Independence of hyperlogarithms over function fields via algebraic combinatorics.

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Abstract. We obtain a necessary and sufficient condition for the linear independence of solutions of differential equations for hyperlogarithms. The key fact is that the multiplier (i.e. the factor M in the differential equation $dS = MS$) has only singularities of first order (Fuchsian-type equations) and this implies that they freely span a space which contains no primitive. We give direct applications where we extend the property of linear independence to the largest known ring of coefficients.

Keywords : Noncommutative differential equations, Hyperlogarithms, Fuchsian-type equations.

1 Introduction

In his 1928 study of the solutions of linear differential equations following Poincaré, Lappo-Danilevski introduced the so-called *hyperlogarithmic functions of order m*, functions of iterated integrals of the following form with logarithmic poles [1] :

$$L(a_0, \ldots, a_n|z, z_0) = \int_{z_0}^{z} \int_{z_0}^{s_n} \ldots \int_{z_0}^{s_1} \frac{ds_0}{s_0 - a_0} \ldots \frac{ds_n}{s_n - a_n}, \tag{1}$$

where z_0 is a fixed point. It suffices that $z_0 \neq a_0$ for this iterated integral to converge. The classical polylogarithm Li_n is a particular case of these integrals [2] :

$$\mathrm{Li}_n(z) = \int_{0}^{z} \int_{0}^{s_n} \ldots \int_{0}^{s_2} \frac{ds_1}{1 - s_1} \frac{ds_2}{s_2} \ldots \frac{ds_n}{s_n} = -L(1, \underbrace{0, \ldots, 0}_{n-1 \text{ times}} |z, 0). \tag{2}$$

These iterated integrals also appear in quantum electrodynamics (see [3, 4] for example). Chen [5] studied them systematically and provided a noncommutative algebraic context in which to treat them. Fliess [6, 7] encoded these iterated integrals by words over a finite alphabet and extended them to a symbolic calculus[1] for nonlinear differential equations of the following form, in the context of noncommutative formal power series :

$$\begin{cases} y(z) &= f(q(z)), \\ \dot{q}(z) &= \displaystyle\sum_{i=0}^{m} \frac{A_i(q)}{z - a_i}, \\ q(z_0) &= q_0, \end{cases} \tag{3}$$

where the state $q = (q_1, \ldots, q_n)$ belongs to a complex analytic manifold of dimension N, q_0 denotes the initial state, the observable f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_N]\!]$, and $\{A_i\}_{i=0,n}$ is the polysystem defined as follows

$$A_i(q) = \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \tag{4}$$

[1] A kind of Feynman-like operator calculus [8].

with, for any $j = 1, \dots, n$, $A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \dots, q_N]\!]$.
By introducing the encoding alphabet $X = \{x_0, \dots, x_m\}$, the method of Fliess consists in exhibiting two formal power series over the monoid X^* :

$$F := \sum_{w\in X^*} \mathcal{A}(w) \circ f_{|_{q_0}}\ w \text{ and } C := \sum_{w\in X^*} \alpha_{z_0}^z(w)\ w \tag{5}$$

in order to compute the output y. These series are subject to convergence conditions (precisely speaking, the convergence of a duality pairing), as follows:

$$y(z) = \langle F||C\rangle := \sum_{w\in X^*} \mathcal{A}(w) \circ f_{|_{q_0}}\ \alpha_{z_0}^z(w), \tag{6}$$

where

– $\mathcal{A}$ is a morphism of algebras from $\mathbb{C}\langle\langle X\rangle\rangle$ to the algebra generated by the polysystem $\{A_i\}_{i=0,n}$:

$$\mathcal{A}(1_{X^*}) = \text{identity}, \tag{7}$$

$$\forall w = vx_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v)A_i \tag{8}$$

– $\alpha_{z_0}^z$ is a shuffle algebra morphism from $(\mathbb{C}\langle\langle X\rangle\rangle, ⧢)$ to some differential field $\mathcal{C}$:

$$\alpha_{z_0}^z(1_{X^*}) = 1, \tag{9}$$

$$\forall w = vx_i, x_i \in X, v \in X^*, \quad \alpha_{z_0}^z(w) = \int_{z_0}^z \frac{\alpha_{z_0}^s(v)}{s - a_i}. \tag{10}$$

Formula (6) also states that the iterated integrals over the rational functions

$$u_i(z) = \frac{1}{z - a_i},\ \ i = 0, .., n, \tag{11}$$

span the vector space $\mathcal{C}$.

