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# CONJUGACY IN THOMPSON'S GROUP $F$

NICK GILL AND IAN SHORT

ABSTRACT. We complete the program begun by Brin and Squier of characterising conjugacy in Thompson's group  $F$  using the standard action of  $F$  as a group of piecewise linear homeomorphisms of the unit interval.

## 1. INTRODUCTION

The object of this paper is to extend the methods of Brin and Squier described in [3] to solve the conjugacy problem in Thompson's group  $F$ .

Let  $f : (a, b) \rightarrow (a, b)$  be a piecewise linear order-preserving homeomorphism of the open interval  $(a, b)$ ; the points at which  $f$  is not locally affine are called the *nodes* of  $f$ . We write  $\text{PLF}^+(a, b)$  to denote the group of all piecewise linear order-preserving homeomorphisms of the open interval  $(a, b)$  which have finitely many nodes. Thompson's group  $F$  is the subgroup of  $\text{PLF}^+(0, 1)$  defined as follows: an element  $f$  of  $\text{PLF}^+(0, 1)$  lies in  $F$  if and only if the nodes of  $f$  lie in the ring of dyadic rational numbers,  $\mathbb{Z}[\frac{1}{2}]$ , and  $f'(x)$  is a power of 2 whenever  $x$  is not a node.

In [3] Brin and Squier analysed conjugacy in  $\text{PLF}^+(a, b)$  for  $(a, b)$  any open interval. For  $(a, b)$  equal to  $(0, 1)$  we can restate their primary result [3, Theorem 5.3] as follows: we have a simple quantity  $\Sigma$  on  $\text{PLF}^+(0, 1)$  such that two elements  $f$  and  $g$  of  $\text{PLF}^+(0, 1)$  are conjugate if and only if  $\Sigma_f = \Sigma_g$ . If  $f$  and  $g$  are elements of  $F$  then  $\Sigma_f$  and  $\Sigma_g$  can be computed and compared using a simple algorithm. Brin and Squier comment on their construction of  $\Sigma$  that, "Our goal at the time was to analyze the conjugacy problem in Thompson's group  $F$ ." In this paper we achieve Brin and Squier's goal by defining a quantity  $\Delta$  on  $F$  such that the following theorem holds.

**Theorem 1.1.** *Let  $f, g \in F$ . Then  $f$  and  $g$  are conjugate in  $F$  if and only if*

$$(\Sigma_f, \Delta_f) = (\Sigma_g, \Delta_g).$$

This is not the first solution of the conjugacy problem in  $F$ . In particular the conjugacy problem in  $F$  was first solved by Guba and Sapir in [5] using diagram groups. More recently, Belk and Matucci [1, 2] have another solution using strand diagrams. Kassabov and Matucci [6] also solved the conjugacy problem, and the simultaneous conjugacy problem. Our analysis is different to all of these as we build on the geometric invariants introduced by Brin and Squier.

Note that Thm. 1.1 reduces the study of roots and centralizers in  $F$  (first completed in [5]) to a set of easy computations; we leave this to the interested reader.

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We will not prove Theorem 1.1 directly. Rather we prove the following proposition which, given [3, Theorem 5.3], implies Theorem 1.1:

**Proposition 1.2.** *Two elements  $f$  and  $g$  of  $F$  that are conjugate in  $\text{PLF}^+(0, 1)$  are conjugate in  $F$  if and only if  $\Delta_f = \Delta_g$ .*

Our paper is structured as follows. In §2 we introduce some important background concepts, including the definition of  $\Sigma$ . In §3 we define  $\Delta$ . In §4 we prove Proposition 1.2. In §5 and §6 we outline formulae which can be used to calculate  $\Delta$ .

## 2. THE DEFINITION OF $\Sigma$

Let  $f$  be a member of Thompson's group  $F$ , embedded in  $\text{PLF}^+(0, 1)$ . Following Brin and Squier [3] we define the invariant  $\Sigma_f$  to be a tuple of three quantities,  $\Sigma_1, \Sigma_2$  and  $\Sigma_3$ , which depend on  $f$ .

