On reciprocal symmetry

M.C. Jones[1]

Department of Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, U.K.

Abstract

On the positive half-line, there are two natural, and complementary, analogues of the single notion of symmetry of distributions on the real line. One is the R-symmetry recently proposed and investigated by Mudholkar and Wang (2007); the other is the 'log-symmetry' investigated here. Log-symmetry can be thought of either in terms of a random variable having the same distribution as its reciprocal or as ordinary symmetry of the distribution of the logged random variable. Various properties, analogies, comparisons and consequences are pursued.

Keywords: Log-location-scale; Log-normal; Log-symmetry; R-symmetry

1. Introduction

On the real line, $\mathbb{R}$, symmetry about a point μ of a distribution with density g can be expressed by

$$g(x + \mu) = g(\mu - x). \tag{1.1}$$

Equivalently, in terms of the random variable $X \sim g$, where $\sim$ denotes 'is distributed as', we can say that

$$X - \mu \approx \mu - X \tag{1.2}$$

where $\approx$ denotes 'has the same distribution as'.

[1] *E-mail address:* m.c.jones@open.ac.uk

Now consider $Y \sim f$ on the positive half line, $\mathbb{R}^+$. The natural analogue of the 'additive/negative' symmetry on $\mathbb{R}$ is a 'multiplicative/reciprocal' symmetry on $\mathbb{R}^+$. Let this kind of symmetry be centred on a point $\theta > 0$. Then a natural analogue of (1.1) is

$$f(\theta y) = f(\theta/y). \tag{1.3}$$

This is the 'R-symmetry' recently investigated in a very interesting paper by Mudholkar and Wang (2007); the 'R' stands for 'Reciprocal'. However, the natural analogue of (1.2), I would say, is

$$Y/\theta \approx \theta/Y. \tag{1.4}$$

Unlike (1.1) and (1.2), (1.3) and (1.4) are, of course, not equivalent. Taking logs, (1.4) is, however, equivalent to

$$\log Y - \log\theta \approx \log\theta - \log Y, \tag{1.5}$$

that is, to (ordinary) symmetry about $\mu = \log\theta$ of the distribution of $\log Y$.

Relationships (1.4) and (1.5) provide an alternative candidate for the nomenclature 'reciprocal symmetry' and it is my purpose in this paper to investigate this alternative notion (Section 2), and to compare and contrast it with Mudholkar and Wang's complementary notion of R-symmetry (Section 3). Because of (1.5), I will use the term 'log-symmetry' to refer to (1.4). Section 4 contains many examples of log-symmetric distributions, in which the log-symmetric subfamily of the log-location-scale family (e.g. Lawless, 2003) comes to the fore. The closing remarks of Section 5 touch on practical consequences of log-symmetry and note that it is only R-symmetry, and not log-symmetry, that has a particular link with the inverse Gaussian distribution.

2. Basic properties of log-symmetry

The defining property (1.4) translates to

$$f(\theta y) = (1/y^2) f(\theta/y). \tag{2.1}$$

In terms of distribution functions, this is

$$F(\theta y) = 1 - F(\theta/y), \tag{2.2}$$

a natural analogue of $G(x+\mu) = 1 - G(\mu - x)$ on $\mathbb{R}$. Defining the quantile function $Q \equiv F^{-1}$, it follows that

$$Q(u) = \theta^2 / Q(1-u). \tag{2.3}$$

Equation (2.3) immediately shows that the median of f is at θ. The hazard function h associated with f mimics (2.1):

$$h(\theta y) = (1/y^2) h(\theta/y). \tag{2.4}$$

As is the case for R-symmetry, if Y is associated with a distribution log-symmetric about θ, the distribution of Y/θ is log-symmetric about 1. The latter, rescaled, distribution is therefore a canonical form for log-symmetric distributions — corresponding to $\mu = \log\theta = 0$ in ordinary symmetric distributions — but despite this I find it clearer and just as simple to continue to work with general $\theta \neq 1$ in the remainder of the paper.

