Bunched black (and grouped grey) swans: Dissipative and non-dissipative models of correlated extreme fluctuations in complex geosystems

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[1] I review the hierarchy of approaches to complex systems, focusing particularly on stochastic equations. I discuss how the main models advocated by the late Benoit Mandelbrot fit into this classification, and how they continue to contribute to cross-disciplinary approaches to the increasingly important problems of correlated extreme events and unresolved scales. The ideas have broad importance, with applications ranging across science areas as diverse as the heavy tailed distributions of intense rainfall in hydrology, after which Mandelbrot named the "Noah effect"; the problem of correlated runs of dry summers in climate, after which the "Joseph effect" was named; and the intermittent, bursty, volatility seen in finance and fluid turbulence. **Citation:** Watkins, N. W. (2013), Bunched black (and grouped grey) swans: Dissipative and non-dissipative models of correlated extreme fluctuations in complex geosystems, *Geophys. Res. Lett.*, *40*, 402–410, doi:10.1002/grl.50103.

1. Introduction

[2] Geophysicists, and other environmental scientists, frequently now need to use nonlinear and stochastic models, particularly in support of time series analysis. This need is further motivated by the increasing importance now attached to areas like extreme events [*Bunde et al.*, 2005; *Sharma et al.*, 2012] and trends [*Franzke et al.*, 2012; *Vyushin et al.*, 2007]. Unless we are also specialists in statistics, or in stochastic processes, though, geophysicists' intuition tends to have been formed on the simplest textbook stochastic time series model: Gaussian, independent identically distributed (iid), stationary, "white" noise [e.g., *Bendat and Piersol*, 2011; *Vyushin et al.*, 2007]. This choice is not just a matter of background and culture, but is frequently supported by a physical argument, that fluctuations in degrees of freedom which couple only weakly to a coordinate, and which also do not perturb each other directly, should be additive. If these fluctuations have a finite variance, and are only short-range correlated, they well approximate the requirements of the central limit theorem [e.g., *Sornette*, 2004], which then leads to the Gaussian form.

[3] In stark contrast, however, *Mandelbrot* [1963], *Mandelbrot and Wallis* [1968], *Mandelbrot and Van Ness* [1968], *Mandelbrot* [1974], identified three effects present in many fluctuating time series drawn from observations in both the natural and the economic sciences, and put forward mathematical models for each. Each exemplified a very strong departure from one of the key properties of Gaussian white noise. The first two were:

- Heavy-tailed probability distributions for the values of a time series, which *Mandelbrot and Wallis* [1968] dubbed the "Noah effect." This was seen in "wild" cotton price fluctuations by *Mandelbrot* [1963], and implies a greatly increased probability of large events compared to a "milder" light-tailed distribution.
- Long-ranged serial dependence of a value on all its predecessors, the "Joseph effect" [*Mandelbrot and Wallis*, 1968], leading to "slow" fluctuations which resemble long-range trends. This was introduced to explain the Hurst effect in hydrology, named after Hurst's observation of the phenomenon of anomalous growth of rescaled range in the levels of the Nile River [*Hurst*, 1957].

[4] Subsequently, Mandelbrot [1974] focused on intermittent processes exhibiting a third effect, identified in the study of *Mandelbrot* [1963], which has since come to be known in finance as "volatility clustering," correlations between the absolute values of a time series.

[5] Although Mandelbrot's original papers are famously idiosyncratic [*Goldenfeld*, 1998], the body of work in his *Selecta* [*Mandelbrot*, 1997, 1999, 2002] has been of enduring importance. Numerous excellent reviews exist by now that encompass both his work on fractals and multifractals, and survey the closely related fields of anomalous diffusion and anomalous time series [e.g., *Feder*, 1998; *Klafter and Sokolov*, 2011; *Sokolov*, 2012]. They include some closely tailored to a geoscience audience [e.g., *Ghil et al.*, 2011; *Lovejoy and Schertzer*, 2012], but with some exceptions [notably *Sornette*, 2004], have necessarily tended to focus on parts of the legacy rather than the whole.

[6] This short Frontiers article aims to complement the detailed reviews by providing a very short tour of how Mandelbrot's key contributions fit into the wider and disparate field of related work in this area, via what is hoped to be accessible resource letter. I hope this will equip newcomers to make their own further exploration, for example in the books highlighted above, and that it will be helpful to others engaged in cross-disciplinary applications.

2. Hierarchy of Complexity Models

[7] Although systems in the geosciences usually have very many degrees of freedom, it is nonetheless usual to define macroscopic variables and attempt to model them relatively simply. An example in climate science would be the

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stochastic energy balance model [e.g., *Padilla et al.*, 2011] where the variable simulated is fluctuations in the globally averaged surface temperature, in order to estimate a transient climate sensitivity. Depending on the complexity of a system, its modeling needs to take into account increasing numbers of terms to describe the observed structure in time series, which can be illustrated by a nested set of simple equations.