As for the linear differential equations, the essential difficulty is to construct the fundamental system of solutions, or the Picard-Vessiot extension, to describe the space of solutions of the differential system (3) algorithmically [9]. For that, one needs to prove the linear independence of the iterated integrals in order to obtain the *universal* Picard-Vessiot extension. The $\mathbb{C}$-linear independence was already shown by Wechsung [10]. His method uses a recurrence based on the total degree. However this method cannot be used with variable coefficients. Another proof was given in [11] based on monodromy. In this note we describe a general theorem in differential computational algebra and show that, at the cost of using variable domains (which is the realm of germ spaces), and replacing the recurrence on total degree by a recursion on the words (with graded lexicographic ordering), one can encompass the previous results mentioned above and obtain much larger rings of coefficients and configuration alphabets (even infinite of continuum cardinality).

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2 Non commutative differential equations.

We recall the Dirac-Schützenberger notation, as in [12–14]. Let X be an alphabet and R be a commutative ring with unit. The algebra of noncommutative polynomials is the algebra $R[X^*]$ of the free monoid X^*. As an R-module, $R^{(X^*)}$ is the set of finitely supported R-valued function on X^* and, as such, it is in natural duality with the algebra of all functions on X^* (the large algebra of X^* [15]), $R^{X^*} = R\langle\langle X\rangle\rangle$, the duality being given, for $f \in R\langle\langle X\rangle\rangle$ and $g \in R[X^*]$, by

$$\langle f|g\rangle = \sum_{w\in X^*} f(w)g(w)\ . \tag{12}$$

The rôle of the ring is played here by a commutative differential k-algebra $(\mathcal{A}, d)$; that is, a k-algebra $\mathcal{A}$ (associative and commutative with unit) endowed with a distinguished derivation $d \in \mathfrak{Der}(\mathcal{A})$ (the ground field k is supposed commutative and of characteristic zero). We assume that the ring of constants $ker(d)$ is precisely k.

An alphabet X being given, one can at once extend the derivation d to a derivation of the algebra $\mathcal{A}\langle\langle X\rangle\rangle$ by

$$\mathbf{d}(S) = \sum_{w\in X^*} d(\langle S|w\rangle)w \ . \tag{13}$$

We now act with this derivation $\mathbf{d}$ on the power series C given in (5). We then get :

$$\mathbf{d}(C) = \Big(\sum_{i=1}^{m} u_i x_i\Big)C \ . \tag{14}$$

We are now in a position to state the main theorem which resolves many important questions, some of which we shall see in the applications.

Theorem 1. *Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($ch(k) = 0$) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation*

$$\mathbf{d}(S) = MS \ ; \ \langle S|1\rangle = 1 \tag{15}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x\in X} u_x x \in \mathcal{C}\langle\langle X\rangle\rangle \ . \tag{16}$$

The following conditions are equivalent :

i) The family $(\langle S|w\rangle)_{w\in X^*}$ of coefficients of S is free over $\mathcal{C}$.
ii) The family of coefficients $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathcal{C}$.
iii) The family $(u_x)_{x\in X}$ is such that, for $f \in \mathcal{C}$ and $\alpha_x \in k$

$$d(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0) \ . \tag{17}$$

iv) The family $(u_x)_{x\in X}$ is free over k and

$$d(\mathcal{C}) \cap \mathrm{span}_k\Big((u_x)_{x\in X}\Big) = \{0\} \ . \tag{18}$$

Proof — (i)$\Longrightarrow$(ii) Obvious.
(ii)$\Longrightarrow$(iii)
Suppose that the family $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ (coefficients taken at letters and the empty word) of coefficients of S is free over $\mathcal{C}$ and let us consider the relation as in eq. (17)

$$d(f) = \sum_{x\in X} \alpha_x u_x \ . \tag{19}$$

We form the polynomial $P = -f1_{X^*} + \sum_{x\in X} \alpha_x x$. One has $\mathbf{d}(P) = -d(f)1_{X^*}$ and

$$d(\langle S|P\rangle) = \langle \mathbf{d}(S)|P\rangle + \langle S|\mathbf{d}(P)\rangle = \langle MS|P\rangle - d(f)\langle S|1_{X^*}\rangle = (\sum_{x\in X} \alpha_x u_x) - d(f) = 0 \tag{20}$$

whence $\langle S|P\rangle$ must be a constant, say $\lambda \in k$. For $Q = P - \lambda.1_{X^*}$, we have

$$\mathrm{supp}(Q) \subset X \cup \{1_{X^*}\} \text{ and } \langle S|Q\rangle = \langle S|P\rangle - \lambda\langle S|1_{X^*}\rangle = \langle S|P\rangle - \lambda = 0 \ .$$

