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The Seidel, Stern, Stolz and Van Vleck Theorems on continued fractions

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Abstract

We unify and extend three classical theorems in continued fraction theory, namely the Stern-Stolz Theorem, the Seidel-Stern Theorem and Van Vleck's Theorem. Our arguments use the group of Möbius transformations both as a topological group and as the group of conformal isometries of three-dimensional hyperbolic space.

1. *Introduction*

This paper is about the convergence or divergence of continued fractions of the form

$$\mathbf{K}(b_n) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}, \tag{1.1}$$

where the b_n are complex numbers (possibly zero). We denote the n-th partial quotient of (1.1) by Z_n, and we begin with three classical results which were proved in 1846, 1860 and 1901, respectively (for more recent publications on these results see [**3**, **4**, **5**, **7**, **8**, **9**, **10**]).

Theorem 1.1 The Seidel-Stern Theorem. *If each b_n is positive then the sequences Z_{2n} and Z_{2n+1} are monotonic and convergent. If, in addition, $\sum_n b_n$ diverges then Z_n converges.*

Theorem 1.2 The Stern-Stolz Theorem. *If Z_n converges then $\sum_n |b_n|$ diverges.*

Theorem 1.3 Van Vleck's Theorem. *Suppose that $0 \leqslant \theta < \pi/2$ and that $|\arg(b_n)| \leqslant \theta$ whenever $b_n \neq 0$. Then the sequences Z_{2n} and Z_{2n+1} converge. Further, Z_n converges if and only if $\sum_n |b_n|$ diverges.*

The usual proofs of these three classical theorems are based on the sequences b_n, A_n and B_n, where $Z_n = A_n/B_n$, and the three-term recurrence relations for A_n and B_n. Our aim is to unify and generalise these results by giving new, geometric, proofs (apart from the elementary monotonicity statement, and subsequent convergence, in Theorem 1.1). First, however, we restate these three results (again, apart from the monotonicity) in a more concise way in the two following theorems.

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Theorem 1.4. *If* $\sup_n\{|\arg(b_n)| : b_n \neq 0\} < \pi/2$ *then the sequences* Z_{2n} *and* Z_{2n+1} *converge.*

Theorem 1.5. (i) *If* Z_n *converges then* $\sum_n |b_n|$ *diverges.*
(ii) *If* $\sum_n |b_n|$ *diverges and* $\sup_n\{|\arg(b_n)| : b_n \neq 0\} < \pi/2$*, then* Z_n *converges.*

Throughout, we work in the extended complex plane $\mathbb{C}_\infty$, the open right half-plane $\mathbb{K}$, which is both $\{x + iy : x > 0\}$ and a model of the hyperbolic plane, and three-dimensional hyperbolic space $\mathbb{H}^3$ (the upper half of $\mathbb{R}^3$). We use χ for the chordal metric, and χ_0 for the metric of uniform convergence on $\mathbb{C}_\infty$, and we use ρ for the hyperbolic metric on $\mathbb{H}^3$. We let $j = (0, 0, 1)$; this is in $\mathbb{H}^3$. These spaces will be discussed in detail later in the paper. We recall that the map $h(z) = 1/z$ is a χ-isometry. For $n = 1, 2 \ldots$, we let $t_n(z) = 1/(b_n + z)$, and T_n be the repeated composition $t_1 \circ \cdots \circ t_n$ or, more briefly, $T_n = t_1 \cdots t_n$. Then $T_n(0) = Z_n$, and $\mathbf{K}(b_n)$ converges to w if and only if $T_n(0) \to w$ as $n \to \infty$. It was suggested in [**2**] that any result on continued fractions should, wherever possible, be replaced by a result on the maps T_n. Such results are stronger, and can usually be proved geometrically (often in all dimensions). Our results are of this type and, throughout, we shall emphasise the maps T_n, and refer to the convergence of the sequence $T_n(0)$ rather than the convergence of $\mathbf{K}(b_n)$ or Z_n. All of the notation introduced in this paragraph will be retained throughout the paper.

First, we give our generalisation of Theorem 1.4.