2.1. Cold Deterministic Dynamics

[8] The first level is a single deterministic degree of freedom, captured conceptually by Newton's falling apple, and mathematically by his equation

$$F = ma = m\ddot{x} = -V'(x) \tag{1}$$

where I have additionally modeled all the deterministic forces on the particle's center of mass x by a single scalar potential function $V(x)$.

[9] Adding a few extra linear degrees of freedom, by replacing x by $\mathbf{x}$, is conceptually relatively simple, so the next important level of complication comes when the interactions are nonlinear but still deterministic, i.e., V' is not just a linear function of $\mathbf{x}$. This allows the possibility of deterministic Hamiltonian chaos, for example in the astrophysical Henon-Heiles problem, where $V(x,y)$ is quadratic in x and cubic in y.

[10] Many physical systems of interest cannot be represented as an isolated particle in a potential. Including an explicit dissipation term (damping) allows for another level of complexity. The simplest such term is a constant, η, linear in the velocity, referred to as "Ohmic":

$$m\ddot{\mathbf{x}} = -\eta\dot{\mathbf{x}} - \mathrm{V}'(\mathbf{x}) \tag{2}$$

which, when the force is linearized by approximating the potential as a quadratic about its minimum $V = (k/2)x^2$, gives another of the most famous equations in physics, the dissipative harmonic oscillator: $m\ddot{x} = -\eta\dot{x} - kx$. A (frequently unstated) assumption here is that dissipation is so strong that thermal fluctuations can be neglected. It is thus a low temperature approximation.

2.2. Warm Stochastic Dynamics

[11] The possible levels of description of complexity are not exhausted by deterministic modeling. Instead we frequently need to go to a stochastic description. This could result from the number of effective degrees of freedom being so large that an explicitly random description becomes necessary, or simply as an economical description of a very complex potential. The simplest modification of the equations above to include stochastic fluctuations is to *add* a noisy forcing term ξ giving

$$m\ddot{\mathbf{x}} = -\eta\dot{\mathbf{x}} + \xi - \mathrm{V}'(\mathbf{x}) \tag{3}$$

the (Ohmic) Langevin equation (Langevin, 1908) with a potential, used to model Brownian motion in physics. It describes the trajectory of a physical Brownian particle, viewed on timescales comparable with the dissipation timescale $\gamma = m/\eta$. When written in the velocity $\mathbf{v}$(t) rather than the position, with damping linear in velocity, white noise, and no external potential this becomes the Ornstein-Uhlenbeck (O-U) equation, whose stochastic solution is the O-U process [*Chorin and Hald*, 2009]. The Ornstein-Uhlenbeck equation can also result from the Langevin equation for overdamped motion in a harmonic potential, where this time acceleration has been neglected, and V' retained and linearized. The variable in the O-U process is then position, $\mathbf{x}$(t).

[12] In the case of thermodynamic equilibrium, statistical physics [e.g., *Reif*, 1965, p. 573] gives a further relation, the fluctuation-dissipation theorem, between the autocorrelation function $\langle\xi(0)\xi(\tau)\rangle$ of the fluctuations ξ, and an integral operator η acting on the velocity, for the dissipation. White noise is delta-correlated, and the integral over the delta function then gives the familiar constant η, and a fluctuation-dissipation relation of the form:

$$\eta = \frac{1}{2k_B T}\int_{-\infty}^{\infty} d\tau < \xi(0)\xi(\tau)>_0 \tag{4}$$

2.3. Hot Diffusive Dynamics

[13] Alternatively, rather than consider individual trajectories in an O-U picture, one may describe the evolving pdf of a particle's position in equilibrium with the heat bath, by making the Smoluchowski approximation [*Lemons*, 2002], and dropping the terms in γ. If, for simplicity, one also neglects the potential V, the remaining additive process [*Bachelier*, 1900; *Einstein*, 1905] is known as the Wiener process in physics, and (somewhat confusingly) as Brownian motion in mathematics [*Chorin and Hald*, 2009]. The pdf is governed by the diffusion equation *Lemons* [2002]:

$$\frac{\partial}{\partial t}p(x,t) = D_0\nabla^2 p(x,t) \tag{5}$$

3. Standard vs. Anomalous Noise Models

3.1. "Mild" and "Fast" White Gaussian Stationary Noise

[14] In either a Wiener-Brownian or Ornstein-Uhlenbeck description, an obvious need is to correctly specify the noise process ξ. A "white" Gaussian random noise process will have a time series with values drawn from the short-tailed probability distribution:

$$p(\xi) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-(\xi-\mu)^2/2\sigma^2\right] \tag{6}$$

where the mean μ is usually taken to be zero.