This implies that $Q = 0$ and, as $Q = -(f + \lambda)1_{X^*} + \sum_{x\in X} \alpha_x x$, one has, in particular, all the $\alpha_x = 0$.
(iii)$\Longleftrightarrow$(iv)
Obvious, (iv) being a geometric reformulation of (iii).

(iii)$\Longleftrightarrow$(i)
Let $\mathcal{K}$ be the kernel of $P \mapsto \langle S|P\rangle$ (a linear form $\mathcal{C}\langle X\rangle \to \mathcal{C}$) i.e.

$$\mathcal{K} = \{P \in \mathcal{C}\langle X\rangle | \langle S|P\rangle = 0\} \ . \tag{21}$$

If $\mathcal{K} = \{0\}$, we are done. Otherwise, let us adopt the following strategy.
First, we order X by some well-ordering $<$ ([16] III.2.1) and X^* by the graded lexicographic ordering $\prec$ defined by

$$u \prec v \Longleftrightarrow |u| < |v| \text{ or } (u = pxs_1 \ , \ v = pys_2 \text{ and } x < y). \tag{22}$$

It is easy to check that $\prec$ is also a well-ordering relation. For each nonzero polynomial P, we denote by $lead(P)$ its leading monomial; i.e. the greatest element of its support $\mathrm{supp}(P)$ (for $\prec$).
Now, as $\mathcal{R} = \mathcal{K} - \{0\}$ is not empty, let w_0 be the minimal element of $lead(\mathcal{R})$ and choose a $P \in \mathcal{R}$ such that $lead(P) = w_0$. We write

$$P = fw_0 + \sum_{u \prec w_0} \langle P|u\rangle u \ ; \ f \in \mathcal{C} - \{0\} \ . \tag{23}$$

The polynomial $Q = \frac{1}{f}P$ is also in $\mathcal{R}$ with the same leading monomial, but the leading coefficient is now 1; and so Q is given by

$$Q = w_0 + \sum_{u \prec w_0} \langle Q|u\rangle u \ . \tag{24}$$

Differentiating $\langle S|Q\rangle = 0$, one gets

$$0 = \langle \mathbf{d}(S)|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle MS|Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle S|M^\dagger Q\rangle + \langle S|\mathbf{d}(Q)\rangle = \langle S|M^\dagger Q + \mathbf{d}(Q)\rangle \tag{25}$$

with

$$M^\dagger Q + \mathbf{d}(Q) = \sum_{x \in X} u_x(x^\dagger Q) + \sum_{u \prec w_0} d(\langle Q|u\rangle)u \in \mathcal{C}\langle X\rangle \ . \tag{26}$$

It is impossible that $M^\dagger Q + \mathbf{d}(Q) \in \mathcal{R}$ because it would be of leading monomial strictly less than w_0, hence $M^\dagger Q + \mathbf{d}(Q) = 0$. This is equivalent to the recursion

$$d(\langle Q|u\rangle) = -\sum_{x \in X} u_x \langle Q|xu\rangle \ ; \ \text{for } x \in X \ , \ v \in X^*. \tag{27}$$

From this last relation, we deduce that $\langle Q|w\rangle \in k$ for every w of length $deg(Q)$ and, because $\langle S|1\rangle = 1$, one must have $deg(Q) > 0$. Then, we write $w_0 = x_0 v$ and compute the coefficient at v

$$d(\langle Q|v\rangle) = -\sum_{x \in X} u_x \langle Q|xv\rangle = \sum_{x \in X} \alpha_x u_x \tag{28}$$

with coefficients $\alpha_x = -\langle Q|xv\rangle \in k$ as $|xv| = \deg(Q)$ for all $x \in X$. Condition (17) implies that all coefficients $\langle Q|xu\rangle$ are zero; in particular, as $\langle Q|x_0 u\rangle = 1$, we get a contradiction. This proves that $\mathcal{K} = \{0\}$.
□

3 Applications

Let V be a connected and simply connected analytic variety (for example, the doubly cut plane $\mathbb{C} - (]-\infty, 0[\cup]1, +\infty[)$, the Riemann sphere or the universal covering of $\mathbb{C} - \{0, 1\}$), and let $\mathcal{H} = C^\omega(V, \mathbb{C})$ be the space of analytic functions on V.