Assume now that f is such that whichever positive and negative moments are required exist. Then, by definition,

$$E\left\{(Y/\theta)^r\right\} = E\left\{(\theta/Y)^r\right\}.$$

(This can alternatively be written as $E\{\sinh(r\log(Y/\theta))\} = 0$.) It follows that $\theta^{2r} = E(Y^r)/E(Y^{-r})$ so that, inter alia, $\theta = \{E(Y)/E(Y^{-1})\}^{1/2}$. Application of Jensen's inequality then shows that $E(Y) \geq \theta$. (Alternatively, $E(Y) = E(e^X) \geq e^\mu = \theta$, the inequality by the same token.)

Suppose that f is differentiable and unimodal with mode y_0. It can readily be shown that $f'(\theta) \leq 0$, implying that $y_0 \leq \theta$. Log-symmetric distributions therefore satisfy the mean $\geq$ median $\geq$ mode inequalities that are usually associated with positively skew distributions (see Abadir, 2005, for clarification of the latter). Log-symmetric distributions do have positive skewness at least in the sense of the quantile-based skewness measure discussed, for example, by Groeneveld and Meeden (1984): for $0 < \alpha < 1/2$ and using (2.3),

$$\frac{Q(1-\alpha) - 2Q(1/2) + Q(\alpha)}{Q(1-\alpha) - Q(\alpha)} = \frac{Q(1-\alpha) - \theta}{Q(1-\alpha) + \theta} \geq 0. \tag{2.5}$$

(An important special case is Bowley's, 1937, skewness measure which corresponds to $\alpha = 1/4$.)

If Y_i, $i = 1, ..., m$, are independent random variables each log-symmetric about θ_i, then their product $\prod_{i=1}^m Y_i$ is, immediately, log-symmetric about

$\prod_{i=1}^{m} \theta_i$. Mixtures of log-symmetric distributions each of which are symmetric about the same θ are, immediately, log-symmetric about θ. As an arbitrary density g_1 on $\mathbb{R}$ can be symmetrised through $\{g_1(x+\mu)+g_1(\mu-x)\}/2$, so an arbitrary density f_1 on $\mathbb{R}^+$ can be log-symmetrised through $\{f_1(\theta y) + (1/y^2)f_1(\theta/y)\}/2$.

In Jones (2007), I argued that it was quite natural to think of the behaviour of a density on $\mathbb{R}^+$ at zero as being 'equivalent to' its behaviour at ∞ if the behaviour at 0 (∞) was the behaviour of the density of the reciprocal of the random variable at ∞ (0). This is the case for log-symmetric densities by their definition. A nice example is that of power tailed densities which then behave like $y^{\gamma-1}$ as $y \to 0$ and as $y^{-(\gamma+1)}$ as $y \to \infty$, for some $\gamma > 0$. The focus of Jones (2007) was the consequent 'tail-preserving' transformation $T(Y) = (1/2)(Y - (1/Y)) = \sinh(\log(Y))$. In Section 3.5 of that paper, I refer to log-symmetry about 1 as "an interesting 'pseudo-symmetry' " which holds if the distribution of $T(Y)$ is symmetric about 0.

3. Log-symmetry and R-symmetry

A comparison of many of the properties of log-symmetry (taken from Section 2) and R-symmetry (taken mainly from Sections 4 and 5 of Mudholkar and Wang, 2007) appears in Table 1. It is particularly noteworthy that θ is

* * * Table 1 about here * * *

the median of log-symmetric distributions and the mode of (unimodal) R-symmetric distributions. The respective quantile and mode emphases of the different types of symmetry extend to positivity of skewness measures based on quantiles and modes, respectively; the latter is a new observation utilising the mode-based skewness measure of Arnold and Groeneveld (1995) given in the table. Indeed, Mudholkar and Wang's R-symmetry lends itself to a deeper analysis of skewness via the density-based skewness functions of Averous, Fougéres and Meste (1996), Critchley and Jones (2005) and Boshnakov (2007). Suppose f is unimodal with, for convenience, $f(0) = 0$ and let $y_L(p)$ be the unique value in $(0, \theta)$ such that $f(y_L(p)) = pf(\theta)$, $0 < p < 1$. Likewise, let $y_R(p)$ satisfy $f(y_R(p)) = pf(\theta)$, but with $y_R(p) > \theta$; then, R-symmetry yields $y_R(p) = \theta^2/y_L(p)$. In that case, and similarly to (2.5), the density-based skewness function

$$\frac{y_R(p) - 2y_0 + y_L(p)}{y_R(p) - y_L(p)} = \frac{\theta - y_L(p)}{\theta + y_L(p)} > 0.$$

Note that in general, for R-symmetric distributions, the position of the mean relative to the median > the mode is not clear.

