Poincaré and the idea of a group

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In many different fields of mathematics and physics Poincaré found many uses for the idea of a group, but not for group theory. He used the idea in his work on automorphic functions, in number theory, in his epistemology, Lie theory (on the so-called Campbell–Baker–Hausdorff and Poincaré–Birkhoff–Witt theorems), in physics (where he introduced the Lorentz group), in his study of the domains of complex functions of several variables, and in his pioneering study of 3-manifolds. However, as a general rule, he seldom appealed to deep results in group theory, and developed no more structural analysis of any group than was necessary to solve a problem. It was usually enough for him that there is a group, or that there are different groups. In this article Jeremy Gray gives a brief history on Poincaré's group idea.

It is well-known that between 1880 and 1884 Poincaré brought together in a completely unexpected way the subjects of complex function theory, linear differential equations, Riemann surfaces, and non-Euclidean geometry (see, for example, [42] and [7]). Cauchy's approach to complex function theory and the theory of differential equations were mainstream topics in the education of a French mathematician at the time, but Riemann surfaces were not, largely because Riemann's way of thinking was not congenial to Charles Hermite, who dominated the scene in the 1870s.

In the spring of 1879 the Académie des Sciences in Paris announced the topic of the Grand prize for the mathematical sciences (see *C.R. Acad. Sci.*, 88, 1879, p. 511), which was "to improve in some important way the theory of linear differential equations in a single independent variable". The topic had been proposed by Hermite, and his intention was to draw young French mathematicians to study the work of Lazarus Fuchs, who was the German expert on the subject; in the aftermath of the Franco-Prussian War catching up with the Germans was on every patriotic Frenchman's mind.

Automorphic functions

In a series of papers in 1866 to 1868, starting with [6], Fuchs had been able to characterise a class of linear ordinary differential equations of arbitrary order that have the property that their solutions are meromorphic and have poles only where the coefficients of the equation themselves have poles. Among this class is the celebrated hypergeometric equation,

$$z(z-1)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\,\frac{dw}{dz} - abw = 0 \qquad (1)$$

and we may confine our attention to it, although by then Fuchs had moved on to study other problems.

Poincaré took from the later work of Fuchs the idea that the quotient of a basis of solutions to equation (1) was an interesting object. If the solutions are denoted $w_1(z)$ and $w_2(z)$, and the quotient as $\zeta(z) = \frac{w_1(z)}{w_2(z)}$, then analytic continuation of the solutions around a path enclosing a singular point returns the quotient in the form

$$\begin{aligned}\zeta(z) &= \frac{a_{11}w_1(z)+a_{12}w_2(z)}{a_{21}w_1(z)+a_{22}w_2(z)} \\ &= \frac{a_{11}\zeta(z)+a_{12}}{a_{21}\zeta(z)+a_{22}},\end{aligned}$$

where the coefficients a_{jk} are constants that depend on the path. As this formula makes clear, the 'function' ζ is a multi-valued function, but its set-theoretic inverse is a generalisation of a periodic function:

$$z(\zeta) = z\left(\frac{a_{11}\zeta+a_{12}}{a_{21}\zeta+a_{22}}\right). \qquad (2)$$

Geometrically, the function $\zeta(z)$ maps the upper half-plane to a triangle, the vertices of which are the images of the singular points $0, 1, \infty$ on the real axis, and the angles of which are determined by the coefficients a, b, c of the hypergeometric equation. Analytic continuation shows that the lower half-plane is mapped to another triangle (which will be congruent if the coefficients are all real) and thereafter the images of the halfplanes form a net of triangles, provided the angles are of the form π/n for some integers n. Moreover, each triangle will have the same three angles, although their sides will be circular arcs and not straight, and Poincaré introduced a simple geometric transformation to straighten them out. This much, but little more, formed the content of the essay Poincaré submitted for the prize in May 1880. The extra concerned a discussion of the net of triangles and made some corrections to Fuchs's papers. He showed, for example, that if the angles of the triangles are $\frac{\pi}{2}$, $\frac{\pi}{3}$ and $\frac{\pi}{6}$, then the net covers the plane, eight suitably chosen triangles form a parallelogram, and $z = z(\zeta)$ is an elliptic function; but if the angles are $\frac{\pi}{4}$ and $\frac{\pi}{2}$ at the image of $z = 0$ and $z = 1$ respectively and $\frac{\pi}{6}$ at ∞, then the net lies inside a certain circle, and the sides of the triangles are circular arcs meeting this circle at right angles.

Fuchsian functions

Poincaré then wrote to Fuchs, and it became obvious that he had a much clearer idea of this net of triangles, which he was beginning to think of as the domain of the inverse function $z = \zeta(z)$, than did Fuchs. The correspondence was very amicable, all the same, and on 12 June 1880 Poincaré wrote to Fuchs to say: "I have found some remarkable properties of the functions you define, and which I intend to publish. I ask your permission to give them the name of Fuchsian functions." Fuchs, of course, agreed. (The correspondence between Poincaré and Fuchs is reproduced in Poincaré's, *Oeuvres* 11, pp. 13–25.) But at this point Poincaré was in fact stuck on the case of second-order linear differential equations with no more than three singular points. What happened next was recalled by Poincaré twenty-eight years later when he addressed the Société de Psychologie in Paris about his experience of mathematical discovery. He explained in the lecture [39, *Science et Méthode*, p.51] that one night coffee had kept him awake, and ideas surged up in his mind, eventually forming stable combinations until: "In the morning I had established the existence of a class of Fuchsian functions, those which are derived from the hypergeometric series. I had only to write up the results, which just took me a few hours." He then followed the analogy with elliptic functions and created his thetafuchsian series. The restless night could have been between 29 May and 12 June, when he discovered the Fuchsian functions, or between 12 and 19 June 1880, by which time he knew much more about them.