Theorem 1.6. *Suppose that* $\sup_n\{|\arg(b_n)| : b_n \neq 0\} < \pi/2$*. Then the two sequences* $T_{2n}(0)$ *and* $T_{2n+1}(0)$*, and the infinite series* $\sum_n \chi(T_n(0), T_{n+2}(0))$*, converge. Further, the sequence* $T_n(0)$ *converges if and only if* T_n *converges uniformly on* $\overline{\mathbb{K}}$ *to a constant.*

The fact that the b_n arise naturally in the geometry of the maps t_n, seems to have been overlooked (or, at least, not used), and our second theorem gives a variety of conditions which are equivalent to the convergence of $\sum_n |b_n|$. This result is the link between various points of view, and it enables us to use many different techniques in our proofs.

Theorem 1.7. *Let* b_n*,* t_n*,* T_n *and* h *be as above. Then the following are equivalent:*

(i) $\sum_n |b_n|$ *converges;*

(ii) $\sum_n \chi(b_n, 0)$ *converges;*

(iii) $\sum_n \chi_0(t_n, h)$ *converges;*

(iv) $\sum_n \rho(T_n(j), T_{n+1}(j))$ *converges;*

(v) $\sum_n \chi_0(T_n, T_{n+2})$ *converges;*

(vi) $\sum_n \chi\big(T_n(0), T_{n+2}(0)\big)$ *converges, and the sequences* $T_{2n}(0)$ *and* $T_{2n+1}(0)$ *converge to two distinct values.*

Further, if (i)–(vi) *hold, then there is a Möbius transformation* g *such that* $T_{2n} \to g$*, and* $T_{2n+1} \to gh$*, uniformly on* $\mathbb{C}_\infty$*, as* $n \to \infty$*.*

An algebraic proof of the equivalence of (i) and (vi) occurs in [**6**, Theorem 2.4], but with the Euclidean metric instead of χ, and without using the T_n. Our geometric approach is completely different, and has the advantage that it is valid in higher dimensions. The last statement in Theorem 1.7 has appeared previously in [**9**].

Finally, we introduce two sets and then prove the following substantial generalisation of Van Vleck's Theorem. For each θ in $[0,\pi)$, let

$$W(\theta)=\{z\in\mathbb{C}\setminus\{0\}:|\arg(z)|\leqslant\theta\}\cup\{0\};$$

this is a 'wedge' which is symmetric about $\mathbb{R}^+$. The wedge $W^*(\theta)$ is the reflection of $W(\theta)$ in the imaginary axis.

THEOREM 1.8. *Suppose that* $\theta\in[0,\pi/2)$ *and, for all* n, $b_n\in W(\theta)$. *Then the two sequences* $T_{2n}(0)$ *and* $T_{2n+1}(0)$, *and the infinite series* $\sum_n\chi(T_n(0),T_{n+2}(0))$, *converge. Further,*
(i) *if* $\sum_n|b_n|$ *converges then there is a Möbius map* g *such that* $T_{2n}\to g$ *and* $T_{2n+1}\to gh$ *uniformly on* $\mathbb{C}_\infty$*;*
(ii) *if* $\sum_n|b_n|$ *diverges then* T_n *converges uniformly on* $\overline{\mathbb{K}}$, *and locally uniformly on* $\mathbb{C}\backslash W^*(\theta)$ *to a constant.*

We claim that Theorems 1.4 and 1.5 are contained in Theorems 1.6, 1.7 and 1.8. First, Theorem 1.6 contains Theorem 1.4, and Theorem 1.8(ii) contains Theorem 1.5(ii). Finally, Theorem 1.7 contains Theorem 1.5(i) for if $T_n(0)$ converges then, by Theorem 1.7, $\sum_n|b_n|$ diverges. In Section 2, we give some facts about Möbius maps and hyperbolic geometry. Then, in Sections 3, 4 and 5, we prove Theorems 1.7, 1.6 and 1.8, respectively.

