Reducing the Effect of Noise on Chaos Synchronization without Capping

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Abstract

Research in electronic communications has developed chaos-based modelling to enable messages to be carried by chaotic spreading sequences. When such systems are used it is necessary to simultaneously know the resulting chaotic sequence at both the transmitting and receiving stations. This is possible using the idea of synchronization providing there is no noise present in the system. When noise is present in the transmission channel, recovery of the spreading sequence may be inaccurate or even impossible and the resulting sequence may no longer lie within the chaotic map range. A usual method of dealing with this problem is to cap iterations lying outside the range at their extremes, a procedure which increases the loss of synchronization. This paper discusses how synchronization can be improved by the transformation of the spreading sequence to be transmitted; the method uses knowledge of the invariant distribution of the chaotic spreading sequence, before noise corrupts it in the transmission channel. An 'inverse' transformation is applied at the receiver station with the result that the noise has a reduced impact on the synchronization and also on the subsequence recovery of the message.

1. Communications using Synchronization

The need to simultaneously know a chaotic sequence at each of the transmitting and receiving stations has led to much research in the area of synchronous chaos, see [5], [3] for relevant background. The basic idea revolves around the difference between two chaotic sequences $\{\varepsilon_t\}$ and $\{\varepsilon'_t\}$. Under synchronization, the absolute difference

$$\Delta\varepsilon_t = |\varepsilon'_t - \varepsilon_t|, \quad t = 0, 1, ...$$

is zero or quickly converges to zero. The idea that two dynamical systems can synchronize involves bivariate or higher dimensional chaotic sequences. The possibility of synchronization was initially noted by [6], and involves one of the two chaotic systems being driven by at least one component of the other system. The idea has been widely exploited in the continuous case; however, there is far less literature concerning the use of this property in relation to discrete maps, the topic of this paper.

A discrete bivariate dynamical system $\{X_t, \varepsilon_t\}$ with $t = 1, 2, \ldots$ is generated by

$$\begin{aligned}\{X_t, \varepsilon_t\} &= \tau\{X_{t-1}, \varepsilon_{t-1}\} \\ &\equiv (\tau_X\{X_{t-1}, \varepsilon_{t-1}\}, \tau_\varepsilon\{X_{t-1}, \varepsilon_{t-1}\}). \quad (1)\end{aligned}$$

The communication system starts by generating X_t from (1), initiated by $\{X_0, \varepsilon_0\}$, which is transmitted to the receiver station. Here the received X_t is used to give $\varepsilon_t^R = \tau_\epsilon\{X_{t-1}, \varepsilon_{t-1}^R\}$, using an arbitrary initial value ε_0^R. The receiver and transmitter will synchronize in ε_t if $|\Delta\varepsilon_t| = |\varepsilon_t^R - \varepsilon_t| \to 0$ as $t \to \infty$ where,

$$\Delta\varepsilon_t = \tau_\varepsilon(X_{t-1}, \varepsilon_{t-1}^R) - \tau_\varepsilon(X_{t-1}, \varepsilon_{t-1}). \quad (2)$$

First order approximation gives

$$|\Delta\varepsilon_t| \approx \left|\frac{\partial\tau_\varepsilon(X_{t-1}, \varepsilon_{t-1})}{\partial\epsilon}\right| |\Delta\varepsilon_{t-1}| \quad (3)$$

leading to

$$\ln|\Delta\varepsilon_t| \approx t\left[\frac{1}{t}\sum_{i=1}^{t-1}\ln\left|\frac{\partial\tau_\varepsilon(X_i, \varepsilon_i)}{\partial\epsilon}\right|\right] + \ln|\Delta\varepsilon_0|. \quad (4)$$

For a theoretical analysis the average term can be replaced with the ensemble expectation, giving synchronization if a quantity called the *conditional Lyapunov exponent*

$$\lambda_{\varepsilon|X} = E\left[\ln\left|\frac{\partial\tau_\varepsilon(X, \varepsilon)}{\partial\epsilon}\right|\right], \quad (5)$$

obeys the condition $\lambda_{\varepsilon|X} < 0$.

