CONNECTEDNESS PROPERTIES OF THE SET WHERE THE ITERATES OF AN ENTIRE FUNCTION ARE UNBOUNDED

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Abstract. We investigate the connectedness properties of the set $I^+(f)$ of points where the iterates of an entire function f are unbounded. In particular, we show that $I^+(f)$ is connected whenever iterates of the minimum modulus of f tend to ∞. For a general transcendental entire function f, we show that $I^+(f) \cup \{\infty\}$ is always connected and that, if $I^+(f)$ is disconnected, then it has uncountably many components, infinitely many of which are unbounded.

1. Introduction

Denote the nth iterate of an entire function f by f^n, for $n \in \mathbb{N}$. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that the family of functions $\{f^n : n \in \mathbb{N}\}$ is normal in some neighbourhood of z, and the *Julia set* $J(f)$ is the complement of $F(f)$. We refer to [4, 5, 8], for example, for an introduction to complex dynamics and the properties of these sets.

For any $z \in \mathbb{C}$, we call the sequence $(f^n(z))_{n\in\mathbb{N}}$ the *orbit* of z under f. This paper is concerned with the set of points whose orbits are unbounded, which we denote by

$$I^+(f) = \{z \in \mathbb{C} : (f^n(z))_{n\in\mathbb{N}} \text{ is unbounded}\}.$$

Clearly, $I^+(f)$ contains the escaping set,

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\},$$

and is the complement of $K(f)$, the set of points whose orbits are bounded. If f is a polynomial, then $K(f)$ is the *filled Julia set* of f, and it is well known that $I^+(f) = I(f)$. However, if f is transcendental, then $I^+(f) \setminus I(f)$ always meets $J(f)$ and may also meet $F(f)$; see [11] and references therein for the properties of $I^+(f) \setminus I(f)$.

For a general transcendental entire function, we show in Section 2 that $I^+(f)$ has many properties in common with $I(f)$. For example, we show that the properties of $I(f)$ proved by Eremenko in [7] also hold for $I^+(f)$, and we prove the following result, which parallels [17, Theorem 4.1].

Theorem 1.1. *Let f be a transcendental entire function. Then $I^+(f) \cup \{\infty\}$ is connected.*

In the paper [7], Eremenko remarked that it is plausible that $I(f)$ has no bounded components. This conjecture has stimulated much research in transcendental dynamics and remains open, though there have been several partial results – see for example [12, 14, 19]. One of the strongest partial results for a general

2010 *Mathematics Subject Classification.* Primary 37F10, Secondary 30D05.

The last two authors are supported by the EPSRC grant EP/K031163/1.

transcendental entire function [14, Theorem 1] was obtained by considering the fast escaping set $A(f)$, a subset of $I(f)$ defined in terms of the iterated maximum modulus function; see Section 4 for a definition. By contrast, in this paper we show that the connectedness properties of the *superset* $I^+(f)$ of $I(f)$ are related to a completely new condition involving the iterated *minimum* modulus function.

We prove the following result, in which

$$m(r) = m(r, f) := \min\{|f(z)| : |z| = r\},$$

and $m^n(r)$ denotes the nth iterate of the function $r \mapsto m(r)$.

Theorem 1.2. *Let f be a transcendental entire function for which*

$$\text{there exists } r > 0 \text{ such that } m^n(r) \to \infty \text{ as } n \to \infty. \tag{1.1}$$

Then $I^+(f)$ is connected.

In fact, we show that $I^+(f)$ is connected for a more general class of functions than is covered by Theorem 1.2. Details are given in Section 3.

It is natural to ask which transcendental entire functions satisfy the condition (1.1). Clearly, there are many that do not – for example, any function bounded on a path to ∞. However, there are also functions that do satisfy the condition. In forthcoming work, we consider the consequences of condition (1.1) for other sets related to $I(f)$ and $I^+(f)$, and show that there are many classes of functions for which condition (1.1) holds. In particular, we show that this is the case for all entire functions of order less than $1/2$, so $I^+(f)$ is connected for such functions. It is an interesting question whether condition (1.1) is also sufficient to ensure that $I(f)$ is connected.

Note that there are transcendental entire functions for which $I^+(f)$ is disconnected. For example, if $f(z) = \sin z$, then f maps the real line $\mathbb{R}$ onto the interval $[-1, 1]$, so $\mathbb{R}$ is a closed, connected set in $K(f)$ that disconnects $I^+(f)$.

In Section 4, we prove a number of results on the components of $I^+(f)$ for a general transcendental entire function, including the following.

Theorem 1.3. *Let f be a transcendental entire function such that $I^+(f)$ is disconnected. Then $I^+(f)$ has uncountably many components, infinitely many of which are unbounded.*

The paper is organised as follows. In Section 2, we prove Theorem 1.1 and some basic properties of $I^+(f)$, and in Section 3 we give the proof of Theorem 1.2 and related results. In Section 4, we prove Theorem 1.3 and a number of other results on the components of $I^+(f)$. Finally, in Section 5, we give some examples related to the hypotheses of Theorem 1.2 and its generalisation in Section 3.

2. Basic properties of $I^+(f)$

In this section we prove a number of basic properties of $I^+(f)$ and discuss the interaction of $I^+(f)$ with the Fatou set and the Julia set. We note first that, for a transcendental entire function f, it follows immediately from the corresponding properties of $K(f)$ that $I^+(f)$ is completely invariant and that $I^+(f^n) = I^+(f)$, for $n \in \mathbb{N}$.

As usual, we refer to components of the Fatou set as *Fatou components*. If U is a Fatou component of f then, for every $n \in \mathbb{N}$, $f^n(U) \subset U_n$ for some Fatou component U_n. A simple normality argument shows that, if $U \cap I^+(f) \neq \emptyset$, then $U \subset I^+(f)$.

The following properties of $I(f)$ were proved by Eremenko [7]:

$$I(f) \neq \emptyset, \quad I(f) \cap J(f) \neq \emptyset, \quad J(f) = \partial I(f),$$

and $\overline{I(f)}$ has no bounded components.

The proofs that these properties also hold for $I^+(f)$ are similar to those for $I(f)$ so we give only brief details.

Theorem 2.1. *Let f be a transcendental entire function. Then*

$$I^+(f) \neq \emptyset, \quad I^+(f) \cap J(f) \neq \emptyset, \quad J(f) = \overline{I^+(f) \cap J(f)}, \quad J(f) = \partial I^+(f), \tag{2.1}$$

and $\overline{I^+(f)}$ has no bounded components.

Remark. In view of the considerable interest in Eremenko's conjecture, mentioned in Section 1, it is natural to ask whether all the components of $I^+(f)$ are unbounded.