2. *Möbius maps*

The extended complex plane $\mathbb{C}\cup\{\infty\}$ is denoted by $\mathbb{C}_\infty$, and is equipped with the *chordal metric* χ given by

$$\chi(z,w)=\frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},\quad \chi(z,\infty)=\frac{2}{\sqrt{1+|z|^2}}.$$

This metric is inherited, via stereographic projection, from the Euclidean metric on the unit (or Riemann) sphere in $\mathbb{R}^3$. A Möbius transformation is a map of the form $z\mapsto(az+b)/(cz+d)$, where $ad-bc\neq0$, and this is a conformal map of $\mathbb{C}_\infty$ onto itself. Throughout this paper I denotes the identity Möbius transformation. A direct calculation shows that h, where $h(z)=1/z$, is a chordal isometry. More generally, a Möbius transformation U is a chordal isometry, that is, for all points z and w,

$$\chi\left(U(z),U(w)\right)=\chi(z,w),$$

if and only if U corresponds (under stereographic projection) to a rotation of the Riemann sphere. We say that such a map U is a *unitary transformation.*

The group $\mathcal{M}$ of Möbius transformations is equipped with the supremum metric χ_0 given by

$$\chi_0(f,g)=\sup\{\chi\big(f(z),g(z)\big):z\in\mathbb{C}_\infty\},\quad f,g\in\mathcal{M}.$$

Clearly, $\chi_0(g_n,g)\to0$ if and only if $g_n\to g$ uniformly on $\mathbb{C}_\infty$ with respect to χ and, in this paper, the convergence of Möbius maps always means uniform convergence on $\mathbb{C}_\infty$. The metric χ_0 is right-invariant: for all Möbius f_1, f_2 and g,

$$\chi_0(f_1g,f_2g)=\chi_0(f_1,f_2).$$

If U is a unitary transformation then, for all Möbius f_1 and f_2,

$$\chi_0(Uf_1,Uf_2)=\chi_0(f_1,f_2).$$

This does not hold for a general Möbius transformation U, but each Möbius transformation is a Lipschitz map of $\mathbb{C}_\infty$ onto itself with respect to χ. Explicitly, if

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1, \quad \|g\| = \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}, \tag{2.1}$$

then, for all z and w,

$$\chi\big(g(z), g(w)\big) \leqslant \|g\|^2 \chi(z, w),$$

(see [**2**, Section 3]). This implies that, for all Möbius f_1, f_2 and g,

$$\chi_0(gf_1, gf_2) \leqslant \|g\|^2\, \chi_0(f_1, f_2),$$

and also, as $\|g^{-1}\| = \|g\|$, that

$$\chi_0(f_1, f_2) = \chi_0(g^{-1}gf_1, g^{-1}gf_2) \leqslant \|g\|^2\, \chi_0(gf_1, gf_2).$$

Thus

$$\frac{1}{\|g\|^2}\, \chi_0(f_1, f_2) \leqslant \chi_0(gf_1, gf_2) \leqslant \|g\|^2\, \chi_0(f_1, f_2).$$

It is easy to see that the group $\mathcal{M}$ of Möbius transformations is, with respect to χ_0, both a complete metric space and a topological group (for more details see, for example, [**2**] and [**1**, Section 3.7]). As the uniform limit of a sequence of Möbius transformations is a Möbius transformation, a χ_0-Cauchy sequence of Möbius transformations converges uniformly on $\mathbb{C}_\infty$ to a Möbius transformation.

Three-dimensional hyperbolic space is the set

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$$

equipped with the hyperbolic metric ρ derived from the line element $ds = |dx|/x_3$. Each Möbius map extends to a map of $\mathbb{R}^3 \cup \{\infty\}$ onto itself in such a way that it leaves $\mathbb{H}^3$ invariant and is an isometry with respect to ρ (see, for example, [**1**, Section 3] and [**2**]). If g is given by (2.1), then we have the basic formula

$$\|g\|^2 = 2\cosh \rho\big(j, g(j)\big), \tag{2.2}$$

where $j = (0, 0, 1)$ (see [**1**, Theorem 4.2.1]). In particular, $\|g\|^2 \geqslant 2$ with equality if and only if g fixes j.

3. *The proof of Theorem 1.7*

We begin our proof of Theorem 1.7 with the following lemma.