Next came the realisation that, more than anything else in this part of his work, was to make Poincaré's name among mathematicians [39, *Science et Méthode*, pp. 51–52], when he realised, as he boarded a bus on a geological expedition organized by the École des Mines, that "the transformations I had made use of to define the Fuchsian functions were identical with those of non-Euclidean geometry". If the bus trip took place before the 12th the two breakthroughs happened in a rush, which is not quite the impression Poincaré's account suggests, so perhaps it happened between the 12th and the 19th.

The realisation on boarding the bus was surely that his simple geometric transformation converts the pictures of a net inside a disc made of triangles with circular-arc sides into a net of triangles inside a disc but with straight sides — and this is the non-Euclidean disc of Beltrami. It follows that the original net of triangles can be regarded as made up of congruent copies of the same triangle, where congruence is to be taken with its non-Euclidean meaning. Now instead of analytic continuation as the basic mechanism, Poincaré had isometries to work with. We do not know how Poincaré first heard of non-Euclidean geometry, but Hoüel had translated the original papers by Bolyai

Lazarus I. Fuchs (1833–1902)

and Lobachevskii into French in the 1860s and Helmholtz had also written on the subject.

On 28 June 1880 Poincaré sent a 70-page supplement to his entry to the Académie des Sciences. In it he regarded successive analytic continuations as rotations, denoted M and N, of the basic triangle, and the role of non-Euclidean geometry [42, p. 35]. Significantly, he defined the geometry through its group, writing that "the group of operations formed by means of M and N is isomorphic to a group contained in the pseudogeometric group. To study the group of operations formed by means of M and N is therefore *to do the geometry of Lobachevskii*". He then described the basic features of this convenient language of non-Euclidean geometry, defining points, lines, angles, and equality of figures — two figures are equal if one is obtained from another by a non-Euclidean transformation. Then he turned to the study of the Fuchsian functions, remarking that "the Fuchsian function is to the geometry of Lobachevskii what the doubly periodic function is to that of Euclid", but at this stage he was unable to establish the convergence theorems for the functions.

Poincaré was still able to work only with triangles until at least 30 July, but at some date in August he was rescued by his earlier work on arithmetic. As he recalled in his lecture in 1908, [39, *Science et Méthode*, pp. 52–53], while walking on the cliffs he realised that "the arithmetical transformations of ternary indefinite quadratic forms were identical with those of non-Euclidean geometry." We shall see below that ternary indefinite quadratic forms (objects of the form $x^2 + y^2 - z^2$) were exactly what he had been studying in his papers on the consequences of Hermite's number theory.

The uniformisation theorem

Poincaré now wrote another supplement to his essay, describing the polygonal case, and making progress with the convergence arguments, and he followed it with a third supplement before the competition closed. Even so he did not win, the prize went to Georges Halphen. But by 1881 he had a lot to publish, and as he began to do so more and more ideas occurred to him. This work brought him international attention of two very different kinds. The young Swedish mathematician Gösta Mittag-Leffler was in the process of setting up a new mathematical journal, to be called *Acta Mathematica*, and he saw immediately that Poincaré's papers would establish his journal as one of international significance. The young German mathematician Felix Klein saw someone entering territory he had staked out as his own, although he did not perhaps foresee at first what a challenge this might be.

Both men entered into a correspondence with Poincaré. Mittag-Leffler got the long papers he wanted in which Poincaré set out the theory of the new functions, and Klein had to realise that the younger, less well-educated Poincaré was moving faster than he ever could. Their exchanges are both scholarly and personal. Klein objected to the name 'Fuchsian' for the new functions on the grounds that some ideas of Schwarz were much closer, and Poincaré agreed when he got round to consulting Schwarz's paper, which he had not known. But he could not agree to change the name, which he had already used in publications, and Klein railed against this, doubtless because Fuchs, as a Berlin-trained mathematician close to Weierstrass and Kummer, was a rival likely to have a better career than him but with less talent. To shut him up Poincaré named the generalisation of Fuchsian functions that require 3-dimensional non-Euclidean geometry 'Kleinian functions'. Klein protested, correctly, that he had had nothing to do with these functions and Schottky's name would be more appropriate; "Name ist Schall und Rauch" Poincaré replied in German ("Name is sound and fury", the quotation comes from Gretchen in Goethe's *Faust*).

