T_n(\partial H)$  for each integer  $n$ . Our geometric approach to the parabola theorem allows us to describe all possible sequences of discs and points that arise in that theorem. Beardon carried out a similar programme for the Śleszyński–Pringsheim theorem in [4].

Given an open disc  $D$  in  $\mathbb{C}_\infty$ , a point  $a$  in  $D$ , a point  $b$  in  $\partial D$ , and a point  $c$  that lies outside  $\bar{D}$ , we define a *parabola sequence* to be a sequence of Möbius transformations  $f_1, f_2, \dots$  that satisfies  $f_n(D) \subseteq D$ ,  $f_n(c) = b$ , and  $f_n(b) = a$  for each integer  $n$ . When  $a = 0$ ,  $b = \infty$ ,  $c = -1$ , and  $D = H_\alpha$  we recover the usual sequence of maps  $t_n$  that arise in the classical parabola theorem. For simplicity, we focus only on *symmetric* parabola sequences, which are parabola sequences such that  $a$  and  $c$  are inverse points with respect to  $\partial D$ . Let  $F_n = f_1 \cdots f_n$ , and let  $F_0$  be the identity map.

**Theorem 7.1** *Let  $f_1, f_2, \dots$  be a symmetric parabola sequence with associated disc  $D$  and points  $a, b$ , and  $c$ . Let  $D_n = F_n(D)$  and  $z_n = F_n(b)$ . Then*

- (i)  $D_0 \supseteq D_1 \supseteq D_2 \supseteq \dots$ ,
- (ii)  $z_n \in \partial D_n$ ,
- (iii)  $z_{n-1} \in \mathbb{C}_\infty \setminus \bar{D}_n$  and  $z_{n+1} \in D_n$ , and these are inverse points with respect to  $\partial D_n$ .

*Conversely, any sequence of discs  $D_n$  and points  $z_n$  satisfying (i), (ii), and (iii) arise as the  $F_n$ -images of  $D$  and  $b$  for some symmetric parabola sequence  $f_n$ .*

There is a similar theorem for general parabola sequences, which has a more elaborate version of statement (iii) involving hyperbolic distance.

Theorem 7.1 is illustrated in Fig. 7.1.

*Proof (of Theorem 7.1)* Suppose that  $f_n$  is a symmetric parabola sequence. Property (i) follows from the inclusion  $F_n(D) \subseteq F_{n-1}(D)$ . Property (ii) holds because  $b \in \partial D$ . Property (iii) follows by preservation of inverse points under Möbius transformations, as  $z_{n-1} = F_n(c)$  and  $z_{n+1} = F_n(a)$ .

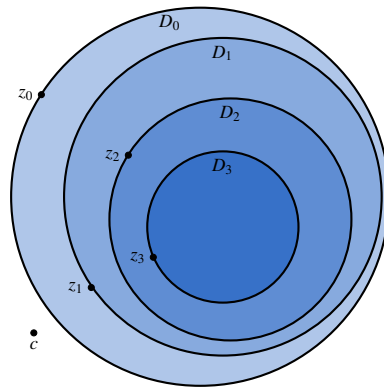


Fig. 7.1 Orbits of discs and points under a symmetric parabola sequence

Now suppose that  $D_n$  and  $z_n$  have been given to satisfy (i), (ii), and (iii). Define  $H = D_0$ ,  $b = z_0$ ,  $a = z_1$ , and let  $c$  be the inverse point of  $a$  in  $\partial D$ . For each positive integer  $n$  choose a Möbius transformation  $F_n$  that satisfies  $F_n(D) = D_n$ ,  $F_n(b) = z_n$ , and  $F_n(c) = z_{n-1}$ . From property (iii), and because Möbius transformations preserve inverse points, we see that  $F_n(a) = z_{n+1}$ . Therefore the sequence  $f_1, f_2, \dots$  defined by  $f_n = F_{n-1}^{-1}F_n$  is a symmetric parabola sequence with  $D_n = F_n(D)$  and  $z_n = F_n(b)$ .  $\square$

There is a subtle geometric fact that is not immediately apparent from Theorem 7.1: if the sequence  $z_n$  converges then the intersection of the nested closed discs  $\overline{D}_n$  is a single point. A proof of this can be extracted from [17]. In general, the intersection of a nested sequence of closed discs is either a single point or a closed disc, and it is usual in continued fraction theory to refer to these two alternatives as the *limit-point case* and the *limit-disc case*, respectively.

For more general parabola sequences, it is no longer true that convergence of  $z_n$  can only arise with the limit-point case. To see why this is so, consider the map

$$t(z) = \frac{-3/16}{1+z},$$

which is a loxodromic Möbius transformation with attracting fixed point  $-1/4$  and repelling fixed point  $-3/4$ . Let  $t_n = t$  for each positive integer  $n$ , so that  $T_n = t^n$ . If  $H$  is the half-plane given by  $\operatorname{Re}[z] > -1/2$  then  $t_1, t_2, \dots$  and  $H$  together form a symmetric parabola sequence, because  $-1$  and  $0$  are inverse points with respect to  $\partial H$ . Because the repelling fixed point of  $t$  lies outside  $\overline{H}$ , the limit-point case occurs for this parabola sequence. The limit point is  $-1/4$ . If instead  $H$  is the half-plane  $\operatorname{Re}[z] > -3/4$  then  $t_n$  and  $H$  again form a parabola sequence, but this time the parabola sequence is not symmetric, because  $-1$  and  $0$  are not inverse points with respect to  $\partial H$ . The repelling fixed point of  $t$  belongs to  $\partial H$  for this parabola sequence, and it follows that the limit-disc case occurs: the limit disc is given by  $|z + 1/2| \leq 1/4$ .