Proof of Theorem 2.1. The first two properties in (2.1) follow immediately from the corresponding properties of $I(f)$, and the fact that $I(f) \subset I^+(f)$. The third property follows from the second by the blowing up property of $J(f)$; see Lemma 4.2.

It follows from the third property in (2.1) that $J(f) \subset \overline{I^+(f)}$. Now since the repelling periodic points of f are dense in $J(f)$ (see [1, Theorem 1]), any open set $G \subset I^+(f)$ satisfies $G \subset F(f)$. Hence $J(f) \subset \partial I^+(f)$. On the other hand, no point of $\partial I^+(f)$ can lie in $F(f)$, since any such point would have a neighbourhood in $I^+(f)$. We conclude that $J(f) = \partial I^+(f)$.

Finally, if $\overline{I^+(f)}$ has a bounded component, E say, then there is an open topological annulus A that surrounds E and lies in the complement of $\overline{I^+(f)}$. Since $\overline{I^+(f)}$ is completely invariant under f, we deduce by Montel's theorem that A lies in a component of $F(f)$, and this component must be multiply connected since $J(f) = \partial I^+(f)$. But any multiply connected Fatou component of f is contained in $I(f)$ (see [2, Theorem 3.1]) and hence in $I^+(f)$, so we obtain a contradiction. This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let f be a transcendental entire function. Then $I^+(f)$ is neither open nor closed.*

Proof. If $I^+(f)$ is open, then this implies that $I^+(f) \subset F(f)$, which is a contradiction since $I^+(f) \cap J(f) \neq \emptyset$. If $I^+(f)$ is closed, then since $J(f) = \partial I^+(f)$ we have $J(f) \subset I^+(f)$, which is again a contradiction. □

Next, we consider Fatou components in $I^+(f)$ and their boundaries. A Fatou component U in $I^+(f)$ must be a Baker domain, a preimage of a Baker domain or a wandering domain. The definitions are as follows, where as before U_n denotes the Fatou component containing $f^n(U)$, for $n \in \mathbb{N}$:

- if $U = U_p$ for some $p \in \mathbb{N}$, so U is periodic with period p, then U is a *Baker domain* and has the property that $f^{np}(z) \to \infty$ as $n \to \infty$ for all $z \in U$;
- if U is not eventually periodic, that is, $U_m \neq U_n$ whenever $m \neq n$, then U is a *wandering domain.*

We refer to [5], for example, for further information on the classification of Fatou components.

We now state a number of results on the boundaries of the possible types of Fatou components in $I^+(f)$, and prove a simple consequence of these results that we use later in the paper.

Clearly, Baker domains and their preimages lie in $I(f)$. A Baker domain U of period p is said to be *univalent* if f^p is univalent in U. Our first lemma is a simple corollary of [18, Theorem 1.1].

Lemma 2.3. *Let f be a transcendental entire function and let U be a univalent Baker domain of f. Then $\partial U \cap I(f) \neq \emptyset$.*

It is an interesting open question whether the conclusion of Lemma 2.3 applies for *any* Baker domain U of a transcendental entire function. The following property of the boundary of a non-univalent Baker domain was proved by Baker and Domínguez [3, Corollary 1.3].

Lemma 2.4. *Let f be a transcendental entire function and let U be a Baker domain of f such that f is not univalent in U. Then ∂U is disconnected.*

The next lemma follows from a general result on the boundaries of wandering domains [11, Theorem 1.5].

Lemma 2.5. *Let f be a transcendental entire function and let U be a wandering domain of f such that $U \subset I^+(f)$. Then $\partial U \cap I^+(f) \neq \emptyset$.*

We now prove the following consequence of Lemmas 2.3, 2.4 and 2.5.

Lemma 2.6. *Let f be a transcendental entire function. Then every component of $I^+(f)$ that is neither a non-univalent Baker domain nor a preimage of such a domain must meet $J(f)$. In particular, every component of $I^+(f)$ with connected boundary meets $J(f)$.*

Proof. Suppose that f is a transcendental entire function, and that some component C of $I^+(f)$ does not meet $J(f)$ and is neither a non-univalent Baker domain nor a preimage of such a domain. Then $C \subset U$ for some Fatou component $U \subset I^+(f)$, and indeed $C = U$ since C is a component of $I^+(f)$. Since C is not a non-univalent Baker domain or a preimage of such a domain, either some iterate of f maps C to a univalent Baker domain, or C is wandering domain.

Using the fact that f maps any boundary point of a Fatou component to a boundary point of a Fatou component, we deduce that

- if C is mapped to a univalent Baker domain, then $\partial C \cap I^+(f) \neq \emptyset$ by Lemma 2.3, and
- if C is a wandering domain, then $\partial C \cap I^+(f) \neq \emptyset$ by Lemma 2.5.

In either case we have a contradiction, since $\partial C \subset J(f)$ and $C \cup \{\zeta\}$ is connected for any $\zeta \in \partial C$.

The final statement of the lemma follows from the fact that non-univalent Baker domains and their preimages have disconnected boundaries, by Lemma 2.4. □

Remark. If we were able to show that $\partial U \cap I(f) \neq \emptyset$ for *any* Baker domain U of a transcendental entire function f, then it would follow that every component of $I^+(f)$ meets $J(f)$.

We now give the proof of Theorem 1.1. In fact, we prove the following, which includes a useful equivalent result.

Theorem 2.7. *If f is a transcendental entire function, then the following statements hold and are equivalent.*

(a) If G is a bounded, simply connected domain such that $G \cap I^+(f) \neq \emptyset$, then $\partial G \cap I^+(f) \neq \emptyset$.
(b) $I^+(f) \cup \{\infty\}$ is connected.

Note that if $E \cup \{\infty\}$ is connected, where E is a subset of $\mathbb{C}$, then it does not follow that the components of E are all unbounded, unless E is closed.

The proof of Theorem 2.7 is similar to that of the corresponding result for $I(f)$ given in [17, Theorem 4.1]. In particular, we use the following lemma.

Lemma 2.8. [17, Lemma 4.1] *Let f be a transcendental entire function. If G is a bounded, simply connected domain such that $G \cap J(f) \neq \emptyset$, then $\partial G \cap I(f) \neq \emptyset$.*

Proof of Theorem 2.7. We first prove (a), and then show that this implies (b). Since it is clear that (b) implies (a), this will prove the theorem.