LEMMA 3.1. *Let $t_n(z) = 1/(z + b_n)$ and $h(z) = 1/z$. Then*

$$|b_n| = 2\sinh \tfrac{1}{2}\rho\,(j, t_n(j)) = 2\sinh \tfrac{1}{2}\rho\big(T_{n-1}(j), T_n(j)\big); \tag{3.1}$$

$$\chi(b_n, 0) = \frac{2|b_n|}{\sqrt{1 + |b_n|^2}}; \tag{3.2}$$

$$\chi_0(t, h) = \begin{cases} 2 & \text{if } |b_n| \geqslant 2, \\ \frac{8|b_n|}{4+|b_n|^2} & \text{if } |b_n| \leqslant 2. \end{cases} \tag{3.3}$$

In particular, (i), (ii), (iii) and (iv) of Theorem 1.4 are equivalent to each other.

Proof. Equation (3.1) follows from (2.2), and equation (3.2) is immediate. As h is a chordal isometry we see that $\chi\big(t(z), h(z)\big) = \chi(z + b_n, z)$, and (3.3) is obtained by finding the minimum of $(1+|z+b_n|^2)(1+|z|^2)$ over $\mathbb{C}_\infty$. This minimum occurs at $z=-b_n/2$, and we omit the details. □

Proof of Theorem 1.7. Since (i), (ii), (iii) and (iv) are equivalent to each other, it remains to add (v) and (vi) to this list. We shall show that (iii) implies (v), (v) implies (vi), and (vi) implies (i).

Suppose that (iii) holds. Then, as $h^2=I$, and h is a chordal isometry, the properties of χ_0 show that

$$\begin{aligned}\chi_0(T_n^{-1},T_{n+2}^{-1}) &= \chi_0(T_n^{-1}T_{n+2},T_{n+2}^{-1}T_{n+2})\\ &= \chi_0(t_{n+1}t_{n+2},I)\\ &\leqslant \chi_0(t_{n+1}t_{n+2},ht_{n+2})+\chi_0(ht_{n+2},I)\\ &= \chi_0(t_{n+1},h)+\chi_0(t_{n+2},h).\end{aligned}$$

Thus, by (iii), the series $\sum_n \chi_0(T_n^{-1},T_{n+2}^{-1})$ converges. This implies that each of the sequences $T_1^{-1},T_3^{-1},\dots$ and $T_2^{-1},T_4^{-1},\dots$ is a Cauchy sequence with respect to χ_0, and so each sequence converges to some Möbius map. We deduce that, for some M,

$$\sup_n \|T_n\| = \sup_n \|T_n^{-1}\| \leqslant M.$$

Finally,

$$\chi_0(T_n,T_{n+2}) = \chi_0(I,T_{n+2}T_n^{-1}) \leqslant \|T_{n+2}\|^2\chi_0(T_{n+2}^{-1},T_n^{-1}) \leqslant M^2\chi_0(T_{n+2}^{-1},T_n^{-1}),$$

so that (v) holds. Thus (iii) implies (v).

Next, we show that (v) implies (vi), so suppose that (v) holds. Clearly, the convergence of the series in (v) implies the convergence of the series in (vi). Moreover, it also implies that $T_1,T_3,T_5,\dots$ is a Cauchy sequence, so that $T_{2n+1}\to g$, say. Therefore $T_{2n+1}(0)\to g(0)$ and $T_{2n}(0)=T_{2n+1}(\infty)\to g(\infty)\neq g(0)$. Thus (vi) holds.

It remains to show that (vi) implies (i), and for this we need the following lemma.

LEMMA 3.2. *Let $z_0,z_1,\dots$ be a sequence of complex numbers such that $\sum_{n\geqslant 1}|1-z_{n-1}z_n|$ converges. Then there is a positive number k such that, for all sufficiently large n, $|z_n|>k$.*

Proof. Since the given series converges we see that $z_n=0$ for only a finite set of n. Thus we may assume that $z_n\neq 0$ for all n. Standard results on infinite products now show that as the series

$$|1-z_0z_1|+|1-z_2z_3|+|1-z_4z_5|\cdots,\quad |1-z_1z_2|+|1-z_3z_4|+|1-z_5z_6|\cdots$$

converge there are positive numbers α and β such that $|z_1z_2\cdots z_{2n}|\to\alpha$ and $|z_0z_1\cdots z_{2n+1}|\to\beta$ as $n\to\infty$. This shows that $|z_{2n}|\to|\alpha z_0|/|\beta|$ and $|z_{2n+1}|\to|\beta|/|\alpha z_0|$ as $n\to\infty$. □