The first quantity,  $\Sigma_1$ , is a list of integers relating to values of the *signature of  $f$* ,  $\epsilon_f$ . We define  $\epsilon_f$  as follows:

$$\epsilon_f : (0, 1) \rightarrow \{-1, 0, 1\}, \quad x \mapsto \begin{cases} 1, & f(x) > x; \\ 0, & f(x) = x; \\ -1, & f(x) < x. \end{cases}$$

If  $f$  is an element of  $\text{PLF}^+(0, 1)$  then there is a sequence of open intervals

$$I_1, I_2, \dots, I_m, \quad I_j = (p_{j-1}, p_j), \quad p_0 = 0, \quad p_m = 1, \quad (2.1)$$

such that  $\epsilon_f$  is constant on each interval, and the values of  $\epsilon_f$  on two consecutive intervals differ. We define  $\Sigma_1 = (\epsilon_f(x_1), \dots, \epsilon_f(x_m))$  where  $x_i \in I_i$  for  $i = 1, \dots, m$ .

Let  $\text{fix}(f)$  be the set of fixed points of  $f$  and observe that the points  $p_0, \dots, p_m$  from (2.1) all lie in  $\text{fix}(f)$ . We say that the interval  $I_j$  is a *bump domain* of  $f$  if  $\epsilon_f$  is non-zero on that interval. Our next two invariants consist of lists with entries for each bump domain of  $f$ .

If  $k$  is a piecewise linear map from one interval  $(a, b)$  to another, then the *initial slope* of  $k$  is the derivative of  $k$  at any point between  $a$  and the first node of  $k$ , and the *final slope* of  $k$  is the derivative of  $k$  at any point between the final node of  $k$  and  $b$ . The invariant  $\Sigma_2$  is a list of positive real numbers. The entry for a bump domain  $I_j = (p_{j-1}, p_j)$  is the value of the initial slope of  $f$  in  $I_j$ .

To define  $\Sigma_3$  we need the notion of a *finite function*; this is a function  $[0, 1) \rightarrow \mathbb{R}^+$  which takes the value 1 at all but finitely many values. The invariant  $\Sigma_3$  is a list of equivalence classes of finite functions. We calculate the entry for a bump domain  $I_j = (p_{j-1}, p_j)$  as follows. Suppose first of all that  $\Sigma_1 = 1$  in  $I_j$ . Define, for  $x \in I_j$ , the *slope ratio*  $f^*(x) = \frac{f'_+(x)}{f'_-(x)}$ . Thus  $f^*(x) = 1$  except when  $x$  is a node of  $f$ . Now define

$$\phi_{f,j} : I_j \rightarrow \mathbb{R}, \quad x \mapsto \prod_{n=-\infty}^{\infty} f^*(f^n(x)).$$

Since  $f$  has only finitely many nodes, only finitely many terms of this infinite product are distinct from 1. Let  $p$  be the smallest node of  $f$  in  $I_j$  and let  $p_*$  be the smallest

node of  $f$  in  $I_j$  such that  $\phi_{f,j}(p_*) \neq 1$  (such a node must exist). Define, for  $s \in [0, 1)$ ,

$$\psi_{f,j}(s) = \phi_f(\lambda^s(r - p_{j-1}) + p_{j-1}).$$

Here  $\lambda$  is the entry in  $\Sigma_2$  corresponding to  $I_j$  and  $r$  is any point in the interval  $(0, p)$  which satisfies the formula  $r = f^n(p_*)$  for  $n$  some negative integer.

Note that  $\psi_{f,j}$  is a finite function; furthermore, in our definition of  $\psi_{f,j}$ , we have chosen a value for  $r$  which guarantees that  $\psi_{f,j}(0) \neq 1$ ; we can do this by virtue of [3, Lemma 4.4].

The entry for  $\Sigma_3$  corresponding to  $I_j$  is the equivalence class  $[\psi_{f,j}]$ , where two finite functions  $c_1$  and  $c_2$  are considered equivalent if  $c_1 = c_2 \circ \rho$  where  $\rho$  is a translation of  $[0, 1)$  modulo 1. If  $f(x) < x$  for each  $x \in I_j$  then the entry for  $\Sigma_3$  corresponding to  $I_j$  is the equivalence class  $[\psi_{f^{-1},j}]$ .

### 3. THE DEFINITION OF $\Delta$

The quantity  $\Delta$  will also be a list, this time a list of equivalence classes of tuples of real numbers. To begin with we need the concept of a *minimum cornered function*.

**3.1. The minimum cornered function.** Take  $f \in \text{PLF}^+(0, 1)$ ; in this subsection we focus on the restriction of  $f$  to one of its bump domains  $D = (a, b)$ . We adjust one of the definitions of Brin and Squier [3]: for us, a *cornered function* in  $\text{PLF}^+(0, 1)$  is an element  $l$  which has a single bump domain  $(a, b)$  and which satisfies the following property:  $\Sigma_1$  takes value 1 (resp.  $-1$ ) in relation to  $(a, b)$  and there exists a point  $x \in (a, b)$  such that all nodes of  $l$  which lie in  $(a, b)$  lie in  $(x, l(x))$  (resp.  $(l(x), x)$ ). We will sometimes abuse notation and consider such a cornered function as an element of  $\text{PLF}^+(a, b)$ .