It is possible to enlarge the range of scalars to coefficients that are analytic functions with variable domains $f : \ dom(f) \to \mathbb{C}$.

Definition 1. *We define a differential field of germs as the data of a filter basis $\mathcal{B}$ of open connected subsets of V, and a map $\mathcal{C}$ defined on $\mathcal{B}$ such that for every $U \in \mathcal{B}$, $\mathcal{C}[U]$ is a subring of $C^\omega(U, \mathbb{C})$ and*

1. $\mathcal{C}$ is compatible with restrictions i.e. if $U, V \in \mathcal{B}$ and $V \subset U$, one has

$$res_{VU}(\mathcal{C}[U]) \subset \mathcal{C}[V]$$

2. if $f \in \mathcal{C}[U] \setminus \{0\}$ then there exists $V \in \mathcal{B}$ s.t. $V \subset U - \mathcal{O}_f$ and f^{-1} (defined on V) is in $\mathcal{C}[V]$.

There are important cases where the conditions (2) are satisfied as shown by the following theorem.

Theorem 2. *Let V be a simply connected non-void open subset of $\mathbb{C} - \{a_1, \cdots a_n\}$ ($\{a_1, \cdots a_n\}$ are distinct points), $M = \sum_{i=1}^n \frac{\lambda_i x_i}{z - a_i}$ be a multiplier on $X = \{x_1, \cdots x_n\}$ with all $\lambda_i \neq 0$ and S be any regular solution of*

$$\frac{d}{dz} S = MS \ . \tag{29}$$

Then, let $\mathcal{C}$ be a differential field of functions defined on V which does not contain linear combinations of logarithms on any domain but which contains z and the constants (as, for example the rational functions). If $U \in \mathcal{B}$ (i.e. U is a domain of $\mathcal{C}$) and $P \in \mathcal{C}[U]\langle X \rangle$, one has

$$\langle S|P \rangle = 0 \Longrightarrow P = 0 \tag{30}$$

Proof — Let $U \in \mathcal{B}$. For every non-zero $Q \in \mathcal{C}[U]\langle X \rangle$, we denote by $lead(Q)$ the greatest word in the support of Q for the graded lexicographic ordering $\prec$. We endow X with an arbitrary linear ordering, and call Q monic if the leading coefficient $\langle Q|lead(Q) \rangle$ is 1. A monic polynomial is then given by

$$Q = w + \sum_{u \prec w} \langle Q|u \rangle u \ . \tag{31}$$

Now suppose that it is possible to find U and $P \in \mathcal{C}[U]\langle X \rangle$ (not necessarily monic) such that $\langle S|P \rangle = 0$; we choose P with $lead(P)$ minimal for $\prec$.

Then

$$P = f(z)w + \sum_{u \prec w} \langle P|u \rangle u \tag{32}$$

with $f \not\equiv 0$. Thus $U_1 = U \setminus \mathcal{O}_f \in \mathcal{B}$ and $Q = \frac{1}{f(z)} P \in \mathcal{C}[U_1]\langle X \rangle$ is monic and satisfies

$$\langle S|Q \rangle = 0 \ . \tag{33}$$

Differentiating eq. (33), we get

$$0 = \langle S'|Q \rangle + \langle S|Q' \rangle = \langle MS|Q \rangle + \langle S|Q' \rangle = \langle S|Q' + M^\dagger Q \rangle \ . \tag{34}$$

Remark that one has

$$Q' + M^\dagger Q \in \mathcal{C}[U_1]\langle X \rangle \tag{35}$$

If $Q' + M^\dagger Q \neq 0$, one has $lead(Q' + M^\dagger Q) \prec lead(Q)$ and this is not possible because of the minimality hypothesis of $lead(Q) = lead(P)$. Hence, one must have $R = Q' + M^\dagger Q = 0$. With $|w| = n$, we now write

$$Q = Q_n + \sum_{|u| < n} \langle Q|u \rangle u \tag{36}$$

where $Q_n = \sum_{|u|=n} \langle Q|u \rangle u$ is the dominant homogeneous component of Q. For every $|u| = n$ we have

$$(\langle Q|u \rangle)' = -\langle M^\dagger Q|u \rangle = -\langle Q|Mu \rangle = 0 \tag{37}$$

thus all the coefficients of Q_n are constant.