Let G be a bounded, simply connected domain that meets $I^+(f)$. If $G \cap J(f) \neq \emptyset$, then $\partial G \cap I^+(f) \neq \emptyset$ by Lemma 2.8. Thus we may assume that $G \subset U$ for some Fatou component $U \subset I^+(f)$, and indeed that $G = U$, because otherwise we again have $\partial G \cap I^+(f) \neq \emptyset$. Since Baker domains and their preimages are unbounded, G must be a wandering domain, and it follows from Lemma 2.5 that we again have $\partial G \cap I^+(f) \neq \emptyset$. This proves statement (a).

To show that (a) implies (b), suppose that $I^+(f) \cup \{\infty\}$ is not connected. Then there exist disjoint open sets $G_1, G_2 \subset \widehat{\mathbb{C}}$ such that

$$I^+(f) \cup \{\infty\} \subset G_1 \cup G_2 \tag{2.2}$$

and

$$G_i \cap (I^+(f) \cup \{\infty\}) \neq \emptyset, \quad \text{for } i = 1, 2.$$

We can assume that G_1 is bounded and simply connected, and that $\infty \in G_2$. Since G_1 meets $I^+(f)$, it follows from (a) that $\partial G_1 \cap I^+(f) \neq \emptyset$, which contradicts (2.2). Thus $I^+(f) \cup \{\infty\}$ is connected, as required. □

3. Proof of Theorem 1.2

Theorem 1.2 states that, if f is a transcendental entire function and there exists $r > 0$ such that $m^n(r) \to \infty$, then $I^+(f)$ is connected. In this section, we prove the following more general result, which shows that $I^+(f)$ is connected for an even wider class of functions.

We use the notation $\mathbb{N}_0$ for the set of non-negative integers $\mathbb{N} \cup \{0\}$, and we say that a set $A \subset \mathbb{C}$ *surrounds* a set $B \subset \mathbb{C}$ if B lies in a bounded component of the complement of A.

Theorem 3.1. *Suppose that f is a transcendental entire function and there exists a sequence of bounded, simply connected domains $(D_n)_{n\in\mathbb{N}_0}$ such that*

$$f(\partial D_n) \text{ surrounds } D_{n+1}, \text{ for } n \in \mathbb{N}_0, \tag{3.1}$$

and

$$\text{every disc centred at } 0 \text{ is contained in } D_n \text{ for sufficiently large } n. \tag{3.2}$$

Then $I^+(f)$ is connected.

Before proving this result we make some remarks.

(1) First we point out that Theorem 1.2 follows from Theorem 3.1. Suppose that, for a transcendental entire function f, there exists $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$. If we define

$$D'_n = \{z \in \mathbb{C} : |z| < m^n(r)\}, \text{ for } n \in \mathbb{N}_0,$$

then it is clear that every disc centred at 0 is contained in D'_n for sufficiently large n. Moreover, it follows from the definition of the minimum modulus function that $f(\partial D'_n)$ lies in the complement of D'_{n+1} for all $n \in \mathbb{N}_0$, and indeed if $N \in \mathbb{N}_0$ is such that D'_N contains the full orbit of some periodic point of f, then we have

$$f(\partial D'_n) \text{ surrounds } D'_{n+1}, \text{ for } n \geq N.$$

Putting

$$D_n = D'_{n+N}, \quad \text{for } n \geq 0,$$

we see that the conditions of Theorem 3.1 are satisfied for the sequence of domains $(D_n)_{n\in\mathbb{N}_0}$. Thus Theorem 1.2 follows from Theorem 3.1. There are, however, transcendental entire functions that meet the conditions of Theorem 3.1 but not those of Theorem 1.2; see Example 5.1.

(2) Many of the functions that meet the conditions of Theorem 3.1 are *strongly polynomial-like*, in the sense defined in [10], and so they have the nice properties of such functions proved in that paper. Strongly polynomial-like functions can be characterised [10, Theorem 1.6] as those transcendental entire functions for which there exists a sequence of bounded, simply connected domains $(D'_n)_{n\in\mathbb{N}_0}$ such that

(i) $f(\partial D'_n)$ surrounds $\overline{D'_n}$, for $n \in \mathbb{N}_0$,
(ii) $\bigcup_{n\in\mathbb{N}_0} D'_n = \mathbb{C}$, and
(iii) $\overline{D'_n} \subset D'_{n+1}$, for $n \in \mathbb{N}_0$.

It is easy to see that f is strongly polynomial-like if it satisfies the conditions of Theorem 1.2, and also if it satisfies the conditions of Theorem 3.1 together with a condition such as '$\overline{D}_n \subset D_{n+1}$, for arbitrarily large values of n'.

Note that not all strongly polynomial-like functions meet the conditions of Theorem 3.1 – indeed there are strongly polynomial-like functions for which $I^+(f)$ is disconnected; see Example 5.2.

The following lemma contains the key induction step in the proof of Theorem 3.1.

Lemma 3.2. *Suppose that f is a transcendental entire function and there exists a sequence of bounded, simply connected domains $(D_n)_{n\in\mathbb{N}_0}$ such that (3.1) and (3.2) hold. Suppose also that, for some $j \in \mathbb{N}_0$, there exists $n_j \in \mathbb{N}_0$ and a continuum Γ_{n_j} with the following properties:*

(i) $\Gamma_{n_j} \subset K(f) \cap (\mathbb{C} \setminus D_{n_j})$;
(ii) there is a point $z_{n_j} \in \Gamma_{n_j} \cap \partial D_{n_j}$;
(iii) there is a point $z'_{n_j} \in \Gamma_{n_j}$ such that $f^n(z'_{n_j}) \in D_{n_j+n}$ for all $n \in \mathbb{N}$.

Then there exists $n_{j+1} > n_j$ and a continuum $\Gamma_{n_{j+1}} \subset f^{n_{j+1}-n_j}(\Gamma_{n_j})$ such that properties (i), (ii) and (iii) hold with n_j replaced by n_{j+1} throughout.

The proof of Lemma 3.2 depends on the following result from plane topology.