2. Invariant Distribution of Bivariate Chaotic Maps

The bivariate invariant distribution of a bivariate map is required for message detection. An outline is given next and more detail can be found in [4]. The distribution function satisfies the equation

$$\begin{aligned} P\{(X,\varepsilon) \le (x,\epsilon)\} &= P\{\tau^{-1}(X,\varepsilon) \le (x,\epsilon)\} \\ &= P\{g_X(X,\varepsilon) \le x, g_\varepsilon(X,\varepsilon) \le \epsilon\} \end{aligned} \tag{6}$$

where $g = (g_X, g_\varepsilon)$ are the pre-image functions such that

$$\{X_{t-1}, \varepsilon_{t-1}\} = \{g_X(X_t,\varepsilon_t), g_\varepsilon(X_t,\varepsilon_t)\}.$$

If the map has k (≥ 1) pre-images there are functions $\{g_{Xi}(x,\epsilon), g_{\varepsilon i}(x,\epsilon)\}$, $i = 1, 2, \ldots, k$, which are the solutions to

$$\tau\{g_{Xi}(x,\epsilon), g_{\varepsilon i}(x,\epsilon)\} = (x,\epsilon)$$

and called multiple pre-image functions. So then (6) can be written

$$P(X \le x, \varepsilon \le \epsilon) = \sum_{i=1}^{k}\{g_{Xi}(X,\varepsilon) \le x, g_{\varepsilon i}(X,\varepsilon) \le \epsilon\}. \tag{7}$$

This leads to a bivariate generalization of the Perron-Frobenius operator for the invariant density for $f(x,\epsilon)$ of (X,ε) as

$$Pf(x,\epsilon) = \sum f\{g_i(x,\epsilon)\}\left|\frac{\partial^2 g_i(x,\epsilon)}{\partial x\partial\epsilon}\right| \tag{8}$$

with the invariant equation being $f(x,\epsilon) = Pf(x,\epsilon)$.

As an illustration of a tractable bivariate map, consider the *Arnold cat map* [1], [4],

$$\{X_t, \varepsilon_t\} = \{(X_{t-1}+\varepsilon_{t-1}), (X_{t-1}+2\varepsilon_{t-1})\} \bmod (1) \tag{9}$$

which has unique pre-images

$$\{X_{t-1}, \varepsilon_{t-1}\} = \{(2X_t - \varepsilon_t), (\varepsilon_t - X_t)\} \bmod (1). \tag{10}$$

It can be shown that this map has an independent bivariate uniform invariant distribution over $[0,1]$ and this knowledge can be used to obtain the joint lagged moments $E[a(X_t,\varepsilon_t)b(X_{t+s},\varepsilon_{t+s})]$. Unusually the lagged and squared lagged auto and cross correlations are all zero. The map is potentially useful in communication systems, though not for systems where synchronization via a drive-response or master-slave system is necessary.

3. Synchronization of the Bivariate Logistic Map without Noise

The bivariate logistic map [2], [4] is suitable for drive-response synchronization. The bivariate logistic map cross couples two standard logistic maps and allows for some auto-memory. With $0 \le c \le 1$, it takes the form

$$\begin{pmatrix} X_t \\ \varepsilon_t \end{pmatrix} = \begin{pmatrix} (1-c)X_{t-1} + 4c\varepsilon_{t-1}(1-\varepsilon_{t-1}) \\ (1-c)\varepsilon_{t-1} + 4cX_{t-1}(1-X_{t-1}) \end{pmatrix} \tag{11}$$