If $n = 0$, $Q \neq 0$ is constant which is impossible by eq. (33) and because S is regular. If $n > 0$, for any word $|v| = n - 1$, we have

$$(\langle Q|v \rangle)' = -\langle M^\dagger Q|v \rangle = -\langle Q|Mv \rangle = -\sum_{i=0}^n \frac{\lambda_i}{z - a_i} \langle Q|x_i v \rangle = -\sum_{i=0}^n \frac{\lambda_i}{z - a_i} \langle Q_n|x_i v \rangle \tag{38}$$

bcause all $x_i v$ are of length n.
Then

$$\langle Q|v\rangle = -\sum_{i=0}^{n}\langle Q_n|x_i v\rangle \int_{\alpha}^{z} \frac{\lambda_i}{s-a_i} ds + const \tag{39}$$

But all the functions $\int_{\alpha}^{z} \frac{\lambda_i}{s-a_i} ds$ are linearly independent over $\mathbb{C}$ and not all the scalars $\langle Q_n|x_i v\rangle$ are zero (write $w = x_k v$ and choose v accordingly). This contradicts the fact that $Q \in \mathcal{C}[U_1]\langle X\rangle$ as $\mathcal{C}$ contains no linear combination of logarithms. □

Corollary 1 *Let V be as above and R be the ring of functions which can be analytically extended to some $V \cup U_{a_1} \cup U_{a_2} \cup \cdots U_{a_n}$ where U_{a_i} are open neighborhoods of $a_i, i = 1 \cdots n$ and have non-essential singularities at these points. Then, the set of hyperlogarithms $(\langle S|w\rangle)_{w\in X^*}$ are linearly independent over R.*

Remark 1. i) If a series $S = \sum_{w\in X^*}\langle S|w\rangle w$ is a regular solution of (29) and satisfies the equivalent conditions of the theorem (2), then so too does every Se^C (with $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$) .

ii) Series such as that of polylogarithms and all the exponential solutions of equation

$$\frac{d}{dz}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z})S \tag{40}$$

satisfy the conditions of the theorem (2) as shown by theorem (2).

iii) Call $\mathcal{F}(S)$ the vector space generated by the coefficients of the series S. One may ask what happens when the conditions for independence are not satisfied.
In fact, the set of Lie series $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle \mathbb{X}\rangle\rangle$ such that there exists a $\phi \in \mathrm{End}(\mathcal{F}(S))$ (thus a derivation) such that $SC = \phi(S)$, is a closed Lie subalgebra of $\mathrm{Lie}_{\mathbb{C}}\langle\langle \mathbb{X}\rangle\rangle$ which we will denote by Lie_S. For example
   - for $X = \{x_0, x_1\}$ and $S = e^{zx_0}$ one has $x_0 \in Lie_S$; $x_1 \notin Lie_S$
   - for $X = \{x_0, x_1\}$ and $S = e^{z(x_0+x_1)}$, one has $x_0, x_1 \notin Lie_S$ but $(x_0 + x_1) \in Lie_S$.

iv) Theorem (2) holds *mutatis mutandis* when the multiplier is infinite i.e.

$$M = \sum_{i\in I} \frac{\lambda_i x_i}{z-a_i}$$

even if I is continuum infinite (say $I = \mathbb{R}$, singularities being all the reals).

v) Theorem (2) no longer holds with singularities of higher order (i.e. not Fuchsian). For example, with

$$M = \frac{x_0}{z^2} + \frac{x_1}{(1-z)^2} \ . \tag{41}$$

Firstly, the differential field $\mathcal{C}$ generated by

$$u_0 = \frac{1}{z^2},\ u_1 = \frac{1}{(1-z)^2} \tag{42}$$

contains

$$\frac{d}{dz}(\frac{1}{2u_0}) = z \tag{43}$$

and hence $\mathcal{C} = \mathbb{C}(z)$, the field of rational functions over $\mathbb{C}$. Condition (ii) of Theorem (2) is not satisfied (as $z^2 u_0 - (1-z)^2 u_1 = 0$). Moreover, one has also $\mathbb{Z}$-dependent relations such as

$$\langle S|x_1x_0\rangle + \langle S|x_0x_1\rangle + \langle S|x_1\rangle - \langle S|x_0\rangle = 0 \ . \tag{44}$$