Rewriting (2.1) as

$$\theta y f(\theta y) = (\theta/y) f(\theta/y),$$

we see that log-symmetry of f equates to R-symmetry of the length-biased version $yf(y)/E(Y)$ of f. Conversely, R-symmetry of f is log-symmetry of the inverse-length-biased version $y^{-1}f(y)/E(Y^{-1})$ of f.

This confirms something that is clear from comparing (1.3) and (2.1): no non-degenerate distribution on $\mathbb{R}^+$ can be both R-symmetric and log-symmetric *about the same centre*. It is, however, possible that a non-degenerate distribution on $\mathbb{R}^+$ can be both R-symmetric and log-symmetric *about different centres*. An example is provided by the log-normal distribution which is both log-symmetric about $\theta_1 = e^{\mu}$ *and* (Mudholkar and Wang, 2007) R-symmetric about $\theta_2 = e^{\mu-\sigma^2} < \theta_1$. It is tempting to conjecture that the log-normal is unique in this respect but I can not prove it. (One of many equivalent ways of looking at f's that are both log- and R-symmetric is that they satisfy $f(\theta^2 y) = y^2 f(y)$ for some $\theta > 0$.)

4. Examples of log-symmetric distributions

Any ordinary symmetric distribution on $\mathbb{R}$, of course, provides a corresponding log-symmetric distribution on $\mathbb{R}^+$. Foremost amongst the many examples, in the sense of popularity in practice, is, of course, the log-normal distribution. Second in popularity, perhaps, is the log-logistic distribution (e.g. Lawless, 2003). We might also mention the log-Laplace (e.g. Johnson, Kotz and Balakrishnan, 1994, Section 24.6), log-hyperbolic-secant and so on.

Turning to distributions not so obviously (but inevitably) linked with log transformations to symmetry, F distributions on equal numerator and denominator degrees of freedom are, by their definition, log-symmetric. Another prominent example is the Birnbaum-Saunders distribution (e.g. Johnson, Kotz and Balakrishnan, 1994, Section 33.3), log-symmetric about its scale parameter β. This can also be interpreted as the log-sinh-normal distribution (Rieck and Nedelman, 1991). Further, one might construct a log-symmetric distribution by splicing together a distribution (of Z, say) on $(0, \theta)$ and the distribution of θ^2/Z (on (θ, ∞)). The simplest unimodal example of this construction employs a power-law distribution on $(0, \theta)$ and leads back to the log-Laplace distribution.

By construction, the distribution of the rth power of a random variable with a distribution log-symmetric about θ is also log-symmetric, with centre of log-symmetry θ^r. On the log scale, r becomes a scale parameter, suggesting a particular role for the log-symmetric subset of the log-location-scale family (Lawless, 2003).

5. Closing remarks

Inference for log-symmetric distributions based on an i.i.d. sample $Y_1, ..., Y_n$ from f proceeds, trivially, by working with the distribution of the logged dataset $X_i = \log(Y_i)$, $i = 1, ..., n$. For instance, testing for the appropriateness of the log-symmetry assumption can be done by employing a standard test of ordinary symmetry (about an unknown centre) of the distribution of $X_1, ..., X_n$ (e.g. Cabilio and Masaro, 1996, and references in Pewsey, 2002). Parametric inference for log-symmetric-location-scale distributions has been explored in the more general case of log-location-scale distributions by Lawless (2003). (Note that the log-symmetric subfamily of log-location-scale distributions does not include the most popular member of the general family, namely the Weibull distribution.) That said, the sample median always provides a particularly natural (and robust) estimate of θ.