The correspondence was also a competition, and it was to cost Klein his health, but not before both men had come to the deepest jewel in the field, what became known as the uniformisation theorem. Klein took the lead, because he knew Riemann's theory of the moduli of algebraic curves on Riemann surfaces according to which a Riemann surface of genus $g > 1$ depends on $3g - 3$ complex parameters. On the other hand, it was possible to show that the non-Euclidean polygons that are mapped around in the disc by a Fuchsian group in such a way that the quotient space is a Riemann surface of genus g also depend on $3g - 3$ complex parameters. The implication was obvious: every Riemann surface arises as a quotient of the disc with its non-Euclidean metric under the action of a Fuchsian group. Not only that, but locally the map from the disc to the Riemann surface will be an isometry, so all but the Riemann surfaces of lowest genus locally carry non-Euclidean geometry. (This copies the case of elliptic functions, which are quotient spaces of the Euclidean plane.) Klein's formulation of this result [14, 15] was perhaps sharper than Poincaré's, but neither man come close to a proof; Poincaré's account [24], generalised to apply to the uniformisation of any many-valued function (not necessarily algebraic), was particularly obscure. The topic was made one of Hilbert's problems in his address at the Paris ICM in 1900 (see [13]), and the first proofs were given independently by Koebe and Poincaré in 1907. (See [38] and Koebe's paper [16] and his many subsequent publications.) Poincaré's many notes and his long papers in *Acta Mathematica* exploited the analogy with elliptic functions. In [22] he defined a Fuchsian group to be the one associated to a polygon with angles of the form π/n where n is an integer, which identifies the edges of the polygon in pairs, and where the edges are arcs of circles perpendicular to the boundary of a circle. In [23] he turned to the task of defining automorphic functions.

New functions (see [23]) were defined by using sums of the form

$$F(z) = \sum_{\gamma \in G} f(\gamma z),$$

where the summation is taken over all the elements of a Fuchsian group G. This would ensure that the function F is G-invariant, but first Poincaré had to confront several problems: the sums had to be convergent, and it was to turn out that non-trivial examples could converge to the zero function. To work round this problem he generalised the idea of theta functions, and introduced functions of the form

$$\theta\left(\frac{az+b}{cz+d}\right) = \theta(z)(cz+d)^m, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

where m is an integer. He established convergence for $m > 2$ by an ingenious argument mixing Euclidean and non-Euclidean geometry, and showed that products of quotients of these 'thetafuchsian' functions were Fuchsian functions. He then showed that the inverse of a Fuchsian function is a quotient of two functions that satisfy a second-order linear differential equation that was intimately connected to the polygon.

What is very striking is that Poincaré offered a very general theory, with a small number of examples to illustrate how local difficulties can be tackled. He did not offer a detailed presentation of Fuchsian groups; if a reader wanted to know what Fuchsian groups were like and if in particular there were any well-known examples in the literature the best that could be said is that Poincaré left the field wide open. This is another way in which he resembled Riemann, and it was to be the pattern throughout his work in mathematics.

Epistemology

In 1891 Poincaré wrote the first of the many popular essays, [27], that were to keep him before a large audience thereafter. He had become famous in 1889 for winning the prize competition organised by King Oscar II of Sweden for his essay, [26], on the stability of the solar system. (For a rich history of this event, see [2].) The success also saw him appointed a Knight of the Legion d'Honnneur, and after a short essay on what he had done on celestial mechanics the next popular essay was a rather cryptic account of geometry. Here again it became clear that he put the group ahead of the space, or, as he preferred to say, the form ahead of the matter. This made him a person with an eagerly sought-after opinion on the burning question of the day: Is space Euclidean or non-Euclidean? His surprising answer was that one could never tell. In his view both geometries were consistent, and he set out a dictionary of terms for translating from one geometry to the other. This showed that any contradiction in one geometry would provide a contradiction in the other, and so the two were relatively consistent (he admitted this did not establish the consistency of either but said he had some ideas of how this could be done).

He asked his readers to imagine some experiment in which a seemingly decisive result had been obtained, for example the construction of a figure with light rays marking out four equal sides meeting at four equal angles for which the sum of the angles was less than 2π. This would seem to suggest that space was non-Euclidean, but, said, Poincaré, there is another interpretation, which was that space was Euclidean and light rays were curved. There could be no way of deciding logically between these two interpretations, and all we could do would be to settle for the geometry we found most convenient, which, indeed, he said would be the Euclidean one. His reasons were, however, unexpected, and will be considered shortly.

Not long after this, Poincaré was drawn into a long dispute with Russell, who began with the intention of establishing on Kantian lines that we must be able to form some idea of geometry, else we could not have a concept of space, and that, furthermore, that concept of geometry was necessarily projective geometry. Once this was established, Russell then sought to graft a concept of distance onto this framework, but he did not assume that space was Euclidean (see [44] and [45]).

Poincaré, in his [30], had little trouble demolishing the clumsy presentation of projective geometry that Russell offered, and Russell endeared himself to Poincaré by his willingness to concede his errors in print (this ability was to remain a charming feature of all Russell's philosophical investigations). But the further the debate went on the clearer it became that the two had fundamentally different starting points. To Russell it was clear "before we begin", as he put it, that the distance from London to Paris is greater than a millimetre. But to Poincaré this was not how one could begin.

The best account he gave of his philosophy he published in an English translation in the *Monist* [29] in 1898. He began by raising the fundamental question of how we construct a sense of space around us at all. This was much discussed by psychologists at the time, and Poincaré observed that we can construct many spaces. A single motionless eye would construct a two-dimensional projective geometry with no sense of distance. A pair of eyes could construct a sense of depth. We have our sense of touch, and we could construct a high-dimensional space by recording the muscular sensations needed to put the tip of a finger somewhere. Out of this welter of experiences and before we are capable of formal instruction, we all construct a sense of three-dimensional space occupied by some bodies with predictable behaviour. These are the rigid bodies, and they are singled out by the fact that we can compensate for the motion of a rigid body by a motion of our own. We can, for example, distinguish the motion of a glass from the motion of the wine swirling around in it. So, he argued, we build up in our minds a sense of what rigid bodies are, and are able to handle them hypothetically. This mental construction gives us our idea of the isometries of a hypothetical rigid body, and from this we construct our concept of space – notice that rigid bodies and their motions came first in this analysis. The concept of distance is derived from the behaviour of rigid bodies, which is why on this account Poincaré could dispute Russell's claim about London, Paris, and the millimetre. For Poincaré the claim is true not because we know what distance in space is, but because we know what isometries are. This knowledge is innate, it has evolved with the human species, and it is triggered by the experiences of every sentient infant.