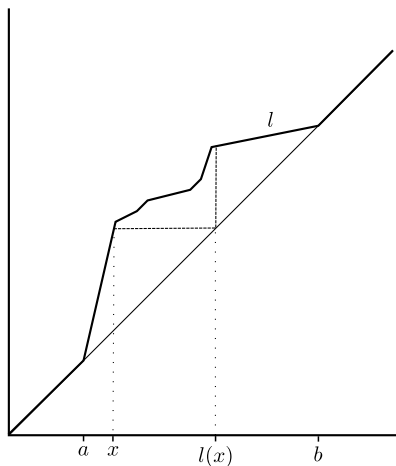


FIGURE 1. A cornered function.

We say that a cornered function  $l$  corresponds to a finite function  $c$  if  $\psi_l = c$  (we drop the subscript  $i$  here, since there is only one bump domain). Roughly speaking this means that the first node of  $l$  corresponds to  $c(0)$ . For a given initial slope  $\lambda$  there is a unique cornered function in  $\text{PLF}^+(a, b)$  such that  $\psi_l = c$  (this follows from [3, Prop. 4.9]; it can also be deduced from the proof of Lemma 5.2).

Now let  $c : [0, 1) \rightarrow \mathbb{R}$  be a finite function such that  $[c]$  is the entry in  $\Sigma_3$  associated with  $D$ . Within this equivalence class  $[c]$  we can define a *minimum finite function*  $c_m$  as follows. First define  $C = \{c_1 \in [c] \mid c_1(0) \neq 1\}$  and define an ordering on  $C$  as follows. Let  $c_1, c_2 \in C$  and let  $x$  be the smallest value such that  $c_1(x) \neq c_2(x)$ . Write  $c_1 < c_2$  provided  $c_1(x) < c_2(x)$ . We define  $c_m$  to be the minimum function in  $C$  under this ordering.

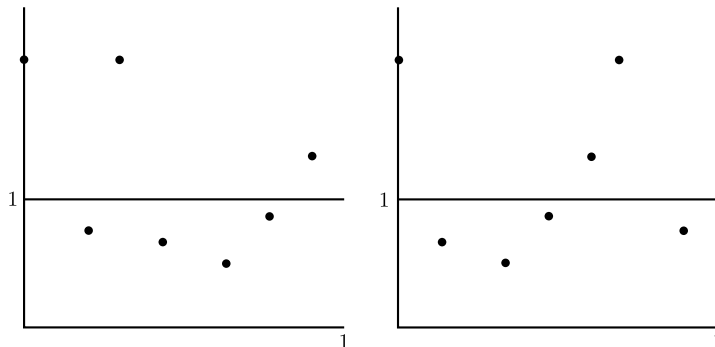


FIGURE 2. Two equivalent finite functions. The minimum finite function is on the left.

Suppose that  $\lambda$  is the entry in  $\Sigma_2$  associated with  $D$ . Suppose that  $l$  is the cornered function in  $\text{PLF}^+(a, b)$ , with initial slope  $\lambda$ , which corresponds to  $c_m$ . We say that  $l$  is the *minimum cornered function* associated with  $f$  over  $D$ .

**3.2. The quantity  $\Delta$ .** Now let us fix  $f$  to be in  $F$  and define  $\Delta$  accordingly. A *bump chain* is a subsequence  $I_t, I_{t+1}, \dots, I_u$  of (2.1) such that each interval is a bump domain, and of the points  $p_{t-1}, p_t, \dots, p_u$  only  $p_{t-1}$  and  $p_u$  are dyadic. Thus  $I_1, I_2, \dots, I_m$  can be partitioned into bump chains and open intervals of fixed points of  $f$  (which have dyadic numbers as end-points).

In [3], conjugating functions in  $\text{PLF}^+(0, 1)$  are constructed by dealing with one bump domain at a time. We will construct conjugating functions in  $F$  by dealing with one bump chain at a time. Consider a particular bump chain  $D_1, \dots, D_s$  and let  $f_j$  be the restriction of  $f$  to  $D_j = (a_j, b_j)$ .