Suppose that (vi) holds. By adjusting the complex number b_1 if necessary we can assume that each of the sequences $T_{2n-1}(0)$ and $T_{2n}(0)$ consists only of complex numbers (not ∞), and converges to a complex number. Since (a) chordal and Euclidean metrics are equivalent within compact subsets of the plane, (b) $\sum_n |T_n(0)-T_{n-2}(0)|$ converges, and (c) $T_{2n-1}(0)$ and $T_{2n}(0)$ converge to distinct limits, we see that

$$\sum_n \frac{|T_n(0)-T_{n-2}(0)|}{|T_n(0)-T_{n-1}(0)|} \tag{3.4}$$

also converges.

We now need to use the *absolute cross-ratio*

$$|a,b,c,d| = \frac{\chi(a,b)\chi(c,d)}{\chi(a,c)\chi(b,d)}.$$

of distinct points a, b, c and d of $\mathbb{C}_\infty$. Observe that

$$|a,b,c,d| = \left|\frac{(a-b)(c-d)}{(a-c)(b-d)}\right|$$

when a, b, c and d are distinct from ∞. As cross-ratios are invariant under Möbius maps,

$$\left|t_n(0),\infty,0,T_{n-1}^{-1}(\infty)\right| = |T_n(0),T_{n-1}(\infty),T_{n-1}(0),\infty|,$$

and as $T_{n-1}(\infty) = T_{n-2}(0)$, this shows that

$$\left(\frac{T_n(0)-T_{n-2}(0)}{T_n(0)-T_{n-1}(0)}\right) = b_n\, T_{n-1}^{-1}(\infty). \tag{3.5}$$

Next, as $T_{n-1}^{-1}(\infty) = t_n\left(T_n^{-1}(\infty)\right)$, and $t_n(z) = 1/(b_n+z)$, we see that

$$b_n\, T_{n-1}^{-1}(\infty) = 1 - T_{n-1}^{-1}(\infty)T_n^{-1}(\infty). \tag{3.6}$$

Let $z_n = T_n^{-1}(\infty)$. Then, from (3.4), (3.5) and (3.6) we see that $\sum_n |1-z_{n-1}z_n|$ converges. Hence, by Lemma 3.2, there is a positive constant k such that $|z_n| > k$ for all sufficiently large n. It now follows from (3.4) and (3.5) that $\sum_n |b_n|$ converges.

The last part of Theorem 1.7 follows easily. Given (iii), we see that $t_n \to h$. Also, as (v) holds, $T_2, T_4, \ldots$ is a χ_0-Cauchy sequence and so converges to some Möbius g. Finally, $T_{2n+1} = T_{2n}t_{2n+1} \to gh$. □

4. *The proof of Theorem 1.6*

Theorem 1.6 is directly comparable with Theorem 1.4. However, it is convenient to first state Theorem 1.6 in a conjugation invariant fashion using an open disc D in $\mathbb{C}_\infty$ (with hyperbolic metric ρ_D) instead of $\mathbb{K}$. To achieve this we map $\mathbb{K}$ onto D by a Möbius map which takes 0 to a, ∞ to b, and $\mathbb{R}^+$ (the positive real axis) to a hyperbolic geodesic γ in D. Now, the maps t_n in Theorem 1.6 map $\mathbb{K}$ into itself (since if $b \in \mathbb{K}\cup\{0\}$ then both $z \mapsto z+b$ and $z \mapsto 1/z$ map $\mathbb{D}$ into itself). Moreover, the hyperbolic interpretation (and the true significance) of the condition $\sup_n\{|\arg b_n| : b_n \neq 0\} < \pi/2$ is that the non-zero b_n are at bounded hyperbolic distance from the hyperbolic geodesic $\mathbb{R}^+$, and hence so too are the points $t_n(0)$. With this in mind, we transform Theorem 1.6 into the following equivalent statement *in which we continue to use the notation t_n and T_n for the transformed maps*.