4 Through the looking glass: passing from right to left.

We are still in the context of analytic functions as above. A series $S \in \mathcal{H}\langle\langle X\rangle\rangle$ is said to be group-like if

$$\Delta(S) = S \otimes S \tag{45}$$

where Δ is the dual of the shuffle product [14] defined on series by $\Delta(S) = \sum_{w\in X^*} \langle S|w\rangle \Delta(w)$ and on the words by the recursion ($x \in X, u \in X^*$)

$$\Delta(1_{X^*}) = 1_{X^*} \otimes 1_{X^*} \ ; \ \Delta(xu) = (x \otimes 1_{X^*} + 1_{X^*} \otimes x)\Delta(u) \tag{46}$$

Let $S \in \mathcal{H}\langle\langle X\rangle\rangle$. We call $\mathcal{F}(S)$ the $\mathbb{C}$-vector space generated by the coefficients of S. One has

$$\mathcal{F}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}\langle X\rangle} \ . \tag{47}$$

We recall that, for $a \in X$ and $w \in X^*$, the partial degree $|w|_a$ is the number of occurrences of a in w, it is defined by the recursion

$$|1_{X^*}|_a = 0 \ ; \ |bu|_a = \delta b, a + |u|_a \ . \tag{48}$$

Or course the length of the word is the sum of the partial degrees i.e. $|w| = \sum_{x\in X} |w|_x$. The function $a \mapsto |w|_a$ belongs to $\mathbb{N}^{(X)}$ (finitely supported functions from X to $\mathbb{N}$). For $\alpha \in \mathbb{N}^{(X)}$, we note $\mathbb{C}_{\leq\alpha}\langle X\rangle$, the set of polynomials $Q \in \mathbb{C}\langle X\rangle$ such that $supp(Q) \subset X^{\leq\alpha}$ i.e.

$$\langle Q|w\rangle \neq 0 \Longrightarrow (\forall x \in X)(|w|_x \leq \alpha(x)) \tag{49}$$

In the same way, we consider the filtration by total degree (length)

$$\mathbb{C}_{\leq n}\langle X\rangle = \sum_{|\alpha|\leq n} \mathbb{C}_{\leq\alpha}\langle X\rangle \ . \tag{50}$$

We use the following increasing filtrations

$$\mathcal{F}_{\leq\alpha}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}_{\leq\alpha}\langle X\rangle} \ . \tag{51}$$

or

$$\mathcal{F}_{\leq n}(S) = \{\langle S|P\rangle\}_{P\in\mathbb{C}_{\leq n}\langle X\rangle} \ . \tag{52}$$

Proposition 1. *We have the following properties :*

i) If $T \in \mathbb{C}\langle\langle X\rangle\rangle$ then $\mathcal{F}(ST) \subset \mathcal{F}(S)$ and one has equality if T is invertible.

ii) If S is group-like, then $\mathcal{F}(S)$ is a unital sub-algebra of $\mathcal{H}$, which is filtered w.r.t. (51) and (52) i.e.

$$\mathcal{F}_{\leq\alpha}(S)\mathcal{F}_{\leq\beta}(S) \subset \mathcal{F}_{\leq\alpha+\beta}(S) \tag{53}$$

Proof — (i) The space $\mathcal{F}(ST)$ is spanned by the

$$\langle ST|w\rangle = \sum_{uv=w} \langle S|u\rangle\langle T|v\rangle \in \mathcal{F}(S)$$

and if T is invertible one has $\mathcal{F}(S) = \mathcal{F}(STT^{-1}) \subset \mathcal{F}(ST)$ which proves the equality.
ii) If S is group-like, one has

$$\langle S|u\rangle\langle S|v\rangle = \langle S \otimes S|u \otimes v\rangle = \langle \Delta(S)|u \otimes v\rangle = \langle S|u \sqcup\!\sqcup v\rangle \tag{54}$$

In the case when all the functions $\langle S|w\rangle$ are $\mathbb{C}$-linearly independent, one has a correspondence between the differential Galois group (acting on the right) of a differential equation of type (40) (acting on the right) and the group of automorphisms of $\mathcal{F}(S)$ compatible with the preceding filtration (they turn out to be unipotent).