According to [3, Theorem 4.18], the centralizer of  $f_j$  within  $\text{PLF}^+(a_j, b_j)$  is an infinite cyclic group generated by a root  $\widehat{f}_j$  of  $f_j$ . We define  $\lambda_j$  to be the initial slope of  $\widehat{f}_j$  and  $\mu_j$  to be the final slope of  $\widehat{f}_j$ . (Let  $m_j$  be the integer such that  $\widehat{f}_j^{m_j} = f_j$ ; then  $\lambda_j$  and  $\mu_j$  are the positive  $m_j$ th roots of the initial and final slopes of  $f_j$ .)

Next, let  $k_j$  be a member of  $\text{PLF}^+(a_j, b_j)$  that conjugates  $f_j$  to the associated minimum cornered function,  $l_j$ , in  $\text{PLF}^+(a_j, b_j)$ . Thus  $k_j$  is some function satisfying the equality  $k_j f_j k_j^{-1} = l_j$ . Let  $\alpha_j$  be the initial slope of  $k_j$  and let  $\beta_j$  be the final slope.

Consider the equivalence relation on  $\mathbb{R}^s$  where  $(x_1, \dots, x_s)$  is equivalent to  $(y_1, \dots, y_s)$  if and only if there are integers  $m, n_1, \dots, n_s$  such that

$$\begin{aligned} 2^m x_1 &= \lambda_1^{n_1} y_1 \\ \mu_1^{n_1} x_2 &= \lambda_2^{n_2} y_2 \\ \mu_2^{n_2} x_3 &= \lambda_3^{n_3} y_3 \\ &\vdots \\ \mu_{s-1}^{n_{s-1}} x_s &= \lambda_s^{n_s} y_s. \end{aligned}$$

It is possible to check whether two  $s$ -tuples of real numbers are equivalent according to the above relation in a finite amount of time because the quantities  $\lambda_i$  and  $\mu_j$  are rational powers of 2. We assign to the chain  $D_1, \dots, D_s$  the equivalence class of the  $s$ -tuple

$$\left( \frac{\alpha_1}{w_1}, \frac{\alpha_2 w_1}{w_2 \beta_1}, \dots, \frac{\alpha_s w_{s-1}}{w_s \beta_{s-1}} \right)$$

where  $w_j = b_j - a_j$ . We define  $\Delta_f$  to consist of an ordered list of such equivalence classes; one per bump chain.

#### 4. PROOF OF PROPOSITION 1.2

We prove Proposition 1.2 after the following elementary lemma.

**Lemma 4.1.** *Let  $f$  and  $g$  be maps in  $F$ , and let  $h$  be an element of  $\text{PLF}^+(0, 1)$  such that  $hfh^{-1} = g$ . Let  $D = (a, b)$  be a bump domain of  $f$  and suppose that the initial slope of  $h$  in  $D$  is an integer power of 2. Then all slopes of  $h$  in  $D$  are integer powers of 2 and all nodes of  $h$  in  $D$  occur in  $\mathbb{Z}[\frac{1}{2}]$ .*

*Proof.* Let  $(a, a + \delta)$  be a small interval over which  $h$  has constant slope; suppose that this slope is greater than 1. We may assume that  $f$  has initial slope greater than 1 otherwise replace  $f$  with  $f^{-1}$  and  $g$  with  $g^{-1}$ . Now observe that  $hf^n h^{-1} = g^n$  for all integers  $n$  and so  $h = g^n h f^{-n}$ .

Now, for any  $x \in (a, b)$  there is an interval  $(x, x + \epsilon)$  and an integer  $n$  so that  $f^{-n}(x, x + \epsilon) \subset (a, a + \delta)$ . Then the equation  $h = g^n h f^{-n}$  implies that, where defined, the derivative of  $h$  over  $(x, x + \epsilon)$  is an integral power of 2. Furthermore any node of  $h$  occurring in  $(x, x + \epsilon)$  must lie in  $\mathbb{Z}[\frac{1}{2}]$  as required.

If  $h$  does not have slope greater than 1 then apply the same argument to  $h^{-1}$  using the equation  $h^{-1}gh = f$ .  $\square$

We have two elements  $f$  and  $g$  of  $F$  and a third element  $h$  of  $\text{PLF}^+(0, 1)$  such that  $hfh^{-1} = g$ . We use the notation for  $f$  described in the previous section, such as the quantities  $I_j, p_j, f_j, \widehat{f}_j, k_j, l_j, \alpha_j, \beta_j, w_j, \lambda_j$ , and  $\mu_j$ . We need exactly the same quantities for  $g$ , and we distinguish the quantities for  $g$  from those for  $f$  by adding a ' after each one. In particular, we choose a bump chain  $D_1, \dots, D_s$  of  $f$  and define  $D'_i = h(D_i)$  for  $i = 1, \dots, s$ . Note that  $D'_1, \dots, D'_s$  are bump domains but need not form a bump chain for  $g$  according to our assumptions, because  $h$  is not necessarily a member of  $F$ .