Theorem 4.1. *Let γ be a hyperbolic geodesic in an open disc D joining a and b on ∂D. Suppose that for some positive number k the Möbius self-maps $t_1, t_2, \ldots$ of D satisfy (i) $t_n(b) = a$ and (ii) either $t_n(a) = b$, or $t_n(a) \in \{z \in D : \rho_D(z,\gamma) \leqslant k\}$. Let $T_n = t_1 \cdots t_n$. Then the infinite series $\sum_n \chi(T_n(a), T_{n+2}(a))$, and the two sequences $T_{2n}(a)$ and $T_{2n+1}(a)$, converge. Further, the sequence $T_n(a)$ converges if and only if the sequence T_n converges uniformly on $\overline{D}$ to a constant.*

We stress that this result is equivalent to Theorem 1.6, for the convergence of the series is preserved when we change between the two models precisely because any Möbius map is a Lipschitz map of $\mathbb{C}_\infty$ onto itself with respect to the chordal metric χ. Our task, then, is to prove Theorem 4.1 and to do this we shall use the following three results on hyperbolic geometry. In all three results, V is a Euclidean disc equipped with the hyperbolic metric ρ_V, and γ is a geodesic in V with end-points p and q on ∂V.

LEMMA 4.2. *Suppose that $u \in V$, and that the hyperbolic geodesic from p that passes through u ends at v. Then $\cosh \rho_V(u, \gamma) = |u, q, v, p|$.*

Proof. It is sufficient to prove this when V, p and q are replaced by $\mathbb{H}$, ∞ and 0, respectively. After this replacement, γ becomes the imaginary axis, and u becomes z, say where $z = x + iy$ is in $\mathbb{H}$, and $v = x$. Since $\cosh \rho(z, \gamma) = |z|/y = |z, 0, x, \infty|$ [**1**, p. 145], the result follows. □

LEMMA 4.3. *Suppose that the two points u_1 and u_2 in V lie on a horocycle in V based at p. Then $\cosh \rho_V(u_1, \gamma) = \cosh \rho_V(u_2, \gamma)|u_1, q, p, u_2|$.*

Proof. Again, we replace V by $\mathbb{H}$, and p, q, u_1 and u_2 by ∞, 0, z_1 and z_2, respectively. Then z_1 and z_2 have the same imaginary part, say y_0, and

$$\frac{\cosh \rho(z_1, \gamma)}{|z_1|} = \frac{1}{y_0} = \frac{\cosh \rho(z_2, \gamma)}{|z_2|}.$$

Thus $\cosh \rho(z_1, \gamma)/\cosh \rho(z_2, \gamma) = |z_1|/|z_2| = |z_1, 0, \infty, z_2|$ as required. □

LEMMA 4.4. *Suppose that z is a point on a horocycle in V based at p, and let the Euclidean radii of V and the horocycle be R and r, respectively. Then*

$$\cosh \rho_V(z, \gamma) = \frac{2|z - q|rR}{|z - p||q - p|(R - r)} \geqslant \frac{|z - q|}{2(R - r)}. \tag{4.1}$$

Proof. Let the Euclidean diameter of V with end-point p meet ∂V at v, and the horocycle at u. Then, from Lemmas 4.2 and 4.3, we see that

$$\begin{aligned}\cosh \rho_V(z, \gamma) &= \cosh \rho_V(u, \gamma)\,|z, q, p, u| \\ &= |u, q, v, p|\,|z, q, p, u| \\ &= \frac{2|z - q|rR}{|z - p||q - p|(R - r)} \\ &\geqslant \frac{|z - q|}{2(R - r)},\end{aligned}$$

because $|z - p| \leqslant 2r$ and $|q - p| \leqslant 2R$. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. By conjugating we can assume that D is the unit disc $\mathbb{D}$. Let T_0 be the identity map and, for $n = 0, 1, 2, \ldots$, let

$$D_n = T_n(\mathbb{D}), \quad a_n = T_n(a), \quad b_n = T_n(b), \quad \gamma_n = T_n(\gamma).$$