Lemma 3.3. [9, page 84] *If E_0 is a continuum in $\hat{\mathbb{C}}$, E_1 is a closed subset of E_0 and C is a component of $E_0 \setminus E_1$, then $\overline{C}$ meets E_1.*

Proof of Lemma 3.2. Since $z_{n_j} \in \Gamma_{n_j} \subset K(f)$ and the domains (D_n) satisfy (3.2), there exists $N \in \mathbb{N}_0$ such that

$$f^n(z_{n_j}) \in D_{n_j+n}, \quad \text{for } n > N. \tag{3.3}$$

By property (ii) and (3.1),

$$f(z_{n_j}) \in \mathbb{C} \setminus D_{n_j+1}, \tag{3.4}$$

so the minimal integer N such that (3.3) holds is at least 1. Define $n_{j+1} = n_j + N$, where N is this minimal integer. Then, by (3.3) and the minimality of N,

$$f^n(z_{n_j}) \in D_{n_j+n}, \quad \text{for } n > n_{j+1} - n_j, \tag{3.5}$$

and

$$f^{n_{j+1}-n_j}(z_{n_j}) \in \mathbb{C} \setminus D_{n_{j+1}}.$$

Moreover, $f^{n_{j+1}-n_j}(z_{n_j}) \notin \partial D_{n_{j+1}}$, by (3.1) and (3.5), so

$$f^{n_{j+1}-n_j}(z_{n_j}) \in \mathbb{C} \setminus \overline{D}_{n_{j+1}}. \tag{3.6}$$

Also, by property (iii),

$$f^{n_{j+1}-n_j}(z'_{n_j}) \in D_{n_{j+1}}. \tag{3.7}$$

It follows from (3.6) and (3.7) that the continuum $f^{n_{j+1}-n_j}(\Gamma_{n_j})$ includes points from both $D_{n_{j+1}}$ and $\mathbb{C} \setminus \overline{D}_{n_{j+1}}$ (see Figure 1).

Now let $\Gamma_{n_{j+1}}$ be the component of the closed set

$$f^{n_{j+1}-n_j}(\Gamma_{n_j}) \cap (\mathbb{C} \setminus D_{n_{j+1}})$$

that contains the point

$$z'_{n_{j+1}} := f^{n_{j+1}-n_j}(z_{n_j}).$$

Then we deduce that $\Gamma_{n_{j+1}}$ meets $\partial D_{n_{j+1}}$ by applying Lemma 3.3 with

$$E_0 = f^{n_{j+1}-n_j}(\Gamma_{n_j}) \cap (\mathbb{C} \setminus D_{n_{j+1}}) \quad \text{and} \quad E_1 = E_0 \cap \partial D_{n_{j+1}}.$$

Thus there exists $z_{n_{j+1}} \in \Gamma_{n_{j+1}} \cap \partial D_{n_{j+1}}$. Therefore, properties (i) and (ii) hold with n_j replaced by n_{j+1}, and property (iii) also holds, since

$$f^n(z'_{n_{j+1}}) = f^{n+n_{j+1}-n_j}(z_{n_j}) \in D_{n_{j+1}+n}, \quad \text{for } n \in \mathbb{N},$$

by (3.5). $\square$

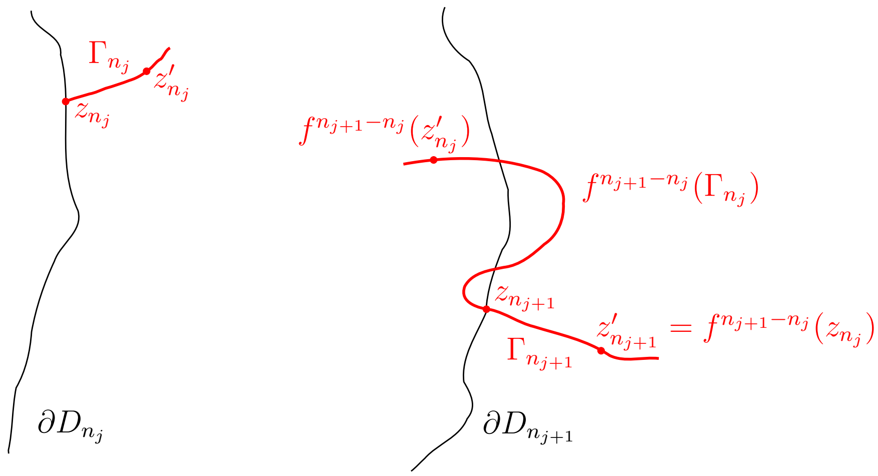

FIGURE 1. Proof of Lemma 3.2.

Remark. Note that property (i) in Lemma 3.2 could be weakened to the property

$$\Gamma_{n_j} \subset \{z : f^n(z) \in D_{n_j+n} \text{ for } n > N = N(z)\} \cap (\mathbb{C} \setminus D_{n_j}),$$

since the only place in the proof where we use the fact that $\Gamma_{n_j} \subset K(f)$ is to deduce (3.3) and it is clear that, if Γ_{n_j} satisfies this weaker property, then any point $z \in \Gamma_{n_{j+1}} \subset f^{n_{j+1}-n_j}(\Gamma_{n_j})$ must satisfy property (iii) with n_j replaced by n_{j+1}.

Next, we state two further topological lemmas that are needed for the proof of Theorem 3.1. The first is a useful characterisation of a disconnected subset of the plane.

Lemma 3.4. [13, Lemma 3.1] *A subset S of $\mathbb{C}$ is disconnected if and only if there exists a closed, connected set $E \subset \mathbb{C}$ such that $S \cap E = \emptyset$ and at least two different components of E^c intersect S.*

We also need the following generalisation of [15, Lemma 1], given in [20]. This result will be used again later in the paper.

Lemma 3.5. *Let $(E_j)_{j\in\mathbb{N}_0}$ be a sequence of compact sets in $\mathbb{C}$, $(m_j)_{j\in\mathbb{N}_0}$ be a sequence of positive integers and f be a transcendental entire function such that $E_{j+1} \subset f^{m_j}(E_j)$, for $j \in \mathbb{N}_0$. Set $p_k = \sum_{j=0}^{k} m_j$, for $k \in \mathbb{N}_0$. Then there exists $\zeta \in E_0$ such that*

$$f^{p_k}(\zeta) \in E_{k+1}, \quad \text{for } k \in \mathbb{N}_0.$$

We now give the proof of Theorem 3.1.

Proof of Theorem 3.1. Suppose that $I^+(f)$ is disconnected. Then, by Lemma 3.4, there is a closed, connected set $E \subset K(f)$ such that two distinct components of E^c, say G_1 and G_2, each meet $I^+(f)$. Evidently the boundaries of G_1 and G_2 are connected and are contained in $K(f)$. By Theorem 2.7(a), we deduce that G_1 and G_2 are both unbounded, so ∂G_1 and ∂G_2 are unbounded, as are their images under the iterates of f, by (3.1) and (3.2).

We now show that there exists $n_0 \in \mathbb{N}$ and a continuum Γ_{n_0} that satisfies properties (i), (ii) and (iii) in Lemma 3.2, with $j = 0$.