Let the function  $h_i = h|_{D_i}$  have initial slope  $\gamma_i$  and final slope  $\delta_i$ . Let  $u$  be the member of  $\text{PLF}^+(0, 1)$  which, for  $i = 1, \dots, m$ , is affine when restricted to  $I_i$ , and maps this interval onto  $I'_i$ . Notice that, restricted to  $D'_i$ ,  $ul_iu^{-1}$  is a cornered function which is conjugate to  $l'_i$  (by the map  $k'_i h_i k_i^{-1} u^{-1}$ ), and which satisfies  $\psi_{l'_i} = \psi_{ul_iu^{-1}}$ . Therefore  $ul_iu^{-1} = l'_i$ . Combine this equation with the equations  $k_i f_i k_i^{-1} = l_i$ ,  $k'_i g_i k_i^{-1} = l'_i$ , and  $h_i f_i h_i^{-1} = g_i$  to yield

$$(k_i^{-1} u^{-1} k'_i h_i) f_i (k_i^{-1} u^{-1} k'_i h_i)^{-1} = f_i.$$

Therefore  $k_i^{-1} u^{-1} k'_i h_i$  is in the centralizer of  $f$ , so there is an integer  $N_i$  such that

$$h_i = (k'_i)^{-1} u k_i \widehat{f_i}^{N_i}$$

for each  $i = 1, \dots, s$ . Then by comparing initial and final slopes in this equation we see that

$$\gamma_i = \lambda_i^{N_i} \frac{\alpha_i w'_i}{w_i \alpha'_i}, \quad \delta_i = \mu_i^{N_i} \frac{\beta_i w'_i}{w_i \beta'_i} \quad (4.1)$$

for  $i = 2, \dots, s$ . We are now in a position to prove Proposition 1.2.

*Proof of Proposition 1.2.* Suppose that  $h \in F$ . Then there are integers  $M_1, \dots, M_s$  such that  $\gamma_1 = 2^{M_1}$  and  $\gamma_i = \delta_{i-1} = 2^{M_i}$  for  $i = 2, \dots, s$ . Substituting these values into (4.1) we see that

$$2^{M_1} \frac{\alpha'_1}{w'_1} = \lambda_1^{N_1} \frac{\alpha_1}{w_1}, \quad \mu_{i-1}^{N_{i-1}} \frac{\alpha'_i w'_{i-1}}{w'_i \beta'_{i-1}} = \lambda_i^{N_i} \frac{\alpha_i w_{i-1}}{w_i \beta_{i-1}},$$

for  $i = 2, \dots, s$ , as required.

Conversely, suppose that  $\Delta_f = \Delta_g$ . We modify  $h$  so that it is a member of  $F$ . If  $I_j$  is an interval of fixed points of  $f$  then modify  $h_j$  so that it is any piecewise linear map from  $I_j$  to  $I'_j$  whose slopes are integer powers of 2, and whose nodes occur in  $\mathbb{Z}[\frac{1}{2}]$ . (It is straightforward to construct such maps, see [4, Lemma 4.2].)

Now we modify  $h$  on a bump chain  $D_1, \dots, D_s$ . Since  $\Delta_f = \Delta_g$  we know that there are integers  $m$  and  $n_1, \dots, n_s$  such that, for  $i = 2, \dots, s$ ,

$$2^m \frac{\alpha_1}{w_1} = \lambda_1^{n_1} \frac{\alpha'_1}{w'_1}, \quad \mu_{i-1}^{n_{i-1}} \frac{\alpha_i w_{i-1}}{w_i \beta_{i-1}} = \lambda_i^{n_i} \frac{\alpha'_i w'_{i-1}}{w'_i \beta'_{i-1}}. \quad (4.2)$$