## 8 Higher dimensions

All our methods generalise to higher dimensions, and the results and their proofs go through virtually unchanged. We give just one example of this, namely a version of Theorem 1.2 in

many dimensions. See [2, 18] for the theory of Möbius transformations in several dimensions.

So far we have only considered Möbius transformations acting on  $\mathbb{C}_\infty$ , and in particular we have studied continued fractions using sequences of maps  $t_n$  given by  $t_n(z) = a_n/(b_n + z)$ . Each of these maps takes  $\infty$  to 0. Now we would like to consider Möbius transformations that act on  $\mathbb{R}^N \cup \{\infty\}$ . Consider a sequence of Möbius maps  $t_1, t_2, \dots$  acting on  $\mathbb{R}^N \cup \{\infty\}$  that satisfies  $t_n(\infty) = 0$  for each integer  $n$ . Let  $\sigma$  be inversion in the unit sphere. Then  $t_n$  can be expressed in the form

$$t_n(x) = a_n \sigma(b_n + x),$$

where  $b_n \in \mathbb{R}^N$ , and  $a_n$  denotes an orthogonal map of  $\mathbb{R}^N$  followed by a dilation. That is, there is a positive scalar  $\lambda_n$  and an orthogonal map  $A_n$  such that  $a_n(x) = \lambda_n A_n(x)$ . In the two dimensional case  $a_n$  is a complex number,  $\lambda_n = |a_n|$ , and  $A_n$  is a rotation by  $a_n/|a_n|$ . Let us denote  $\lambda_n$  by  $|a_n|$  even in higher dimensions. Because  $\sigma$  is anticonformal, we must also declare that  $A_n$  is anticonformal in order for  $t_n$  to be conformal. This is not strictly necessary, as we have not needed conformality so far, but it tallies with the two-dimensional case in which all the maps  $t_n$  are conformal. The Stern–Stolz series in higher dimensions is the series

$$|b_1| \left( \frac{1}{|a_1|} \right) + |b_2| \left( \frac{|a_1|}{|a_2|} \right) + |b_3| \left( \frac{|a_2|}{|a_1||a_3|} \right) + |b_4| \left( \frac{|a_1||a_3|}{|a_2||a_4|} \right) + \dots$$

Now, for the parabola theorem we need all the coefficients  $b_n$  to be equal, so let  $b_n = (1, 0, \dots, 0)$  for each integer  $n$ , and we write this more simply as  $b_n = 1$ . Our maps  $t_n$  now have the form  $t_n(x) = a_n \sigma(1 + x)$ , and as usual we let  $T_n = t_1 \cdots t_n$ . Let  $-1$  denote the point  $(-1, 0, \dots, 0)$  and let  $H$  be a Euclidean half-space that contains 0, but does not contain  $-1$ , even its closure. As before, we let  $u$  be the point inverse to  $-1$  in  $\partial H$ , and let us also define  $v = \sigma(1 + u)$ . Using the argument of Section 2 we see that  $t_n(H) \subseteq H$  if and only if  $t_n(u)$  lies in the region  $\{x \in \mathbb{R}^N : |x| \leq d(x, \partial H)\}$  bounded by a paraboloid. Notice that  $t_n(u) = a_n v$ . We now have a strong version of the parabola theorem, valid in several dimensions.

**The paraboloid theorem** *Let  $H$  be a Euclidean half-space in  $\mathbb{R}^N$  that contains 0, but does not contain  $-1$ , even in its closure, and suppose that  $a_n v \in \{z \in \mathbb{C} : |z| \leq d(z, \partial H)\}$  for each positive integer  $n$ . If the Stern–Stolz series*

$$\left( \frac{1}{|a_1|} \right) + \left( \frac{|a_1|}{|a_2|} \right) + \left( \frac{|a_2|}{|a_1||a_3|} \right) + \left( \frac{|a_1||a_3|}{|a_2||a_4|} \right) + \dots$$

*converges, then there are distinct points  $p$  and  $q$  such that  $T_{2n-1} \rightarrow p$  and  $T_{2n} \rightarrow q$  locally uniformly on  $\mathbb{R}^N$ , and  $T_{2n-1}(\infty) \rightarrow q$  and  $T_{2n}(\infty) \rightarrow p$ . If the Stern–Stolz series diverges then  $T_n$  converges locally uniformly on  $H$ , and almost everywhere on  $\mathbb{R}_\infty^N$ , to a single point.*

This framework for describing continued fractions in several dimensions can be used to generalise many other results on continued fractions. For instance, the Śleszyński–Pringsheim theorem says that  $\mathbf{K}(a_n|b_n)$  converges if  $|b_n| \geq 1 + |a_n|$  for each integer  $n$ . In higher dimensions, using our notation  $t_n(x) = a_n \sigma(b_n + x)$  and  $T_n = t_1 \cdots t_n$ , the same sequence of inequalities  $|b_n| \geq 1 + |a_n|$  guarantees convergence of  $T_1(0), T_2(0), \dots$

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