for $0 \le X_t, \varepsilon_t \le 1$. The general conditional Lyapunov exponent is $\lambda_{\varepsilon|X} = \ln(1-c)$. If c is taken large enough, say $c = 0.9999$ then, $\lambda_{\varepsilon|X} = \ln 0.0001 \approx -9.21$ and hence the systems starting from nearby points should quickly synchronize in ε and ε^R. As an illustration, if the X variable is used to drive the response system, starting with $(X_0, \varepsilon_0) = (0.6, 0.4)$ and $\varepsilon_0^R = 0.7$, say, then ε in the transmitter and receiver are perfectly synchronized to six decimal places after only two iterations, with

$$\begin{aligned} \varepsilon_{0,\ldots,5} &= (0.400000, 0.959944, 0.153813, 0.520786) \\ \varepsilon^R_{0,\ldots,5} &= (0.700000, 0.959974, 0.153813, 0.520786). \end{aligned}$$

The invariant distribution is only available from numerical simulations, although for c near 1 it is very close to $beta(\frac{1}{2}, \frac{1}{2})$.

4. Synchronization of the Bivariate Logistic Map with Noise, Capping Adjustment

The fast synchronization of the bivariate logistic map, seen in Section 3, quickly deteriorates once channel noise is incorporated. By simply adding noise n_t to the transmitter variable X_t, the resulting signal $X'_t = X_t + n_t$ may escape the $[0,1]$ bounds of X_t and hence X'_t will be outside $[0,1]$ and no longer acceptable as input to the receiver map $\varepsilon^R_t = (1-c)\varepsilon^R_{t-1} + 4cX'_{t-1}(1-X'_{t-1})$. To avoid this problem a usual practice is to cap the variable X'_t at its bounds, [5]. For the bivariate logistic map, $(0,0)$ is a stationary point and therefore X'_t is capped at 1×10^{-10} and 1 and therefore generated as

$$X'_t = \begin{cases} X_t + n_t & \text{if} \quad 0 \le X_t + n_t \le 1 \\ 1 & \text{if} \quad X_t + n_t > 1 \\ 1\times10^{-10} & \text{if} \quad X_t + n_t < 0. \end{cases} \tag{12}$$

Figure 1 illustrates the synchronization problem when the transmitted variable is subject to strong AWGN with variance $\sigma_n^2 = 1/8$, that of X; thus the *signal to noise ratio*, $SNR = 20\log(\sigma_X/\sigma_n)$, is 0. In practice, the signal to noise ratio can be adjusted by applying an amplification factor to $\{X_t\}$ on the left hand side of (12) as it is transmitted and suitably redefining the capping. When X'_t exceeds the bounds $[0,1]$ the capping results in an enforced period of desynchronization of ε^R and ε. There are less severe problems with synchronization in the middle of the range where the variable is only slightly perturbed. Thus if an alternative to capping could be used, the synchronization should be improved, and this is the subsequent direction of the contribution.

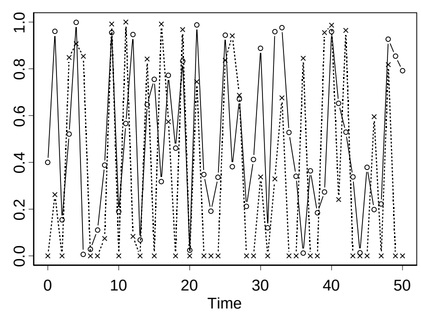


Figure 1: Synchronization of ε ($\circ$) and ε^R ($\times$) using capping with $SNR = 0$ and the double logistic map.

5. Reducing the Effect of Noise by Transformation

This section is concerned in a general way with transformation as an alternative to capping. To avoid a bounded variable escaping its bounds $[a, b]$, say, after adding noise, one alternative to capping is to first transform the variable from its closed interval $[a, b]$ to an infinite range $(-\infty, \infty)$. Adding noise to this transformed variable would obviously have no effects on the bounds. A reverse transformation could then be applied, so the resulting variable was again in the original $[a, b]$ range. Suitable inverse transformations will result in a variable X'_t which has the same invariant distribution as the original X_t variable.