Consider the piecewise linear map  $h'_i : D_i \rightarrow h_i(D_i)$  given by  $h'_i = h_i \widehat{f_i}^{-n_i - N_i}$ . The initial slope  $\gamma'_i$  of  $h'_i$  is  $\gamma_i \lambda_i^{-n_i - N_i}$  and the final slope  $\delta'_i = \delta_i \mu_i^{-n_i - N_i}$ . From (4.1) and (4.2) we see that

$$\gamma'_1 = 2^{-m}, \quad \gamma'_i = \delta'_{i-1}.$$

for  $i = 2, \dots, s$ . We modify  $h$  by replacing  $h_i$  with  $h'_i$  on  $D_i$ . Then  $h$  does not have a node at any of the end-points of  $D_1, \dots, D_s$  other than the first and last end-point. By Lemma 4.1, the nodes of  $h_1$  occur in  $\mathbb{Z}[\frac{1}{2}]$  and the slopes of  $h_1$  are all powers of 2. Since the initial slope of  $h_1$  coincides with the final slope of  $h_1$ , the same can be said of  $h_2$ . Similarly, for  $i = 2, \dots, s$ , the initial slope of  $h_i$  coincides with the final slope of  $h_{i-1}$ . We repeat these modifications for each bump chain of  $f$ ; the resulting conjugating map is a member of  $F$ .  $\square$

## 5. CALCULATING $\alpha_i$ AND $\beta_i$

It may appear that, in order to calculate  $\Delta$ , it is necessary to construct various conjugating functions. In particular to calculate  $\alpha_i$  one might have to construct the function in  $\text{PLF}^+(a_i, b_i)$  which conjugates  $f_i$  to the conjugate minimum cornered function in  $\text{PLF}^+(a_i, b_i)$ .

It turns out that this is not the case. The values for  $\alpha_i$  and  $\beta_i$  can be calculated simply by looking at the entries in  $\Sigma_1, \Sigma_2$  and  $\Sigma_3$  which correspond to  $D_i$ . In this section we give a formula for  $\alpha_i$ ; we then observe how to use the formula for  $\alpha_i$  to calculate  $\beta_i$ .

In what follows we take  $f$  to be a function in  $\text{PLF}^+(a, b)$  such that  $f(x) \neq x$  for  $x \in (a, b)$ . Let  $l$  be the minimum cornered function which is conjugate to  $f$  in  $\text{PLF}^+(a, b)$ .

**5.1. Calculating  $\alpha_i$ .** Suppose first that  $f(x) > x$  for  $x \in (a, b)$ . Let  $y_j$ , for  $j = 0, \dots, t$  be the points at which the finite function  $\psi_f$  does not take value 1; let  $\psi_f$  take the positive value  $z_j$  at the point  $y_j$  and assume that  $0 = y_0 < y_1 < \dots < y_t < 1$ . We will denote  $\psi_f$  by  $c_t$  and define  $c_j = c_t(x + y_{j+1})$ . Then  $c_j$  is a translation of  $c_t$  under which  $y_j$  is mapped to the last point of  $c_j$  which does not take value 1.

Let  $u_j$  be the cornered function corresponding to  $c_j$  and let  $x_j$  be the final node of  $u_j$ . Note that  $u_j$  is conjugate to  $f$  and, for  $j$  equal to some integer  $n$ ,  $u_j$  equals  $l$ , the minimum cornered function. Define the *elementary function*  $h_{x,r}$  to be the function which is affine on  $(0, x)$  and  $(x, 1)$  and which has slope ratio  $r$  at  $x$ . We define  $\zeta_j$  to be the initial slope of the elementary function  $h_{x_j, z_j}$ .

Let  $p$  be the first node of  $f$  and let  $q$  be the first node of  $u_t$ .

**Lemma 5.1.** *There exists  $k$  in  $\text{PLF}^+(a, b)$  such that  $kfk^{-1} = l$  and the initial slope of  $k$  is*

$$(\zeta_t \zeta_{t-1} \dots \zeta_{n+1}) \left( \frac{q - a}{p - a} \right).$$

Note that, in the formula just given,  $p$  and  $q$  stand for the  $x$ -coordinates of the corresponding nodes. Before we prove Lemma 5.1 we observe that we can calculate values for the  $\zeta_j$  and  $q$  simply by looking at  $\Sigma_2$  and  $\Sigma_3$  and using the following lemma:

**Lemma 5.2.** *Let  $l$  be a cornered function in  $\text{PLF}^+(a, b)$  with initial slope  $\lambda > 1$ , and suppose that the corresponding finite function  $c$  takes the value 1 at all points in  $[0, 1)$  except  $0 = s_0 < s_1 < \dots < s_k < 1$ , at which  $c(s_i) = z_i$ . Then the first node  $q_0$  of  $l$  is given by the formula*

$$q_0 = a + \frac{(b-a)(1 - [\lambda z_0 \dots z_k])}{[\lambda(1-z_0)] + [\lambda^{s_1+1} z_0(1-z_1)] + \dots + [\lambda^{s_k-1+1} z_0 \dots z_{k-2}(1-z_{k-1})] + [\lambda^{s_k+1} z_0 \dots z_{k-1}(1-z_k)]}, \quad (5.1)$$

and the initial slope  $\zeta$  of the elementary function  $h_{q_k, z_k}$ , where  $q_k$  is the final node of  $l$ , is given by

$$\zeta = \frac{b - a}{\lambda^{s_k}(q_0 - a)(1 - z_k) + (b - a)z_k}. \quad (5.2)$$