For Poincaré it is only via the introduction of the group that the non-measurable 'space' of Helmholtz and Lie becomes a measurable magnitude "that is to say, a veritable space". Therefore [29, Sections 21 and 22]:

"What we call a geometry is nothing but the formal properties of a certain continuous group ... so that we may say, space is a group. But the assertion is no less true of the notion of many other continuous groups; for example, that which corresponds to the geometry of Lobachevskii. There are, accordingly, several geometries possible, and it remains to be seen how a choice is made between them."

Poincaré was adamant that different creatures, with a different history, might be non-Euclidean in the sense that their brains would find non-Euclidean geometry convenient. If we met such creatures, we would not share their sense of what is easy of natural, nor would they share ours, but neither side would be able to trap the other in a contradiction. Where we and they would differ would be in our innate understandings of rigid bodies, and we would be the ones whose brains appreciated that the translations in the group of isometries formed a normal subgroup, a statement that is false in the group of non-Euclidean isometries.

This is fundamental epistemology (and quite different from his other conventionalisms). It accounts for the one wry observation Poincaré had on Hilbert's foundations of geometry as they were set out in his *Grundlagen der Geometrie* [12], in 1899. Hilbert gave a purely axiomatic formulation, with no pretence to being an account of how we can have knowledge of the external world, and when Poincaré reviewed them he commented [33, p. 272] that:

"The objects which he calls points, straight lines, or planes become thus purely logical entities which it is impossible to represent to ourselves. We should not know how to picture them as sensory images ... Each of his geometries is still the study of a group. The logical point of view alone appears to interest him. With the foundation of the first proposition, with its psychological origin, he does not concern himself. His work is then incomplete; but this is not a criticism ... Incomplete one must indeed resign one's self to be."

Groups in number theory

When Poincaré began to publish in 1880 it was not only on linear differential equations in the complex domain (as discussed above) and on the curves defined by real differential equations, but on the number theory of forms in several variables, as well as some other papers on number theory, and the subject remained a lifelong interest of his. His second paper, [21], picked up on a remark of Hermite's that there is a group of transformations mapping an indefinite ternary quadratic form to itself. Poincaré showed how these transformations can be regarded as isometries of the non-Euclidean disc, and then how this illuminates the reduction of these forms to canonical form.

The most significant of Poincaré's papers on number theory were on Fuchsian functions and arithmetic, and grew out of his previous work on ternary forms. In [25] he observed that the linear transformations that map the ternary quadratic form $\Phi = y^2 - xz$ to itself have an eigenvalue $+1$ or -1, and those with eigenvalue $+1$ can all be written in the form

$$\begin{pmatrix} \delta^2 & -\delta\gamma & \gamma^2 \\ -2\delta\beta & \alpha\delta+\beta\gamma & -2\alpha\gamma \\ \beta^2 & -\alpha\beta & \alpha^2 \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are four arbitrary quantities such that $(\alpha\delta - \beta\gamma)^2 = 1$. Poincaré restricted his attention to the case when they are all real and moreover $\alpha\delta - \beta\gamma = 1$. Moreover, the groups of Fuchsian transformations

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta},$$

and of 3 by 3 matrices defined above are isomorphic.

If, however, T is a linear transformation of the variables x, y, z that does not map the form Φ to itself, then the new form, which Poincaré denoted $F = \Phi T$, that is mapped to itself by all the transformations of the form $\Sigma = T^{-1}ST$, and so the group of self-transformations of F and of Φ are isomorphic (indeed, conjugate).

On the further assumption that the coefficients of the form F are integers and that the coefficients of the self-transformations of F are likewise integers, number theorists such as Hermite had already been led, he remarked, to interesting discontinuous groups. Poincaré had earlier called the corresponding Fuchsian functions 'arithmetic Fuchsian functions', and he now proposed to show that these functions satisfied a theorem generalising the addition theorem for elliptic functions, which more general Fuchsian functions do not.

To establish this result, Poincaré examined the Fuchsian groups that are associated to different (ternary quadratic) forms, and divided them into four families according as they have no elliptic or parabolic elements, elliptic but no parabolic elements, parabolic but no elliptic elements, or both elliptic and parabolic elements. He showed that it was always possible to determine which of these four cases any given ternary form belonged to. He then looked at the geometry of the corresponding Fuchsian polygon — the number of its sides, the size of its angles, whether the vertices lie inside or on the boundary of the non-Euclidean disc — and showed that in each case this information was determined by the quadratic form.