It will be shown in Section 6 that in communication use the transformed transmitted variable reduces the loss of synchronization of ε and ε^R caused by capping at the limits, and avoids boundary values, shown in Figure 1. The general method will be demonstrated with noise added to uniform and beta distributed values.

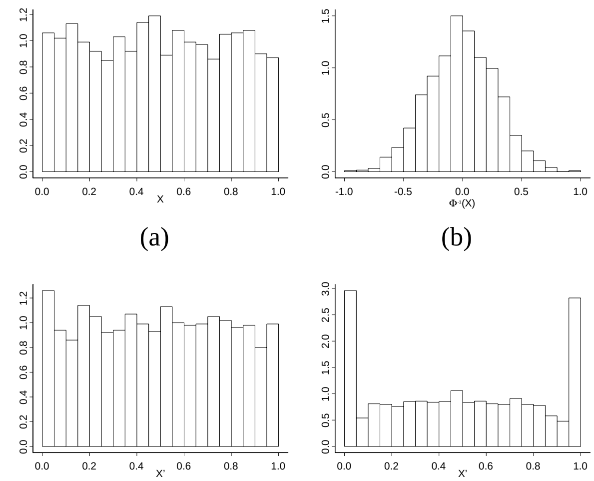
5.1. Uniform distribution

Consider a general random variable X having a marginal distribution which is uniform over $[0, 1]$. This variable is transformed to $[-\infty, \infty]$, using an inverse $N(0, \sigma_X^2)$ distribution function, where σ_X^2 is the variance of the marginal distribution of X. Normally distributed noise, $n \sim N(0, \sigma_n^2)$, is added; σ_n^2 is assumed to be known. The resulting variable also follows a Normal distribution. This can then be transformed back to $[0, 1]$. So the original variable with noise added is

$$X' = \Phi_{N(0,\sigma_X^2+\sigma_n^2)}\{\Phi^{-1}_{N(0,\sigma_X^2)}(X) + n\}, \qquad (13)$$

where $\Phi_{N(0,\sigma^2)}$ is the distribution function of a Gaussian random variable with mean 0 and variance σ^2. X' has a uniform $[0, 1]$ distribution like X.

Consider the following example in Figure 2 where a random sample of size 2000 is drawn from a uniform $[0, 1]$ distribution, $\sigma_X^2 = 1/12$ and $n \sim N(0, 1/12)$ giving $SNR = 0$, as shown in panels (a) and (b); the resulting distribution of X' in panel (c) appears to be uniformly distributed. Hence the form of the distribution is preserved. However, when the capping method is used, the resulting distribution of X' is far from uniform, as illustrated by panel (d).



(a) (b) (c) (d)

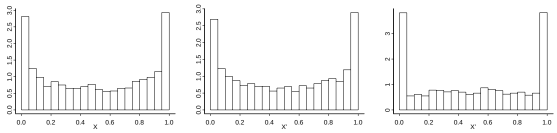
Figure 2: Distribution of (a) x, (b) $\Phi^{-1}_{N(0,1/12)}(x) + n$, (c) x' under transformation, (d) x' under capping.

5.2. Beta distribution

The transformation approach always works for any distribution. Suppose X has a marginal invariant distribution which is $beta(\frac{1}{2}, \frac{1}{2})$ over $[0, 1]$. This can be initially transformed to a uniform distribution, applying the earlier procedure, then finally transforming back, resulting in a beta distribution; thus corresponding to (13),

$$X' = beta^{-1}[\Phi_{N(0,\sigma_X^2+\sigma_n^2)}\{\Phi^{-1}_{N(0,\sigma_X^2)}(beta(X)) + n\}] \qquad (14)$$

where $beta(.)$ is the distribution function of the $beta(\frac{1}{2}, \frac{1}{2})$ distribution. Again using $SNR = 0$, Figure 3 panel (a) shows a random sample of 2000 from a $beta(\frac{1}{2}, \frac{1}{2})$ distribution. Panel (b) shows the preserved beta distribution using the transformation method, compared to the capping method in panel (c).