[15] In this time series, departures in magnitude from the mean by more than ± 3 times the standard deviation σ will be rare, occurring in only about 0.2% of values. The iid property of the series means its fluctuations are fast, and short-lived, so its autocorrelation function is theoretically a δ-function and its power spectrum "white." Its moments, including the familiar mean and standard deviation, are finite and constant, rather than growing with time. From beginnings associated with names such as R A Fisher, who coined the term "variance" as recently as 1918, a highly developed theory of statistical inference has grown up around white, Gaussian noise.

3.2. Anomalously "Slow" Time Series and the Joseph Effect

[16] Not all natural time series are spectrally white, or even short-range correlated. Some power spectral densities $S(f) \sim f^{-2d}$, estimated for example by Fourier methods, are

singular at the origin [*Beran*, 1994], reflecting a "pile up" of energy at low frequencies and long wavelengths. Such "1/f" noise exhibits long correlated runs of values with similar sizes, something that is essentially never seen in white noise. The paradigmatic geophysical example is the annual minimum level of the Nile River [*Hurst*, 1957], while the colorful coinage by *Mandelbrot and Wallis* [1968] of the name "Joseph effect" came from the correlation implied by the 7 years of drought and 7 years of plenty in Pharoah's dream.

3.3. Anomalously "Wild" Time Series and the Noah Effect

[17] The tails of a Gaussian pdf are light, and the distribution's only non-zero moments are its first two, giving the mean and standard deviation. Many important real-world time series, however, show kurtosis. Some show a mildly increased kurtosis compared to the Gaussian, while some have very high kurtosis, "heavy tailed" distributions, named after their property that most of the probability mass of the distribution resides in the tail [*Newman*, 2005; *Clauset et al*., 2009]. Celebrated examples in astrophysics and geoscience include the Gutenberg-Richter law for earthquake magnitudes [*Sethna et al*., 2001], and the magnitudes of solar flares.

[18] Introduced first as a wealth distribution by Pareto, after whom it was named, the archetypal distribution for modeling this effect has a pdf whose tail decays asymptotically as a pure power law:

$$p(\xi) \sim \xi^{-(1+\alpha)} \tag{7}$$

When α lies in the range between 0 and 2, sums of random numbers drawn from such a distribution do not flow to the Gaussian. Instead, they follow the corresponding α-stable (or "Lévy-stable") law [*Lévy*, 1937], which also has a power law tail with the same exponent. Power law distributions with α exactly 2, or greater, still flow to the Gaussian.

[19] Distinguishing between the Gaussian and an α-stable distribution with exponent very close to 2, when the sample size of available data is small, can be difficult. This is partly because the limited dynamic range explored by the samples seen so far means that the asymptotic power law shape of the latter distribution may not yet be so pronounced as to be obvious. Weron [2001] and *Burnecki et al*. [2012a] demonstrate that it is nonetheless possible. The latter authors applied their method to 'L-H' mode transitions in fusion plasmas.

[20] Some other aspects of best practice for measurement of power laws have also been controversial in recent years [e.g., *Clauset et al*., 2009; *Buchanan*, 2008; *Edwards et al*., 2007; *Viswanathan et al*., 2011]. In the light of this, it is important not to lose track of the distinction between two questions: "am I seeing a power law" versus "am I seeing a distribution with a heavy tail?" From the viewpoint of the risk of large events, the second question is the key one, because *any* heavy tailed distribution will give significantly larger probabilities of large events than a Gaussian does.

[21] If a tail is heavy, the first question, of its specific functional form, i.e., whether it is for example a power law, truncated power law, lognormal or stretched exponential is important for two other reasons. One reason is that differing shapes for the upper tail will translate into a different probability for (as yet unobserved) large "grey swan" events (e.g., *Weitzman, M. L., A precautionary tale of uncertain tail fattening, Harvard University Economics Department, Working paper, March 10th, 2012*). In coining his term "black swan," [*Taleb*, 2007] distinguished between these events, which could never have been anticipated, and the "grey swans" which would result from observing a tail which is much heavier than the model that was anticipated. The second reason is that different pdf shapes may each indicate different physical mechanisms [*Sornette*, 2004].

4. "Anomalous" Extensions to the Wiener Process: Additive and Undamped Models

[22] We have seen so far that a full description of natural complexity often requires the use of stochastic components in a model. Many natural time series describe an integrated property, as emphasized by Klemeš [2011], and so are more appropriately modeled as an additive, and so possibly nonstationary aggregation Y of fluctuations, than a stationary noise ξ. We have also seen that the simplest and best-understood stochastic processes are white Gaussian noise and the integrated models derived from it such as the Wiener or Ornstein-Uhlenbeck processes. Integration here means a stochastic integral [*Lemons*, 2002], which is formally represented as $Y = \int x\mathrm{d}t \equiv \int \mathrm{d}L$ as x is no longer a deterministic variable. The increment dL thus also represents a stochastic variable here. Space prevents discussion of the meaning of stochastic integration, but see *Lemons* [2002] among many excellent discussions available at widely varying levels of sophistication and rigor. In what follows, we will consider increments taken from a unit normal process L_2, and from an α-stable process L_α. The identification of the Noah and Joseph effects by Mandelbrot, which are anomalous with respect to white noise, has motivated the search for stochastic models to capture them. In this section, I will concentrate on extensions to the self similar Wiener process.