Without loss of generality we can assume, by (3.2), that for some point $\alpha \in \partial G_1$ every domain D_n, $n \in \mathbb{N}_0$, contains the entire orbit of α. By Lemma 3.3, there exists a continuum $\Gamma \subset \partial G_1$ such that $\alpha \in \Gamma$ and $\Gamma \cap \partial D_0 \neq \emptyset$.

Now take $z_0 \in \Gamma \cap \partial D_0$ and choose $N \in \mathbb{N}$ such that $f^n(z_0) \in D_N$, for $n \in \mathbb{N}_0$. Since $f(z_0) \in \mathbb{C} \setminus D_1$, by (3.1), it follows that the maximal value n_0 of $n \in \mathbb{N}$ such that $f^n(z_0) \in \mathbb{C} \setminus D_n$ satisfies $1 \leq n_0 < N$. Note that $z'_{n_0} := f^{n_0}(z_0)$ lies outside $\overline{D}_{n_0}$, so $f^{n_0}(\Gamma) \cap \partial D_{n_0} \neq \emptyset$.

Now let Γ_{n_0} be the component of $f^{n_0}(\Gamma) \setminus D_{n_0}$ that contains z'_{n_0}. Then Γ_{n_0} meets ∂D_{n_0}, by Lemma 3.3 again. It follows that the continuum Γ_{n_0} satisfies

(i) $\Gamma_{n_0} \subset K(f) \cap (\mathbb{C} \setminus D_{n_0})$;
(ii) there is a point $z_{n_0} \in \Gamma_{n_0} \cap \partial D_{n_0}$;
(iii) there is a point $z'_{n_0} \in \Gamma_{n_0}$ such that $f^n(z'_{n_0}) \in D_{n_0+n}$ for all $n \in \mathbb{N}$.

Thus by Lemma 3.2 there is a strictly increasing sequence $(n_j)_{j\in\mathbb{N}_0}$ and a sequence of continua $(\Gamma_{n_j})_{j\in\mathbb{N}_0}$ such that, for each $j \in \mathbb{N}_0$,

(i) $\Gamma_{n_j} \subset K(f) \cap (\mathbb{C} \setminus D_{n_j})$;
(ii) there is a point $z_{n_j} \in \Gamma_{n_j} \cap \partial D_{n_j}$;
(iii) there is a point $z'_{n_j} \in \Gamma_{n_j}$ such that $f^n(z'_{n_j}) \in D_{n_j+n}$ for all $n \in \mathbb{N}$;
(iv) $f^{n_{j+1}-n_j}(\Gamma_{n_j}) \supset \Gamma_{n_{j+1}}$.

We now apply Lemma 3.5 with

$$E_j = \Gamma_{n_j} \quad \text{and} \quad m_j = n_{j+1} - n_j, \quad \text{for } j \in \mathbb{N}_0.$$

Then, by property (iv),

$$f^{m_j}(E_j) \supset E_{j+1}, \quad \text{for } j \in \mathbb{N}_0.$$

We deduce that there exists $\zeta \in E_0 = \Gamma_{n_0}$ such that

$$f^{p_k}(\zeta) \in E_{k+1}, \quad \text{for } k \in \mathbb{N}_0,$$

where $p_k = m_0 + \cdots + m_k = n_{k+1} - n_0$; that is,

$$f^{n_{k+1}-n_0}(\zeta) \in \Gamma_{n_{k+1}}, \quad \text{for } k \in \mathbb{N}_0.$$

Thus, by (3.2) and property (i) of the sequence of continua (Γ_{n_j}),

$$f^{n_{k+1}-n_0}(\zeta) \to \infty \text{ as } k \to \infty,$$

so $\zeta \in I^+(f)$, which contradicts the fact that $\zeta \in \Gamma_{n_0} \subset K(f)$. This completes the proof of Theorem 3.1. □

Remark. This proof shows that, under the hypotheses of Theorem 3.1, if K is any closed connected set in $K(f)$ and $\alpha \in K$, then there is a positive constant $C(K, \alpha)$ such that $K \subset \{z : |z| \leq C(K, \alpha)\}$. It follows that, under the hypotheses of Theorem 3.1, any Fatou component of f contained in $K(f)$ is bounded.

If f is strongly polynomial-like, then this conclusion about Fatou components is already known and moreover such Fatou components cannot be wandering domains [10, Theorem 1.4]. However, under this hypothesis $K(f)$ may contain an unbounded closed connected set; see Example 5.2.

4. Components of $I^+(f)$

In this section we prove Theorem 1.3 and a number of other results about the components of $I^+(f)$.

We begin by proving the following, which parallels the result for $I(f)$ in [18, Theorem 1.2]. Several of the results in this section follow from this result with appropriate choices of the set E.

Theorem 4.1. *Let f be a transcendental entire function and let E be a set such that $E \subset I^+(f)$ and $J(f) \subset \overline{E}$. Then either $I^+(f)$ is connected or it has infinitely many components that meet E.*

It follows immediately from Theorem 4.1, by taking $E = I^+(f)$ and using (2.1), that either $I^+(f)$ is connected or it has infinitely many components. However, Theorem 1.3 is considerably stronger than this statement.

The proof of Theorem 4.1 uses the well known *blowing up property* of the Julia set, stated as the next lemma. Here $E(f)$ is the *exceptional set* of f, that is, the set of points with a finite backwards orbit under f (which for a transcendental entire function contains at most one point).

Lemma 4.2. *Let f be an entire function, let K be a compact set such that $K \subset \mathbb{C} \setminus E(f)$ and let G be an open neighbourhood of $z \in J(f)$. Then there exists $N \in \mathbb{N}$ such that $f^n(G) \supset K$, for all $n \geq N$.*

The proof of Theorem 4.1 also uses the following result. This was proved in the special case that $F = I(f)$ in [17, Theorem 5.1(a)].

Lemma 4.3. *Let f be a transcendental entire function, and let E and F be sets such that $E \subset F$, F is backwards invariant, and $J(f) \subset \overline{E}$. If E meets only finitely many components of F, then $F \cap J(f)$ lies in a single component of F.*

Proof. Suppose that E is contained in the union of finitely many components of F, say $F_1, F_2, \ldots, F_m$. Take any $z \in F \cap J(f)$. Since $J(f) \subset \overline{E}$, there exist $z_n \in E$ such that $z_n \to z$ as $n \to \infty$. Without loss of generality all terms of this sequence (z_n) lie in a single component, F_j say. Since $z \in F$, we have $z \in F_j$. Hence

$$F \cap J(f) \subset F_1 \cup F_2 \cup \cdots \cup F_m. \tag{4.1}$$

We now assume that $F_1, F_2, \ldots, F_m$ is the minimal set of components of F such that (4.1) holds. Then $F_j \cap J(f) \neq \emptyset$, for $j = 1, 2, \ldots, m$. Note that if the exceptional set $E(f)$ is non-empty, then

$$(F_j \setminus E(f)) \cap J(f) \neq \emptyset, \quad \text{for } j = 1, 2, \ldots, m. \tag{4.2}$$

Indeed, if $E(f) = \{\alpha\} \subset F_j \cap J(f)$, then it follows from Lemma 4.2 that α is a limit point of the backwards orbit of any non-exceptional point in $F \cap J(f)$ and hence α is the limit of a sequence in $F_i \cap J(f)$, say, by (4.1). Thus $i = j$ and so (4.2) holds.