Proposition 2. *Let S be a group-like series. The following conditions are equivalent:*

i) For every $x \in X$, $ker_{\mathbb{C}}(S) \subset ker_{\mathbb{C}}(Sx)$.

ii) For every $x \in X$, there is a derivation $\delta_x \in \mathfrak{Der}(\mathcal{F}(S))$ such that

$$\delta_x(S) = Sx \tag{55}$$

iii) For every $x \in X$, there is a one-parameter group of automorphisms $\phi_x^t \in Aut(\mathcal{F}(S))$; $t \in \mathbb{R}$ such that

$$\phi_x^t(S) = Se^{tx} \tag{56}$$

iv) For every $C \in \mathrm{Lie}_C\langle\langle X\rangle\rangle$, there is $\delta \in \mathfrak{Der}(\mathcal{F}(S))$ such that

$$\delta(S) = SC \tag{57}$$

v) For every $C \in \mathrm{Lie}_C\langle\langle X\rangle\rangle$, there is $\phi \in Aut(\mathcal{F}(S))$ such that

$$\phi(S) = Se^C \tag{58}$$

vi) The functions $(\langle S|w\rangle)_{w\in X^*}$ are $\mathbb{C}$-linearly independant.

Proof — $i) \Longrightarrow ii)$ From the inclusion, we deduce that, for all $x \in X$ there exists a $\mathbb{C}$-linear mapping $\phi \in \mathrm{End}(\mathcal{F}(S))$ such that for all $w \in \mathcal{M}$, $\phi(\langle S|w\rangle) = \langle Sx|w\rangle$. It must be a derivation of $\mathcal{F}(S)$ as

$$\begin{aligned} \phi(\langle S|u\rangle\langle S|v\rangle) = \phi(\langle S|u \sqcup\!\sqcup v\rangle) = \langle Sx|u \sqcup\!\sqcup v\rangle = \langle S|(u \sqcup\!\sqcup v)x^{-1}\rangle = \\ \langle S|(ux^{-1} \sqcup\!\sqcup v) + (u \sqcup\!\sqcup vx^{-1})\rangle = \langle S|(ux^{-1} \sqcup\!\sqcup v)\rangle\langle S|(u \sqcup\!\sqcup vx^{-1})\rangle = \\ \langle Sx|u\rangle\langle S|v\rangle + \langle S|u\rangle\langle Sx|v\rangle = \phi(\langle S|u\rangle)\langle S|v\rangle + \langle S|u\rangle\phi(\langle S|v\rangle) \end{aligned} \tag{59}$$

from the fact that $(\langle S|w\rangle)_{w\in X^*}$ spans $\mathcal{F}(S)$.
$ii) \Longrightarrow iv)$ As $(\langle S|w\rangle)_{w\in X^*}$ spans $\mathcal{F}(S)$, the derivation ϕ is uniquely defined. We denote it by δ_x, and notice that, in so doing, we have constructed a mapping $\Phi : X \to \mathfrak{Der}(\mathcal{F}(S))$, which is Lie algebra. Therefore, there is a unique extension of this mapping as a morphism $\mathrm{Lie}_{\mathbb{C}}\langle X\rangle \to \mathfrak{Der}(\mathcal{F}(S))$. This correspondence, which we denote by $P \to \delta(P)$, is (uniquely) recursively defined by

$$\delta(x) = \delta_x \ ; \ \delta([P,Q]) = [\delta(P), \delta(Q)] \ . \tag{60}$$

For $C = \sum_{n\geq 0} C_n \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$ with $C_n \in \mathrm{Lie}_{\mathbb{C}}\langle X\rangle_n$, we remark that the sequence $\langle S\sum_{0\leq n\leq N} C_n|w\rangle$ is stable (for large N). Set $\delta_{\leq N} := \delta(\sum_{0\leq n\leq N} C_n)$. We see that $\delta_{\leq N}$ is stable (for large N) on every $\mathcal{F}_\alpha$; we call its limit $\delta(C)$. It is clear that this limit is a derivation and that it corresponds to C.
$iv) \Longrightarrow v)$ For every $C = \sum_{n\geq 0} C_n \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$, the exponential e^C defines a mapping $\phi \in \mathrm{End}(\mathcal{F}(S))$ as indeed $e^{\delta_{\leq N}}$ is stationnary. It is easily checked that this mapping is an automorphism of algebra of $\mathcal{F}(S)$.
$v) \Longrightarrow iii)$ For $C_i \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle; i = 1,2$ which commute we have

$$Se^{C_1}e^{C_2} = \phi_{C_1}(S)e^{C_2} = \phi_{C_1}(Se^{C_2}) = \phi_{C_1}\phi_{C_2}(S). \tag{61}$$