4.1. Self-Similar and α-Stable Models: The Fractional Stable Motions

[23] The first group of models represents fractional integrations of stable noises, of which the Gaussian is the basic example.

[24] *"Grey swans" modeled as Lévy flights*: $d = 0, \alpha \neq 2$: An additive random walk time series model where power law steps, based on the concept of the α-stable laws discovered by Lévy, replaced Gaussian steps, was proposed by *Mandelbrot* [1963] to describe the "wild" fluctuations of cotton prices. By the time of the English edition [*Mandelbrot*, 1983] of his book *Les Objects Fractals*, he had introduced the term "Lévy flight" for the equivalent spatial diffusion process, using it in relation to the trajectory of an imaginary spacecraft travelling randomly between the stars of a galaxy, and in distinction to shorter-tailed "Rayleigh" flights.

[25] By the mid 1960s, he had become intrigued by the pioneering work on the growth of rescaled range in annual minimum levels of the Nile River [*Hurst*, 1957; *Feder*, 1998; *Beran*, 1994], and in his autobiographical notes [*Mandelbrot*, 2002, pp. 218–219] recounted that he had at first thought that an α-stable noise model could account for them. On seeing the data, however, he realized that it was relatively light tailed.

[26] *Bunching and the "Joseph effect" modeled via fractional Brownian motion*, $d \neq 0, \alpha = 2$. The above epiphany led him, in *Mandelbrot and Wallis* [1968, 1969a], to advocate the

use of a different type of self-similar process, Y_H, which he called fractional Brownian motion (fBm),

$$Y_{H,2}(t) = \frac{1}{C_{H,2}} \int_R dL_2(s) K_{H,2}(t-s) \tag{8}$$

which extends the Wiener process to include a long-range-dependent, self-similar, memory kernel $K_{H,2}(t-s)$,

$$K_{H,2}(t-s) = \left[(t-s)_+^{H-1/2} - (-s)^{H-1/2}\right] \tag{9}$$

thus giving a decaying, non-zero weight to all of the values in the time integral over dL. fBm was given the name "Wiener's spiral" by its discoverer, *Kolmogorov* [1940]. The first derivative of the above process is fractional Gaussian noise (fGn), which provides a stationary, long-range-dependent model, when that is what is needed. Many of the first applications of fBm actually used fGn.

[27] The study of diffusion in the last century was greatly aided by the development and comparison of several descriptions: the central limit theorem, the diffusion equation, and the Langevin equation, and particularly by their equivalence in the limit of long times. The extent to which such an equivalence is possible is more subtle in the case of fBm, because the presence of long-range dependence means that information is present in a Langevin trajectory that is not captured by the pdf. The non-Markovian temporal dependence, for example, is needed to accurately predict the distribution of first passage times [*Lim and Muniandy*, 2002]. An equation of the diffusion type can be retrieved [*Lutz*, 2001] from a fractional Langevin equation (c.f. section 6). It remains Markovian in structure but has a diffusion coefficient that grows nonlinearly in time as t^{2d}, or more precisely :

$$D(t) = D_0 2Ht^{2H-1} \tag{10}$$

[28] The same diffusion coefficient was adapted empirically in hydrology by *Wheatcraft and Tyler* [1988] to model anomalous diffusion. Their equation can be seen in refs. [*Wang and Lung*, 1990; Lutz, 2001] to reproduce the pdf of fBm, a Gaussian whose width progressively stretches as its variance grows with time as t^{2H}.

[29] "*Grouped grey swan" models with both LRD and heavy tails*: Once Mandelbrot had appreciated that each of the above models exemplified one of the Noah and Joseph effects at the expense of the other, he proposed a more general model that would exhibit both. In *Mandelbrot and Wallis* [1969b] he looked at a fractional noise with power law distributed steps, calling it a "fractional hyperbolic" process. This paper presented the model in the context of demonstrating how the "R/S" diagnostic [*Mandelbrot*, 2002] was a measure of long-range dependence d rather than the self-similarity exponent H, which may have contributed to its relative neglect compared to his papers on fBm and Lévy flights. Mandelbrot himself seems not to have pursued the fractional hyperbolic model further, preferring to develop multifractal models [e.g., *Mandelbrot*, 1974], as will be discussed in section 7. This seems in part to have been because of their more obvious volatility clustering properties-a visual feature of many turbulence and finance series. However, interest in fractional stable models continues in the stochastic processes community. fBm is the most general Gaussian random walk that keeps the analytically desirable self-similar and stable property. Fractional stable models, such as linear fractional stable motion (LFSM) [*Samorodnitsky and Taqqu*, 1994; *Burnecki and Weron*, 2010], have subsequently extended fBm to the infinite variance case. LFSM generalizes fBm by combining a stochastic integral over α-stable variables with a self-similar kernel, so:

$$Y_{H,\alpha}(t) = \frac{1}{C_{H,\alpha}} \int_R dL_\alpha(s) K_{H,\alpha}(t-s) \tag{11}$$

where $K_{H,\alpha}(t-s) = (t-s)_+^{H-1/\alpha} - (-s)^{H-1/\alpha}$ which we can now see to be the generalization of the fBm kernel. The memory parameter d, self-similarity exponent H, and stability exponent α are related to each other [*Weron et al.*, 2005] by

$$d = H - \frac{1}{\alpha} \tag{12}$$

Taqqu [1987] pointed out the potential relevance of LFSM to geophysical modeling, and some applications have followed [e.g., *Watkins et al.*, 2009a].

4.2. Fractal Sums of Pulses, and Generalized Shot Noise

[30] The derivative of the Wiener process defines Gaussian white noise, whereas the derivative of the Poisson process yields white "shot" noise [*Haenggi and Jung*, 1995, page 244]. Recalling that fBm is the self-similar process whose *fractional* derivative gives Gaussian white noise raises the question of what kind of self-similar point process would have white shot noise (or related pulse models) as its fractional derivative. This was studied in a series of papers reviewed in *Eliazar and Klafter* [2011]. The book by Lowen and Teich [2005] is a very useful complementary survey of the field of fractal point processes, with a notably comprehensive set of problem solutions.

4.3. Continuous Time Random Walk

[31] fBm and LFSM have the convenience of preserving a mapping to the CLT or extended CLT, but as noted above they have a relatively uninformative diffusion equation. An alternative approach has been the continuous time random walk (CTRW) paradigm, in which one specifies both a distribution of jump lengths and of waiting times, but the stable property of the pdf is lost [*Kolokoltsov et al.*, 2001]. The most widely studied version has a factorizing pdf $P(x,t) = \Theta(x)\Phi(t)$, with heavy tailed pdfs for both jump size and waiting time. A particularly accessible heuristic treatment of it is given in the supplementary information of *Brockmann et al*. [2006], who christened it the "ambivalent" process to emphasize its competing subdiffusive and superdiffusive elements.

5. Fractality and Its Discontents: Common Questions about the Use of Fractal Models

5.1. Kadanoff: Where's the Physics?

[32] One question about the above models has been a perception that the anomalous effects lacked a physical mechanism. A widely cited source for this is *Kadanoff* [1986], a *Physics Today* editorial entitled "Fractals: where's the physics?" but it has often not been appreciated that this was not so much a criticism of the fractal idea but more a

plea for their increased physical understanding. A particularly noteworthy and influential attempt to address this was the stimulating proposal by *Bak et al*. [1987] of self-organized criticality (SOC). Its frequent presentation in statistical terms has meant that it is not always appreciated that SOC is a physical *mechanism* intended to account for, and unify, spacetime fractals, and thus much more than simply a statistical description of complex systems. Instead it took its name from the space and time correlation seen in the "critical phenomena" [*Wilson*, 1979], the prototypes of which were seen at phase transitions. SOC posited a tendency of slowly driven, thresholded and interaction-dominated systems to organize via "avalanches" of energy release into such a state.

5.2. Klemeš: Is the Physics of Fractals the Wrong Kind of Physics?

[33] An unfortunate corollary to the misperceptions noted above has been that stochastic models such those of *Mandelbrot* [1963] and *Mandelbrot and Wallis* [1968] are also sometimes decried as being "purely statistical." When examined more closely, as I have above, one sees that they all imply physics, as they are modifications to the established physics of noisy inertial near-equlibrium dynamics. In consequence the more important issue is whether the physics that they embody is a good match to the natural system being modeled.

[34] *Joseph*: Mandelbrot's advocacy of fGn, a model with an infinite correlation length, as a model for the Joseph effect in hydrology led to a response that the spacetime structure that they implied for any time series would be *a priori* unphysical. For example, as argued by Vít Klemeš in a series of papers collected as Klemeš [2011], the Hurst phenomenon could either be due to the presence of long-range dependence in a stationary noise *or* of nonstationarity. Klemeš took the view that physical arguments implied that the latter position was more conservative, from an Occam's razor point of view. Such a position leaves open the question of whether fGn or similar models might be *operationally* better than short-range-dependent ones.

[35] *Noah*: Similarly, the infinite variance for the amplitude of jumps in the Lévy flight model became a key perceived barrier to their wider application. There are physical variables for which infinite variance has *not* been a priori controversial, because they are not bounded by an obvious conservation law, e.g., waiting times in cold trapped atoms. Infinite variance is already present even in standard Brownian motion, where the distribution of times taken to return for the first time to the starting point decay as $\sim t^{-3/2}$. A finite variance modification was provided by "truncated Lévy" flights [*Mantegna and Stanley* [1994] and initially applied to financial indices. LFSM also exhibits infinite variance jumps, and so Fractional Tempered Stable Motion (FTSM) has been introduced by *Houdre and Kawai* [2006] to combine LFSM with truncation.