If $m = 1$, then $F \cap J(f)$ is contained in one component of F, as required. If $m > 1$, then we can take $z_1 \in F_1 \cap J(f)$ and an open disc D centred at z_1 so small that

$$D \cap (F_2 \cup \cdots \cup F_m) = \emptyset. \tag{4.3}$$

Consider F_j, $j \geq 2$. Then there exists $N \in \mathbb{N}$ such that $f^N(D)$ meets both $F_1 \cap J(f)$ and $F_j \cap J(f)$, by (4.2) and Lemma 4.2. Hence there exist $w_1, w_j \in D$ such that

$$f^N(w_1) \in F_1 \cap J(f) \quad \text{and} \quad f^N(w_j) \in F_j \cap J(f),$$

so $w_1, w_j \in F_1$ by the backwards invariance of $F \cap J(f)$ and (4.3). Thus $f^N(F_1)$ is a connected subset of F that meets both F_1 and F_j, which is a contradiction. This completes the proof. □

We now deduce Theorem 4.1 from Lemma 4.3.

Proof of Theorem 4.1. Let E be a set such that $E \subset I^+(f)$ and $J(f) \subset \overline{E}$. Suppose that E meets only finitely many components of $I^+(f)$. Then, since $I^+(f)$ is backwards invariant, it follows from Lemma 4.3 with $F = I^+(f)$ that $I^+(f) \cap J(f)$ lies in a single component of $I^+(f)$. We shall deduce from this that $I^+(f)$ is connected.

It follows from Lemma 2.6 that, if there is a component of $I^+(f)$ that does not meet $J(f)$, then it must be a Baker domain with a disconnected boundary, or a preimage of such a Baker domain. Suppose then that U is such a component. Then U has more than one complementary component, each of which is closed and unbounded, and meets $J(f)$.

All the points of $I^+(f) \cap J(f)$ lie in these complementary components of U, and $I^+(f) \cap J(f)$ cannot be contained in a single complementary component of U because $J(f) = \overline{I^+(f) \cap J(f)}$, by Theorem 2.1. Hence the component C_1 of $I^+(f)$ that contains $I^+(f) \cap J(f)$ must meet at least two complementary components of U and so it must meet the boundaries of these two complementary components, which are subsets of ∂U. This contradicts the fact that $\partial U \cap I^+(f) = \emptyset$. Hence such a component U of $I^+(f)$ cannot exist. Thus any component of $I^+(f)$ must meet $J(f)$ and hence must lie in C_1; that is, $I^+(f)$ is connected. This completes the proof. □

We now show that several connectedness properties of $I^+(f)$ follow easily from Theorem 4.1. First, noting Eremenko's result [7] that $J(f) = \partial I(f)$, we apply Theorem 4.1 with $E = I(f)$ to give the following.

Corollary 4.4. *Let f be a transcendental entire function. If $I(f)$ is connected, then $I^+(f)$ is connected.*

Next, we give conditions for $I^+(f)$ and $I^+(f) \cap J(f)$ to be spiders' webs. We say that a connected set E is a *spider's web* if there exists a sequence $(G_n)_{n \in \mathbb{N}}$ of bounded, simply connected domains such that

$$G_n \subset G_{n+1} \text{ and } \partial G_n \subset E, \text{ for each } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.$$

Clearly, any connected set that contains a spider's web is itself a spider's web, so it follows from Corollary 4.4 that if $I(f)$ is a spider's web, then $I^+(f)$ is a spider's web. In fact, we prove the following more general result.

Corollary 4.5. *Let f be a transcendental entire function.*

(a) If $I^+(f)$ contains a spider's web, then $I^+(f)$ is a spider's web.
(b) If $I^+(f) \cap J(f)$ contains a spider's web, then $I^+(f) \cap J(f)$ is a spider's web.

We prove Corollary 4.5 by using various properties of the subset of $I(f)$ known as the *fast escaping set* $A(f)$. This set, introduced in [6] and studied in detail in [14, 16], may be defined as follows:

$$A(f) = \{z \in \mathbb{C} : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\},$$

where

$$M(r, f) = \max_{|z|=r} |f(z)|,$$

$R > 0$ is such that $M(r, f) > r$ for $r \geq R$, and $M^n(r, f)$ denotes the nth iterate of the function $r \mapsto M(r, f)$. In particular, we use the facts that $J(f) = \partial A(f)$, that all components of $A(f)$ are unbounded, and that all components of $A(f) \cap J(f)$ are unbounded whenever f has no multiply connected Fatou components. For proofs of these properties we refer to [16], for example.

Proof of Corollary 4.5. To prove part (a), suppose that $I^+(f)$ contains a spider's web and let C_1 be the component of $I^+(f)$ containing the spider's web. Then, since all the components of $A(f)$ are unbounded, we have $A(f) \subset C_1$. Also, since $J(f) = \partial A(f)$, we can apply Theorem 4.1 with $E = A(f)$. It follows that $I^+(f)$ is connected and hence that $I^+(f)$ is a spider's web.

The proof of part (b) is similar. If $I^+(f) \cap J(f)$ contains a spider's web, then f has no multiply connected Fatou components by [2, Theorem 3.1]. Hence every component of $A(f) \cap J(f)$ is unbounded, so if C_2 is the component of $I^+(f) \cap J(f)$ that contains the spider's web, we have $A(f) \cap J(f) \subset C_2$. The result then follows by applying Theorem 4.1 with $E = A(f) \cap J(f)$. □

Finally in this section, we prove Theorem 1.3. In fact, we prove the following slightly stronger result.

Theorem 4.6. *Let f be a transcendental entire function such that $I^+(f)$ is disconnected. Then:*

(a) $I^+(f)$ has infinitely many unbounded components;
(b) every neighbourhood of a point in $J(f)$ meets uncountably many components of $I^+(f)$.

The proof of Theorem 4.6(b) is closely related to other proofs of this type (for example, [10, Theorem 1.3]), but we give it in full for the convenience of the reader.

