Discussiones Mathematicae
Graph Theory xx (xxxx) 1–6

NOTE

MAXIMAL BUTTONINGS OF TREES

IAN SHORT[1]

Department of Mathematics and Statistics
The Open University
Milton Keynes MK7 6AA
United Kingdom

e-mail: ian.short@open.ac.uk

Abstract

A *buttoning* of a tree that has vertices $v_1, v_2, \ldots, v_n$ is a closed walk that starts at v_1 and travels along the shortest path in the tree to v_2, and then along the shortest path to v_3, and so forth, finishing with the shortest path from v_n to v_1. Inspired by a problem about buttoning a shirt inefficiently, we determine the maximum length of buttonings in trees.

Keywords: Centroid, graph metric, tree, walk, Wiener distance.

2010 Mathematics Subject Classification: Primary: 05C05, 05C38; Secondary: 05C85.

1. INTRODUCTION

At the retirement meeting of Jenny Piggott as director of the mathematics education project NRICH, Bernard Murphy proposed the following problem (paraphrased).

Problem 1. My shirt has eight buttons in a vertical line with a spacing of one unit between each adjacent pair. Usually I button them from top to bottom, so that my hands move a distance of seven units. Suppose I button them in a different order; what is the maximum number of units my hands may travel?

In this partly expository note we address the more general problem of identifying, for each finite tree T with graph metric d, the maximum value of the sum

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \qquad (1)$$

[1]The author thanks Jozef Širáň for helpful suggestions.

among all lists $v_1, v_2, \ldots, v_n$ of the vertices of T. Problem 1 is a particular case of this more general problem when T is the linear graph of order 8. (To be precise, we must remove the final term $d(v_n, v_1)$ from (1) to recover Problem 1, but we shall see that this is an insignificant complication.) Our problem is itself a special case of the maximum travelling salesman problem. To see this, observe that the sum (1) is the length of a Hamilton cycle in the weighted complete graph that has vertices $v_1, v_2, \ldots, v_n$ and has, for each distinct pair i and j, an edge of weight $d(v_i, v_j)$ between v_i and v_j.

All trees throughout the paper are finite. Further, T will always denote a tree with graph metric d. We denote by V_T the vertex set of T. Let $[u, v]$ denote the unique shortest path from one vertex u to another vertex v in T. A *buttoning* of T is a closed walk in T consisting of n paths $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1]$, where $v_1, v_2, \ldots, v_n$ are the vertices of T. The *length* of this buttoning is the sum (1). A *centroid* of T is a vertex v such that the sum $\sum_{u \in V_T} d(v, u)$ is minimized. Each tree has either one centroid or two adjacent centroids. Given a centroid v we define

$$\Phi(T) = 2 \sum_{u \in V_T} d(v, u).$$

The theory of centroids is covered briefly in [1, Section 1] and [2, Section 3]. The authors of [1] emphasise the importance of centroids in distance calculations, and our work supports this assertion. We can now state our main theorem.

Theorem 2. *Given a tree T with vertices $v_1, v_2, \ldots, v_n$ we have*

$$2n - 2 \leq d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \leq \Phi(T), \tag{2}$$

and the upper and lower bounds are each attained by the lengths of certain buttonings of T.

The lower inequality in (2) has been proven already, in [4, Theorem 1] (including proof that the lower bound is attainable). There are results of a similar nature to Theorem 2 in [3].

A *maximal buttoning* of a tree T is a buttoning of maximum length $\Phi(T)$. When T is the linear tree of order 8, the two middlemost vertices of T are both centroids, and one can check that $\Phi(T) = 32$. We show in Lemma 5 that you can choose $d(v_n, v_1) = 1$ in a maximal buttoning of such a tree, and so the solution to Problem 1 is 31.

The quantity $\Phi(T)$ is closely related to the *Wiener distance* $W(T)$, which is given by $W(T) = \sum_{a,b \in V_T} d(a, b)$. It is known (see, for example, [2]) that, among trees of order n, $W(T)$ is minimized when T is the star with n vertices and $W(T)$ is maximized when T is the linear graph with n vertices. The same is true of $\Phi(T)$, and we state this as a theorem (which is easily proven). Let $\lfloor x \rfloor$ denote the integer part of a real number x.