[36] A second key perceived problem with Levy-type models was the possibility of infinite velocities. A modification of the Lévy flight idea to ensure finite velocity, known as the "Lévy walk" was proposed by *Shlesinger and Klafter* [1985]. Their approach was to make the length of a CTRW step an explicit function of its duration. Applications of Lévy walks have since included the modeling of diffusing particles in turbulent media, and (more controversially) animal foraging.

5.3. Avnir et al.: Are Fractals a Solution Looking for a Non-existent Problem ?

[37] A third issue has been the question of the range over which fractals really existed in nature, and how well scaling exponents are measurable. Space and time present different challenges here, because the frequency range over which a temporal fractal is seen can in principle always be increased by waiting longer, but a spatial fractal occupies a fixed area. An important paper was *Avnir et al*. [1998], and the exchange of letters (including Mandelbrot's) that resulted from it, which still repay reading.

6. But Do We Always Need Purely Fractal Models Anyway?

[38] A key point that is often lost in discussions like those cited above is that the presence of damping in a system, as for example via the η term in the Ohmic Langevin equation, immediately changes its character. Rather than a self-similar unbounded nonstationary walk, it becomes a mean reverting process at high frequencies. This can occur without a loss of asymptotic self-similarity, as for example seen in the Auto Regressive Fractionally Integrated Moving Average (ARFIMA) model [*Beran*, 1994] (also known as FARIMA). ARFIMA retains a singular behavior in its power spectrum at low frequencies. ARFIMA generalizes the more familiar autoregressive processes such as AR(1), to provide a universal model for subdiffusive dynamics in engineering and the basic sciences [*Burnecki et al*., 2012b] and can be seen as adding realism in the noise driver. When the basic ARFIMA model is extended to allow α-stable Lévy jumps, it becomes a universal model for fractional mean-reverting dynamics in general, e.g., its use by *Franzke et al*. [2012] to assess the behavior of diagnostics of H on such asymptotically self-similar time series, and by *Vyushin et al*. [2012] in contrast to AR(1) as a temperature time series model.

[39] In view of this, I will now re-emphasize the contrast between self-similar but nondissipative Wiener types of process and the dissipative Ornstein-Uhlenbeck class of processes.

[40] *Wiener-type processes* are models of the integrated displacement of free Brownian motion, in the Smoluchowski limit. They embody a mathematical result, the central limit theorem, about the limiting behavior of sums of iid finite variance random variables. They naturally connect time series modeling to idealized models of diffusion, because a 1D time series from a random walk is a projection of a diffusion in a multidimensional space. They are known in mathematics as Brownian motion, and ever since their invention by Bachelier has remained a natural paradigm in finance where dissipation is not necessarily a priori obviously present. The standard deviation of the displacement of an ensemble of their trajectories will grow even though its mean remains fixed. The Wiener process is thus *non*-stationary, but is self similar.

[41] *Ornstein-Uhlenbeck processes* in contrast cannot be completely self-similar, as the presence of an explicit dissipation timescale breaks this property. Conversely they can be stationary, and the effect of the dissipation is to make them mean-reverting. This can be an essential property for the modeling of a physical system on timescales comparable with the dissipation time. In finance mean reversion can be

motivated as a model of how a stock would recover the price supported by its fundamentals after an external "shock."

[42] The discussion of the Langevin equation above made the usual assumption that the noise is very short-range correlated and so the ACF can be approximated as a delta function giving the most familiar form (equation 3), with constant damping. However, many physical scenarios [*Haenggi and Jung* 1995] result in the tail of a heat bath's autocorrelation function being non-negligible in time, necessitating the use of the less approximate form of the Langevin equation, where the constant η is replaced by an integral over the damping kernel ρ:

$$m\ddot{\mathbf{x}} = -m\int_0^t dt' \rho(t-t')\dot{\mathbf{x}}(t') + \xi - \mathbf{V}'(\mathbf{x}) \tag{13}$$

[43] It must be stressed that the noise ξ, while Gaussian, is no longer white (a fluctuation dissipation theorem is still obeyed). A well-studied example is when a Langevin equation is driven by Ornstein-Uhlenbeck noise, which has an exponential acf [*Padilla et al.*, 2011; *Haenggi and Jung*, 1995]. The extreme case is where the heat bath's correlation function has the slowest possible, power law, decay [*Lutz*, 2001], so, for large τ:

$$\rho(\tau) \sim \tau^{-(1+2d)} \tag{14}$$

where we are taking the non-Brownian case, when d is not equal to zero.