After a consideration of the geometry of the Fuchsian polygons and its connection to the arithmetic properties of the form F, Poincaré turned to the classical example of the modular function $J(z)$, which is invariant under the group $PSL(2, \mathbb{Z})$. The transformation S given by $z \mapsto z/n$ is not in this group, and the relationship between $J(z)$ and $J(z/n)$ is governed by the celebrated modular equation. Poincaré noted that the groups Γ and $S^{-1}\Gamma S$ are what he called commensurable, that is to say their intersection is of finite index in each of them. Indeed, the elements of $S^{-1}\Gamma S$ are of the form

$$z \mapsto \frac{\alpha z + \beta/n}{\gamma n z + \delta},$$

where $\alpha, \beta, \gamma, \delta$ are integers and $\alpha\delta - \beta\gamma = 1$. The required subgroup of Γ and $S^{-1}\Gamma S$ consisted of elements of the form

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

where $\gamma \equiv 0 \bmod n$. A repeat of the same argument showed that $J(z)$ and $J(pz/n)$ are algebraically related, and more generally that $J(z)$ and $J\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)$ are algebraically related.

To generalise the modular function, Poincaré returned to the three groups that map a given ternary form to itself: those where the corresponding matrices have real, rational, or integer coefficients. Each of these group gives rise to a corresponding Fuchsian group; when the coefficients are integers Poincaré called the corresponding Fuchsian group the principal group. When, however, one starts from the group of matrices with rational coefficients that preserved a given form, the corresponding Fuchsian group is not discontinuous, and Poincaré chose an element S in it with rational, non-integral coefficients. This element gave rise to a Fuchsian transformation s that was not in the principal Fuchsian group, and by his earlier results this means that a Fuchsian function for the principal group is algebraically related to its transform by s. Poincaré continued on this theme in [35], which he extended and corrected in [41] (as a footnote observes, Poincaré posted this paper 7 July 1912, the day he went in to hospital for his fatal operation).

Poincaré and Lie theory

Helmholtz was the first person to draw attention to the importance of rigid-body motions in geometry, and there is no doubt that his ideas influenced Poincaré. But his investigation of the nature of the corresponding groups, the types of elements they could possess, and the number of points of space that could be fixed before the group element was also fixed was not mathematically rigorous, and Felix Klein asked Sophus Lie to improve it, which he did in the third volume of his *Theorie der Transformationsgruppen*, 1893. This, and all of his other investigations, impressed Poincaré, who had a high opinion of Lie and assisted in the effort to send promising students to work with him so that his ideas could be written up. But Poincaré did not work on Lie's ideas himself until after Lie's death in 1899. What he then did has remained obscure and unappreciated until the work of Schmid [46] in 1982, Grivel [9] in 2006, and most recently Achilles and Bonfiglio [1], and can only be briefly noted here. (See these authors for references to the original papers.)

Lie's third fundamental theorem

For Lie, a transformation from a space to itself was typically given by an expression of the form $y = f(x, a_1, \ldots, a_n)$, where x and y belong to the space and the a's are parameters. For example, the rotation of a sphere about a fixed axis defines a transformation with the angle of the rotation as the single parameter a. Composition of transformations Lie presumed produced a third transformation of the same type,

$$\begin{aligned} &f(f(x, a_1, \ldots, a_n), b_1, \ldots, b_n) \\ &\quad = f(x, c_1, \ldots, c_n) \end{aligned} \tag{3}$$

where the values of the parameters c are determined by the values of the a's and b's. The composition is valid for rotations of a sphere about an axis, and any pair of parameters determine a third, but in general this is false.

Sophus Lie (1842-1899)

When it is valid Lie said the transformations formed a group (the modern term is a group germ). Lie did not demand inverses, nor indeed did he specify that there be an identity transformation of this form.

Lie considered the transformations obtained as the parameters vary by an arbitrarily small amount. He often thought of an infinitesimal transformation of this kind as a directional derivative, and wrote it in in the form $y = x + t\xi(x)$, where t is arbitrarily small and ξ depends on the parameters a. For, any smooth function φ of the x's, now regarded as k-tuples $x_1, \ldots, x_k$, can be given a Taylor series expansion

$$\varphi(y_1, \ldots, y_k) = \varphi(x_1, \ldots, x_k) + t\left(\xi_1 \frac{\partial\varphi}{\partial x_1} + \cdots + \xi_k \frac{\partial\varphi}{\partial x_k}\right).$$

Lie denoted $\xi_1 \frac{\partial}{\partial x_1} + \cdots + \xi_k \frac{\partial}{\partial x_k}$ by X; he called X an infinitesimal operator, and wrote

$$\varphi(y_1, \ldots, y_k) = 1 + tX\varphi(x_1, \ldots, x_k).$$

These infinitesimal operators combine according to rules already studied by Poisson and Jacobi. The composition of two infinitesimal operators X_j and X_k corresponding to the same group germ, written $[X_j, X_k]$, is given by an expression of the form

$$[X_j, X_k] = \sum_l c_{jkl} X_l, \qquad (4)$$

where the c_{jkl} are constants and the X_l run through a basis for the infinitesimal operators corresponding to the same group germ. Lie claimed the converse, that given partial differential operators $X_1, \ldots, X_n$ for which equation (4) holds and for which the constants c_{jkl} must satisfy some trivial identities, there are transformations (3) which have the X_j as their infinitesimal counterparts and a rule for determining the $c_1, \ldots, c_n$ as functions of the $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$, such that equation (3) holds. It is this converse, known as Lie's third fundamental theorem, that attracted the most interest. In more modern language, it concerns the exponential map from a Lie algebra of a Lie group to the corresponding group. It asks: Given $X, Y \in \mathfrak{g}$, find Z such that $\exp(X).\exp(Y) = \exp(Z)$.