This proves the existence, for a $C \in \mathrm{Lie}_{\mathbb{C}}\langle\langle X\rangle\rangle$, of a one-parameter (rational) group ϕ_C^t in $\mathrm{Aut}(\mathcal{F}(S))$ such that $Se^{tC} = \phi_C^t(S)$. This one-parameter (rational) group can be extended to $\mathbb{R}$ as continuity is easily checked by taking the scalar products $\langle\phi_C^t(S)|w\rangle = \langle Se^{tC}|w\rangle$ and it suffices to specialize the result to $C = x$.
$iii) \Longrightarrow ii)$ By stationary limits one has

$$\langle Sx|w\rangle = \lim_{t\to 0} \frac{1}{t}(\langle Se^{tx}|w\rangle - \langle S|w\rangle) = \lim_{t\to 0} \frac{1}{t}(\langle\phi_x^t(S)|w\rangle - \langle S|w\rangle) \tag{62}$$

$v) \Longrightarrow i)$ Let $x \in X, t \in \mathbb{R}$, we take $C = tx$ and $\phi_t \in \mathrm{Aut}(\mathcal{F}(S))$ s.t. $\phi_t(S) = Se^{tx}$. It there is $P \in \mathbb{C}\langle X\rangle$ such that $\langle S|P\rangle = 0$ one has

$$0 = \langle S|P\rangle = \phi_t(\langle S|P\rangle) = \langle\phi_t(S)|P\rangle = \langle Se^{tx}|P\rangle = \sum_{n=0}^{\deg(P)} \frac{t^n}{n!}\langle Sx^n|P\rangle \tag{63}$$

and then, for all $z \in V$, the polynomial

$$\sum_{n=0}^{\deg(P)} \frac{t^n}{n!} \langle S(z)x^n|P\rangle \tag{64}$$

is identically zero over $\mathbb{R}$ hence so are all of its coeefficients in particular $\langle S(z)x|P\rangle$ for all $z \in V$. This proves the claim.
i) $\Longrightarrow$ *vi)* Let $P \in ker_{\mathbb{C}}(S)$ if $P \neq 0$ take it of minimal degree with this property. For all $x \in X$, one has $P \in ker_{\mathbb{C}}(Sx)$ which means $\langle Sx|P\rangle = 0$ and then $Px^\dagger = 0$ as $\deg(Px^\dagger) = \deg(P) - 1$. The reconstruction lemma implies that

$$P = \langle P|1\rangle + \sum_{x\in X} (Px^\dagger)x = \langle P|1\rangle \tag{65}$$

Then, one has $0 = \langle S|P\rangle = \langle S|1\rangle\langle P|1\rangle = \langle P|1\rangle$ which shows that $ker_{\mathbb{C}}(S) = \{0\}$. This is equivalent to the statement (vi).
vi) $\Longrightarrow$ *i)* Is obvious as $ker_{\mathbb{C}}(S) = \{0\}$.
□

Remark 2. The derivations δ_x cannot in general be expressed as restrictions of derivations of $\mathcal{H}$. For example, with equation (40), one has $\delta_{x_0}(\frac{\log(z)^{n+1}}{(n+1)!}) = \frac{\log(z)^n}{n!}$ but $\delta_{x_0}(\langle S|ux_1\rangle) = 0$.

5 Conclusion

In this paper we showed that by using fields of germs, some difficult results can be considerably simplified and extended. For instance, polylogarithms were known to be independant over either $\mathbb{C}[z, 1/z, 1/(1-z)]$ or, presumably, over "functions which do not involve monodromy"; these two results are now encompassed by Theorem (2). We believe that this procedure is not only of theoretical importance, but can be taken into account at the computational level because every formula (especially analytic) carries with it its domain of validity. As a matter of fact, having at hand the linear independence of coordinate functions over large rings allows one to express uniquely solutions of systems like (3) in the basis of hyperlogarithms.
A valuable prospect would be to determine the asymptotic expansion at infinity of the Taylor coefficients of the $y(z)$ as given in (6) for the general case. This has been done already for the case of singularities at $\{0, 1\}$ and for different purposes (see arXiv:1011.0523v2 and http://fr.arxiv.org/abs/0910.1932).

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