[44] In this maximally long-range-dependent case the power law form of the correlations allows an alternative form for the integral in Equation (13). By introducing the Riemann-Liouville fractional derivative:

$$\frac{\partial^\lambda f(t)}{\partial t^\lambda} = \frac{1}{\Gamma(-\lambda)}\int_0^t d\tau f(\tau)(t-\tau)^{-(1+\lambda)} \tag{15}$$

the Langevin equation can be rewritten as

$$m\ddot{\mathbf{x}} = -m\rho_{(1+2d)}\frac{\partial^{2d}}{\partial t^{2d}}\dot{\mathbf{x}}(t') + \xi - \mathrm{V}'(\mathbf{x}) \tag{16}$$

[45] Note that the heat bath's memory parameter d is equivalent to $(\alpha - 1)/2$ in the notation of *Lutz* [2001]. It runs from $-1/2$ to $1/2$. *Lutz* [2001] should be consulted for further details including the definition of the constant $\rho_{(1+2d)}$ and a summary of a (quantum mechanical) microphysical derivation of (13) using a random matrix approach.

7. Multiplicative Models, Volatility Bunching, and Multifractals

[46] All the above sections have discussed additive stochastic models. This is true both for classic models like the Langevin equation, where white noise fluctuations are added via the ξ term, and for more recent ones like Mandelbrot's Lévy flight and fBm models which exemplified the Noah and Joseph effects. The additive nature of the Langevin equation, the near equilibrium and weak coupling assumptions made, and the Gaussian form of the noise used, are related, see, e.g., the introductory discussions in *Reif* [1965] or the more advanced [*Chorin and Hald*, 2009]. In the path-integral formulation, *Feynman and Hibbs* [2005] discuss how this picture arises "*...[as] a good approximation in a much wider class of situations, namely, where the effect itself is the result of a very large number of influences, each of which by itself has little effect ...*."

[47] However, additive models are not the only ones possible, and in stochastics the motivation for a multiplicative model has frequently been to capture the unknown effects of unresolved scale [c.f. *Majda et al.*, 1999]. It modifies the form of the Langevin equation to allow state-dependent noise or a noisy potential, e.g., by a form such as:

$$m\ddot{\mathbf{x}} = -\eta\dot{\mathbf{x}} + f(x)\xi \tag{17}$$

Of particular note are some linear multiplicative processes, including the Kesten process, which offer alternative models for heavy tailed time series [*Sardeshmukh and Sura*, 2009; *Sornette*, 2004]. By the early 1970s, Mandelbrot also came to question the applicability of additive models in some circumstances, and began to advocate multiplicative, *multi*fractal models, particularly for the study of fluid turbulence and financial markets [*Mandelbrot*, 1974, 1997, 1999]. *Chang and Wu* [2008] have noted that for intermittent turbulence, one may visualize the fluctuations to be composed of many types, each being characterized by a particular fractal dimension. Two questions arise: (i) What are the different types of fractal dimensions? (ii) How are they distributed in the turbulent medium? Recently, a new technique of analyzing intermittent fluctuations has been developed to specifically address these questions [*Chang and Wu*, 2008]. The technique, known as Rank-Ordered Multifractal Analysis (ROMA), retains the spirit of the traditional structure function analysis and combines it with the idea of one-parameter scaling of monofractals. ROMA maps the complete set of non-self-similar Probability Distribution Functions (PDF) and determines the fractal spectrum in terms of the concept of generalized crossover invariant functions, connecting the understanding of intermittent turbulence one-step closer to the concept of the dynamic renormalization group.

[48] The long flights in space captured by the Noah effect have the effect of linking points in the system together over longer distances than would be seen in a more traditional model. However, in their early work on SOC, *Bak and Chen* [1989] also made a strong case that models which separate spatial and temporal correlation may be artificial: "*Actually, for those (like us) who are brought up as condensed matter physicists it is hard to believe that long-range spatial and temporal correlations can exist independently. A local signal cannot be "robust" and remain coherent over long times in the presence of any amount of noise, unless stabilized by the interactions with its environment. And a large, coherent spatial structure cannot disappear (or be created) instantly. For an illustration, think of the temporal distribution of sunshine, which must be correlated with the spatial distribution of clouds, through the dynamics of meteorology*."

[49] The issues of mean reversion and multifractality noted above have arisen in the study of complexity in the Earth's magnetosphere. A much-studied example here is the Auroral Electrojet [AE] index [*Davis and Sugiura*, 1966], a proxy for energy dissipation in the system. Early work on complexity in AE used low-dimensional chaos as a paradigm, but the act of *AE* reveals an exponential decay on timescales of less than about 2 hours, and a much slower power law decay on longer timescales [*Takalo and Timonen*,

1994]. It has thus been a candidate for stochastic descriptions, including fBm [*Takalo and Timonen*, 1994] and self-organized criticality (see, for example, the reviews of Watkins [2002]; Chapman and Watkins [2001]; *Freeman and Watkins* [2002]). With hindsight, however, it seems clearer that stochastic modeling of the *AE* index is (or should be) broadly of the Langevin type because the intent is to model its evolution on timescales from hours down to minutes, on which energy is being dissipated. A recent development in modeling both the mean-reversion and volatility bunching noted above, in both ionospheric physics and finance, has come in a series of papers by Martin Rypdal and coworkers. They have used models of Ornstein–Uhlenbeck form with multifractal driving noises, for example fBm in multifractal time [*Rypdal and Rypdal*, 2010].