*Proof.* If  $q_0, \dots, q_k$  are the nodes of  $l$  we have equations

$$\lambda^{s_i}(q_0 - a) + a = q_i, \quad i = 0, 1, 2, \dots, k. \quad (5.3)$$



Define  $q_{k+1} = b$  and let  $\lambda_i$  be the slope of  $l$  between the nodes  $q_{i-1}$  and  $q_i$  for  $i = 1, \dots, k+1$ . Then  $z_i = \lambda_i/\lambda_{i-1}$  for  $i > 1$ , and we obtain

$$\lambda_i = \lambda z_0 \dots z_{i-1}, \quad i = 1, \dots, k+1. \quad (5.4)$$

If we substitute (5.3) and (5.4) into the equation

$$b - a = \lambda(q_0 - a) + \lambda_1(q_1 - q_0) + \lambda_2(q_2 - q_1) + \dots + \lambda_{k+1}(b - q_k),$$

then we obtain (5.1). To obtain (5.2), notice that  $z_k \zeta$  is the final slope of  $h_{q_k, z_k}$ , therefore  $b - a = \zeta(q_k - a) + z_k \zeta(b - q_k)$ . Substitute the value of  $q_k$  from (5.3) into this equation to obtain (5.2).  $\square$

Before we prove Lemma 5.1, we make the following observation. Let  $g$  be a function such that  $g(x) > x$  for all  $x \in (a, b)$  and suppose that  $g$  has nodes  $p_1 < \dots < p_s$ . Now let  $h = h_{p_s, g^*(p_s)}$ . Then  $hgh^{-1}$  has nodes  $h(p_1), \dots, h(p_{s-1}), hg^{-1}(p_s)$  with  $(hgh^{-1})^*$  taking on values  $g^*(p_1), \dots, g^*(p_t)$  at the respective nodes. If  $hg^{-1}(p_s) = h(p_i)$  for some  $i$ , then  $(hgh^{-1})^*$  has value  $g^*(p_i)g^*(p_s)$ .

*Proof of Lemma 5.1.* The formula given in Lemma 5.1 arises as follows. We start by finding the conjugator from  $f$  to the cornered function  $u_t$ ; then we cycle through the cornered functions  $u_j$  until we get to  $u_n = l$ . Thus the  $\frac{q-a}{p-a}$  part of the formula arises from the initial conjugation to a cornered function, and the  $\zeta_j$ 's arise from the cycling.

Consider this cycling part first and use our observation above on the cornered functions,  $u_j$ : we have  $h_{x_j, z_j} u_j (h_{x_j, z_j})^{-1} = u_{j-1}$ . Thus in order to move from  $u_t$  to  $u_n$  we repeatedly conjugate by elementary functions with initial gradient  $\zeta_t, \dots, \zeta_{n+1}$ .

We must now explain why we can use  $\frac{q-a}{p-a}$  for the first conjugation which moves from  $f$  to  $u_t$ . It is sufficient to find a function which conjugates  $f$  to  $u_t$  and which is linear on  $[a, p]$ .

Consider the effect of applying an elementary conjugation to a function  $f$  that is not a cornered function. Suppose that  $f$  has nodes  $p_1 < \dots < p_s$ . So  $p = p_1$ . We consider the effect of conjugation by an elementary function  $h = h_{p_s, f^*(p_s)}$  as above. To reiterate, we obtain a function with nodes

$$h(p_1), \dots, h(p_{s-1}), hf^{-1}(p_s)$$

Now observe that, since  $f^*(p_s) < 1$ ,  $h(x) > x$  for all  $x$  and  $h$  is linear on  $[a, p_s]$ . So clearly  $h$  is linear on the required interval. There are three possibilities:

- If  $hf^{-1}(p_s) < h(p)$  then  $f$  was already a cornered function; in fact  $f = u_t$ . We are done.
- If  $hf^{-1}(p_s) > h(p)$  then we simply iterate. We replace  $f$  with  $hfh^{-1}$ ,  $p$  with  $h(p)$  etc. We conjugate by another elementary function exactly as before. It is clear that the next elementary conjugation will be linear on  $[a, h(p)]$  which is sufficient to ensure that the composition is linear on  $[a, p]$ .
- If  $hf^{-1}(p_s) = h(p)$  then we need to check if  $hfh^{-1}$  is a cornered function. If so then  $hfh^{-1} = u_t$ , the corner function we require. If  $hfh^{-1}$  is not a cornered function then we iterate as above, replacing  $f$  with  $hfh^{-1}$ . It is possible that  $h(p)$  will no longer be the first node of  $hfh^{-1}$ , but in this case we replace  $p$  by

$h(p_2)$ . Since  $[a, h(p_2)] \supset [a, h(p)]$  this is sufficient to ensure that the composition is linear on  $[a, p]$ .