As the above authors show, Lie himself had offered two proofs of his third fundamental theorem, one in the special case when the group has no centre, and one in the language of differential equations. In 1890 Friedrich Schur had given a different proof, which showed that the way the c's depended on the a's and b's was determined by an infinite series of a universal form (independent of the parameters) in which the coefficients were polynomial functions of the a's and b's. In 1901 the Oxford mathematician John Edward Campbell addressed the question in its exponential form directly, and dealt successfully with the formal, algebraic side of the problem but did not deal with the convergence of the power series he exhibited.

Convergence of the power series

In [31] and [32] Poincaré offered his proof, as part of a volume celebrating the work of Sir George Stokes, only to find that his work was very close to that of Campbell and Schur, although he hoped that his ideas had enough in them still to merit publication. He claimed to have reduced the problem to the solution of simple differential equations, a process that could, he said, be carried out in finite terms. His argument is too long and technical to describe here, but unlike Campbell he did tackle the convergence of the power series, making an ingenious use of the residue calculus of complex function theory. He otherwise adhered to a purely formal point of view, abstracting, as he put it, completely from the 'matter' of the group, the same phrase he had used earlier in discussing how we use our innate concept of a group to construct space in his [29], and found formulae applicable to all isomorphic groups. These formulae gave relations between the parameters in the transformations from which he could construct a group isomorphic to the one he was studying, and so establish Lie's third theorem. The same question was then tackled by Felix Hausdorff [11], who noted the existence of a more recent paper by Baker written when he was finishing his own. Hausdorff gave an account that he regarded as a considerable simplification of Poincaré's, as it is, algebraically (convergence is established much as Poincaré had done). Hausdorff's account met with approval of Bourbaki (see [3]), and that is one reason the result is today known as the Campbell–Baker–Hausdorff formula.

Poincaré–Birkhoff–Witt theorem

The other major result in this area to which Poincaré contributed has been known since Cartan and Eilenberg's [5] as the Poincaré–Birkhoff–Witt theorem. Ton-That and Tran, in their thorough account of the history of this theorem [47], concluded that: "But by a careful analysis of Poincaré [31] one must conclude without a shade of doubt that Poincaré had discovered the concept of the universal algebra of a Lie algebra and gave a complete and rigorous proof of the so-called Birkhoff–Witt theorem." Poincaré again displayed his liking for form over matter in proving this result, which is a major result about the structure of a Lie algebra. (See [47] for a thorough account of Poincaré's work and the convoluted history of the names for this theorem.)

It is only possible to be brief here about what Poincaré accomplished. Given a finite-dimensional family of vector fields $X_1, X_2, \ldots, X_n$ or infinitesimal transformations with a multiplication, so

$$[X_j, X_k] = \sum_l c_{jkl} X_l,$$

where the so-called structure constants c_{jkl} satisfy certain identities that follow from the properties the multiplication, Poincaré drew attention to the largest algebra one can usefully construct from these symbols. He formed equivalence classes of these relations, and showed that the set of equivalence classes is the universal enveloping algebra of $\mathcal{L}$. He then gave a detailed analysis if the best form for the representatives of each equivalence class, which was highly symmetrical; the modern statement of this result is the Poincaré–Birkhoff–Witt theorem. It implies that there is a canonical injective map from $\mathcal{L}$ to its universal enveloping algebra, and therefore any Lie algebra over a field is isomorphic to a Lie subalgebra of an associative algebra and some problems about Lie algebras can be transferred to problems about associative algebras.

Poincaré's Lorentz group

Lorentz, in his *Electromagnetic Phenomena* [19] had discussed the failure of attempts to detect the motion of the Earth relative to the ether that could have detected effects of the order of $\frac{v^2}{c^2}$. He explained this null result by the hypothesis of a contraction later named after him, on supposing that bodies contract in the direction of motion of the Earth, thus making any attempt to detect motion relative to the ether impossible. Poincaré, in his short note [34] said that this was so important that he had been inspired to take up the question, and had been able to come to results that agreed in all important respects with those of Lorentz, although he wished to modify and complete them in some respects.

Poincaré did not respond to Lorentz's arguments, but took from him the observation that the equations of the electro-magnetic field were unaltered by transformations of the form

$$x' = kl(x + \varepsilon t),\ y' = ly,\ z' = lz,$$
$$t' = kl(t + \varepsilon x),$$

where x, y, z are the coordinates and t the time before the transformation and x', y', z' and t' afterwards. Furthermore, ε is a constant that depends on the transformation, and $k = 1/\sqrt{1-\varepsilon^2}$. Poincaré insisted that these transformations must form a group, and deduced from this that $l = 1$, a conclusion, he observed, that Lorentz had reached in another way. Poincaré's argument relied on the fundamental principle of relativity: two observers in constant relative velocity must make measurements that differ in a way that only takes note of their different situations (positions and velocities). Poincaré had first explained this argument to Lorentz in a letter in 1904–1905 (see Miller [20, p. 81] for a facsimile of the letter).