The complementary notion of R-symmetry has a particular link with the inverse Gaussian distribution (and its concept of IG-symmetry, Mudholkar and Natarajan, 2002) through the R-symmetry of the root reciprocal inverse Gaussian distribution (Mudholkar and Wang, 2007). It appears that no such link can be made between the inverse Gaussian distribution and log-symmetry.

Acknowledgement

I am very grateful to Professor Govind Mudholkar for his encouraging comments.

References

Abadir, K.M., 2005. The mean-median-mode inequality: counterexamples. Economet. Theory 21, 477–482.

Arnold, B.C., Groeneveld, R.A., 1995. Measuring skewness with respect to the mode. Amer. Statist. 49, 34–38.

Averous, J., Fougéres, A.L., Meste, M., 1996. Tailweight with respect to the mode for unimodal distributions. Statist. Probab. Lett. 28, 367–373.

Boshnakov, G.N., 2007. Some measures for asymmetry of distributions. Statist. Probab. Lett., to appear.

Bowley, A.L., 1937. Elements of Statistics, sixth ed. Scribner, New York.

Cabilio, P., Masaro, J., 1996. A simple test of symmetry about an unknown median. Canad. J. Statist. 24, 349–361.

Critchley, F., Jones, M.C., 2005. Density-based skewness and kurtosis. Open University Department of Statistics Technical Report 05/06 available from `http://statistics.open.ac.uk/TechnicalReports/TechnicalReportsIntro.htm`.

Groeneveld, R.A., Meeden, G., 1984. Measuring skewness and kurtosis. Statist. 33, 391–399.

Johnson, N.L., Kotz, S., Balakrishnan, N., 1994. Continuous Univariate Distributions, vol. 2, second ed. Wiley, New York.

Jones, M.C., 2007. Connecting distributions with power tails on the real line, the half line and the interval. Internat. Statist. Rev. 75, 58–69.

Lawless, J.F., 2003. Statistical Models and Methods for Lifetime Data, second ed. Wiley, Hoboken, NJ.

Mudholkar, G.S., Natarajan, R., (2002). The inverse Gaussian models: analogues of symmetry, skewness and kurtosis. Ann. Inst. Statist. Math. 54, 138–154.

Mudholkar, G.S., Wang, H., 2007. IG-symmetry and R-symmetry: interrelations and applications to the inverse Gaussian theory. J. Statist. Planning Inference, to appear.

Pewsey, A., 2002. Testing circular symmetry. Canad. J. Statist. 30, 591–600.

Rieck, J.R., Nedelman, J.R., 1991. A log-linear model for the Birnbaum–Saunders distribution. Technometrics 33, 51–60.

Table 1
Comparison of properties of log-symmetry and R-symmetry

	Log-Symmetry	R-Symmetry
Density relation	$f(\theta y) = (1/y^2) f(\theta/y)$	$f(\theta y) = f(\theta/y)$
Distribution relation	$F(\theta y) = 1 - F(\theta/y)$	$F(\theta y) = \theta^2 \int_{\theta/y}^{\infty} w^{-2} f(w) dw$
Quantile relation	$Q(u) = \theta^2/Q(1-u)$	not explicitly available
Hazard relation	$h(\theta y) = (1/y^2) h(\theta/y)$	not so simple
Median	θ	$\geq \theta$
Mode y_0 (if unimodal)	$\leq \theta$	θ
Moment relation	$E\{(Y/\theta)^r\} = E\{(\theta/Y)^r\}$	$E\{(Y/\theta)^r\} = E\{(\theta/Y)^{r+2}\}$
Mean	$\geq \theta$	$\neq \theta$
Positive skewness measure	$\dfrac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$	$1 - 2F(y_0)$
$\prod_{i=1}^m Y_i$, each about θ_i	about $\prod_{i=1}^m \theta_i$	about $\prod_{i=1}^m \theta_i$
Mixtures about same θ	about θ	about θ
Symmetrise arbitrary f_1	$\frac{\theta}{2}\{f_1(\theta y) + (1/y^2) f_1(\theta/y)\}$	$\dfrac{\theta\{f_1(\theta y) + f_1(\theta/y)\}}{1 + E_{f_1}\{(\theta/Y)^2\}}$
Specific Khintchine-type theorem	not available	mixture of $U(1/a, a)$ distributions