Proof of Theorem 4.6. For part (a), we again apply Theorem 4.1 with $E = A(f)$ and conclude that, if $I^+(f)$ is disconnected, then there are infinitely many components of $I^+(f)$ that meet $A(f)$. Since, as noted earlier, all components of $A(f)$ are unbounded, the result follows.

To prove part (b), note first that, since $I^+(f)$ is disconnected, it follows from Lemma 3.4 that there exists a closed connected set $\Gamma \subset K(f)$ with two complementary components, say G_0 and G_1, each containing points in $I^+(f)$.

Now since the boundaries of G_0 and G_1 are connected and lie in $K(f)$, both G_0 and G_1 must meet $J(f)$. For suppose that G_0, say, lies in $F(f)$. Then since $\partial G_0 \subset K(f)$, it follows that G_0 is a Fatou component that is also a component of $I^+(f)$. However, since ∂G_0 is connected, Lemma 2.6 shows that this is impossible. So, for $i = 0, 1$, there exist $z_i \in J(f)$ and a bounded open neighbourhood H_i of z_i such that $\overline{H_i} \subset G_i \setminus E(f)$.

Also, since $J(f)$ is unbounded, for each $n \geq 2$ there is a point $z_n \in J(f)$ and a bounded open neighbourhood H_n of z_n, with the properties that $H_n \cap E(f) = \emptyset$ and $\inf_{z \in H_n} |z| \to \infty$ as $n \to \infty$.

Now let z be an arbitrary point in $J(f)$ and let V be a bounded open neighbourhood of z. Then, by Lemma 4.2, there exists $k \in \mathbb{N}$ such that

$$f^k(V) \supset \overline{H}_0 \cup \overline{H}_1, \tag{4.4}$$

and, for any $n \geq 2$, there exists $m_n \in \mathbb{N}$ such that

$$f^{m_n}(H_0) \supset \overline{H}_n, \; f^{m_n}(H_1) \supset \overline{H}_n \text{ and } f^{m_n}(H_n) \supset \overline{H}_0 \cup \overline{H}_1. \tag{4.5}$$

Now let $s = s_1 s_2 s_3 \ldots$ be an infinite sequence of 0s and 1s. We will show that there is an uncountable set of such sequences that encode the orbits of points that lie in distinct components of $I^+(f)$ that meet $\overline{V}$.

To show this, put $E_0 = \overline{V}$ and, for $n \in \mathbb{N}$, set

$$E_{2n} = \overline{H}_{n+1} \text{ and } E_{2n-1} = \overline{H}_{s_n}.$$

Then, for each sequence $s = s_1 s_2 s_3 \ldots$, it follows from (4.4), (4.5) and Lemma 3.5 that there is a corresponding sequence $(p_n)_{n \in \mathbb{N}}$ and a point $\zeta_s \in \overline{V}$ such that $f^{p_n}(\zeta_s) \in E_n$ for $n \in \mathbb{N}$. Furthermore, all such points must lie in $I^+(f)$.

We now claim that points in $\overline{V} \cap I^+(f)$ whose orbits are encoded by different infinite sequences of 0s and 1s must lie in different components of $I^+(f)$. For if two such sequences differ, then some iterate of f will map one point to G_0 and the other to G_1. Thus, if the two points are in the same component C of $I^+(f)$, then some iterate of C meets $\Gamma \subset K(f)$, which is a contradiction.

Evidently, the set of all sequences $s = s_1 s_2 s_3 \ldots$ of 0s and 1s can be put in one-to-one correspondence with the binary representations of points in the unit interval. We have therefore shown that every neighbourhood of an arbitrary point in $J(f)$ meets uncountably many components of $I^+(f)$, and this proves part (b). □

Remark. It follows by a similar argument to the proof of Theorem 4.6(b) that, for a transcendental entire function f, every neighbourhood of a point in $J(f)$ meets uncountably many components of $I^+(f) \cap J(f)$. The proof uses Lemma 3.5 with the compact sets E_n, $n \geq 0$, in the above proof replaced by $E_n \cap J(f)$, $n \geq 0$.

5. Examples

In this section, we give details of the two examples referred to in the remarks after the statement of Theorem 3.1.

First, we give an example of a transcendental entire function that satisfies the conditions of Theorem 3.1 but not those of Theorem 1.2.

Example 5.1. *Let f be the transcendental entire function defined by*

$$f(z) = -10ze^{-z} - \tfrac{1}{2}z.$$

Then f satisfies the conditions of Theorem 3.1, so $I^+(f)$ is connected, but there is no $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$.

Proof. Since

$$m(r) \le |f(r)| = \tfrac{1}{2}r(1 + o(1)) \quad \text{as } r \to \infty,$$

it is clear that there is no $r > 0$ such that $m^n(r) \to \infty$ as $n \to \infty$.

We show, nevertheless, that $I^+(f)$ is connected because (3.1) and (3.2) in the statement of Theorem 3.1 are satisfied.

Let $(D_n)_{n\in\mathbb{N}}$ be the sequence of nested domains defined by

$$\begin{aligned} D_n = &\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 4n\pi,\ |\operatorname{Im} z| < 4n\pi\} \\ &\cup \{z \in \mathbb{C} : -n\pi < \operatorname{Re} z \le 0,\ |\operatorname{Im} z| < n\pi\}. \end{aligned}$$

(Here, for convenience, we have labelled the domains with subscripts in $\mathbb{N}$ rather than in $\mathbb{N}_0$.) Each domain D_n is the union of two rectangles, the larger in the right half-plane and the smaller in the left half-plane (see Figure 2). It is clear that (3.2) is satisfied by this sequence of domains.

We now show that (3.1) is also satisfied for $n > 1$. To see this, consider Figure 2, in which sections of the boundary of D_n are labelled with lower case letters and their images under f with the corresponding upper case letters. The following brief notes discuss the images of different sections of the boundary of D_n using the same labelling as in the figure.

<u>Section a</u> On this section $z = 4n\pi + iy$, where $-4n\pi \le y \le 4n\pi$, so

$$|f(z) - (-\tfrac{1}{2}z)| = |f(z) - (-2n\pi - \tfrac{1}{2}iy)| \le 40\pi\sqrt{2}\exp(-4\pi) < 10^{-3},$$

and hence $f(z)$ evidently lies outside D_{n+1} for $n > 1$.