The postulate of relativity

In his long paper [36] Poincaré set forth a more elaborate position. He raised the impossibility of detecting motion with respect to the ether to the status of a law of nature and dignified it with the name of the *postulate of relativity*. He agreed that Lorentz transformations do imply the postulate of relativity, but now the new Lorentz contraction must be explained, and the deep problem, said Poincaré, was not that of explaining something, but knowing what has to be explained. The speed of light, said Poincaré, appears in many branches of physics, each time with the same size, if we admit the principle of relativity. This might be because everything in the world is electro-magnetic in origin — a grand claim much discussed at the time — or simply from the way we measure everything, which is based on the assumption that two lengths are equal if it takes light the same time to traverse them. That said, Poincaré also apologised for bringing forward partial results at a time when the whole theory might be in danger from fresh discoveries in the study of cathode rays (i.e., fast-moving electrons).

To study the effect of a Lorentz transformation on electro-magnetic phenomena, Poincaré wrote Maxwell's equations in the potential form, and in units in which $c = 1$. He then observed (see [36, p. 499]) that if in one frame the equation of a moving sphere is

$$(x - \xi t)^2 + (y - \eta t)^2 + (z - \zeta t)^2 = r^2,$$

then the equation in transformed coordinates described a moving ellipsoid, the shape of which depends on the velocity of the sphere. He next obtained the components of the electric and magnetic fields in the new coordinate frame, noting that his results agreed with those of Lorentz, and observed that Maxwell's equations were still satisfied. He also wrote down the new addition law for velocities [36, p. 500]: if two electrons have velocities ξ and ξ' relative to an observer, one has a velocity of $\frac{\xi+\xi'}{1+\xi\xi'}$ relative to the other. Finally he returned to the problem of the shape of a moving electron and the stresses required to ensure that the electron is in equilibrium. (See [20, p. 55 et seq.] for a discussion of this.)

Poincaré then re-obtained the invariance of Maxwell's equations under Lorentz transformations, and in Section 4 of his paper Poincaré showed that the Lorentz transformations form a group provided that $l = 1$. His argument is notable for using Lie's method of infinitesimal generators. Poincaré began by considering transformations of the form

$$x' = kl(x + \varepsilon t),\ y' = ly,\ z' = lz,$$
$$t' = kl(t + \varepsilon x), \tag{5}$$

where $k^{-2} = 1 - \varepsilon^2$. He wrote down the composition law, and then found the infinitesimal transformations that generate the group. He argued that when $l = 1$ and ε is infinitely small, there is a transformation, T_1,

$$\delta x = \varepsilon t,\ \delta y = 0 = \delta z,\ \delta t = \varepsilon x,$$

which can be written in Lie's fashion as

$$T_1 = t\frac{\partial\varphi}{\partial x} + x\frac{\partial\varphi}{\partial t}.$$

There are analogous transformations T_2 and T_3 found by replacing x with y and z respectively. Poincaré also defined the transformation found T_0 by setting $\varepsilon = 0$ and $l = 1 + \delta l$, when

$$\delta x = x\delta l,\ \delta y = y\delta l,\ \delta z = z\delta l,\ \delta t = t\delta l,$$

so

$$T_0 = x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z} + t\frac{\partial\varphi}{\partial t}.$$

These infinitesimal generators give rise to transformations that can all be written as composites of transformations of the form

$$x' = lx,\ y' = ly,\ z' = lz,\ t' = lt,$$

and those that preserve the quadratic form $x^2+y^2+z^2-t^2$. Poincaré now stipulated that these transformations form a group, in which l is a function of ε. The transformation that consists of a rotation of π around the y-axis forces equation (5) to imply that $l(\varepsilon) = l(-\varepsilon)$. But this transformation is its own inverse, and so $l = \frac{1}{l}$, and so $l = 1$.

In Sections 7 and 8 Poincaré showed that Lorentz's hypothesis was the only one compatible with the impossibility of detecting absolute motion, but he gave a different reason, true to lifelong emphasis on the idea of a group: the Lorentz transformations must form a group and therefore necessarily $l = 1$. This left the possibility of detecting absolute motion to those phenomena that were not of electro-magnetic origin, such as gravitation. In [18] Lorentz had argued that gravitation behaved under Lorentz transformations as the electro-magnetic forces do. To analyse this idea, Poincaré considered the effect of a Lorentz transformation on any function of time, position, and velocity such as would arise in any analysis of time, under the extra assumptions that any suitable law of attraction would reduce to Newton's law for bodies at rest, and would not disagree with astronomical observations of slowly moving objects. He found that if speeds faster than light are allowed, then time can pass negatively, and when he excluded this possibility he was left with the proposition that gravity would travel at the same speed as light, but he concluded that further investigations were called for.

Space and time

The most surprising consequence of this work is that it did not open the way for Poincaré to embrace space-time. In his opinion in 1912 [40, see *Dernières pensées* 108] what changed with the work of Lorentz was that there were now two principles that could serve to define space: the old one involving rigid bodies, and a new one to do with the transformations that do not alter our differential equations. The first one had been a fundamental epistemological principle for Poincaré, but the new one is an experimental truth. Poincaré had always insisted that geometry belonged to mathematics, and could not be subject to revision by experimenters, and so physical relativity must become a convention, but whereas our conventional knowledge of geometry had formerly been rooted in the group of Euclidean isometries, it could now be rooted in the Lorentz group. The Lorentz group would guarantee our equations, at the price of placing us in a four-dimensional space. In an unmistakable reference to the ideas of Minkowski, although he did not mention his name, Poincaré went on [40, *Dernières pensées*, p. 108]:

"Time itself must be profoundly modified. Here are two observers, the first linked to fixed axes, and the second to moving axes, but each believing themselves to be at rest. Not only any figure, which the first one considers as a sphere, appear to the second as an ellipsoid; but two events which the first will consider as simultaneous will not be so for the second. Everything happens as if time were a fourth dimension of space, and as if four-dimensional space resulting from the combination of ordinary space and of time could rotate not only around an axis of ordinary space in such a way that time were not altered, but around any axis whatever. For the comparison to be mathematically accurate, it would be necessary to assign purely imaginary values to this fourth coordinate of space. [...] the essential thing is to notice that in the new conception space and time are no longer two entirely distinct entities which can be considered separately, but two parts of the same whole, two parts which are so closely knit that they cannot be easily separated."