Theorem 3. *If T is a tree of order n then*

$$2n - 2 \le \Phi(T) \le \left\lfloor \tfrac{1}{2}\, n^2 \right\rfloor . \tag{3}$$

Furthermore, the lower bound is attained when T is a star and the upper bound is attained when T is a linear graph.

2. PROOF OF THEOREM 2

Theorem 2 concerns the maximum and minimum lengths of buttonings of a tree T of order n. Let us briefly summarize the proof from [4, Theorem 1] of the lower bound in (2). Because a buttoning is a closed walk that visits every vertex, each edge must be traversed at least twice, and this proves that each buttoning has length at least $2n-2$. To see that this lower bound can be attained, between any two adjacent vertices in T introduce a new edge. By 'opening out' the resulting graph to form a cycle it is straightforward to construct a buttoning of T of length $2n-2$. The remainder of this section concerns the upper bound of (2).

Lemma 4. *Let $[v_1, v_2], [v_2, v_3], \dots, [v_{n-1}, v_n], [v_n, v_1]$ be a buttoning of a tree T. Then*

$$d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n) + d(v_n, v_1) \le \Phi(T),$$

with equality if and only if each centroid of T is contained in every path $[v_i, v_{i+1}]$ (including $[v_n, v_1]$).

Proof. Let v be a centroid of T and let $v_{n+1} = v_1$. Then the triangle inequality gives

$$\sum_{i=1}^{n} d(v_i, v_{i+1}) \le \sum_{i=1}^{n} \big(d(v_i, v) + d(v, v_{i+1})\big) = \Phi(T).$$

Equality is attained in this inequality if and only if $d(v_i, v_{i+1}) = d(v_i, v) + d(v, v_{i+1})$ for $i = 1, 2, \dots, n$. This occurs if and only if v is contained in each path $[v_i, v_{i+1}]$. ■

We must now prove that the upper bound $\Phi(T)$ in (2) can always be attained. We deal separately with trees that contain two centroids and trees that contain just one centroid. It is an old result of C. Jordan (see [2, Theorem 1]) that a tree with two centroids u and v has even order $2k$, and there is an edge connecting u and v which, once removed, leaves two disconnected subtrees U and V each of order k, where u is a leaf of U and v is a leaf of V. We use this notation in the next lemma.

Lemma 5. *Suppose that a tree T has two centroids u and v and corresponding subtrees $U = \{u_1, u_2, \ldots, u_k\}$ and $V = \{v_1, v_2, \ldots, v_k\}$. Then the buttoning $[u_1, v_1], [v_1, u_2], [u_2, v_2], \ldots, [v_k, u_1]$ of T is a maximal buttoning, and all maximal buttonings arise in this fashion.*

Proof. By Lemma 4, each buttoning $[u_1, v_1], [v_1, u_2], [u_2, v_2], \ldots, [v_k, u_1]$ is a maximal buttoning because the paths $[u_i, v_i]$ and $[v_i, u_{i+1}]$ all contain u and v. Furthermore, in any buttoning $[w_1, w_2], [w_2, w_3], \ldots, [w_{2k-1}, w_{2k}], [w_{2k}, w_1]$ not of this form there must be two consecutive vertices w_i and w_{i+1} that both lie in U, in which case $[w_i, w_{i+1}]$ does not contain v, and so, by Lemma 4, the buttoning is not maximal. ■

All the maximal buttonings of T are described explicitly in Lemma 5, so we have the following corollary.

Corollary 6. *A tree T that has two centroids and is of order $2k$ has $2(k!)^2$ maximal buttonings.*

Next we turn to trees with a single centroid. A preliminary lemma is needed.