We also let r_n be the Euclidean radius, and ρ_n the hyperbolic metric, of the disc D_n. It is clear that $D_0, D_1, D_2, \ldots$ is a nested decreasing sequence of discs, and D_n is internally tangent to D_{n-1} at a_{n-1}, because $a_{n-1} = T_{n-1}(a) = T_n(b) = b_n$. If we note that the disc $t_n(\mathbb{D})$ is internally tangent to $\mathbb{D}$ at the point b, and then apply the map T_{n-1} to this configuration, we obtain the situation illustrated in Figure 1. By condition (ii) we know that either $t_n(a) \in \mathbb{D}$ or $t_n(a) = b$. Suppose for the moment that $t_n(a) \in \mathbb{D}$. As T_{n-1} is a (hyperbolic) isometry from (D_0, ρ_0) to (D_{n-1}, ρ_{n-1}), we see that

$$\rho_{n-1}(a_n, \gamma_{n-1}) = \rho_0\big(t_n(a), \gamma\big) \leqslant k. \tag{4.2}$$

Since ∂D_n is a horocycle in the hyperbolic plane D_{n-1} that is based at b_n, we can apply Lemma 4.4 with $z = a_n = b_{n+1}$, $p = a_{n-1} = b_n$ and $q = a_{n-2} = b_{n-1}$ to deduce that

$$|a_n - a_{n-2}| \leqslant 2(r_{n-1} - r_n)\cosh k. \tag{4.3}$$

When $t_n(a) = b$, we have

$$a_n = T_n(a) = T_{n-1}(b) = T_{n-2}(a) = a_{n-2};$$

thus (4.3) holds whether $t_n(a) \in \mathbb{D}$ or $t_n(a) = b$. If we sum the inequality of (4.3) over n we see that $\sum_n |a_n - a_{n+2}|$ converges. Since $a_n = T_n(a)$, and $\chi(z, w) \leqslant 2|z - w|$ throughout $\mathbb{C}$, we conclude that $\sum_n \chi(T_n(a), T_{n+2}(a))$ converges.

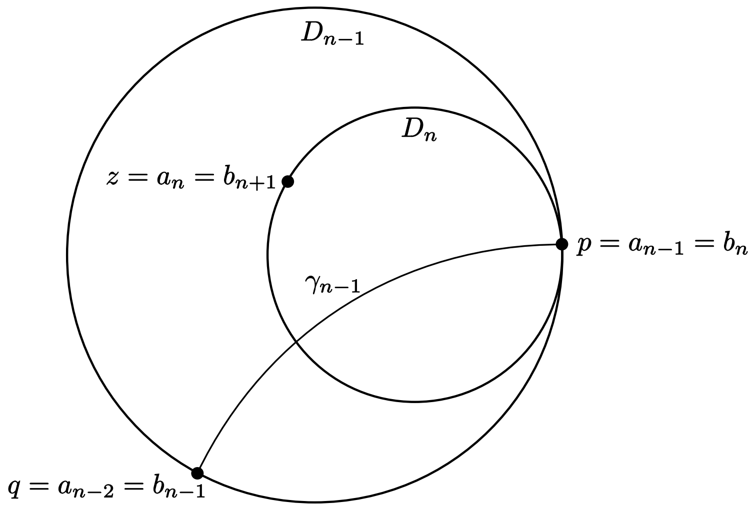

Figure 1.

It remains to show that $T_n(a)$ converges if and only if T_n converges uniformly on $\overline{\mathbb{D}}$ to a constant or, equivalently, if and only if the nested sequence D_n of discs decreases to a single point. Certainly, if these discs decrease to a point then the sequence $T_n(a)$ converges. We prove the reverse implication in the contrapositive form: if the discs decrease to a disc of positive radius then the sequence $T_n(a)$ does not converge. We assume, then, that the discs D_n decrease to a limiting disc of positive radius R, say. Thus $r_n \geqslant R$ for all n.