So, Poincaré concluded [40, *Dernières pensées*, p. 109]:

"What shall be our position in view of these new conceptions? Shall we be obliged to modify our conclusions? Certainly not; we had adopted a convention because it seemed convenient and we had said that nothing could constrain us to abandon it. Today some physicists want to adopt a new convention. It is not that they are constrained to do so; they consider this new convention more convenient; that is all. And those who are not of this opinion can legitimately retain the old one in order not to disturb their old habits. I believe, just between us, that this is what they shall do for a long time to come."

Poincaré chose to be a Galilean to the end.

'Conformal' maps of several variables

Another topic that occupied Poincaré for many years was the function theory of several complex variables, and he was instrumental in extending the methods of potential theory. By the time he published his paper [37] on the subject in 1907, the German mathematician Friedrich Hartogs had done important work on what domains in $\mathbb{C}^n$ can be the domains of holomorphic functions [10]. This inspired Poincaré to study the nature of maps between domains in two complex variables, and he was able to show conclusively that the boundaries of some domains are such that there can be no regular map between the interiors of these domains. This showed that the Riemann mapping theorem cannot be extended to two complex dimensions, but Poincaré gave no explicit examples, which were only exhibited for the first time in Reinhardt's [43] in 1921.

Poincaré first observed that in single variable complex function theory, one can ask for a map that takes a curve ℓ and a point m on ℓ to a curve $\mathcal{L}$ with a point M on $\mathcal{L}$. The map is required to map m to M and to be regular in a neighbourhood of m. Or, one can ask for a map taking a closed curve ℓ bounding a domain d to a closed curve $\mathcal{L}$ bounding a domain D. The first of these problems, which Poincaré called the local problem, is always solvable in infinitely many ways, and the second, 'extended', problem has a unique solution via the Dirichlet principle.

Poincaré now looked at the analogous problems for analytic functions of two complex variables, and found that they behave very differently. The local problem asks for a map of a three-dimensional 'surface' (a hypersurface) s with a point m to a three-dimensional hypersurface S with a point M that takes m to M and is regular in a neighbourhood of m. The extended problem asks for a map of a closed hypersurface s bounding a domain d to a closed hypersurface S bounding a domain D, and asks if there is a regular function that maps s to S and d to D.

Now, the extended problem in two complex dimensions always has a solution. Poincaré noted that this follows directly from one of Hartogs's theorems, and Poincaré also sketched his own proof of that result. However, as Poincaré showed, the local problem will not always have a solution because it asks for three functions that satisfy four differential equations. He deduced that the local question is one about types of surfaces, which can be classified according to their groups of analytic automorphisms (for, if two surfaces correspond under an analytic automorphism, their groups are necessarily conjugate, and so the surfaces belong to the same class). In particular, if a surface s admits only the identity analytic automorphism, then the local problem has at most one solution, else the automorphism can be used to generate a second solution. Poincaré relied on Lie's theory of transformation groups in Lie's *Theorie der Transformationsgruppen*, vol. 3, and Campbell's *Introductory Treatise* [4] to establish that there are 27 possible groups. He then showed explicitly that for most groups there is a hypersurface having that group as its analytic automorphism group, but some groups correspond to two-dimensional surfaces, so there are hypersurfaces that not analytically equivalent, and the main result of the paper is established. But Poincaré's account was very unspecific. He described the group of the hypersurface (hypersphere) with equation $z\bar{z} + z'\bar{z}' = 1$ explicitly (in Section 7), but all he did to exhibit a hypersurface with a different group was to indicate how its equation could be found by means of Lie's theory.

Conclusions

Poincaré thought deeply about what it is to do good mathematics, to find what he once called the 'soul of the fact' around which a body of theory grouped itself in the most perspicacious way and enabled to mathematician to do the most effective work. (A fuller account of Poincaré's work will appear in [8].) In both his pure mathematics and his work on mathematical physics he advocated the study of what he called form over matter, by which he meant the abstract system of relations rather than specific objects that obey those relations. He looked for analogies that would enable a system of relations to be moved, with a degree of fidelity, from one field to another. In all of these ways the group idea was vital to him. In number theory it enabled him to place the modular equation in a richer setting, one that opened the way for some types of Fuchsian functions to do arithmetic work. The group idea was crucial to his analysis of surfaces at the start of his career, and his work on 3-manifolds towards the end. It allowed him to classify domains of holomorphic functions in $\mathbb{C}^2$. It allowed him to give an account of how we construct space around us, and also of the space and time of contemporary physics (but not space-time, a step he refused to take). But in none of these areas did he then pause to study the groups in any detail. It was the idea of a group, not group theory, that animated him.

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