(a) (b) (c)

Figure 3: Distribution of (a) x, (b) x' for transformation, (c) x' for capping.

6. Chaos Synchronization with the Bivariate Logistic Map - Comparing Capping and Transformation to Reduce Noise Effects

The method of capping, used in Section 4 to keep X_t' in the range of the bivariate map τ, is replaced by the preferable method of transformation discussed in Section 5. Continuing to use the bivariate logistic map as the basis of the communication system, in which $\{\varepsilon_t\}$ and $\{\varepsilon_t^R\}$ are synchronized by $\{X_t\}$, the marginal distribution of $\{X_t\}$ is taken as a $beta(\frac{1}{2}, \frac{1}{2})$ distribution [4]. Thus to generate the received values $\{X_t'\}$ and hence generate the $\{\varepsilon_t^R\}$ as discussed in Section 1, (14) is adapted to the form

$$X_t' = beta^{-1}[\Phi_{N(0,\sigma_X^2+\sigma_n^2)}\{\Phi^{-1}_{N(0,\sigma_X^2)}(beta(X_t)) + n_t\}]. \quad (15)$$

In a communication context, as when using capping, an adjustable signal to noise ratio needs to be incorporated into the transmission process. Thus an amplification factor a is included in the transformation of X_t before its transmission. The equation (15) in the communications context then becomes

$$X_t' = beta^{-1}[\Phi_{N(0,a^2\sigma_X^2+\sigma_n^2)}\{\Phi^{-1}_{N(0,a^2\sigma_X^2)}(beta(X_t))+n_t\}]. \quad (16)$$

This gives a signal to noise variance ratio of $a^2(\sigma_X^2/\sigma_n^2)$; a specific SNR requires an amplification factor of $a = (\sigma_n/\sigma_X)10^{(SNR/20)}$. The improved synchronization of $(\varepsilon_t, \varepsilon_t^R)$ using the transformation method is demonstrated by Figure 4. Table 1 illustrates the improvement in synchronization from transformation over capping, both in regard to mean and standard deviation of $|\varepsilon - \varepsilon^R|$. From Table 1 reasonable synchronization is seen to need a SNR of at least 10-15 decibels.

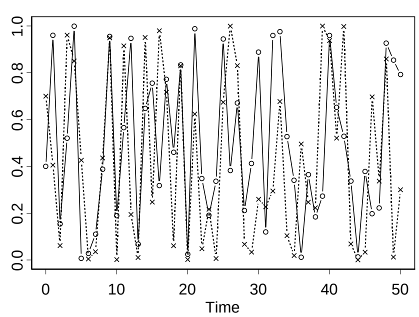


Figure 4: Synchronization by transformation of ε (○) and ε^R (×) with $SNR = 0$, using the double logistic map.

SNR	a	Capping		Transformation	
		mean	sd	mean	sd
-5	$10^{\frac{-5}{20}}$	0.4166	0.3157	0.3458	0.2901
0	1	0.3835	0.2858	0.3187	0.2532
5	$10^{\frac{5}{20}}$	0.2588	0.2422	0.2314	0.2119
10	$10^{\frac{10}{20}}$	0.1773	0.1708	0.1528	0.1463
15	$10^{\frac{15}{20}}$	0.0910	0.0927	0.0723	0.0669

Table 1: Mean and standard deviation for $|\varepsilon - \varepsilon^R|$ using the double logistic map.

7. Conclusions

It has been shown how using a method of transforming the transmitted variable increases the ability of the receiver system to synchronize with the transmitter system and avoids the standard method of capping the transmitter variable at its limits. Of further importance is the preservation of the underlying invariant distribution of the spreading values at the receiver.

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