<u>Section b</u> On this section $z = x + 4n\pi i$, where $0 \le x \le 4n\pi$, so

$$f(z) = -\tfrac{1}{2}x - 10xe^{-x} + i\left(-2n\pi - 40n\pi e^{-x}\right),$$

lies in the left half-plane, below the line $\operatorname{Im} z = -2n\pi$, and hence outside D_{n+1}. The image of this section meets the imaginary axis at $f(4n\pi i) = -42n\pi i$ (off the scale in Figure 2).

<u>Section c</u> Here $z = iy$, where $n\pi \le y \le 4n\pi$. The term $-10ze^{-z}$ is dominant, and since this section has length $3n\pi$, the image of the section winds around the origin, the factor of 10 ensuring that it stays outside D_{n+1}. The image is thus a spiralling curve joining $f(4n\pi i) = -42\pi i$ to $f(n\pi i) = (19/2)n\pi i$.

<u>Section d</u> This section of the boundary is the union of the three line segments

$$\begin{aligned} z &= x + n\pi i, \quad -n\pi \le x \le 0, \\ z &= -n\pi + iy, \quad -n\pi \le y \le n\pi, \\ z &= x - n\pi i, \quad -n\pi \le x \le 0. \end{aligned}$$

On all three segments the modulus of $10ze^{-z}$ is at least $10n\pi$ and exceeds the modulus of $\frac{1}{2}z$ by a factor of at least 20, so the images of these segments lie well outside D_{n+1}. Most of these images lies off the scale in Figure 2.

<u>Sections e and f</u> The images of these sections of the boundary are the reflections in the real axis of the images of sections c and b, respectively.

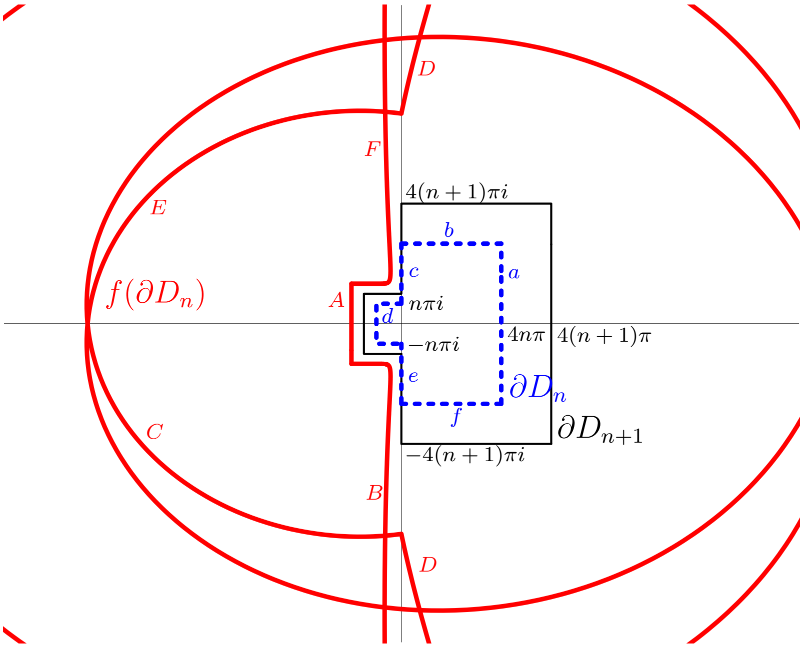

FIGURE 2. The boundary of the domain D_n and its image under the function $f : z \mapsto -10ze^{-z} - \frac{1}{2}z$ (see Example 5.1), illustrated for $n = 2$. Sections of the boundary of D_n are labelled with lower case letters and their images with the corresponding upper case letters. Only the immediate vicinity of D_n is shown.

We have now shown that the whole of $f(\partial D_n)$ lies outside D_{n+1}, so the conditions of Theorem 3.1 are satisfied and hence $I^+(f)$ is connected. □

Next, we show that not all transcendental entire functions that are strongly polynomial-like in the sense defined in [10] meet the conditions of Theorem 3.1. The authors are grateful to Dave Sixsmith for suggesting this example.

Example 5.2. *Let f be the transcendental entire function defined by*

$$f(z) = \cos z + z.$$

Then f is strongly polynomial-like and $I^+(f)$ is disconnected.

Proof. By [10, Theorem 1.6], a transcendental entire function f is strongly polynomial-like if and only if there exists a sequence of bounded, simply connected domains $(D_n)_{n\in\mathbb{N}_0}$ such that

(i) $f(\partial D_n)$ surrounds $\overline{D}_n$, for $n \in \mathbb{N}_0$,
(ii) $\bigcup_{n\in\mathbb{N}_0} D_n = \mathbb{C}$, and
(iii) $\overline{D}_n \subset D_{n+1}$, for $n \in \mathbb{N}_0$.

Let $(D_n)_{n\in\mathbb{N}_0}$ be the sequence of nested open rectangles defined by

$$D_n = \left\{ z \in \mathbb{C} : -\left(2n + \frac{11}{4}\right)\pi < \text{Re } z < \left(2n + \frac{9}{4}\right)\pi,\ |\text{Im } z| < 2(n+1)\pi \right\}.$$

Then properties (ii) and (iii) are evidently satisfied.

To show that property (i) is also satisfied, consider first the images under f of the two sides of the rectangle D_n parallel to the real axis. Writing

$$f(z) = \tfrac{1}{2}(e^{iz} + e^{-iz}) + z,$$

we see that f maps both these sides into the annulus

$$\{z : \tfrac{1}{2}e^{2(n+1)\pi} - 4(n+1)\pi < |z| < \tfrac{1}{2}e^{2(n+1)\pi} + 4(n+1)\pi\},$$

which clearly lies outside $\overline{D}_n$ for all $n \in \mathbb{N}_0$.

Next, if z lies on the vertical side of D_n in the right half-plane, we have

$$\text{Re } f(z) = \text{Re } z + \frac{1}{2\sqrt{2}}(e^y + e^{-y}),$$

where $y = \text{Im } z$. Similarly, for points on the vertical side of D_n in the left half-plane,

$$\text{Re } f(z) = \text{Re } z - \frac{1}{2\sqrt{2}}(e^y + e^{-y}).$$

It follows that $f(\partial D_n)$ surrounds $\overline{D}_n$, for $n \in \mathbb{N}_0$, so f is strongly polynomial-like.

Now, f has fixed points at $z = (k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$. These fixed points are repelling if k is odd and superattracting if k is even, and all points on the real axis except for the repelling fixed points tend under iteration towards the nearest superattracting fixed point. It follows that the real axis lies in $K(f)$, and indeed that the real axis is a closed, connected set in $K(f)$ that disconnects $I^+(f)$. $\square$

Remark. It can be shown that $K(f)$ is connected for the function f in Example 5.2. We omit the details.

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