8. Implications of the Above Models for Diagnostics of, and Approaches to, Extreme Events

[50] Now that I have reviewed the above diverse collection of models and paradigms, one can compare them with the approaches used so far to the problems of extreme events. The approaches fall roughly into two main groups, which we will briefly note, and attempt to compare. Space prevents more than a brief sketch which we hope will inspire readers towards new research problems.

8.1. Threshold Exceedance Approaches

[51] The first group might be called extreme "Joseph problems," and are based around the periods during which a continuous time quantity exceeds a threshold. A paradigmatic problem here would be the number of hot days above a temperature threshold, a statistical definition of a "heat wave." This has an immediate link to weather derivatives [*Jewson et al*., 2005].

[52] The stochastic processes community have contributed the idea of sojourns [*Berman*, 1992], the times $L_t(u)$ spent above a threshold u by a stochastic process,

$$\int_0^t I_{[x(s)>u]} ds \tag{18}$$

and level sets, the areas defined by threshold crossings [*Azaïs and Wschebor*, 2009].

[53] From a more physical perspective, self-organized criticality has contributed the idea of a ($3+1$ dimensional) avalanche or "burst" in space time. This has a 4-dimensional surface, the space and timescaling exponents of which have been calculated for some SOC models by *Paczuski et al*. [1996]. The "burst problem" becomes more ambiguous, though, when only a time series is accessible, because one is trying to distinguish a $1+1$ dimensional cut through a candidate SOC model, from other possible stochastic models. *Watkins et al*. [2009a] gives further references on the burst problem and *Watkins et al*. [2012] derive burst scaling relations for such a null model and make a preliminary comparison to data. Inspired in part by SOC, and more generally by application of fractals and LRD to time series modeling, Bunde and his colleagues have produced a series of papers [e.g., *Bunde et al*., 2005] on level crossings in fractional noises.

8.2. Extensions of the Extreme Value approach

[54] By contrast, one can look at extreme "Noah problems," like the bunching of severe point like events, e.g., hurricanes. A key step has been the use of the idea of bunching of events into "clumps" with a characteristic size, to modify the main distributions of extreme value theory via an extremal index [*Coles*, 2001]. The Extremal index is effectively the inverse of the mean size of a clump, which poses the obvious problem that an LRD time series may not have a finite theoretical mean size of bunch.

9. Conclusions: How Can We Fairly Compare Models and Theories of Complex Systems?

[55] Even from the brief survey above, I hope two things will be clear to the reader. One is the evidence that stochastic models of much greater complexity than the familiar random walk will need to be considered in order to span the full range of extreme events of the earth system [*Sharma et al*., 2012; *Moffatt and Shuckburgh*, 2011]. The second is that there are typically several possible explanations for any scaling property of a complex system [*Sornette*, 2004], raising the obvious question of how to distinguish models in the presence of both large intrinsic variability in the data itself, and the bias and uncertainty of the inferred scaling exponents. A topical example from another area of complex systems research has been the debate around the Lévy foraging hypothesis (LFH) in ecology, where researchers had proposed that early indicative observations, evidence for optimal properties of Lévy flights, and the fractal properties of some environments, all motivated non-Brownian models for animal foraging [e.g., *Viswanathan et al*., 2011]. Experimental indications of heavy-tailed pdfs and long-range memory were taken to be direct evidence for the LFH, reminiscent to the way in which researchers (including the present author) had directly interpreted the ionospheric *AE* index within the SOC paradigm (Section 7). An increased awareness of statistical hypothesis testing as a tool for model comparison [e.g., *Buchanan*, 2008; *Clauset et al*., 2009; *Edwards et al*., 2007] has contributed to bringing a broader range of possible foraging models into the discussion, which do not all have exactly the same signatures. Rather than being specific to ecology, this issue of the interface between statistical inference, statistical mechanics and stochastic processes has been recognized for at least a decade as a frontier problem in other complex systems. *Mandelbrot et al*. [1997]) for example, wrote of their own Multifractal Model of Asset Returns (MMAR) that: "*the main disadvantage ... is the dearth of applicable statistical methods. We propose that new econometric methods are needed for models which are both time-invariant and scale-invariant*." This, and other challenges, exemplifies why the interface of complexity science and the more traditional mathematical disciplines continues to be so stimulating and fruitful.

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