A direct application of Lemma 4.4 to the configuration in Figure 1 gives

$$\frac{|a_n - a_{n-2}|}{|a_n - a_{n-1}|} = \frac{(r_{n-1} - r_n)|a_{n-1} - a_{n-2}| \cosh \rho_{n-1}\big(a_n, \gamma_{n-1}\big)}{2r_{n-1}r_n}.$$

(Lemma 4.4 only applies when $a_n \in D_{n-1}$, or, equivalently, when $a_n \neq a_{n-2}$; however, equality holds trivially when $a_n = a_{n-2}$ since then $r_n = r_{n-1}$ and both sides of the equation are zero.) This with (4.2), and the fact that each $a_n \in \overline{\mathbb{D}}$ (so that $|a_n| \leqslant 1$), gives

$$\frac{|a_n - a_{n-2}|}{|a_n - a_{n-1}|} \leqslant \frac{\cosh k}{R^2}(r_{n-1} - r_n).$$

We deduce that

$$\sum_n \frac{|a_n - a_{n-2}|}{|a_n - a_{n-1}|} < +\infty.$$

This means that the product

$$\prod_n \left(1 + \frac{|a_n - a_{n-2}|}{|a_n - a_{n-1}|}\right) \tag{4.4}$$

converges to a positive value, and so is bounded above by M, say. Now,

$$\begin{aligned} 1 + \frac{|a_n - a_{n-2}|}{|a_n - a_{n-1}|} &= \frac{|a_n - a_{n-1}| + |a_n - a_{n-2}|}{|a_n - a_{n-1}|} \\ &\geqslant \frac{|a_{n-1} - a_{n-2}|}{|a_n - a_{n-1}|}. \end{aligned}$$

We deduce from (4.4) that for all N we have

$$\frac{|a_1 - a_0|}{|a_N - a_{N-1}|} = \prod_{n=2}^{N} \frac{|a_{n-1} - a_{n-2}|}{|a_n - a_{n-1}|} \leqslant M,$$

and this shows that the sequence $|T_n(a) - T_{n-1}(a)|$ is bounded below. Therefore $T_n(a)$ does not converge. □

5. *The proof of Theorem 1.8*

The first assertion in Theorem 1.8 follows directly from Theorem 1.6, and the second assertion (labelled (i)) follows from Theorem 1.7. In order to prove the third assertion (labelled (ii)) we assume that $\sum_n |b_n|$ diverges. First we show that T_n converges uniformly on $\overline{\mathbb{K}}$ to a constant. Given that each b_n lies in $W(\theta)$, we have from Theorem 1.6 that $\sum_n \chi(T_n(0), T_{n+2}(0))$ converges, and the two sequences $T_{2n}(0)$ and $T_{2n+1}(0)$ each converge. We are also given that $\sum_n |b_n|$ diverges, thus by comparing (i) and (vi) in Theorem 1.7 we deduce that $T_{2n}(0)$ and $T_{2n+1}(0)$ converge to the same value. Again we appeal to Theorem 1.6 to conclude that T_n converges uniformly on $\overline{\mathbb{K}}$ to a constant p, and since $T_n(\mathbb{K}) \subseteq \mathbb{K}$, p belongs to $\overline{\mathbb{K}}$.

Next we prove that T_n converges locally uniformly on $\mathbb{C} \setminus W^*(\theta)$ to p. Choose a compact subset X of $\mathbb{C} \setminus W^*(\theta)$. Choose two distinct points a and b in $\mathbb{K}$, and a point w in the interior of $W^*(\theta)$. Since $t_n^{-1}(W^*(\theta)) \subseteq W^*(\theta)$ for each n, we see that $T_n^{-1}(w) \in W^*(\theta)$. We restrict to sufficiently large integers n that $\chi(T_n(a), w) > \frac{1}{2}\chi(p, w)$. For a point x in X we see from the cross-ratio identity

$$|a, b, T_n^{-1}(w), x| = |T_n(a), T_n(b), w, T_n(x)|$$

that

$$\begin{aligned} \chi(T_n(x), T_n(b)) &= \frac{\chi(b, x)\, \chi(a, T_n^{-1}(w))\, \chi(w, T_n(x))}{\chi(a, b)\, \chi(T_n^{-1}(w), x)\, \chi(T_n(a), w)}\, \chi(T_n(a), T_n(b)) \\ &\leqslant \frac{8}{\chi(a, b)\, \chi(W^*(\theta), X)\, \frac{1}{2}\chi(p, w)}\, \chi(T_n(a), T_n(b)). \end{aligned}$$

Hence T_n converges uniformly on X to p.

Finally, we remark that there are examples that show that the sequence T_n of Theorem 1.8 may diverge everywhere on $W^*(\theta) \setminus \{0\}$.

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