We can proceed like this until the process terminates at a cornered function. Since conjugating a non-cornered function by  $h$  preserves  $\psi$  we can be sure that we will terminate at  $u_t$  as required. What is more the composition of these elementary functions is linear on  $[a, p]$ .  $\square$

Suppose next that  $f(x) < x$  for all  $x \in (a, b)$  and  $kfk^{-1} = l$ , a minimum cornered function. Observe that  $kf^{-1}k^{-1} = l^{-1}$  and  $f^{-1}(x) > x$  for all  $x \in (a, b)$ . We can now apply the formula in Lemma 5.1, replacing  $f$  with  $f^{-1}$  and  $l$  with  $l^{-1}$ , to get a value for the initial slope of  $k$ .

**5.2. Calculating  $\beta_i$ .** The method we have used to calculate  $\alpha_i$  can also be used to calculate  $\beta_i$ . Define

$$\tau : [a, b] \rightarrow [a, b], x \mapsto b + a - x.$$

Now  $\tau$  is an automorphism of  $\text{PLF}^+(a, b)$ ; the graph of a function, when conjugated by  $\tau$ , is rotated  $180^\circ$  about the point  $(\frac{b+a}{2}, \frac{b+a}{2})$ . Consider the function  $\tau f \tau$  and let  $k$  be the conjugating function from earlier, so that  $kfk^{-1} = l$ . Then

$$(\tau k \tau)(\tau f \tau)(\tau k \tau)^{-1} = (\tau l \tau).$$

The initial slope of  $\tau k \tau$  equals the final slope of  $k$ . Thus we can use the method outlined above – replacing  $f$  with  $\tau f \tau$  and  $l$  with  $\tau l \tau$  – to calculate the initial slope of  $\tau k \tau$ . Note that, for this to yield  $\beta_i$ , we must make an adjustment to the integer  $n$  in the formula in Lemma 5.1: the function  $\tau l \tau$  is not necessarily the *minimum* cornered function which is conjugate to  $\tau f \tau$ . Thus we choose  $n$  to ensure that  $l$  is minimum rather than  $\tau l \tau$ .

## 6. CALCULATING $\lambda_i$ AND $\mu_i$

Let  $f$  be a fixed-point free element of  $\text{PLF}^+(a, b)$ . Let  $\widehat{f}$  be a generator of the centralizer of  $f$  within  $\text{PLF}^+(a, b)$ . The formula for  $\Delta$  requires that we calculate the initial slope and the final slope of  $\widehat{f}$ . It turns out that this is easy—thanks to the work of Brin and Squier [3].

Let  $c, c' : [0, 1) \rightarrow \mathbb{R}$  be finite functions. We say that  $c'$  is the  $p$ -th root of  $c$  provided that, for all  $x \in [0, 1)$ , we have  $c(x) = c'(px)$ . The property of having a  $p$ -th root is preserved by the equivalence used to define  $\Sigma_3$ . Thus we may talk about the equivalence class  $[c]$  having a  $p$ -th root, provided any representative of  $[c]$  has a  $p$ -th root.

Now [3, Theorem 4.15] asserts that  $f$  has a  $p$ -th root in  $\text{PLF}^+(a, b)$ , for  $p$  a positive integer, if and only if the single equivalence class in  $\Sigma_3$  is a  $p$ -th power (following Brin and Squier we say that this class has  $p$ -fold symmetry). What is more [3, Theorem 4.18] asserts that  $\widehat{f}$  must be a root of  $f$ .

Thus if  $p$  is the largest integer for which the single class in  $\Sigma_3$  has  $p$ -fold symmetry then  $\widehat{f}$  is the  $p$ -th root of  $f$ . The initial slope of  $\widehat{f}$  is the positive  $p$ -th root of the initial slope of  $f$ , and the final slope of  $\widehat{f}$  is the positive  $p$ -th root of the final slope of  $f$ .

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