Lemma 7. *Let $X_1, X_2, \ldots, X_m$, where $m \geq 2$, be a collection of disjoint finite sets such that $\sum_{i \neq j} |X_i| \geq |X_j|$ for each j. Then we can list the elements $v_1, v_2, \ldots, v_n$ of $X_1 \cup X_2 \cup \cdots \cup X_m$ in such a way that no two consecutive terms v_i and v_{i+1} both lie in the same set X_j.*

Sketch of proof. Remove the elements of $X_1 \cup X_2 \cup \cdots \cup X_m$ one by one and place them in the sequence $v_1, v_2, \ldots, v_n$, each time choosing the element v_i from a set X_j of largest current size (excluding the set X_k from which v_{i-1} was chosen). When $m = 2$, this strategy clearly gives a suitable list. When $m > 2$, the strategy preserves the inequality $\sum_{i \neq j} |X_i| \geq |X_j|$ (until only two elements, in two distinct sets X_j, remain), and hence eventually exhausts the sets X_j. ■

If a tree T has a single centroid v, then removing v from T, and removing all edges connected to v, leaves a number of disconnected subtrees of T, say $X_1, X_2, \ldots, X_m$. Again, it was proven by C. Jordan (see [2, Theorem 1]) that no one of these subtrees has order larger than the sum of the orders of all the others; in other words $\sum_{i \neq j} |X_i| \geq |X_j|$ for each j. We use this notation in the next lemma.

Lemma 8. *Suppose that a tree T has a single centroid v_0, and removing v_0 and its edges from T leaves disconnected subtrees $X_1, X_2, \ldots, X_m$. Then we can label the vertices of $T \setminus \{v_0\}$ as $v_1, v_2, \ldots, v_n$ in such a way that no pair v_i and v_{i+1} both lie in the same set X_j, and $[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_0]$ is a maximal buttoning of T.*

Proof. Lemma 7 shows that it is possible to choose the vertices $v_1, v_2, \ldots, v_n$ in the described fashion, and, because each path $[v_i, v_{i+1}]$ passes through v_0, we see from Lemma 4 that the resulting buttoning is maximal. ∎

In fact, Lemma 4 shows that all maximal buttonings of T are of the form described in Lemma 8, up to cyclic permutations of the paths $[v_i, v_{i+1}]$ in the buttoning $[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_n], [v_n, v_0]$. In contrast to Corollary 6, however, there does not appear to be a simple general formula for the number of maximal buttonings.

We proved in Lemma 4 that the length of a buttoning of a tree T is less than or equal to $\Phi(T)$, and Lemmas 5 and 8 show that this bound can always be attained. This completes the proof of Theorem 2.

3. CONCLUDING REMARKS

The concept of a buttoning extends to all finite connected graphs, and we finish with brief remarks about extremal buttoning lengths in this more general context.

From (2), a buttoning of a tree of order n has length at least $2n - 2$. For more general connected graphs of order n, however, the lower bound for buttoning lengths is n, rather than $2n-2$. This is because every buttoning has n constituent paths each of length at least 1, which implies that the total length is at least n. Furthermore, the lower bound of length n is achieved by any buttoning of the complete graph of order n.

On the other hand, by (3), a buttoning of a tree of order n has length at most $\left\lfloor \frac{1}{2}\, n^2 \right\rfloor$, and this is also an upper bound for the length of a buttoning of a graph of order n. This is because the length of a buttoning of a graph is less than or equal to the length of the same buttoning on a spanning tree of the graph. It follows that among connected graphs of order n, the linear graph has the largest maximal buttoning length. In particular, the maximal buttoning length in Problem 1 remains 31 even when we rearrange the eight buttons to form a more general connected graph.

REFERENCES

[1] C. A. Barefoot, R. C. Entringer and L. A. Székely, *Extremal values for ratios of distances in trees*, Discrete Appl. Math. **80** (1997) 37–56.

[2] A. A. Dobrynin, R. Entringer and I. Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math **66** (2001) 211–249.

[3] L. Johns and T. C. Lee, *S-distance in trees*, in: Computing in the 90's (Kalamazoo, MI, 1989), Lecture Notes in Comput. Sci., 507, ed(s), N. A. Sherwani, E. de Doncker and J. A. Kapenga (Springer, Berlin, 1991) 29–33.

[4] T. Lengyel, *Some graph problems and the realizability of metrics by graphs*, Congr. Numer. **78** (1990) 245–254.